String diagrams

a functorial semantics of proofs and programs

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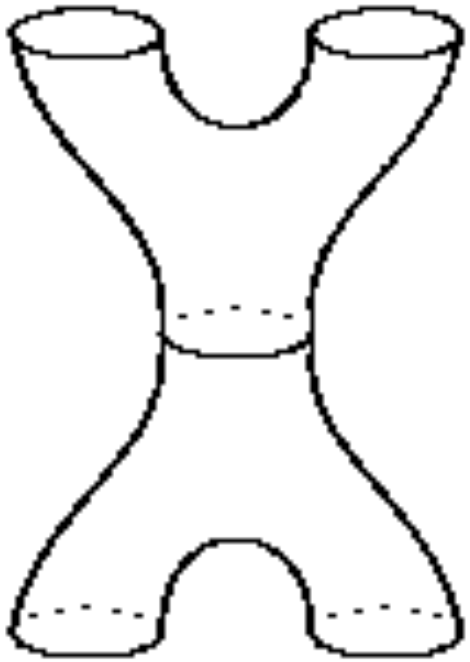
Connecting 2-dimensional cobordism and logic



Part 1

Categories as monads

Starting point: categories as monads

Depending on the bicategory, the point of view federates:

Enriched categories	a bicategory of $\mathcal{V}$-matrices
Internal categories	a bicategory of spans
Algebra	a bicategory of comodules

The last connection appears in Marcelo Aguiar's PhD thesis

The bicategory $\overrightarrow{Cat}(\mathcal{B})$

– the 0**-cells** are the monads (A, s) of the bicategory $\mathcal{B}$

– the 1**-cells**

$$(f, \tilde{f}) \quad : \quad (A, s) \longrightarrow (B, t)$$

are the pairs consisting of a 1-cell

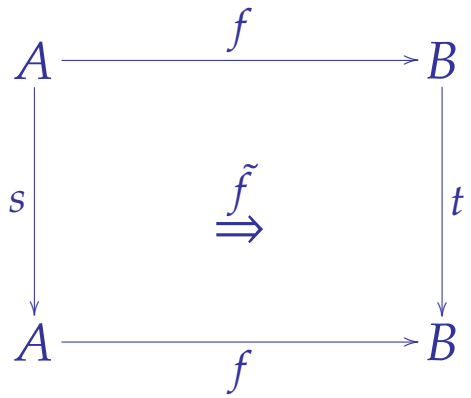
$$f \quad : \quad A \quad \longrightarrow \quad B$$

and of a 2-cell

$$\tilde{f} \quad : \quad f \otimes s \quad \Rightarrow \quad t \otimes f \quad : \quad A \quad \longrightarrow \quad B$$

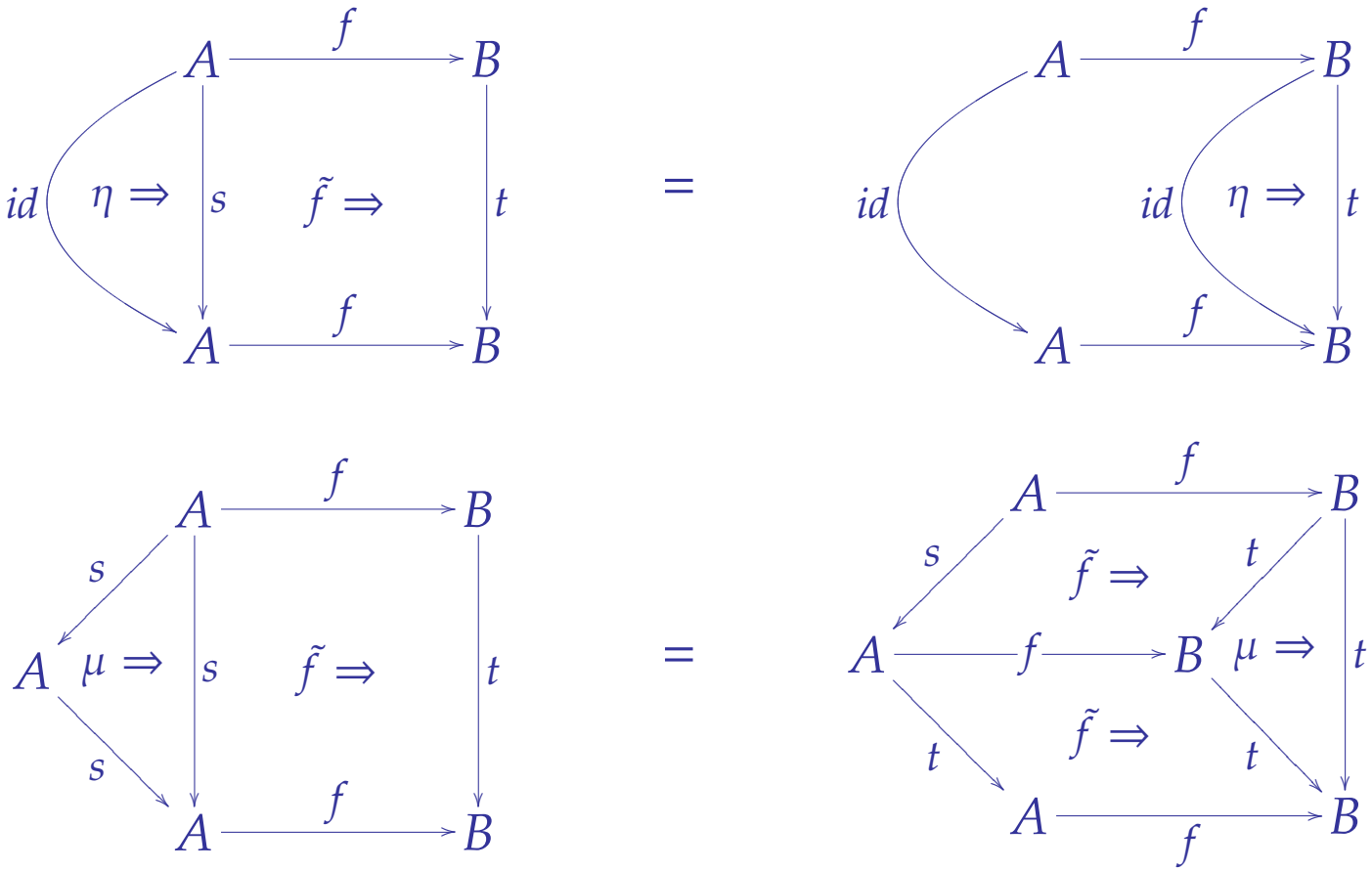
The bicategory $\overrightarrow{Cat}(\mathcal{B})$

Diagrammatically:

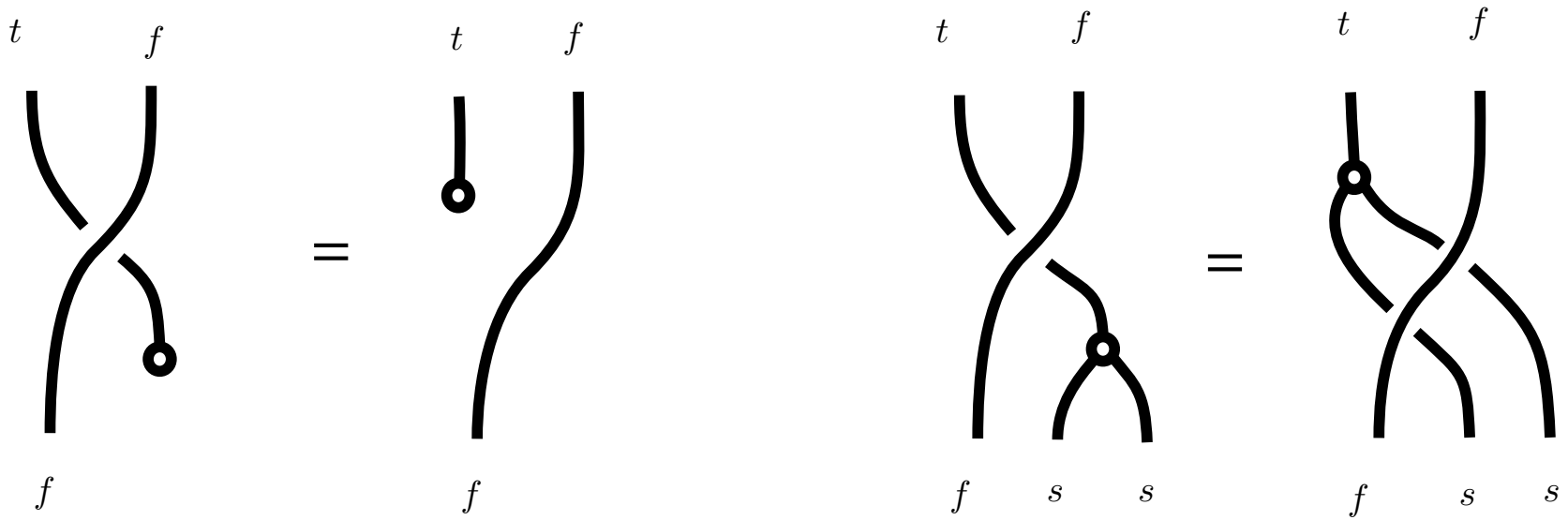


satisfying the equalities below:

