Dialogue categories up to deformation

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Abstract

In this paper, we introduce the notion of dialogue chirality, a relaxed notion of dialogue category defined as an adjunction between a monoidal category $\mathscr{A}$ and a monoidal category $\mathscr{B}$ equivalent to its opposite category $\mathscr{A}^{op(0,1)}$. The comparison between dialogue categories and dialogue chiralities is based on the construction of a 2-dimensional adjunction between a 2-category of dialogue categories and a 2-category of dialogue chiralities. The resulting coherence theorem clarifies in what sense every dialogue chirality may be strictified to an equivalent dialogue category.

Forewords. This paper is part of a larger research program at the interface of proof theory and of programming language semantics, whose purpose is to investigate the interactive nature of *continuations* in programming languages. The paper is guided by the idea that this interactive nature already lies (although hidden) in the traditional description of continuations in categorical semantics. This motivates to reformulate a dialogue category $\mathscr{C}$ as a pair consisting of a category $\mathscr{A}$ of proofs (or programs) confronted to a category $\mathscr{B}$ of refutations (or environments). The reader should probably keep this basic intuition in mind when reading the paper.

1 Introduction

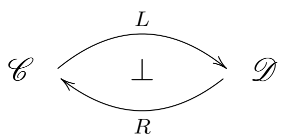
Deformation of algebraic structures. A strictly monoidal category is defined as a category $\mathscr{C}$ equipped with a functor

$$\otimes \quad : \quad \mathscr{C} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and an object I satisfying the associativity and unity equations

$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \qquad\qquad I \otimes x = x = x \otimes I \tag{1}$$

for all objects x, y, z of the category $\mathscr{C}$. A strictly monoidal category may be alternatively defined as a monoid object in the cartesian category ***Cat*** of categories and functors. An interesting question is to characterize the algebraic structure inherited by a category $\mathscr{D}$ equivalent to a strictly monoidal category $\mathscr{C}$. Here, by equivalence, one means an adjunction



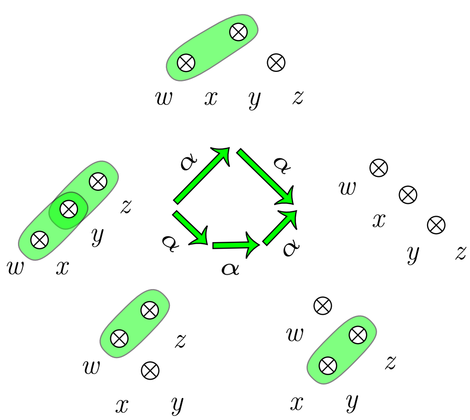
whose unit and counit

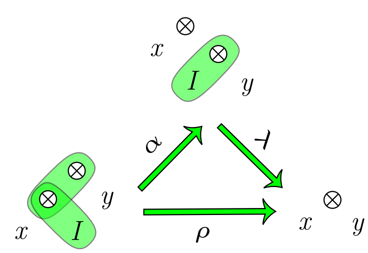
$$\eta \quad : \quad Id \Rightarrow R \circ L \qquad\qquad \varepsilon \quad : \quad L \circ R \Rightarrow Id$$

are invertible. The answer is provided by MacLane's coherence theorem, which states that a category $\mathscr{D}$ is equivalent to a strict monoidal category precisely when it is a monoidal category – that is, a category equipped with a functor $\otimes$ and an object I together with three families of isomorphisms

$$(x \otimes y) \otimes z \xrightarrow{\alpha} x \otimes (y \otimes z) \qquad\qquad I \otimes x \xrightarrow{\lambda} x \xleftarrow{\rho} x \otimes I$$

natural in x, y, z and making the two diagrams





commute for all objects w, x, y, z of the category $\mathscr{D}$. From a conceptual point of view, it should be observed that a monoidal category is the same thing as a pseudo-monoid in the cartesian 2-category ***Cat*** of categories, functors and natural transformations. Now, in every monoidal 2-category, every object $\mathscr{D}$ equivalent to a monoid object $\mathscr{C}$ is a pseudo-monoid object whose structure is inherited from $\mathscr{C}$. The converse property is not true in general: a pseudo-monoid object is not necessarily equivalent to a monoid object. However, the property holds in the particular case of the cartesian 2-category ***Cat***: this is the difficult part of the coherence theorem, which states that every monoidal category $\mathscr{D}$ is equivalent to a strictly monoidal category $\mathscr{C}$. A purely homotopic account of the coherence theorem is possible: the idea is to identify the theorem as an instance of the Boardman-Vogt W-construction of an algebraic theory modulo deformation, performed in the category ***Cat*** of categories equipped with the 'folk' model structure, whose weak equivalences are provided by the categorical equivalences see for instance [2] and [14, Section 4.2].

Deformation of dual structures. The purpose of this article is to understand how the idea of deformation may be applied to dialogue categories and other notions of categories equipped with a duality. A dialogue category is defined as a monoidal category $\mathscr{C}$ equipped with an object $\bot$ together with two functors

$$\begin{array}{ccc} \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\ x & \mapsto & x \multimap \bot \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\ x & \mapsto & \bot \multimapinv x \end{array}$$

and two families of bijections

$$\mathscr{C}(y, x \multimap \bot) \quad \cong \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(x, \bot \multimapinv y)$$

natural in x and y. The notion of dialogue category is preserved by equivalence, this meaning that every category $\mathscr{D}$ equivalent to a dialogue category $\mathscr{C}$ is also a dialogue category. So, the idea of relaxing the notion of dialogue category by deformation is apparently meaningless... unless one applies an even stronger notion of deformation on them!

A first step in that direction is to observe that any notion of self-dual category relates the category $\mathscr{C}$ to its opposite category $\mathscr{C}^{op}$. From that point of view, it makes sense to think of the ambient 2-category ***Cat*** as an "involutive" 2-category equipped with a 2-functor

$$(-)^{op} \quad : \quad \boldsymbol{Cat} \quad \longrightarrow \quad \boldsymbol{Cat}^{op(2)}$$

which transports every category $\mathscr{C}$ to its opposite category $\mathscr{C}^{op}$. Here, the target 2-category $\boldsymbol{Cat}^{op(2)}$ is the 2-category $\boldsymbol{Cat}$ where the 2-cells have been reversed, this reflecting the fact that $(-)^{op}$ transports every natural transformation

$$\mathscr{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{D} \qquad \Downarrow \theta$$

to a natural transformation in the opposite direction:

$$\mathscr{C}^{op} \underset{G^{op}}{\overset{F^{op}}{\rightrightarrows}} \mathscr{D}^{op} \qquad \Uparrow \theta^{op}$$

In the case of a dialogue category, the category $\mathscr{C}$ is related to its opposite category $\mathscr{C}^{op}$ by an adjunction

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}^{op} \qquad \perp \tag{2}$$

defined by the two functors

$$L(x) \quad = \quad x \multimap \perp \qquad \text{and} \qquad R(x) \quad = \quad \perp \multimapinv x$$

and the families of bijections

$$\mathscr{C}^{op}(x \multimap \perp, y) \quad \cong \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(x, \perp \multimapinv y).$$

natural in x and y. From this follows the central idea of the paper that the deformation of the category $\mathscr{C}$ should be *decorrelated* from the deformation of its opposite category $\mathscr{C}^{op}$. This means that we will study and characterize the pairs $(\mathscr{A}, \mathscr{B})$ of categories equivalent to a pair $(\mathscr{C}, \mathscr{C}^{op})$ consisting of a dialogue category $\mathscr{C}$ and of its opposite category $\mathscr{C}^{op}$. In other words, the deformation of a dialogue category $\mathscr{C}$ will not be performed inside the 2-category $\boldsymbol{Cat}$... but inside the 2-category $\boldsymbol{Cat} \times \boldsymbol{Cat}^{op(2)}$. We will see that this additional degree of freedom in the deformation reveals hidden features of dialogue categories, in the same way as traditional deformation does for strict monoidal categories. This decorrelated point of view also enables to think of the two categories $\mathscr{C}$ and $\mathscr{C}^{op}$ in a symmetric and unbiased way.

