# Polynomial equivalence among systems LLNC, $\text{LLNC}_a$ and $\text{LLNC}_0$

François Métayer*

**Abstract**

We investigate three fragments of cyclic linear logic, respectively LLNC containing all propositional variables, $\text{LLNC}_a$ built on a single variable and the constant-only fragment $\text{LLNC}_0$.

By using non-commutative proofnets, we show that the decision problems of these fragments are polynomially equivalent.

Keywords: *linear logic, exchange, proofnet, complexity.*

## 1 Cyclic linear logic

Recall the usual presentation of cyclic linear logic as a sequent calculus: the formulas of LLNC are built on propositional variables $a_1, a_2, \ldots$ and $a_1^\perp, a_2^\perp, \ldots$ with the connectives *tensor* ($\otimes$) and *par* ($\wp$). The linear negation is extended to formulas by

$$u^{\perp\perp} = u \qquad (u \otimes v)^\perp = (u)^\perp \wp (v)^\perp \qquad (u \wp v)^\perp = (u)^\perp \otimes (v)^\perp$$

The rules are as follows [1], where sequents are *sequences* of formulas:

$$\frac{}{x, x^\perp}\,(\text{axiom}) \qquad \frac{\Gamma, u \qquad u^\perp, \Delta}{\Gamma, \Delta}\,(\text{cut})$$

$$\frac{\Gamma, u \qquad v, \Delta}{\Gamma, u \otimes v, \Delta}\,(\text{tensor}) \qquad \frac{\Gamma, u, v}{\Gamma, u \wp v}\,(\text{par})$$

Plus the rule of circular exchange:

$$\frac{\Gamma, u}{u, \Gamma}\,(\sigma)$$

*Équipe de Logique, Université Paris VII–Cnrs, 45-55, 5ème étage, 2 place Jussieu 75251 Paris Cedex 05 France. e-mail: metayer@logique.jussieu.fr & Université Paris X

[1] The prooftrees have been realized with the help of Paul Taylor's macros.

In fact we will be interested in two more fragments of LLNC: the formulas of $\text{LLNC}_a$ (resp.$\text{LLNC}_0$) are built on variables $a$ and $a^\perp$ only (resp. $1, \perp (= 1^\perp)$) with the connectives *tensor* ($\otimes$) and *par* ($\wp$). The *atoms* of a formula $u$ are the subformulas of $u$ which are variables (resp. constants). The linear negation is extended as above, and the logical rules are the same, but for $\text{LLNC}_0$ where axiom and weakening look like

$$\frac{}{1}\ (\text{axiom}) \qquad \frac{\Gamma}{\Gamma, \perp}\ (\text{w})$$

We will show that the decision problems of the three fragments are polynomially equivalent.

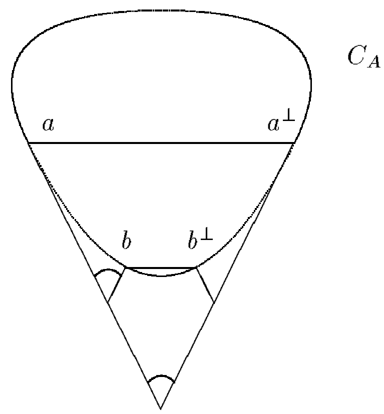


Figure 1: $T_A$

To each formula $A$ we associate a *tree* $T_A$ where the leaves are labeled by the atoms of $A$ and the root by $A$ itself. Each connective is represented by a pair of convergent edges in $T_A$.
We see $T_A$ as embedded in the euclidian plane in such a way that its leaves are on a circle $C_A$, and $T_A$ is exterior to $C_A$ (figure 1). Then, to each pair of leaves we may associate a chord of $C_A$. We know that $A$ is provable if and only if there is a pairing $P$ of the leaves where each atom $x$ is paired with an atom $x^\perp$ and:

- the reunion of $T_A$ and the chords associated to $P$ is a proofnet $\Pi$.
- two distinct chords never intersect.