The bicategory $\overrightarrow{Cat}(\mathcal{B})$

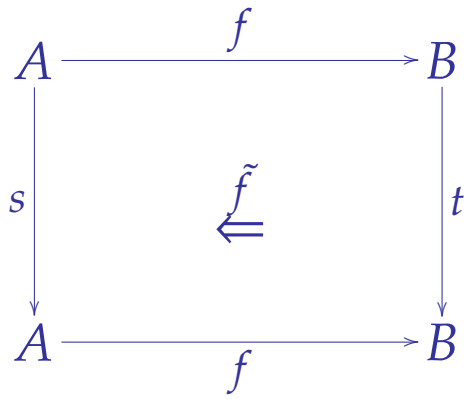


An alternative formulation in string diagrams



The bicategory $\overleftarrow{\mathcal{C}at}(\mathcal{B})$

Same definition, except that the direction of the 2-cell changes:



A kind of "fibration without its functor"

The 2-cells of the bicategory $\overrightarrow{Cat}(\mathcal{B})$

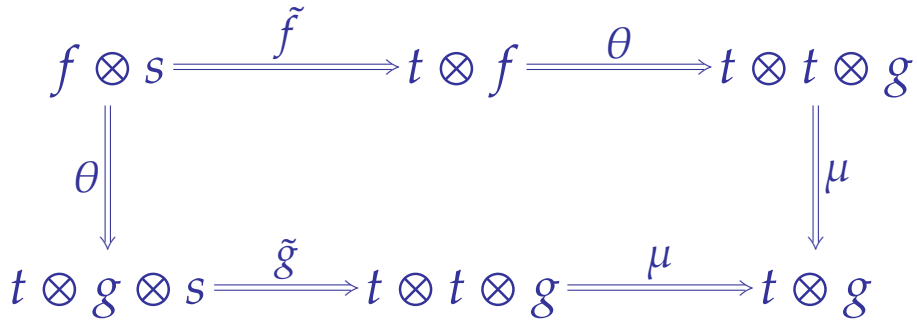
– the 2-**cells**

$$\theta \quad : \quad f \Rightarrow g \quad : \quad s \longrightarrow t$$

are the 2-cells

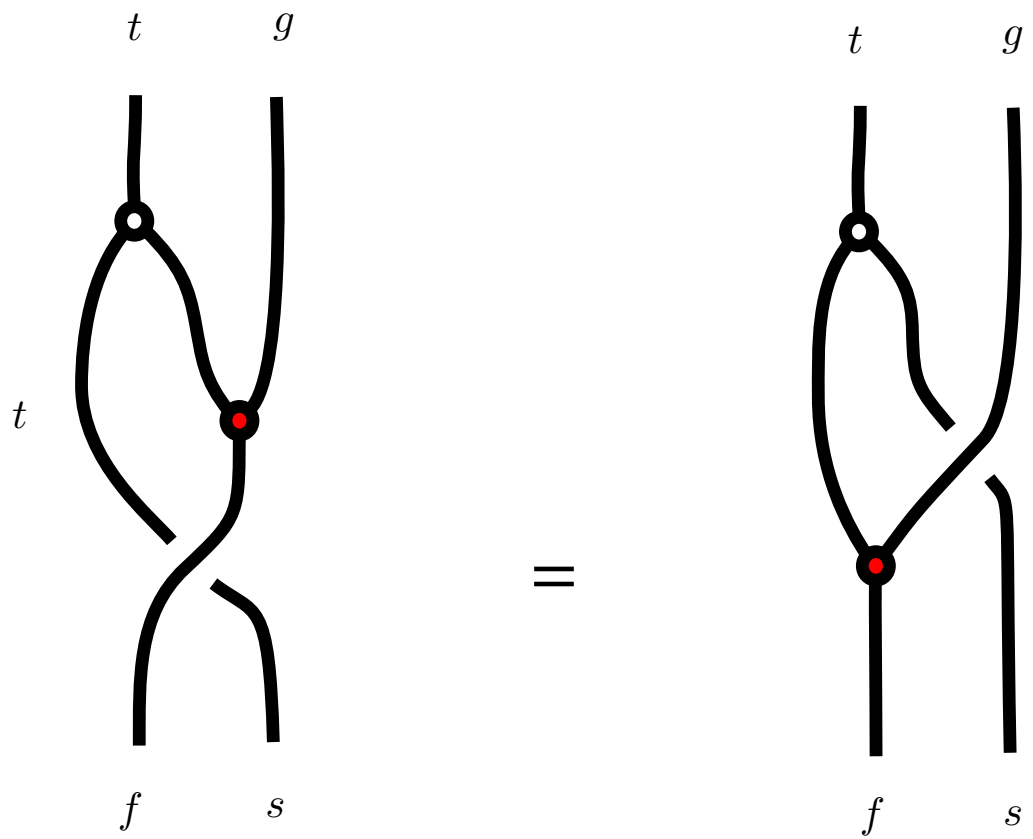
$$\theta \quad : \quad f \Rightarrow t \otimes g \quad : \quad A \longrightarrow B$$

such that the diagram of 2-cells commute:



Reduced form

An alternative formulation in string diagrams



Reduced form

The 2-cells of the bicategory $\overrightarrow{Cat}(\mathcal{B})$

– the 2-**cells**

$$\theta \quad : \quad f \Rightarrow g \quad : \quad s \longrightarrow t$$

are the 2-cells

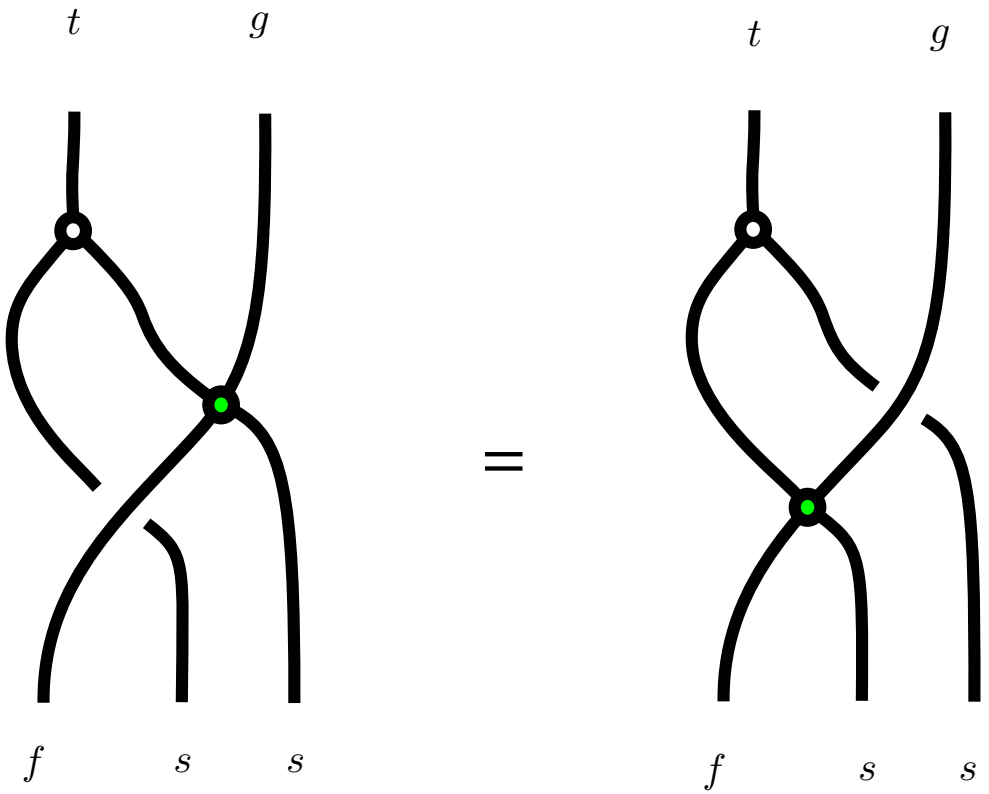
$$\theta \quad : \quad f \otimes s \Rightarrow t \otimes g \quad : \quad A \longrightarrow B$$

making the diagram of 2-cells commute:

$$\begin{array}{ccccc} f \otimes s \otimes s & \xRightarrow{\tilde{f}} & t \otimes f \otimes s & \xRightarrow{\theta} & t \otimes t \otimes g \\ \Big\Downarrow \theta & & & & \Big\Downarrow \mu \\ t \otimes g \otimes s & \xRightarrow{\tilde{g}} & t \otimes t \otimes g & \xRightarrow{\mu} & t \otimes g \end{array}$$