Cartesian closed chiralities. It should be noted that the method is not limited to dialogue categories, as we illustrate below with cartesian closed categories. To that purpose, we define a *cartesian closed chirality* as a pair $(\mathscr{A},\mathscr{B})$ where $\mathscr{A}$ is equivalent to a cartesian closed category $\mathscr{C}$ (and thus a cartesian closed category itself) and $\mathscr{B}$ is equivalent to its opposite category $\mathscr{C}^{op}$. A cartesian closed chirality $(\mathscr{A},\mathscr{B})$ is then easily characterized as a pair consisting of

- a category $\mathscr{A}$ with finite products noted $(a_1, a_2) \mapsto a_1 \wedge a_2$ for the binary products and $\mathtt{true}$ for the terminal object,
- a category $\mathscr{B}$ with finite sums noted $(b_1, b_2) \mapsto b_1 \vee b_2$ for the binary sums and false for the initial object,

equipped with:

- an equivalence of category between $\mathscr{A}$ and $\mathscr{B}^{op}$, which transports every object a of $\mathscr{A}$ to an object noted $\sim a$ of $\mathscr{B}$ and every object b of $\mathscr{B}$ to an object noted $\sim b$ of $\mathscr{A}$,
- a pseudo-action
$$\vee \quad : \quad \mathscr{B} \quad \times \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{A} \tag{3}$$
of the monoidal category $(\mathscr{B}, \vee, \text{false})$ on the category $\mathscr{A}$,
- a bijection
$$\mathscr{A}(a_1 \wedge a_2, a_3) \quad \cong \quad \mathscr{A}(a_2, (\sim a_1) \vee a_3) \tag{4}$$
natural in a_1, a_2 and a_3.

Here, the pseudo-action (3) is inherited from the functor

$$\Rightarrow \quad : \quad \mathscr{C}^{op} \quad \times \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

which transports every pair (x, y) of objects to the hom-object $x \Rightarrow y$. In this unbiased formulation, the two canonical isomorphisms

$$(x_1 \times x_2) \Rightarrow y \quad \cong \quad x_1 \Rightarrow (x_2 \Rightarrow y) \qquad\qquad 1 \Rightarrow x \quad \cong \quad x$$

of the cartesian closed category $\mathscr{C}$ are translated as the two isomorphisms

$$(b_1 \vee b_2) \vee a \quad \cong \quad b_1 \vee (b_2 \vee a) \qquad\qquad \text{false} \vee a \quad \cong \quad a$$

which make the operation $(b, a) \mapsto b \vee a$ into a pseudo-action of $\mathscr{B}$ over $\mathscr{A}$. An outcome of the deformation of $\mathscr{C}$ into $(\mathscr{A}, \mathscr{B})$ is that the functor $\Rightarrow$ factors as

$$\mathscr{A}^{op} \times \mathscr{A} \xrightarrow{\ \sim \times \mathscr{A}\ } \mathscr{B} \times \mathscr{A} \xrightarrow{\ \vee\ } \mathscr{A}$$

in the same way as the implication $P \Rightarrow Q$ of two logical propositions is decomposed in classical logic as the disjunction $(\sim P) \vee Q$ of the negation $\sim P$ of the proposition P with the proposition Q. This phenomenon is familiar in monoidal categories equipped with a strong notion of self-duality. Typically, the hom-object $x \multimap y = [x, y]$ is defined as

- the object ${}^*x \otimes y$ in a ribbon category, where ${}^*x$ is the right dual of x,
- the object ${}^*x \wp y$ in a $*$-autonomous category, where $x \wp y$ is itself defined as ${}^*(y^* \otimes x^*)$ where x^* is the left dual of x.

This unbiased formulation of cartesian closed categories reveals that the decomposition of implication $P \Rightarrow Q$ as $(\sim P) \vee Q$ is not limited to these self-dual categories but also applies to situations where the category $\mathscr{C}$ is not necessarily equivalent to its opposite category $\mathscr{C}^{op}$.

It is also worth mentioning at this point that the definition of cartesian closed chirality requires moreover that the two coherence diagrams

$$\begin{array}{ccc}
\mathscr{A}(a_1 \wedge a_2 \wedge a_3, a_4) & \longrightarrow & \mathscr{A}(a_3, \sim (a_1 \wedge a_2) \vee a_4) \\
\downarrow & & \downarrow {\scriptstyle (*)} \\
\mathscr{A}(a_2 \wedge a_3, \sim a_1 \vee a_4) & \longrightarrow & \mathscr{A}(a_3, \sim a_2 \vee \sim a_1 \vee a_4)
\end{array} \qquad (5)$$

$$\begin{array}{ccc}
\mathscr{A}(\mathbf{true} \wedge a_1, a_2) & \longrightarrow & \mathscr{A}(a_1, \sim \mathbf{true} \vee a_2) \\
\downarrow & & \downarrow {\scriptstyle (*)} \\
\mathscr{A}(a_1, a_2) & \longrightarrow & \mathscr{A}(a_1, \mathbf{false} \vee a_2)
\end{array} \qquad (6)$$

commute, where the isomorphisms $(*)$ from $\sim (a_1 \wedge a_2)$ to $\sim a_2 \vee \sim a_1$ and from $\sim$ true to false are deduced from the fact that the equivalence $\sim$ transports finite products of $\mathscr{A}$ into finite sums of $\mathscr{B}$.

A useful convention. Before carrying on our investigation of dialogue categories, we would like to introduce a useful convention. Since negation tends to reverse the orientation of the tensors, we find convenient to replace the opposite category $\mathscr{C}^{op(1)}$ by the category $\mathscr{C}^{op(0,1)}$ where the 0-dimensional cells (the objects) as well as the 1-dimensional cells (the morphisms) have been reversed. By "reversing the objects", we simply mean that the orientation of tensors is reversed in the following way:

$$x \otimes^{op(0,1)} y \quad := \quad y \otimes x.$$

The terminology reflects the fact that a monoidal category $\mathscr{C}$ may be seen as a 2-category $\Sigma\mathscr{C}$ (more precisely, a bicategory) with one object, called its suspension. Now, the 1-cells of the suspension 2-category $\Sigma\mathscr{C}$ are the 0-cells of the category $\mathscr{C}$. Hence, reversing the 0-cells in $\mathscr{C}$ means reversing the 1-cells in $\Sigma\mathscr{C}$, which amounts to reversing the orientation of the tensor product. Note in particular that the expected equation holds:

$$(\Sigma\mathscr{C})^{op(1,2)} \quad = \quad \Sigma(\mathscr{C}^{op(0,1)})$$

where $(\Sigma\mathscr{C})^{op(1,2)}$ is the 2-category $\Sigma\mathscr{C}$ where the orientation of the 1-cells and of the 2-cells have been reversed.

Dialogue chiralities. The main purpose of the article is to characterize the pairs $(\mathscr{A},\mathscr{B})$ obtained by deforming a dialogue category $\mathscr{C}$ into a category $\mathscr{A}$, and at the same time but independently, its opposite category $\mathscr{C}^{op}$ into a category $\mathscr{B}$. Such a pair $(\mathscr{A},\mathscr{B})$ is called a *dialogue chirality* because of the mirror-symmetry between the two components $\mathscr{A}$ and $\mathscr{B}$. A first observation is that in every dialogue chirality:

- the category $\mathscr{A}$ inherits a tensor product $\owedge$ and a unit true, reflecting the tensor product $\otimes$ and the unit I of the category $\mathscr{C}$,
- the category $\mathscr{B}$ inherits a tensor product $\ovee$ and a unit false from the very same monoidal structure, but considered this time in the opposite category $\mathscr{C}^{op(0,1)}$ where the orientation of objects and morphisms have been reversed.