Of course these proofnets correspond to *cut-free* proofs in sequent calculus. The latter condition is precisely non-commutativity. When such a proofnet $P$ exists, it is of course embedded in the plane, and delimits certain regions on it, exactly one of them unbounded. For each connective *par* we put a mark ($\wp$) in the region which has the *two edges* of this connective on its border (there is exactly one region with this property). Then the following holds:

- Each bounded region contains exactly one mark ($\wp$).

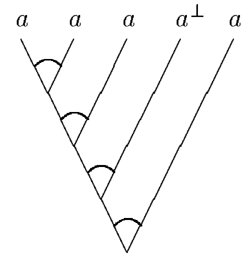


Figure 2: $\phi_3^3$

From now on, we simply call *proof* every proofnet obtained as explained above, and we denote $x \sim x'$ when two leaves $x$ and $x'$ of $T_A$ are related by a chord of $C_A$ in $P$. Figure 1 represents a proof of $A = (a \wp b) \wp (b^\perp \otimes a^\perp)$. We finally recall that a formula $A$ of LLNC is *balanced* when, for each variable $v$, $v$ has the same number of occurences as $v^\perp$ in $A$. As regards LLNC and LLNC$_a$ we may restrict to balanced formulas, since provable formulas are necessarily balanced.

## 2 Equivalence of LLNC and LLNC$a$

We first define a family of formulas $\phi_i^n$ of LLNC$_a$ which help encoding LLNC in LLNC$_a$. For each integer $n$,and each $i \in \{1, \ldots, n\}$, we define $\phi_i^n$ by:

$$\phi_i^n = (\ldots(\ldots(x_1 \wp x_2)\wp \ldots)\wp x_j)\wp \ldots \wp x_{n+2})$$

where $x_j = a$ for $j \neq i+1$ and $x_{i+1} = a^\perp$.
Let $A \in$ LLNC be a balanced formula with variables $a_1, \ldots, a_n, a_1^\perp, \ldots, a_n^\perp$ (we suppose that the variables of all formulas are ordered, once and for all). To $A$ is associated $A^\circ \in$ LLNC$_a$defined by

$$A^\circ = A[\phi_1^n/a_1, \ldots, \phi_n^n/a_n, (\phi_1^n)^\perp/a_1^\perp, \ldots, (\phi_n^n)^\perp/a_n^\perp]$$

$\phi_i^n$ will be simply denoted by $\phi_i$ or even $\phi$ when no confusion occurs. ($\phi_3^3$ is shown on figure 2) The translation $()^\circ$ is clearly sound. But we also have

**Proposition 2.1** $()^\circ$ *is faithful.*

**Proof.** Let $A \in$ LLNC be balanced and consider a proof of $B = A^\circ \in$ LLNC$_a$. The central idea is that the chords of $C_B$ joining the leaves of $T_B$ necessarily join *all* the leaves of a subtree of the form $T_{\phi_i}$ with the leaves of a subtree of the form $T_{\phi_i^\perp}$ (see figure 3). The vertices of $T_B$ split in four classes:

| | $a$ | $a^\perp$ |
|---|---|---|
| $\phi$ | $x$ | $x^\perp$ |
| $\phi^\perp$ | $x_*$ | $x_*^\perp$ |

For instance a leaf of type $x$ corresponds to an atom $a$ in a $T_\phi$. We denote $x(i)$ to point out that this leaf $x$ belongs to a formula $\phi_i$. The only possible configurations are: $x \sim x^\perp$, $x \sim x_*^\perp$, $x_* \sim x^\perp$ and $x_* \sim x_*^\perp$.