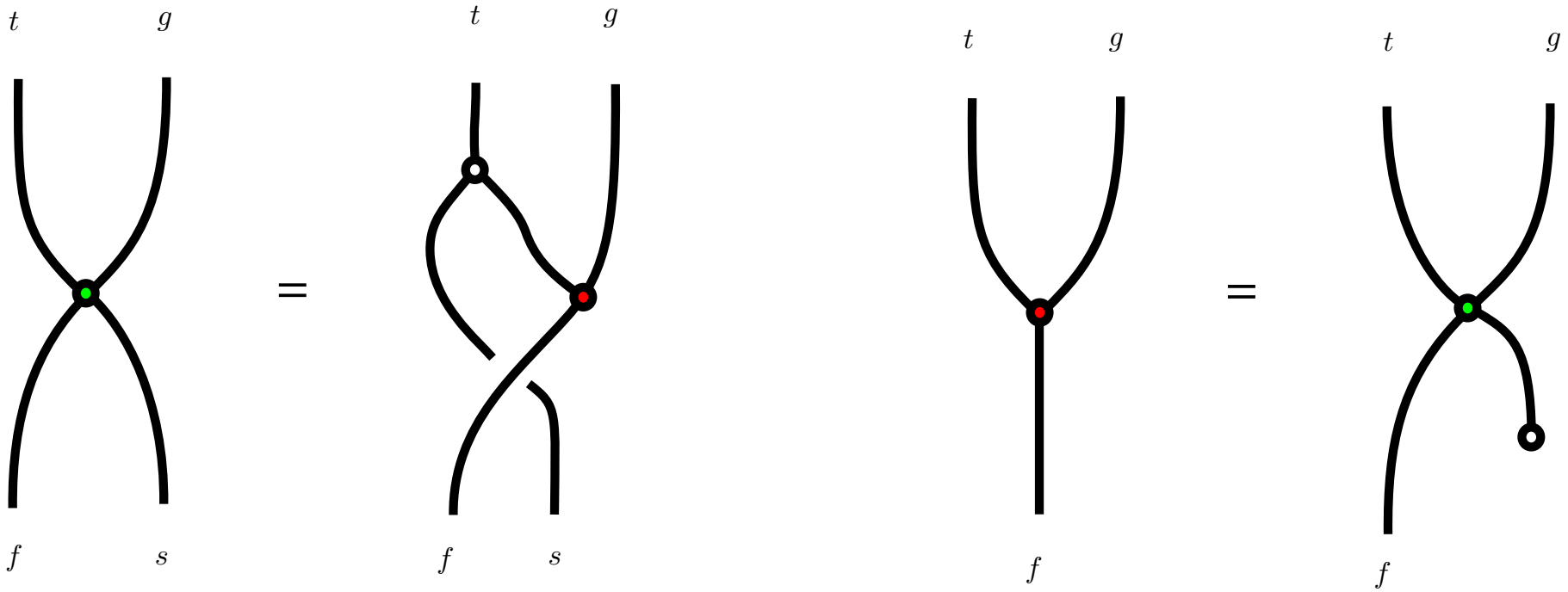
Non reduced form

Alternative formulation in string diagrams

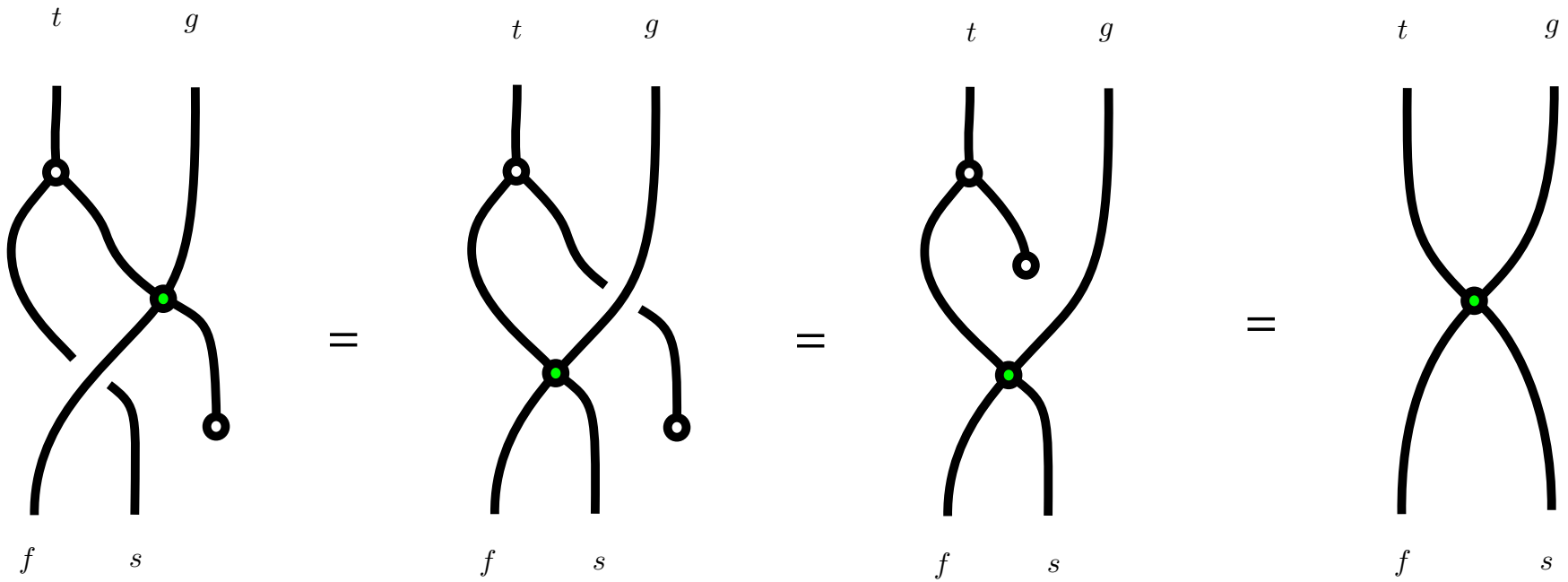


Non reduced form

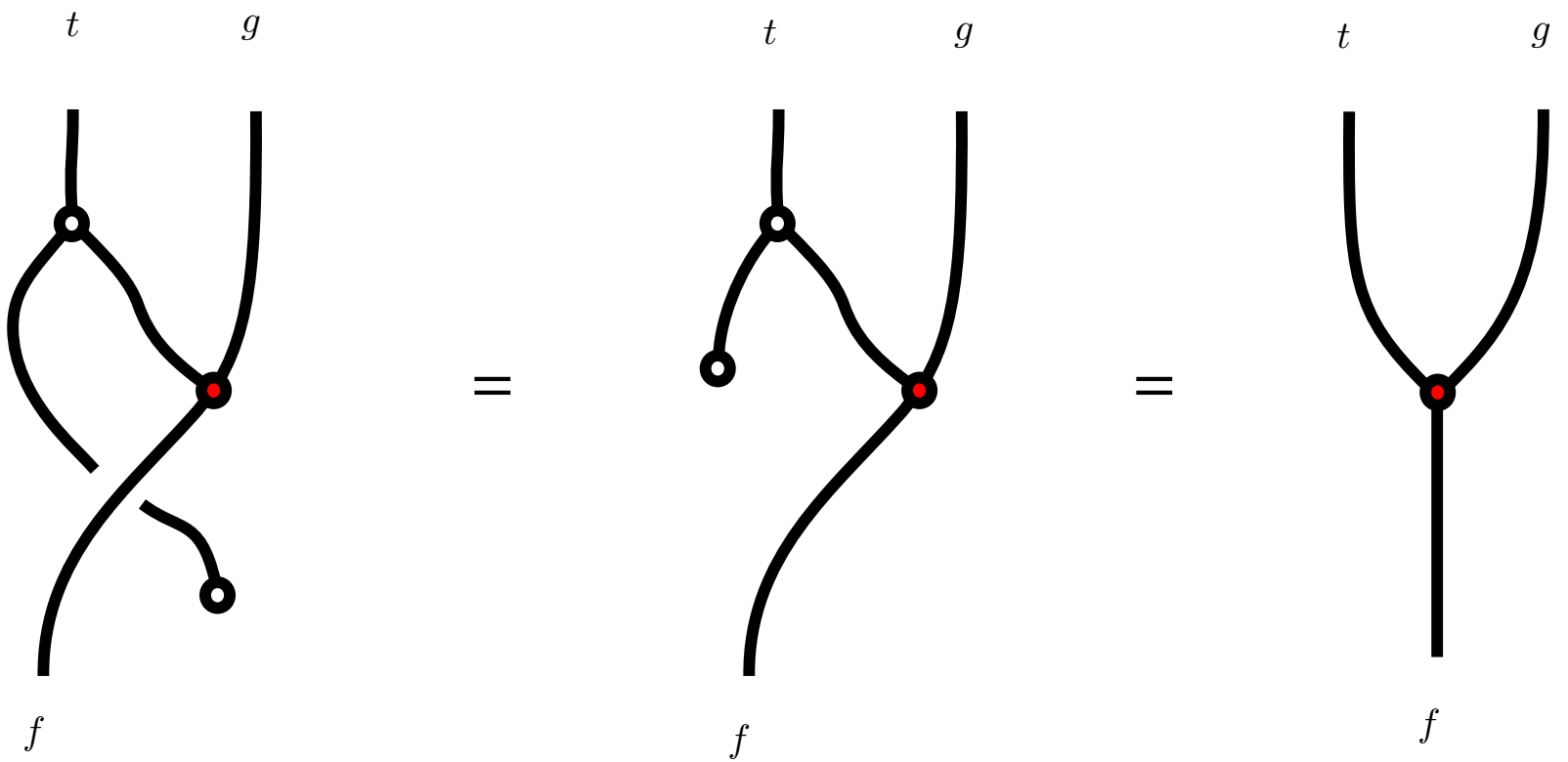
From the reduced form to the non reduced form (and conversely)