This induces a monoidal equivalence

$$(\mathscr{A},\owedge,\mathbf{true}) \underset{(-)^{*}}{\overset{{}^{*}(-)}{\rightleftarrows}}_{\textit{monoidal equivalence}} (\mathscr{B},\ovee,\mathbf{false})^{op(0,1)} \qquad (7)$$

which transports every object a of the category $\mathscr{A}$ into an object ${}^*a$ of the category $\mathscr{B}$, and symmetrically, every object b of the category $\mathscr{B}$ into an object b^* of the category $\mathscr{A}$. By monoidal equivalence, one means that the functors ${}^*(-)$ and $(-)^*$ are equipped with natural isomorphisms

$$\begin{array}{rclcrcl} {}^*(a_1 \owedge a_2) & \cong & {}^*a_2 \ovee {}^*a_1 & \qquad & {}^*\mathbf{true} & \cong & \mathbf{false} \\ (b_1 \ovee b_2)^* & \cong & {b_2}^* \owedge {b_1}^* & & \mathbf{false}^* & \cong & \mathbf{true} \end{array}$$

making the expected coherence diagrams commute. It should be stressed that the notations are directly inspired by logic, just as in the case of cartesian closed chiralities. The idea is that the functors $(a \mapsto {}^*a)$ and $(b \mapsto b^*)$ are involutive forms of negation transporting the objects of $\mathscr{A}$ into the objects of $\mathscr{B}$ and conversely. Accordingly, the tensor product $\owedge$ is interpreted in the category $\mathscr{A}$ as a conjunction with its unit true, whereas the tensor product $\ovee$ is interpreted in the category $\mathscr{B}$ as a disjunction with its unit false.

A second observation on dialogue chiralities is that the two categories $\mathscr{A}$ and $\mathscr{B}$ are related by an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R) \tag{8}$$

inherited from the original adjunction (2) between the categories $\mathscr{C}$ and $\mathscr{C}^{op}$. This adjunction enables to construct the functor

$$\langle - \,|\, - \rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

also called distributor or $\mathscr{A}\mathscr{B}$-module, defined as

$$\langle a \,|\, b \rangle \quad = \quad \mathscr{A}(a, R\,b). \tag{9}$$

For essentially aesthetic reasons, we will also consider the more general notion of *reflection chiralities* where the pair of adjoint functors $L \dashv R$ is replaced by the distributor. A dialogue chirality is then defined as a reflection chirality where the distributor $\langle - \,|\, - \rangle$ is generated by an adjunction $L \dashv R$ in the sense that Equation (9) holds. In particular, the coherence theorem established in Section 6 states that every dialogue chirality may be strictified to a dialogue category.

Relaxed dialogue categories. One motivation for deforming strict monoidal categories into monoidal categories is to encompass interesting examples arising from mathematics. Typically, a cartesian category like Set is monoidal, but not strictly monoidal, because the two sets $X \times (Y \times Z)$ and $(X \times Y) \times Z$ are isomorphic, but not equal. Similarly, one feels the need for a non-strict version of dialogue categories. As we have just explained, every dialogue category $\mathscr{C}$ may be seen as a dialogue chirality $(\mathscr{C}, \mathscr{C}^{op})$ where the two functors $(-)^*$ and ${}^*(-)$ are simply the identity on the category $\mathscr{C}$. This general principle applies more specifically to the self-dual case of $*$-autonomous category $\mathscr{C}$. However, the notion of dialogue chirality enables us to think of $*$-autonomous categories in a slightly different way. Indeed, let us declare that a dialogue chirality $(\mathscr{A}, \mathscr{B})$ is *self-dual* when the two sides $\mathscr{A}$ and $\mathscr{B}$ are equal to the same category $\mathscr{C}$, and when the two functors L and R are identity functors.

$$\mathscr{A} = \mathscr{C} \underset{R = id}{\overset{L = id}{\rightleftarrows}} \mathscr{C} = \mathscr{B} \qquad (\bot)$$

Obviously, every $*$-autonomous category $\mathscr{C}$ may be seen as such a self-dual dialogue chirality $(\mathscr{C}, \mathscr{C})$. It is worth mentioning that the category $\mathscr{A} = \mathscr{C}$ on the lefthand-side of the adjunction is *equivalent* (but not equal!) to the opposite $\mathscr{B}^{op} = \mathscr{C}^{op}$ of the category $\mathscr{B} = \mathscr{C}$ on the righthand side. The important point here is that the shift from dialogue categories to dialogue chiralities enables us to view the two operations $a \mapsto {}^*a$ and $b \mapsto b^*$ as *logical negations*. On the other hand, the notion of dialogue category we started from would require that the category $\mathscr{A} = \mathscr{C}$ is *equal* to $\mathscr{B}^{op} = \mathscr{C}^{op}$. This interpretation requires to "deform" the traditional notion of dialogue category, since the two operations $a \mapsto {}^*a$ and $b \mapsto b^*$ are defined as *categorical identities* when the $*$-autonomous category $\mathscr{C}$ is translated as a "strict" dialogue chirality $(\mathscr{C}, \mathscr{C}^{op})$ where the category $\mathscr{A} = \mathscr{C}$ is *equal* (and not just equivalent) to the opposite of the category $\mathscr{B} = \mathscr{C}^{op}$. So, just as in the case of monoidal categories, relaxing the notion of dialogue category enables to encompass new interesting examples – or at least to understand them better. The ongoing discussion may be summarized in a table:

strict notions	**deformed / relaxed notions**
strict monoidal categories	monoidal categories
cartesian closed categories	cartesian closed chiralities
dialogue categories	dialogue chiralities

A symmetrization of logic. One motivation for the present article is to provide a categorical explanation to the notion of polarity in logic. This notion emerged in the early 1990s in the work by Girard on classical logic [4] and then became prominent in the linear logic circles, with the definition of ludics [5] and of polarized linear logic [9]. The basic principle of polarization is to distinguish two classes of formulas, called *positive* and *negative*, and to apply logical connectives only when the formulas are of the appropriate polarity. However, because of its origins, there is a widespread belief that the notion of polarity is intrinsically connected to classical logic. This is not the case however, and one purpose of the present work is to clarify this issue. Indeed, we advocate here that polarities offer a *symmetric point of view* on logic, rather than an additional structure. Indeed, the basic principle of polarities in logic is to see the category $\mathscr{C}$ as a pair $(\mathscr{A}, \mathscr{B})$ where $\mathscr{A}$ is equivalent to $\mathscr{C}$ and where $\mathscr{B}$ is equivalent to its opposite category $\mathscr{C}^{op}$. One benefit of this unbiased and symmetric perspective is that it enables us to think of $\mathscr{A}$ as the category of proofs (and programs) and of $\mathscr{B}$ as the category of refutations of these proofs (and environments of these programs) – or sometimes conversely, depending on the evaluation strategy. We claim that the idea is extremely simple and general, and may be applied to any notion of category $\mathscr{C}$ with structure – as already illustrated with the chiral formulation of cartesian closed categories given above. It should be mentioned that, in that case, the pseudo-action of $\mathscr{B}$ over $\mathscr{A}$ in the definition of a cartesian closed chirality reflects the structure of an intuitionistic environment of type $b_1 \vee \cdots \vee b_n \vee a$ as a finite list of terms of type $b_1, \ldots, b_n$ appended to (that is, acting on) an environment tail (or continuation) of type a.