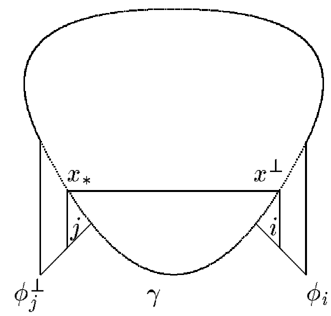


Figure 3: $T_B$

- $x \sim x^\perp$ is impossible: the nodes of $T_B$ which lie immediately under $x$ and $x^\perp$ are *par*-nodes. Then we may chose a switching disconnecting the chord $xx^\perp$, a contradiction.
- As a consequence, every leaf $x$ is related to a leaf $x_*^\perp$ but we have the same number of $x$ and $x_*^\perp$, because $A$ is balanced. This reduces the possible configurations to $x \sim x_*^\perp$ and $x_* \sim x^\perp$.
- We finally show that if $x_*(j) \sim x^\perp(i)$, then $i = j$. We consider on $C_B$ the arc $\gamma = ]x_*(j)x^\perp(i)[$. If it contains all the leaves of $p$ subtrees $T_\phi$ (distinct from $T_{\phi_i}$) it contains $p$ leaves of type $x^\perp$ hence also $p$ leaves of type $x_*$ and therefore also $p$ subtrees $T_{\phi^\perp}$ distinct from $T_{\phi_j^\perp}$. $\gamma$ contains exactly $p(n+1)+i$ leaves of type $x$, and $p(n+1)+j$ leaves of type $x_*^\perp$. But the leaves of type $x$ are in bijection with those of type $x_*^\perp$, such that

$$p(n+1)+i = p(n+1)+j$$

  and clearly $i = j$.
- We now easily construct a proof of $A$. We chose in each tree $T_\phi$ (resp. $T_{\phi^\perp}$) the only switching connecting the root to $x^\perp(i)$ (resp. to $x_*(i)$). The resulting graph is correct, and can be transformed into a proof by retracting useless edges.

◇

**Proposition 2.2** *The decision problems of* LLNC *and* $\mathrm{LLNC}_a$ *are polynomially equivalent.*

**Proof.** Clearly every decision procedure for $L$ applies to $\mathrm{LLNC}_a$. Conversely, the translation $()^\circ$ is polynomial: if $l$ is the length of the formula $A$, the length of $A^\circ$ is $O(l^2)$. It is sound, and faithful by 2.1. Hence the result. ◇

## 3 Equivalence of LLNC$a$ and $\mathrm{LLNC}_0$

We first define a translation $()^*$ of $\mathrm{LLNC}_0$ in $\mathrm{LLNC}_a$ by

$$A^* = A[a \wp a^\perp/1, a \otimes a^\perp/\perp]$$

Likewise, for each sequent $\Gamma = A_1, \ldots, A_n$ we define $\Gamma^* = A_1^*, \ldots, A_n^*$. We denote by $\vdash$ the provability in $\mathrm{LLNC}_0$ and $\vdash_a$ the provability in $\mathrm{LLNC}_a$.

**Lemma 3.1** $()^*$ *is sound.*

**Proof.** By induction of the height of a cut-free proof of $\Gamma$ in $\mathrm{LLNC}_0$.
We first notice that weakening commutes with the rules for *par* and *tensor*. It will be convenient to see successive applications of circular exchange as a single rule:

$$\frac{\Gamma, \Delta}{\Delta, \Gamma}(\epsilon)$$

Suppose then that a proof ends like

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \Delta, u, v}}{\Gamma, \Delta, u \wp v}(\text{par})}{\Delta, u \wp v, \Gamma}(\epsilon)}{\Delta, u \wp v, \Gamma, \perp}(\text{w})$$

The same endsequent is proved by:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \Delta, u, v}}{\Delta, u, v, \Gamma}(\epsilon)}{\Delta, u, v, \Gamma, \perp}(\text{w})}{\Gamma, \perp, \Delta, u, v}(\epsilon)}{\Gamma, \perp, \Delta, u \wp v}(\text{par})}{\Delta, u \wp v, \Gamma, \perp}(\epsilon)$$