First equation



Second equation



Property of the non reduced form

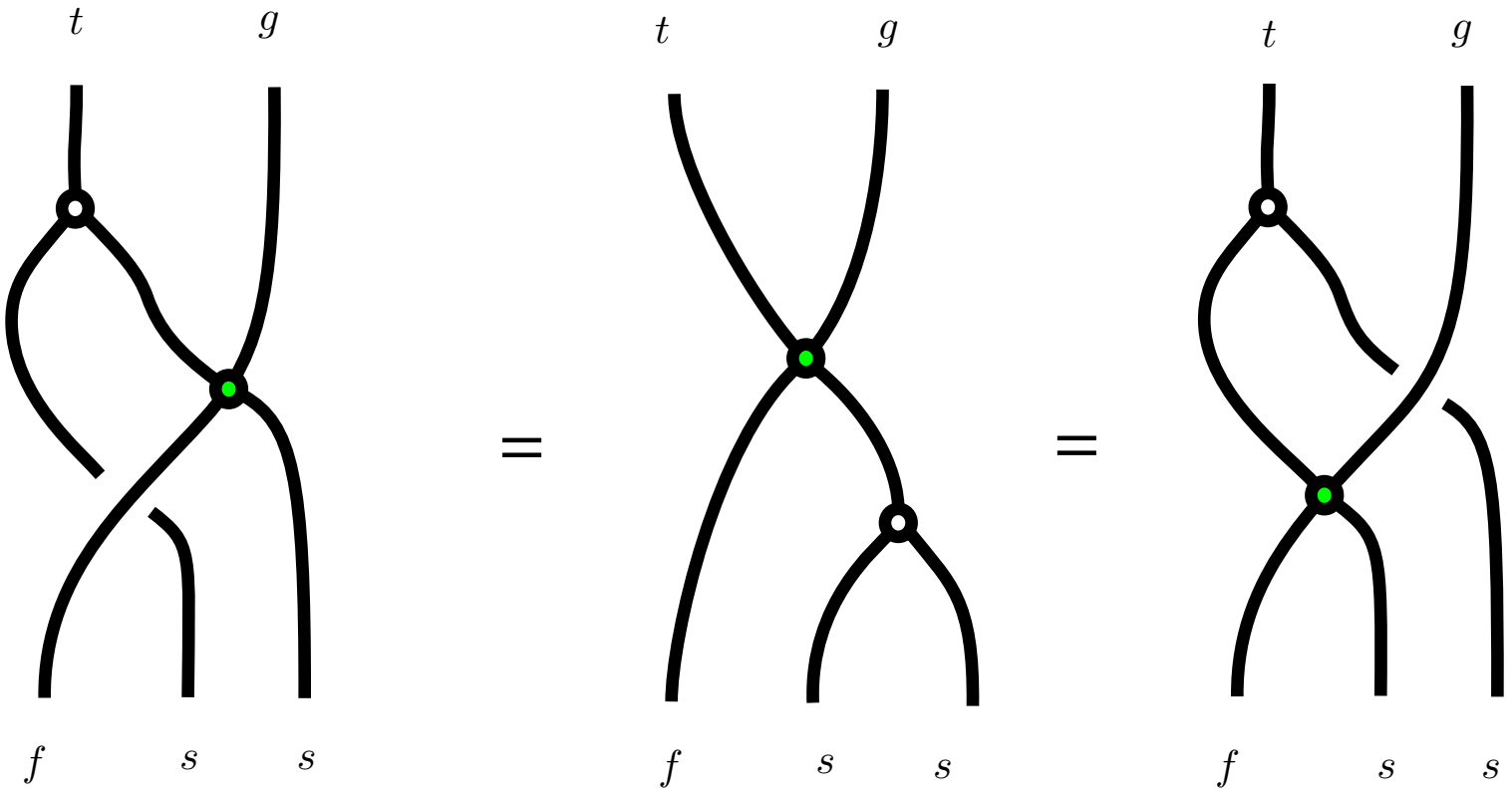


Illustration: internal categories

Given a category $\mathcal{C}$ with finite limits...

the bicategory $\overrightarrow{Cat}(\mathbf{Span})$ has

– the same 0-cells as the category $\mathcal{C}$,

– the 1-cells are the spans,

– the 2-cells are morphisms between spans.

Fact: a monad in $\mathbf{Span}$ is an internal category in $\mathcal{C}$

Part 2

Modules between categories

categories seen as rings with several objects

Representation principle

Every monad (in the bicategorical sense)

$$t \quad : \quad A \quad \longrightarrow \quad A$$

induces a monad (in the categorical sense)

$$\mathcal{B}(X,t) \quad : \quad \mathcal{B}(X,A) \quad \longrightarrow \quad \mathcal{B}(X,A)$$

defined by post-composition

$$X \xrightarrow{f} A \qquad \mapsto \qquad X \xrightarrow{f} A \xrightarrow{t} A$$

for every 0-cell X of the bicategory $\mathcal{B}$.

Representation principle (dual)

Dually, every monad (in the bicategorical sense)

$$t \quad : \quad A \quad \longrightarrow \quad A$$

induces a monad (in the categorical sense)

$$\mathcal{B}(t, X) \quad : \quad \mathcal{B}(A, X) \quad \longrightarrow \quad \mathcal{B}(A, X)$$

defined this time by pre-composition:

$$A \xrightarrow{f} X \qquad \mapsto \qquad A \xrightarrow{t} A \xrightarrow{f} X$$

for every 0-cell X of the bicategory $\mathcal{B}$.

Representation principle (on both sides)

Every pair of monads (in the bicategorical sense)

$$s \;:\; A \longrightarrow A \qquad t \;:\; B \longrightarrow B$$

induces a monad (in the categorical sense)

$$\mathcal{B}(s,t) \;:\; \mathcal{B}(A,B) \longrightarrow \mathcal{B}(A,B)$$

defined by pre- and post- composition:

$$A \xrightarrow{f} B \qquad \mapsto \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle t} \\ A & & B \end{array}$$

for every 0-cell X of the bicategory $\mathcal{B}$.