A microcosm principle for duality. Another motivation for this article is to investigate a "microcosm principle" for dialogue categories and similar notions of categories with a duality. Indeed, an important aspect of dialogue categories is that the two negation functors $A \mapsto A \multimap \bot$ and $A \mapsto \bot \multimapinv A$ are contravariant, and thus cannot be expressed in the 2-category ***Cat*** without the self-duality 2-functor $\mathscr{C} \mapsto \mathscr{C}^{op}$. This phenomenon is similar to the fact that one needs the monoidal structure of ***Cat*** provided by finite products of categories in order to define the notion of monoidal category. And that, more generally, one needs a monoidal category in order to define a monoid object in it. This "microcosm principle" for monoidal categories has been recognized and extensively studied on n-dimensional categories equipped with various monoidal (or algebraic) structures, see for instance

Baez and Dolan [1]. One point of the article is that this principle is not limited to monoidal structures, and that it also regulates the definition of dialogue categories and other algebraic structures equipped with a duality. In particular, there exists an operation $\mathscr{C} \mapsto \mathscr{C}^{op(k)}$ which transforms every n-dimensional category $\mathscr{C}$ into the n-dimensional category $\mathscr{C}^{op(k)}$ where the directions of the k-dimensional cells has been formally reversed, for $k \leq n$. An interesting question is thus to understand what are the needed dualities at higher dimensions in order to define the dual structures at lower dimensions. One purpose of this article is to investigate this microcosm principle in the elementary case of dialogue categories – and to clarify along the way how the traditional dualities of logic based on negation are incorporated inside the first ladders (dimensions 2 and 3) of higher dimensional algebra.

Related works. The fact that negation induces an adjunction between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$ is an old observation, already mentioned by Kock in his study of dualities in monoidal categories [8]. The idea was rediscovered and promoted by Thielecke [13] in his study of continuations in programming language semantics. The idea of describing a dialogue category $\mathscr{C}$ as a pair of opposite categories $\mathscr{A} = \mathscr{C}$ and $\mathscr{B} = \mathscr{C}^{op}$ related by an adjunction comes in our case from the algebraic analysis of game semantics and linear continuations developed by the author in [11]. A similar line of research on polarized categories and game semantics was independently taken by Cockett and Seely [3].

Plan of the article. Before analyzing the case of dialogue categories, we find useful to study the simpler case of general categories. We thus establish in §2. a coherence theorem for categories and chiralities. The theorem is formulated as a biequivalence of 2-categories which provides us with the organizing pattern of the article. We carry on in this 2-categorical spirit, and define the 2-category ***DiaCat*** of dialogue categories in §3, the 2-category ***RefChi*** of reflection chiralities in §4. and the 2-category ***DiaChi*** of dialogue chiralities in §5. The next section is entirely devoted to the construction of a biequivalence between the 2-categories ***DiaCat*** and ***DiaChi***. This leads to our main theorem (Theorem 2) which is stated at the end of §6. We then do some reverse engineering in §7 and introduce the notion of reflection category corresponding to the notion of reflection chirality. We finally conclude the article with a series of postliminary remarks in §8.

2 The basic case: categories and chiralities

In order to clarify the nature of the coherence theorem for dialogue chiralities established in §6, we find convenient and clarifying to start with the simpler example provided by categories and chiralities.

Definition 1 (chirality) *A chirality is defined as a pair* $(\mathscr{A},\mathscr{B})$ *of categories equipped with an equivalence of categories:*

$$\mathscr{A} \quad \underset{(-)^{*}}{\overset{{}^{*}(-)}{\underset{equivalence}{\rightleftarrows}}} \quad \mathscr{B}^{op}$$

An elementary coherence theorem would state that every chirality $(\mathscr{A},\mathscr{B})$ is equivalent to the pair $(\mathscr{A},\mathscr{A}^{op})$ in the 2-category $\boldsymbol{Cat}\times\boldsymbol{Cat}^{op(2)}$. However, this result would be essentially straightforward, and not particularly useful for the applications we have in mind. The reason is that we would like to understand what notions of 1-cell and 2-cell between chiralities should replace the notions of functor and natural transformation between categories. This leads us to introduce the 2-category ***Chi*** of chiralities, defined as follows.

The 1-dimensional cells. A 1-cell in ***Chi***

$$(\mathscr{A}_1,\mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2,\mathscr{B}_2)$$

is defined as a triple $(F_\bullet, F_\circ, \widetilde{F})$ consisting of two functors

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2 \qquad\qquad F_\circ \quad : \quad \mathscr{B}_1 \quad \longrightarrow \quad \mathscr{B}_2$$

and a natural isomorphism

$$\begin{array}{ccc}
\mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\
{\scriptstyle {}^{*}(-)}\downarrow & \widetilde{F} \;\; \Swarrow & \downarrow{\scriptstyle {}^{*}(-)} \\
\mathscr{B}_1^{op} & \xrightarrow[\quad F_\circ^{op} \quad]{} & \mathscr{B}_2^{op}
\end{array}$$

Note that an alternative and unbiased formulation of the same notion of 1-dimensional cell would be to equip the pair of functors $(F_\bullet, F_\circ)$ with a pair of natural isomorphisms

$$^*(-) \circ F_\bullet \Rightarrow F_\circ^{op} \circ {}^*(-) \qquad\qquad F_\bullet \circ (-)^* \Rightarrow (-)^* \circ F_\circ^{op}$$

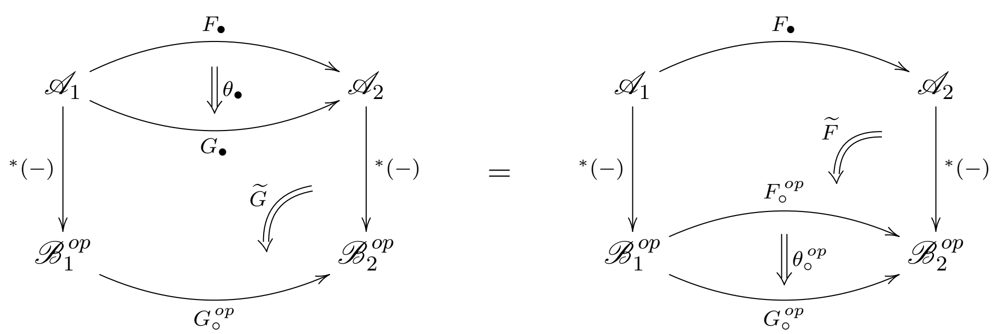
together with a coherence diagram ensuring that the second natural isomorphism coincides with the mate of the first one, in the sense of Kelly and Street [7]. The two definitions are equivalent, and we thus pick the simplest formulation.

The 2-dimensional cells. A 2-cell in ***Chi***

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair of natural transformations

satisfying the equality below:



(10)

This defines a 2-category ***Chi*** with the expected composition and identity laws.

An equivalence of 2-categories. At this point, we are ready to establish that the 2-category ***Chi*** is equivalent to the 2-category ***Cat*** in an appropriate 2-dimensional sense. To that purpose, we define the 2-functor $\mathcal{F} : \boldsymbol{Chi} \longrightarrow \boldsymbol{Cat}$ which transports

- every chirality $(\mathscr{A},\mathscr{B})$ to the category $\mathscr{A}$
- every 1-cell $F = (F_\bullet, F_\circ, \widetilde{F})$ to the functor $F_\bullet$
- every 2-cell $\theta = (\theta_\bullet, \theta_\circ)$ to the natural transformation $\theta_\bullet$

as well as the 2-functor $\mathcal{G} : \boldsymbol{Cat} \longrightarrow \boldsymbol{Chi}$ which transports

- every category $\mathscr{C}$ to the chirality $(\mathscr{C}, \mathscr{C}^{op})$ with $(-)^* = {}^*(-)$ defined as the identity functor on $\mathscr{C}$
- every functor F to the 1-cell (F, F^{op}, id_F)
- every natural transformation θ to the 2-cell (θ, θ^{op}).