Likewise, if a proof ends like:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \Delta, u} \quad \dfrac{\vdots}{v, \Lambda}}{\Gamma, \Delta, u \otimes v, \Lambda}(\text{tensor})}{\Delta, u \otimes v, \Lambda, \Gamma}(\epsilon)}{\Delta, u \otimes v, \Lambda, \Gamma, \perp}(\text{w})$$

The same endsequent is proved by:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \Delta, u}}{\Delta, u, \Gamma}(\epsilon)}{\Delta, u, \Gamma, \bot}(\mathrm{w})}{\Gamma, \bot, \Delta, u}(\epsilon) \quad \dfrac{\vdots}{v, \Lambda}}{\dfrac{\Gamma, \bot, \Delta, u \otimes v, \Lambda}{\Delta, u \otimes v, \Lambda, \Gamma, \bot}(\epsilon)}(\mathrm{tensor})$$

Of course there is a symmetrical case where the weakening rule has to be performed on the branch containing $v$.
Thus we may suppose that the weakenings come before the logical rules, which amounts to suppose that the axioms are

$$\vdash 1, \bot, \ldots, \bot$$

and that the only rules are *tensor*, *par* and *$\epsilon$-exchange*. It is now easy to construct a proof of

$$1^*, \bot^*, \ldots, \bot^* = a \wp a^\bot, a \otimes a^\bot, \ldots, a \otimes a^\bot$$

which settles out the axiom case. The other rules are straightforward. ⋄

On the other hand,

**Lemma 3.2** *$()^*$ is faithful.*

**Proof.** Let $A \in \mathrm{LLNC}_0$ such that $\vdash_a A^*$. By substituting 1 for $a$ and $\bot$ for $a^\bot$ in $A^*$, we obtain a new formula $(A^*)'$ of $\mathrm{LLNC}_0$, clearly provable. We verify that $B = (A^*)'$ is equivalent to $A$, hence the result. ⋄

The decision problem for $\mathrm{LLNC}_0$ now reduces polynomially to the corresponding problem in $\mathrm{LLNC}_a$. To prove the converse, we examine a certain class $\mathcal{C}$ of formulas in $\mathrm{LLNC}_a$. We define $u = (a \wp a^\bot) \wp (a \wp a^\bot)$ and call $\mathcal{C}$ the set of formulas of $\mathrm{LLNC}_a$ of the form

$$B = A[u/a, u^\bot/a^\bot]$$

for any balanced formula $A$ of $\mathrm{LLNC}_a$. If $B \in \mathcal{C}$, the leaves of $T_B$ split in groups of four, according to the subformulas $u$ and $u^\bot$ where they belong. We call $X_1, \ldots, X_i, \ldots$ (resp. $X_1^\bot, \ldots, X_j^\bot, \ldots$) the groups corresponding to subtrees $T_u$ (resp. $T_{u^\bot}$). Let

$$X_i = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\}$$

and

$$X_j^\bot = \{x_{j1}^\bot, x_{j2}^\bot, x_{j3}^\bot, x_{j4}^\bot\}$$

We suppose also that, when traveling clockwise on $C_B$, we encounter the $x_{ik}$'s in the order $(1, 2, 3, 4)$ and the $x_{jk}^\bot$'s in reverse order $(4, 3, 2, 1)$. We denote

$$X_i \sim X_j^\bot$$

when *for each* $k \in \{1, 2, 3, 4\}$, $x_{ik} \sim x_{jk}^\bot$. It is now possible to prove

**Lemma 3.3** *Consider a proof of $B \in \mathcal{C}$. Then, for all $i$, there is a $j$ such that $X_i \sim X_j^\perp$.*