The bicategory $\mathbf{Module}(\mathcal{B})$

– the 0-**cells** are the monads (A, s) of the bicategory $\mathcal{B}$

– the 1-**cells**

$$(A, s) \longrightarrow (B, t)$$

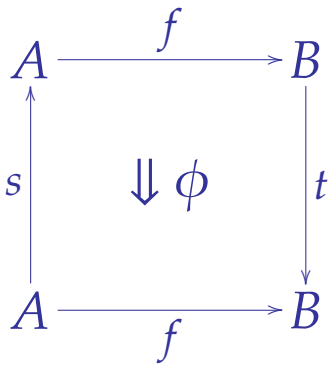
are the algebras of the monad

$$\mathcal{B}(s, t) \;\; : \;\; \mathcal{B}(A, B) \;\; \longrightarrow \;\; \mathcal{B}(A, B).$$

So, they are pairs (f, ϕ) consisting of a 1-cell

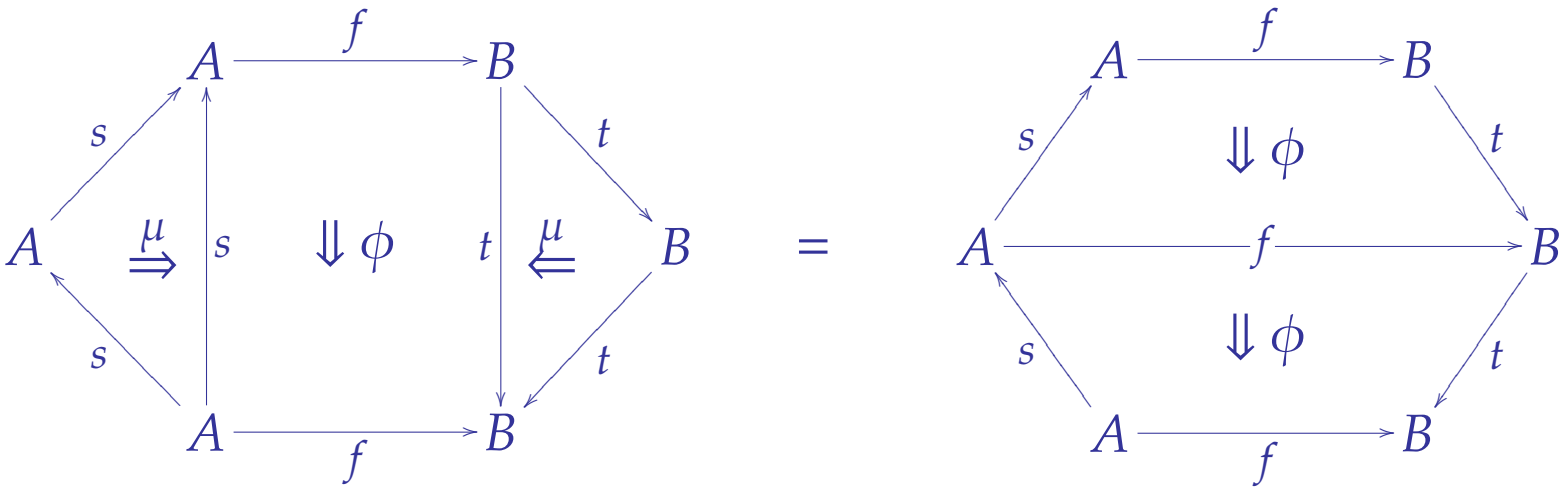
$$A \xrightarrow{\;f\;} B$$

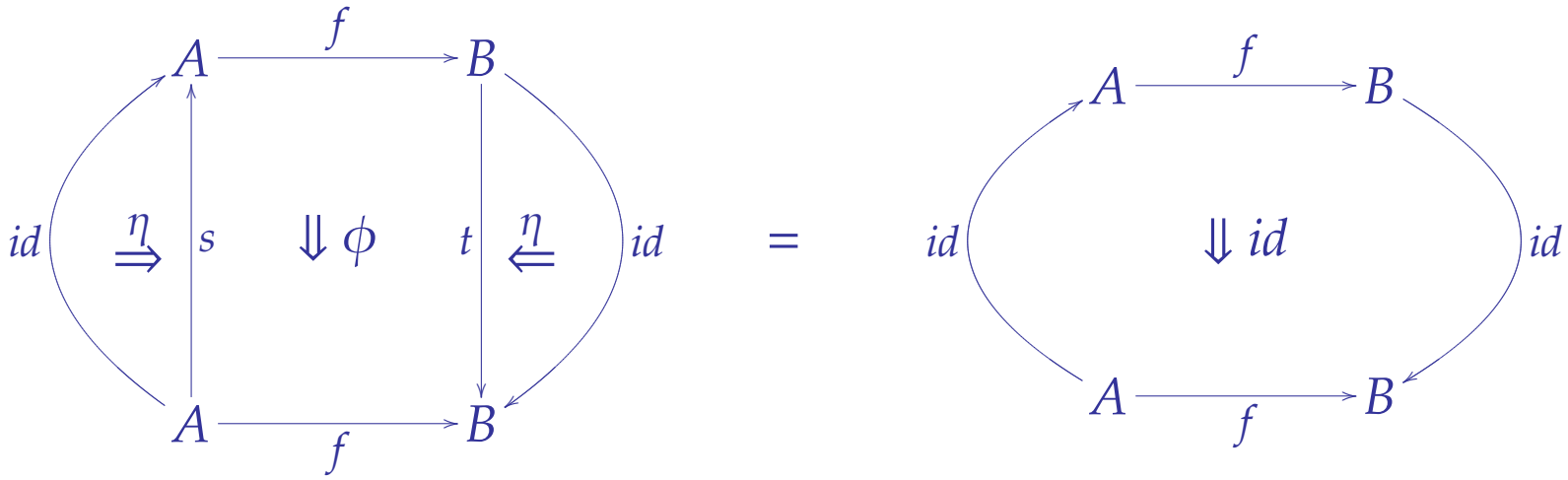
and a 2-cell



in the bicategory $\mathcal{B}$, satisfying the coherence diagrams:

The bicategory $\mathbf{Module}(\mathcal{B})$

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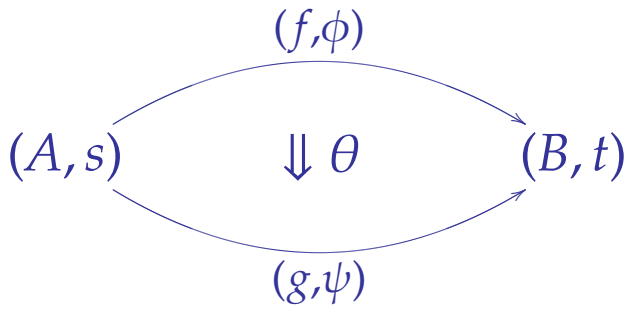
The bicategory $\mathbf{Module}(\mathcal{B})$

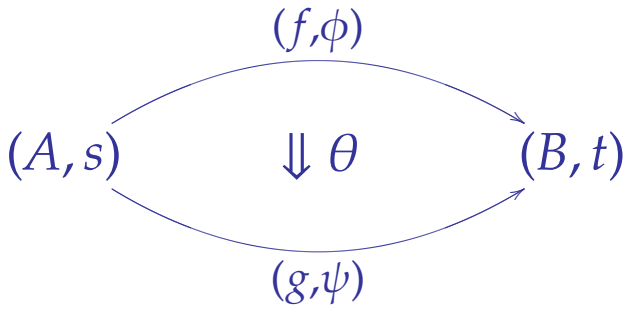
– the 2-**cells**



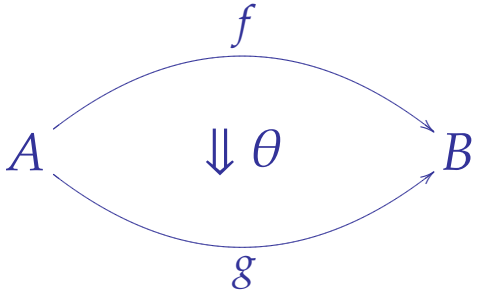
are the morphisms of $\mathcal{B}(s,t)$-algèbres in the category $\mathcal{B}(A,B)$.

The bicategory $\mathbf{Module}(\mathcal{B})$

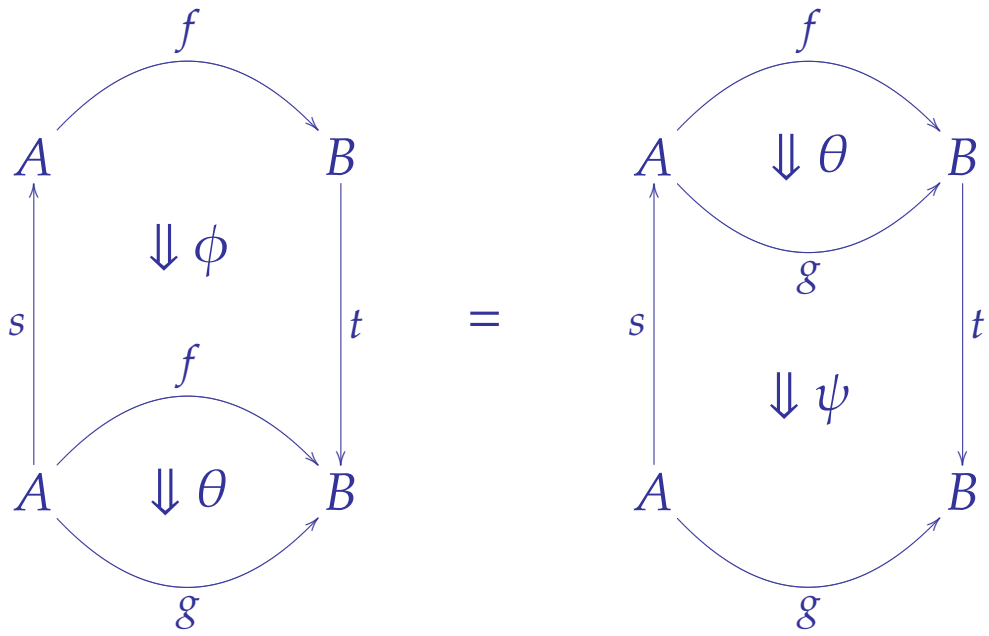
In other words, a 2-cell



is a 2-cell



satisfying the equality:

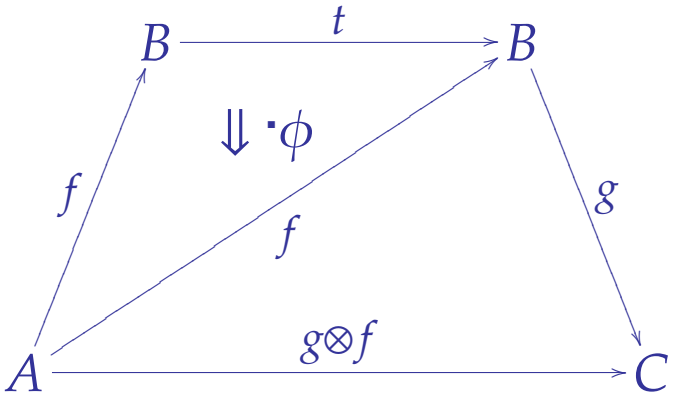


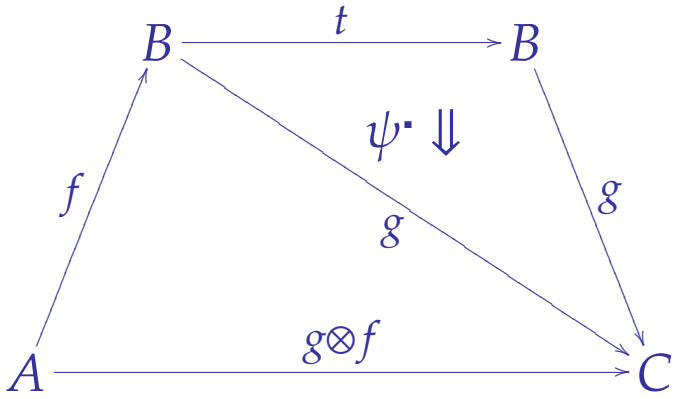
The bicategory $\mathbf{Module}(\mathcal{B})$

The composite of the two 1-cells

$$(A,s) \xrightarrow{(f,\phi)} (B,t) \xrightarrow{(g,\psi)} (C,u)$$

is defined as the co-equalizer of the 2-cells described by the two different ways to compose:





We make the hypothesis

(1) that the category $\mathcal{B}(A, B)$ has coequalizers, for every A, B,

(2) the horizontal composition $\otimes$ in $\mathcal{B}$ preserves these co-equalizers.

Part 3

Game cobordism

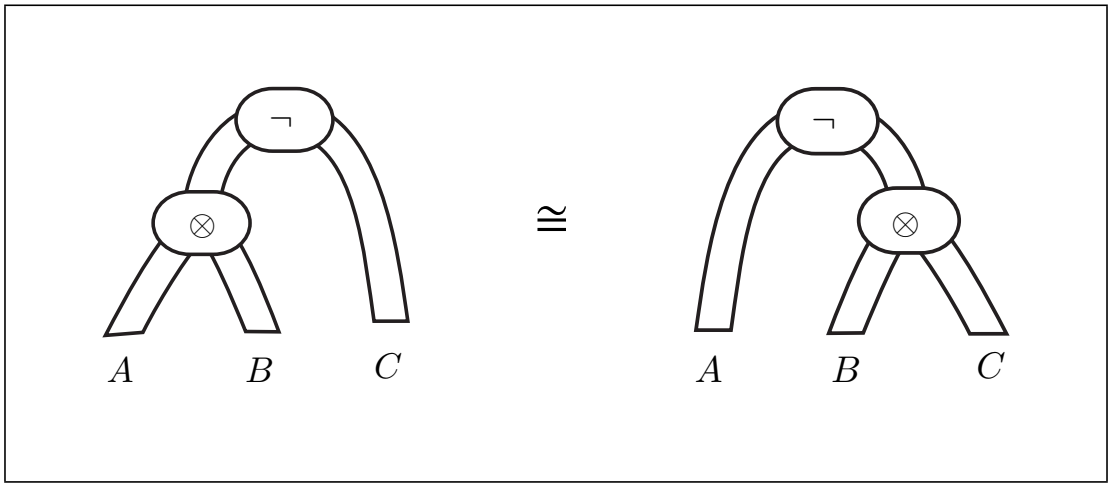
Dialogue categories

A **symmetric monoidal** category $\mathcal{C}$ equipped with a functor

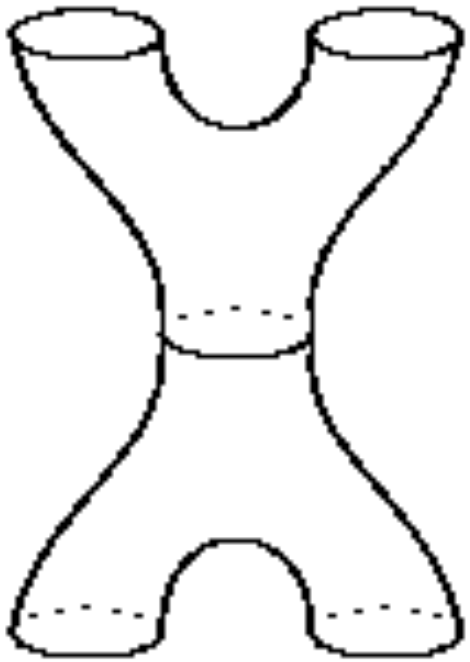
$$\neg \quad : \quad \mathcal{C}^{op} \quad \longrightarrow \quad \mathcal{C}$$

and a natural bijection

$$\varphi_{A,B,C} \quad : \quad \mathcal{C}(A \otimes B\,,\, \neg\, C) \quad \cong \quad \mathcal{C}(A\,,\, \neg\,(\,B \otimes C\,)\,)$$