This leads to a 2-categorical coherence theorem for categories and chiralities:

Theorem 1 (coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence of 2-categories **Cat** and **Chi**.*

Proof. The composite 2-functor $\mathcal{F} \circ \mathcal{G}$ is equal to the identity on the 2-category ***Cat***. So, in order to establish the coherence property, it is sufficient to construct a pair of pseudo-natural transformations

$$\Phi \;:\; Id \;\longrightarrow\; \mathcal{G}\circ\mathcal{F} \qquad\qquad \Psi \;:\; \mathcal{G}\circ\mathcal{F} \;\longrightarrow Id$$

between the identity 2-functor on ***Chi*** and the 2-functor $\mathcal{G}\circ\mathcal{F}$, and to show that their components $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ define together an equivalence in the 2-category ***Chi***.

The pseudo-natural transformation Φ is defined as follows. To every chirality $(\mathscr{A},\mathscr{B})$, one associates the 1-cell $\Phi_{(\mathscr{A},\mathscr{B})}$ defined as the pair of functors

$$(\Phi_{(\mathscr{A},\mathscr{B})})_\bullet \;:\; \mathscr{A} \xrightarrow{id} \mathscr{A} \qquad\qquad (\Phi_{(\mathscr{A},\mathscr{B})})_\circ \;:\; \mathscr{B} \xrightarrow{((-)^*)^{op}} \mathscr{A}^{op}$$

equipped with the natural transformation

$$\widetilde{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{array}{ccc} \mathscr{A} & \xrightarrow{\quad id \quad} & \mathscr{A} \\ {\scriptstyle {}^*(-)}\downarrow & \overset{\eta}{\Swarrow} & \downarrow{\scriptstyle id} \\ \mathscr{B}^{op} & \xrightarrow[\quad (-)^* \quad]{} & (\mathscr{A}^{op})^{op} \end{array}$$

where η denotes the unit of the adjunction ${}^*(-) \dashv (-)^*$. To every 1-dimensional cell $F : (\mathscr{A}_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \mathscr{B}_2)$ one associates the reversible 2-cell

$$\Phi_F \quad : \quad \mathcal{GF}(F) \circ \Phi_{(\mathscr{A}_1,\mathscr{B}_1)} \quad \Rightarrow \quad \Phi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ F$$

defined as the pair of natural transformations

$$(\Phi_F)_\bullet \quad = \quad \xymatrix{\mathscr{A}_1 \ar@/^/[r]^{F_\bullet} \ar@/_/[r]_{F_\bullet} \ar@{}[r]|{\Downarrow\, id} & \mathscr{A}_2}$$

$$(\Phi_F)_\circ \quad = \quad \xymatrix{ & {\mathscr{A}_1}^{op} \ar[r]^{(F_\bullet)^{op}} \ar[dr]^{({}^*(-))^{op}} & {\mathscr{A}_2}^{op} \ar[r]^{id} \ar[dr]^{({}^*(-))^{op}} & {\mathscr{A}_2}^{op} \\ \mathscr{B}_1 \ar[ur]^{((-)^*)^{op}} \ar[r]_{id} \ar@{}[ur]|{\varepsilon^{op}\Uparrow} & \mathscr{B}_1 \ar[r]_{F_\circ} \ar@{}[ur]|{\Uparrow \widetilde{F}^{op}} & \mathscr{B}_2 \ar[ur]_{((-)^*)^{op}} \ar@{}[ur]|{\Uparrow \eta^{op}} & }$$

It is not difficult to check that Φ defines a pseudo-natural transformation, this meaning that

- the 2-cell $\Phi_{G\circ F}$ associated to the composite of two 1-cells F and G pasted along the 0-cell $(\mathscr{A}, \mathscr{B})$ is the composite of the 2-cells Φ_G and Φ_F pasted along the 1-cell $\Phi_{(\mathscr{A},\mathscr{B})}$,
- the 2-cell Φ_{id} associated to an identity 1-cell is an identity 2-cell,
- for every 2-cell $\theta : F \Rightarrow G$, the 2-cell Φ_F pasted to the 2-cell $\mathcal{GF}(\theta)$ along the 1-cell $\mathcal{GF}(F)$ is equal to the 2-cell Φ_G pasted to the 2-cell θ along the 1-cell G. Note that establishing this last property requires the coherence diagram (10).

The pseudo-natural transformation Ψ is defined as follows. To every chirality $(\mathscr{A}, \mathscr{B})$, one associates the 1-cell $\Psi_{(\mathscr{A},\mathscr{B})}$ defined as the pair of functors

$$(\Psi_{(\mathscr{A},\mathscr{B})})_\bullet \quad : \quad \mathscr{A} \xrightarrow{id} \mathscr{A} \qquad\qquad (\Psi_{(\mathscr{A},\mathscr{B})})_\circ \quad : \quad \mathscr{A}^{op} \xrightarrow{({}^*(-))^{op}} \mathscr{B}$$

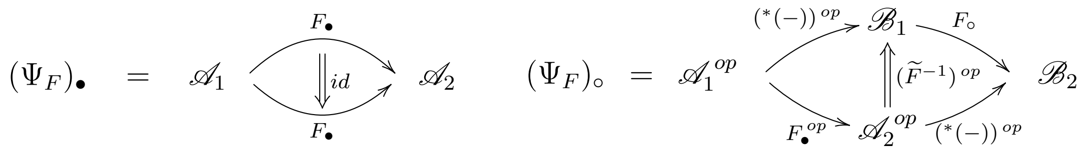
equipped with the natural transformation

$$\widetilde{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \xymatrix{\mathscr{A} \ar[r]^{id} \ar[d]_{id} & \mathscr{A} \ar[d]^{{}^*(-)} \ar@{}[dl]|{id \,\swarrow} \\ (\mathscr{A}^{op})^{op} \ar[r]_{{}^*(-)} & \mathscr{B}^{op}}$$

To every 1-dimensional cell $F : (\mathscr{A}_1, \mathscr{B}_1) \longrightarrow (\mathscr{A}_2, \mathscr{B}_2)$ one associates the reversible 2-cell

$$\Psi_F \quad : \quad F \circ \Psi_{(\mathscr{A}_1,\mathscr{B}_1)} \quad \Rightarrow \quad \Psi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ \mathcal{G}\mathcal{F}(F)$$

defined as the pair of natural transformations



It is not difficult to check that Ψ defines a pseudo-natural transformation in the same way as for Φ. At this point, there simply remains to show that the pair $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ defines an equivalence in the 2-category ***Chi*** in order to establish the statement of the theorem. The proof is essentially immediate and we conclude.

Remark. It should be mentioned that there exists a simpler proof that the 2-categories ***Cat*** and ***Chi*** are equivalent. To that purpose, one needs to prove first that the 2-functor $\mathcal{G}$ is a local equivalence, this meaning that every functor

$$\mathcal{G}(\mathscr{C}, \mathscr{D}) \quad : \quad \boldsymbol{Cat}(\mathscr{C}, \mathscr{D}) \quad \longrightarrow \quad \boldsymbol{Chi}(\mathcal{G}\mathscr{C}, \mathcal{G}\mathscr{D})$$

is an equivalence of categories. Then, one needs to prove that every chirality $(\mathscr{A}, \mathscr{B})$ is equivalent in ***Chi*** to a chirality of the form $\mathcal{G}\mathscr{C} = (\mathscr{C}, \mathscr{C}^{op})$. Both facts are easy to establish, and they imply that the 2-functor $\mathcal{G}$ is a biequivalence, see [6]. On the other hand, this alternative proof does not exhibit the 2-functor $\mathcal{F}$ nor the pseudo-natural transformations Φ and Ψ as in the proof of Theorem 1.