**Proof.** The proof amounts to show that certain configurations of chords are forbidden in a proof of $B$.

- We never have $x_{ik} \sim x_{i'k'}$ because otherwise we would have a switching disconnecting the graph. As a consequence, every $x_{ik}$ is related to a $x_{jl}^\perp$, and conversely since $A$ is balanced.
- Consider a chord $x_{i1}x_{jl}^\perp$, where $l \in \{1,2,3,4\}$ and let $\gamma$ be the one of the two arcs $]x_{i1}x_{jl}^\perp[$ of $C_B$ not containing the $x_{ik}$'s. The leaves on $\gamma$ split in: $m$ groups of type $X$, $n$ groups of type $X^\perp$, plus $x_{j1}^\perp \dots x_{j(l-1)}^\perp$. Thus it contains $4m$ leaves of type $x$ and $4n+l-1$ leaves of type $x^\perp$. As chords do not intersect, we must have $4m = 4n+l-1$, hence $l=1$.
- The same argument shows that if $x_{ik} \sim x_{jl}^\perp$, then $k = l$.
- Suppose now that $x_{i1} \sim x_{j1}^\perp$ and that $x_{j2}^\perp \sim x_{i'2}$ with $i \neq i'$. The region $R$ having both chords $x_{i1}x_{j1}^\perp$ and $x_{j2}^\perp x_{i'2}$ on its border would contain two marks $\wp$: contradiction (see fig.4). Therefore $x_{i1} \sim x_{j1}^\perp$ and $x_{j2}^\perp \sim x_{i2}$. The same argument shows that $x_{i3} \sim x_{j3}^\perp$ and $x_{i4} \sim x_{j4}^\perp$.

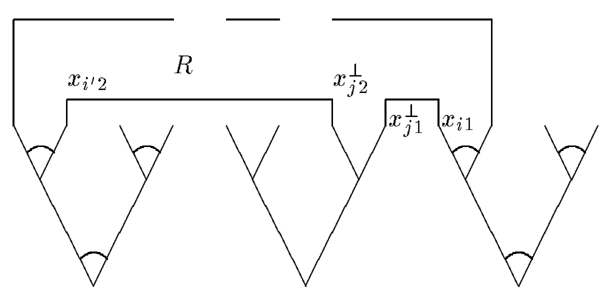


Figure 4: region $R$

We deduce, keeping the notations of (1):

**Lemma 3.4** *If $\vdash_a B$ then $\vdash_a A$.*

**Proof.** By the result 3.3 we may transform a proof of $B$ into a proof of $A$, by collapsing each subtree $T_u^i$ (resp. $T_{u^\perp}^j$) on its root $s_i$ (resp. $s_j^\perp$) and by drawing the chord $s_i s_j^\perp$ if and only if $X_i \sim X_j^\perp$. $\diamond$

Let us translate each formula $A$ of LLNC$_a$ into the formula $A^\dagger$ of LLNC$_0$ defined by

$$A^\dagger = A[1\wp 1/a, \bot \otimes \bot / a^\perp]$$

We show again that

**Lemma 3.5** $A \longrightarrow A^\dagger$ *is sound and faithful.*

**Proof.** Soundness is clear. Suppose conversely that $A$ is a balanced formula of $\text{LLNC}_a$ such that $\vdash A^\dagger$. $B = (A^\dagger)^*$ belongs to the class $\mathcal{C}$ and $\vdash_a B$. But $B$ is also $A[u/a, u^\perp/a^\perp]$ and 3.4 shows that $\vdash_a A$. $\diamond$

**Remarks** Of course the exact complexity remains open while in the commutative case, the three corresponding fragments are known to be NP-complete (see [7] and [6]) hence also polynomially equivalent; translations between the single-variable and the constant-only fragment still work in that case. Precisely, 3.4 still holds, but not 3.3 (see [8]). On the other hand, we know no *simple* translation of the complete fragment into the single-variable one in the commutative case.
Notice finally that the labels $a$, $a^\perp$ play no role in the previous arguments, so that the decision problem reduces to a purely geometrical one.

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