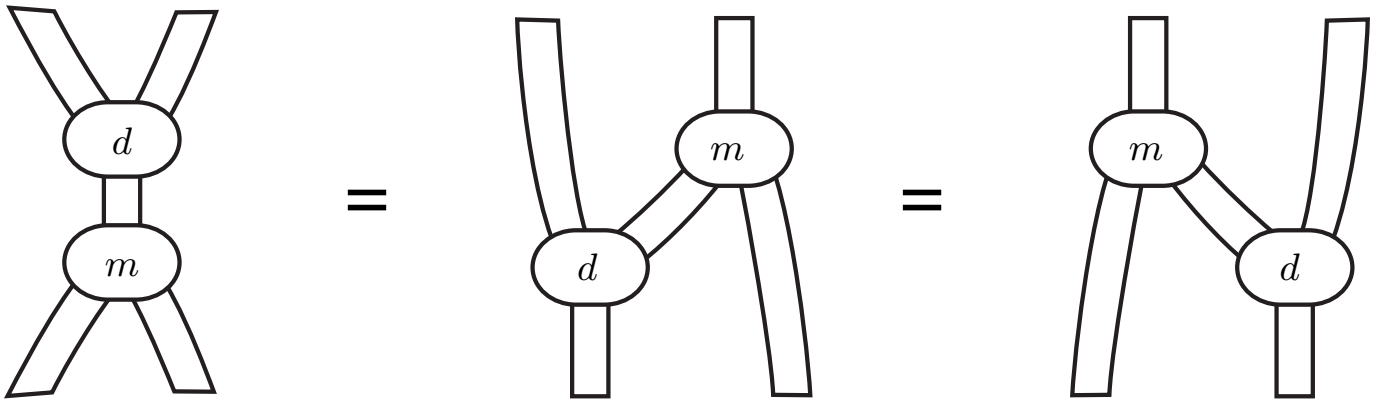


Cobordism



Frobenius objects

A Frobenius object F is a monoid and a comonoid satisfying



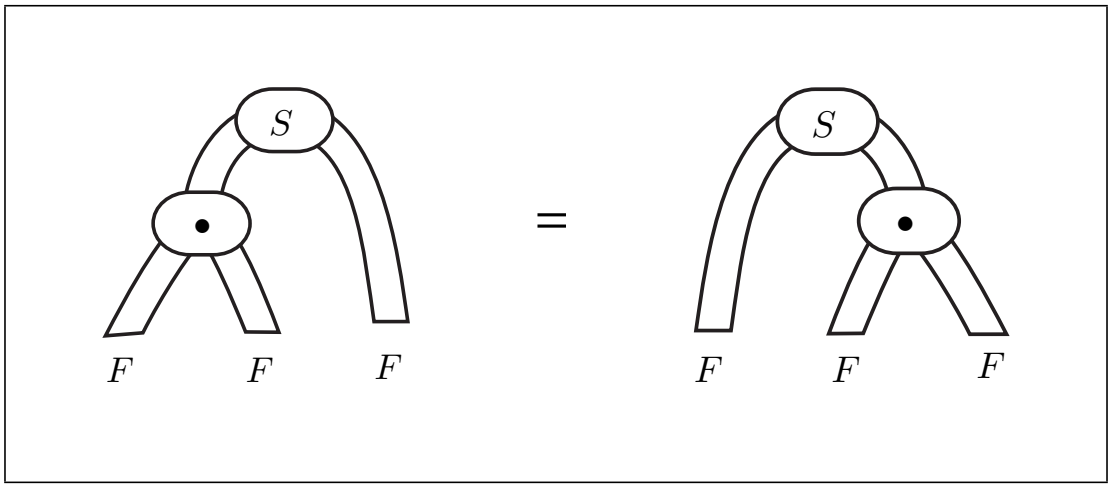
an alternative formulation of cobordism

Frobenius objects

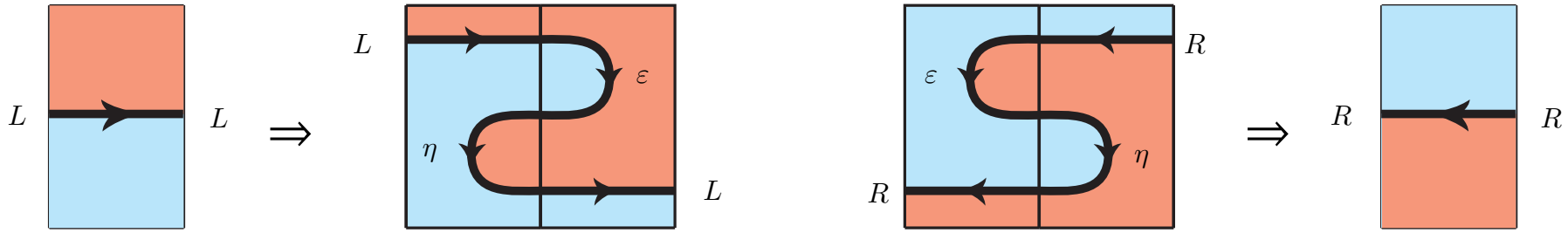
Equivalently, a Frobenius object F is a monoid with an isomorphism

$$S \quad : \quad F \quad \longrightarrow \quad F^*$$

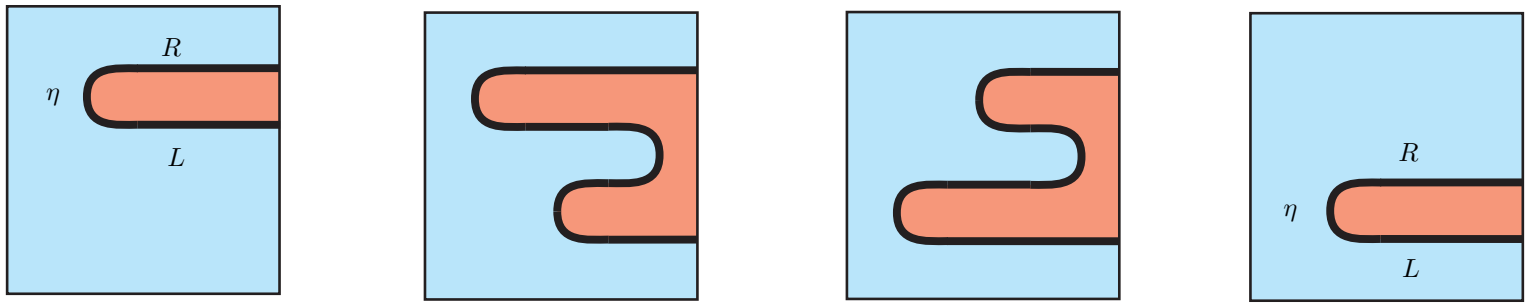
to its dual object F^* such that



Lax dualities in a 3-dimensional category

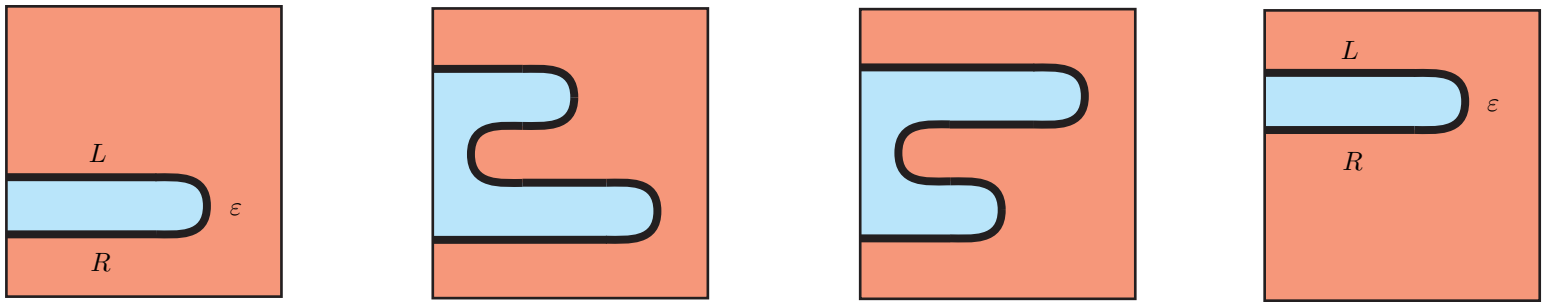


satisfying two coherence properties (a)



is the identity 3-cell on the unit η of the 2-adjunction $L \dashv R$,

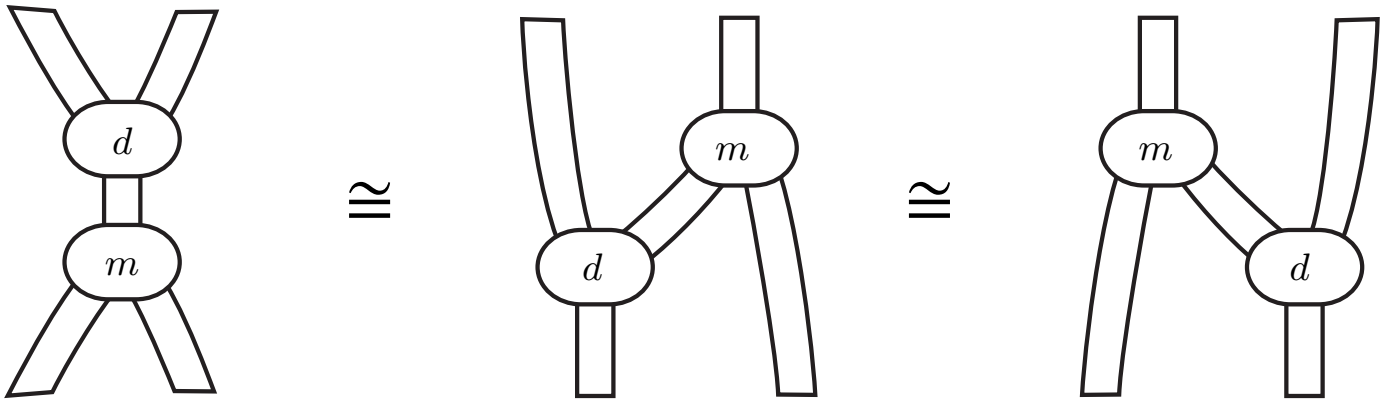
satisfying two coherence properties (b)



is the identity 3-cell on the counit ε of the 2-adjunction $L \dashv R$.

Pseudo Frobenius objects

A **pseudo Frobenius object** in the bicategory of modules



is the same thing as a $*$**-autonomous category**... when the two modules m and e are functors.