Strictification. Theorem 1 is inspired by a similar coherence theorem for monoidal categories, which establishes a biequivalence between the 2-category of strict monoidal categories, strict monoidal functors and monoidal natural transformations, and the 2-category of monoidal categories, monoidal functors and monoidal natural transformations. In particular, the 1-cell $\Phi_{(\mathscr{A},\mathscr{B})}$ should be understood as the operation of *strictifying* the chirality $(\mathscr{A}, \mathscr{B})$ into the category $\mathscr{A}$. It is also important to notice that the category of 1-cells between two chiralities $(\mathscr{A}_1, \mathscr{B}_1)$ and $(\mathscr{A}_2, \mathscr{B}_2)$ is equivalent but not isomorphic (in general) to the category of functors between the categories $\mathscr{A}_1$ and $\mathscr{A}_2$. In particular, the functor $\mathcal{G}$ is faithful, but not full ; conversely, the functor $\mathcal{F}$ is full, but not faithful.

Remark. Since the proof that the 2-categories ***Cat*** and ***Chi*** are biequivalent is purely equational, the argument could be performed in any 2-category ***Kat*** instead of ***Cat***. The only requirement on ***Kat*** is that it is equipped with a 2-functor

$$\dagger \quad : \quad \boldsymbol{Kat} \quad \longrightarrow \quad \boldsymbol{Kat}^{op(2)}$$

satisfying the equality

$$\dagger \circ \dagger = id.$$

Once recast in ***Kat***, the coherence theorem states that the 2-category ***Kat*** is biequivalent to a 2-category of chiralities defined in exactly the same way as the 2-category ***Chi*** except that (1) categories are replaced by objects of the 2-category ***Kat*** and (2) the operation $(-)^{op}$ on categories is replaced by the operation † on objects of ***Kat***. A typical example of such a 2-category ***Kat*** is provided by the 2-category ***MonCat*** of monoidal categories, lax monoidal functors and monoidal natural transformations, equipped with the 2-functor

$$\begin{array}{ccccc} \dagger & : & \boldsymbol{MonCat} & \longrightarrow & \boldsymbol{MonCat}^{op(2)} \\ & & \mathscr{C} & \mapsto & \mathscr{C}^{op(0,1)} \end{array}$$

which transports every monoidal category $\mathscr{C}$ to the monoidal category $\mathscr{C}^{op(0,1)}$ obtained by reversing the orientation of tensors and morphisms in $\mathscr{C}$. From this follows that the 2-category ***MonCat*** is biequivalent to the 2-category ***MonChi*** of monoidal chiralities defined in the expected way.

3 Dialogue categories

We have just established a coherence theorem (Theorem 1) for categories and chiralities in the previous section §2. In the remainder of the article, we will adapt the coherence theorem to dialogue categories. To that purpose, we follow the same pattern as in §2 and thus construct below a 2-category ***DiaCat*** of dialogue categories, dialogue functors and dialogue transformations.

3.1 Definition

We start by recalling the definition of a dialogue category.

Definition 2 (tensorial pole) *A tensorial pole in a monoidal category $\mathscr{C}$ is an object $\perp$ equipped with a representation*

$$\varphi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(y, x \multimap \perp)$$

of the functor

$$y \mapsto \mathscr{C}(x \otimes y, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object x, and with a representation

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \quad \cong \quad \mathscr{C}(x, \perp \multimapinv y)$$

of the functor

$$x \mapsto \mathscr{C}(x \otimes y, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object y.

Definition 3 (dialogue category) *A dialogue category is a monoidal category equipped with a tensorial pole.*

3.2 The 2-category *DiaCat* of dialogue categories

The 2-category ***DiaCat*** has dialogue categories as 0-cells, dialogue functors as 1-cells and dialogue transformations as 2-cells.

The 1-dimensional cells. A dialogue functor

$$(F, \perp_F) \quad : \quad (\mathscr{C}, \perp_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \perp_{\mathscr{D}})$$

between dialogue categories is defined as a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a morphism

$$\perp_F \quad : \quad F(\perp_{\mathscr{C}}) \quad \longrightarrow \quad \perp_{\mathscr{D}}.$$

The 2-dimensional cells. A dialogue transformation

$$\theta \quad : \quad (F, \bot_F) \quad \Rightarrow \quad (G, \bot_G)$$

is defined as a monoidal natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

making the diagram

$$\begin{array}{ccc} F(\bot_{\mathscr{C}}) & & \\ & \searrow^{\bot_F} & \\ \downarrow{\scriptstyle \theta_{\bot_{\mathscr{C}}}} & & \bot_{\mathscr{D}} \\ & \nearrow_{\bot_G} & \\ G(\bot_{\mathscr{C}}) & & \end{array} \tag{11}$$

commute.

Remark. It may be surprising at first sight that the two operations $(x \mapsto x \multimap \bot)$ and $(x \mapsto \bot \multimapinv x)$ as well as the two natural bijections φ and ψ are not mentioned in the definition of dialogue functors and dialogue transformations. This appears to be not necessary, because the map

$$F(\bot_{\mathscr{C}} \multimapinv x) \otimes Fx \longrightarrow F((\bot_{\mathscr{C}} \multimapinv x) \otimes x) \longrightarrow F(\bot_{\mathscr{C}}) \longrightarrow \bot_{\mathscr{D}}$$

induces a map

$$F(\bot_{\mathscr{C}} \multimapinv x) \quad \longrightarrow \quad \bot_{\mathscr{D}} \multimapinv F(x)$$

and similarly for

$$F(x \multimap \bot_{\mathscr{C}}) \quad \longrightarrow \quad F(x) \multimap \bot_{\mathscr{D}}.$$

Moreover, these two maps make the expected coherence diagram

$$\begin{array}{ccc} \mathscr{C}(x \otimes y, \bot_{\mathscr{C}}) & \xrightarrow{\psi_{x,y}} & \mathscr{C}(x, \bot_{\mathscr{C}} \multimapinv y) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathscr{D}(F(x \otimes y), F(\bot_{\mathscr{C}})) & & \mathscr{D}(Fx, F(\bot_{\mathscr{C}} \multimapinv y)) \\ \downarrow & & \downarrow{\scriptstyle (*)} \\ \mathscr{D}(Fx \otimes Fy, \bot_{\mathscr{D}}) & \xrightarrow{\psi_{F(x),F(y)}} & \mathscr{D}(Fx, \bot_{\mathscr{D}} \multimapinv Fy) \end{array}$$

commute. This last point is established by replacing the map $(*)$ by its definition as the unique function making the diagram

$$
\begin{array}{ccc}
\mathscr{D}(Fx, F(\bot_{\mathscr{C}} \multimapinv y)) & \xrightarrow{(*)} & \mathscr{D}(Fx, \bot_{\mathscr{D}} \multimapinv Fy) \\
\Big\downarrow {\scriptstyle -\otimes Fy} & & \Big\uparrow {\scriptstyle \psi_{Fx,Fy}} \\
\mathscr{D}(Fx \otimes Fy, F(\bot_{\mathscr{C}} \multimapinv y) \otimes Fy) & & \mathscr{D}(Fx \otimes Fy, \bot_{\mathscr{D}}) \\
\Big\downarrow {\scriptstyle monoidality\, of\, F} & & \Big\uparrow {\scriptstyle \bot_F} \\
\mathscr{D}(Fx \otimes Fy, F((\bot_{\mathscr{C}} \multimapinv y) \otimes y)) & \xrightarrow{F(eval)} & \mathscr{D}(Fx \otimes Fy, F(\bot_{\mathscr{C}}))
\end{array}
$$

commute.