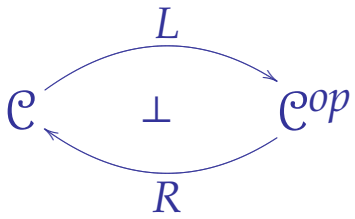
An observation by Brian Day and Ross Street (2003)

Lax Frobenius objects

Relax the self-duality equivalence

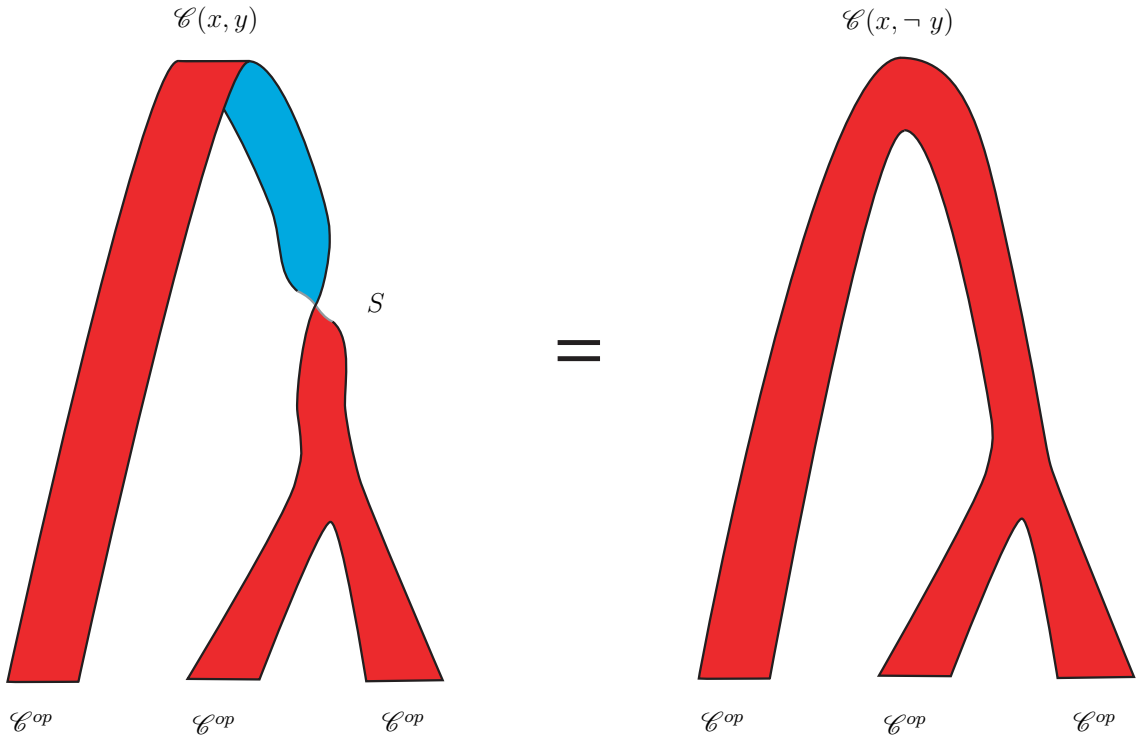
$$\mathcal{C} \quad \cong \quad \mathcal{C}^{op}$$

into an adjunction



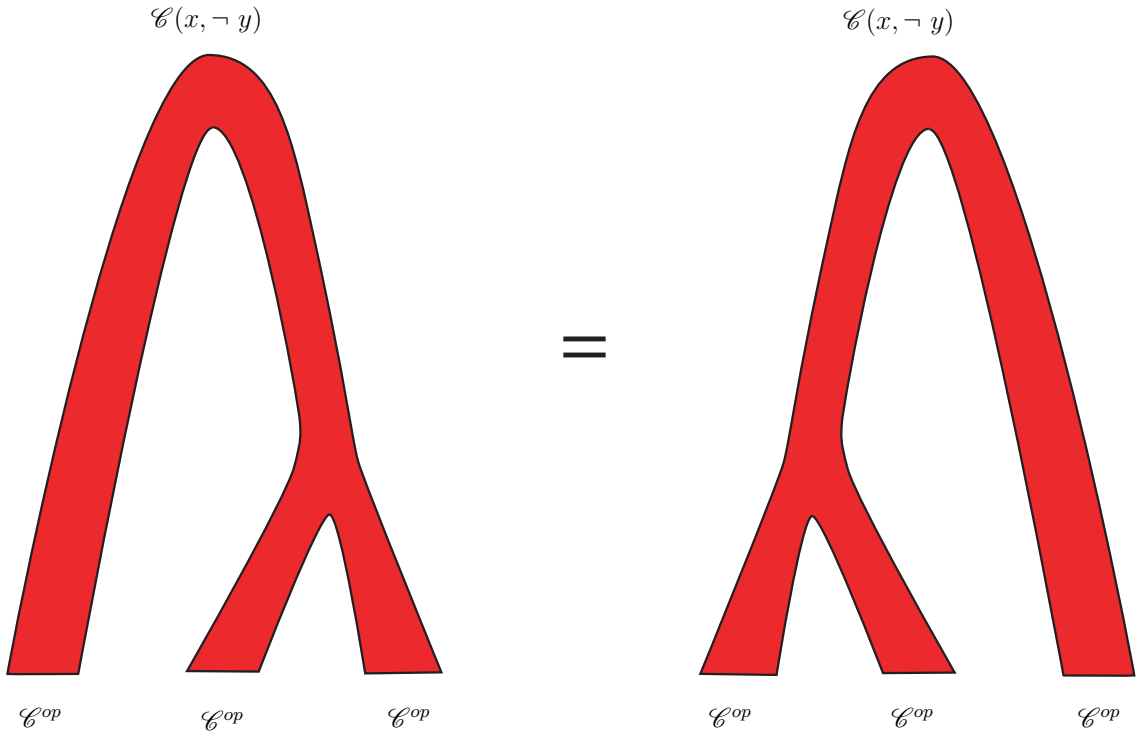
this connects game semantics and quantum groups

Game semantics in ribbon diagrams



Idea: replace the elementary particles by the game boards

Game semantics in ribbon diagrams



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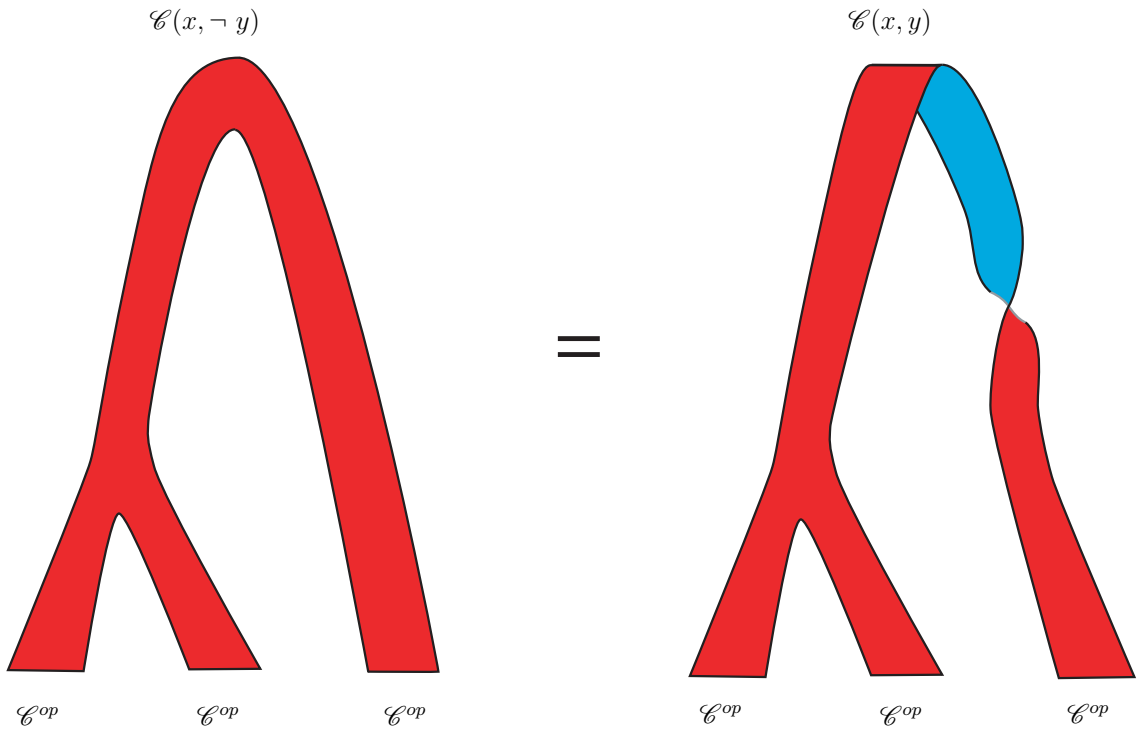
Game semantics in ribbon diagrams



Idea: replace the elementary particles by the game boards

Conclusion

Logic = Data Structure + Duality

This point of view is accessible thanks to 2-dimensional algebra