3.3 An adjunction between negation and itself

In every dialogue category, the family of objects $(x \multimap \bot)_{x \in obj(\mathscr{C})}$ defines a functor

$$x \mapsto (x \multimap \bot) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{op}$$

uniquely determined by the requirement that the bijection $\varphi_{x,y}$ is natural in x and y. This property is established by a simple argument, based on the Yoneda lemma. Similarly, the family of objects $(\bot \multimapinv y)_{y \in obj(\mathscr{C})}$ defines a functor

$$y \mapsto (\bot \multimapinv y) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{op}$$

uniquely determined by the requirement that the bijection $\psi_{x,y}$ is natural in x and y. Moreover, the two functors

$$L(x) \quad = \quad x \multimap \bot \qquad \text{and} \qquad R(x) \quad = \quad \bot \multimapinv x$$

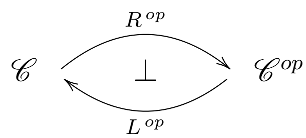
are related by an adjunction

$$
\mathscr{C} \;\underset{R}{\overset{L}{\rightleftarrows}}\; \mathscr{C}^{op} \qquad (L \dashv R) \tag{12}
$$

induced by the series of natural bijections

$$
\begin{array}{rcll}
\mathscr{C}(x, \bot \multimapinv y) & \cong & \mathscr{C}(x \otimes y, \bot) & \text{defined by } \psi_{x,y} \\
& \cong & \mathscr{C}(y, x \multimap \bot) & \text{defined by } \varphi_{x,y} \\
& = & \mathscr{C}^{op}(x \multimap \bot, y) & \text{by definition of } \mathscr{C}^{op}.
\end{array}
$$

Remark. The choice of the functor $(x \mapsto x \multimap \bot)$ as a left adjoint is somewhat arbitrary, because the 2-functor $(-)^{op}$ transports the adjunction (12) into its companion adjunction



where the role of the two functors $(x \mapsto x \multimap \bot)$ and $(x \mapsto \bot \multimapinv x)$ have been interchanged. This second adjunction is witnessed by the series of natural bijections

$$\begin{array}{rcll} \mathscr{C}(y, x \multimap \bot) & \cong & \mathscr{C}(x \otimes y, \bot) & \text{defined by } \varphi_{x,y} \\ & \cong & \mathscr{C}(x, \bot \multimapinv y) & \text{defined by } \psi_{x,y} \\ & = & \mathscr{C}^{op}(\bot \multimapinv y, x) & \text{by definition of } \mathscr{C}^{op}. \end{array}$$

4 Reflection chiralities

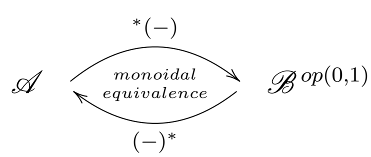
In this section, we recall the notion of *reflection chirality* discussed in the introduction, and construct a 2-category ***RefChi*** of reflection chiralities.

4.1 Definition

Definition 4 (reflection chirality) *A reflection chirality is defined as a pair of monoidal categories*

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with a monoidal equivalence



with a distributor, or categorical bimodule:

$$\langle - \,|\, - \rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee {}^*m \,\rangle$$

natural in a and b. The family χ is moreover required to make the diagram below

$$\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\quad \chi_{m \owedge n} \quad} & & \langle\, a \,|\, b \ovee {}^*(m \owedge n) \,\rangle \\
\Big\downarrow {\scriptstyle associativity} & & & & \Big\uparrow {\scriptstyle associativity \atop monoidality\ of\ negation} \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \owedge m \,|\, b \ovee {}^*n \,\rangle & \xrightarrow{\chi_m} & \langle\, a \,|\, (b \ovee {}^*n) \ovee {}^*m \,\rangle
\end{array} \tag{13}$$

commute.

Remark. The reader may find unexpected that our definition of reflection chirality does not include the coherence diagram

$$\begin{array}{ccc}
\langle\, a \owedge \mathbf{true} \,|\, b \,\rangle & \xrightarrow{\quad associativity \quad} & \langle\, a \,|\, b \,\rangle \\
\Big\downarrow {\scriptstyle \chi_{\mathbf{true}}} & & \Big\downarrow {\scriptstyle associativity} \\
\langle\, a \,|\, b \ovee {}^*\mathbf{true} \,\rangle & \xleftarrow[\quad]{monoidality \atop of\ negation} & \langle\, a \,|\, b \ovee \mathbf{false} \,\rangle
\end{array}$$

which provides a nullary counterpart to Diagram (16). The reason is that this diagram always commutes. This is established by instantiating the coherence diagram (16) at $m = n =$ **true** and by applying the naturality of the bijection χ and the coherence properties of the monoidal categories $\mathscr{A}$ and $\mathscr{B}$.

4.2 The 2-category *RefChi* of reflection chiralities

The 2-category ***RefChi*** has reflection chiralities as 0-dimensional cells, and the following notions of 1-dimensional and 2-dimensional cells.

The 1-dimensional cells. A 1-dimensional cell in ***RefChi***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a quadriple $(F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2$$

an oplax monoidal functor

$$F_\circ \quad : \quad \mathscr{B}_1 \quad \longrightarrow \quad \mathscr{B}_2$$

a monoidal natural isomorphism

$$\begin{array}{ccc}
\mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\
{\scriptstyle {}^*(-)}\downarrow & \widetilde{F} & \downarrow{\scriptstyle {}^*(-)} \\
\mathscr{B}_1^{op(0,1)} & \xrightarrow[\quad F_\circ^{\,op(0,1)} \quad]{} & \mathscr{B}_2^{op(0,1)}
\end{array}$$

and a natural transformation

$$\begin{array}{ccc}
 & Set & \\
{\scriptstyle \langle -|-\rangle_1}\nearrow & \overset{\Longrightarrow}{\overline{F}} & \nwarrow{\scriptstyle \langle -|-\rangle_2} \\
\mathscr{A}_1^{op} \times \mathscr{B}_1 & \xrightarrow[\quad F_\bullet^{\,op} \times F_\circ \quad]{} & \mathscr{A}_2^{op} \times \mathscr{B}_2
\end{array}$$

making the diagram

$$\begin{array}{ccc}
\langle\, a \oslash m \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, b \obslash {}^*m \,\rangle \\
{\scriptstyle \overline{F}}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\
\langle\, F_\bullet(a \oslash m) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(b \obslash {}^*m) \,\rangle \\
 & & \downarrow{\scriptstyle monoidality\, of\, F_\circ} \\
\downarrow{\scriptstyle monoidality\, of\, F_\bullet} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \obslash F_\circ({}^*m) \,\rangle \\
 & & \downarrow{\scriptstyle \widetilde{F}} \\
\langle\, F_\bullet(a) \oslash F_\bullet(m) \,|\, F_\circ(b) \,\rangle & \longrightarrow & \langle\, F_\bullet(a) \,|\, F_\circ(b) \obslash {}^*F_\bullet(m) \,\rangle
\end{array} \qquad \textbf{(14)}$$

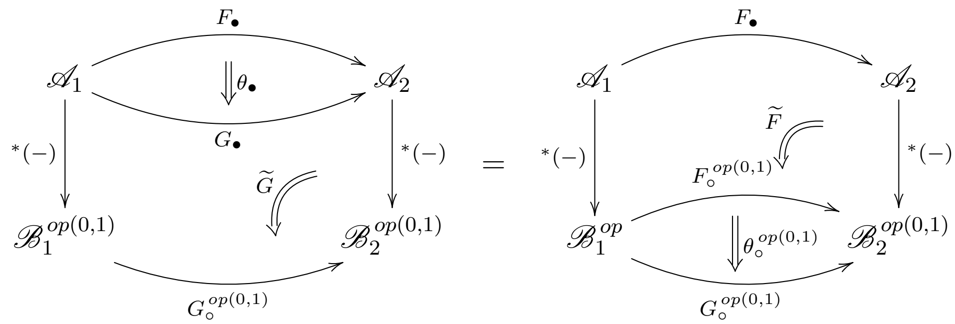
commute for all objects a, m in $\mathscr{A}$ and b in $\mathscr{B}$.

The 2-dimensional cells. A 2-dimensional cell in ***RefChi***

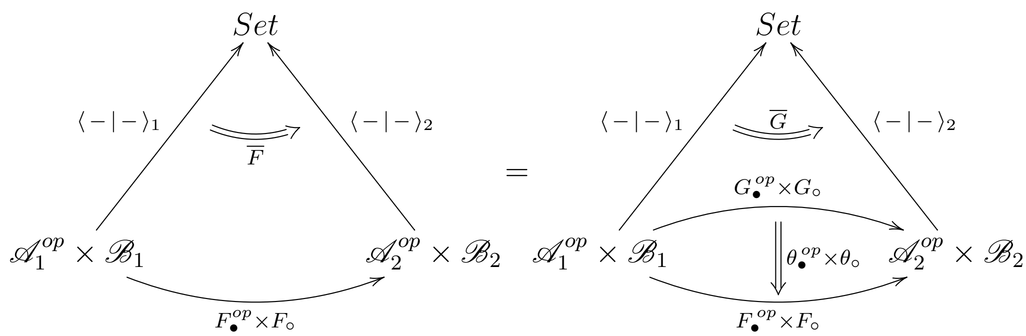
$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations

satisfying the two equalities below:



(15)



The 1-dimensional and 2-dimensional cells are composed by pasting the functors and natural transformations defining them in the expected way. This defines a 2-category of reflection chiralities, which will be denoted ***RefChi***.

Remark. The 2-category ***RefChi*** may be understood as a refinement of the 2-category ***MonChi*** where the structure of monoidal chiralities is extended with an evaluation bracket. In particular, it is already the case in ***MonChi*** that a 1-cell F consists of a lax monoidal functor $F_\bullet$ and an oplax monoidal functor $F_\circ$ and a monoidal natural isomorphism $\widetilde{F}$ relating them.

Remark. The coherence diagram (14) ensures that the natural transformation $\overline{F}$ may be recovered from the natural transformation

$$\langle\, a \,|\, \mathbf{false} \,\rangle_1 \xrightarrow{\overline{F}} \langle\, F_\bullet(a) \,|\, F_\circ(\mathbf{false}) \,\rangle_2 \xrightarrow{\text{monoidality}} \langle\, F_\bullet(a) \,|\, \mathbf{false} \,\rangle_2$$

together with the monoidal functor $F_\bullet$, the monoidal natural transformation $\widetilde{F}$ and the natural bijection χ^R.

5 Dialogue chiralities

The notion of reflection chirality is worth studying from an aesthetic point of view, but we will focus in this article on the specific case of dialogue chiralities, which provides a relaxed notion of dialogue category.

Definition 5 (dialogue chiralities) *A dialogue chirality is a pair of monoidal categories*

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with a monoidal equivalence

$$\mathscr{A} \underset{(-)^*}{\overset{{}^*(-)}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \qquad \text{(monoidal equivalence)}$$

with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R)$$

and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee {}^*m \,\rangle$$

natural in a and b, where $\langle\, a \,|\, b \,\rangle$ is defined as

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a\, ,\, R\, b\,).$$

The family χ is moreover required to make the diagram below

$$\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\quad \chi_{m \owedge n} \quad} & & \langle\, a \,|\, b \ovee {}^*(m \owedge n) \,\rangle \\
\downarrow \scriptstyle{associativity} & & & & \uparrow \scriptstyle{associativity,\ monoidality\ of\ negation} \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \owedge m \,|\, b \ovee {}^*n \,\rangle & \xrightarrow{\chi_m} & \langle\, a \,|\, (b \ovee {}^*n) \ovee {}^*m \,\rangle
\end{array} \qquad (16)$$

commute.

So, in short, a dialogue chirality is a reflection chirality whose evaluation bracket induced by an adjunction. One should be careful however that the adjunction $L \dashv R$ is an additional structure on top of the reflection chirality, rather than an additional property satisfied by the chirality. It is possible to reformulate the notions of 1-cells and 2-cells between reflection chiralities in order to define a 2-category ***DiaChi*** of dialogue chiralities.

The 1-dimensional cells. A 1-cell in ***DiaChi***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a quadruple $F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor $F_\bullet : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_\circ : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism $\widetilde{F} : {}^*(-) \circ F_\bullet \Rightarrow (F_\circ)^{op(0,1)} \circ {}^*(-)$ as in the general case formulated in Section 4.2, together with a natural transformation:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\ {\scriptstyle R}\uparrow & \overset{\overline{F}}{\Rightarrow} & \uparrow{\scriptstyle R} \\ \mathscr{B}_1 & \xrightarrow[\quad F_\circ \quad]{} & \mathscr{B}_2 \end{array}$$

making the diagram

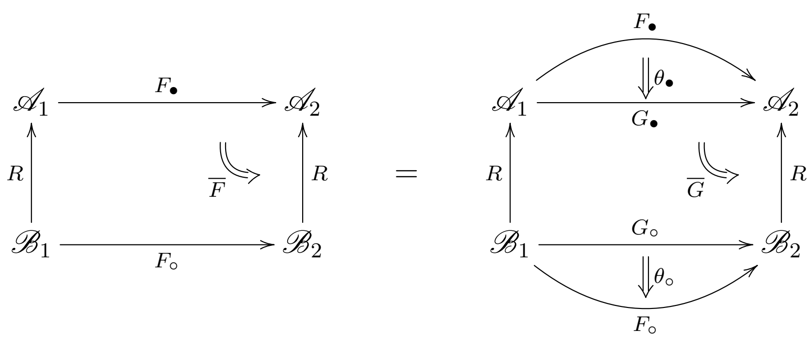
$$\begin{array}{ccc}
\mathscr{A}_1(a \oslash m, R\,b) & \longrightarrow & \mathscr{A}_1(a, R(b \obslash {}^*m)) \\
{\scriptstyle F_\bullet}\downarrow & & \downarrow{\scriptstyle F_\bullet} \\
\mathscr{A}_2(F_\bullet(a \oslash m), F_\bullet\, R\,b) & & \mathscr{A}_2(F_\bullet(a), F_\bullet\, R\,(b \obslash {}^*m)) \\
{\scriptstyle \overline{F}}\downarrow & & \downarrow{\scriptstyle \overline{F}} \\
\mathscr{A}_2(F_\bullet(a \oslash m), R\,F_\circ(b)) & & \mathscr{A}_2(F_\bullet(a), R\,F_\circ(b \obslash {}^*m)) \\
 & & \downarrow{\scriptstyle monoidality\,of\,F_\circ} \\
{\scriptstyle monoidality\,of\,F_\bullet}\downarrow & & \mathscr{A}_2(F_\bullet(a), R\,(F_\circ(b) \obslash F_\circ({}^*m))) \\
 & & \downarrow{\scriptstyle \widetilde{F}} \\
\mathscr{A}_2(F_\bullet(a) \oslash F_\bullet(m), R\,F_\circ(b)) & \longrightarrow & \mathscr{A}_2(F_\bullet(a), R\,(F_\circ(b) \obslash {}^*F_\bullet(m)))
\end{array} \qquad (17)$$

commute for all objects a, m in $\mathscr{A}_1$ and b in $\mathscr{B}_1$.

The 2-dimensional cells. A 2-cell in ***DiaChi***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations $\theta_\bullet : F_\bullet \Rightarrow G_\bullet$ and $\theta_\circ : G_\circ \Rightarrow F_\circ$ satisfying the equation (15) in Section 4.2 as well as the equation (18) below.



(18)

The following proposition is essentially straightforward:

Lemma 1 *The operation of forgeting the adjunction $L \dashv R$ in a dialogue chirality defines a 2-functor*

$$U \quad : \quad \textbf{\textit{DiaChi}} \quad \longrightarrow \quad \textbf{\textit{RefChi}}$$

which is fully faithful in the sense that the hom-functors

$$\textbf{\textit{DiaChi}}\,((\mathscr{A}_1, \mathscr{B}_1), (\mathscr{A}_2, \mathscr{B}_2)) \quad \longrightarrow \quad \textbf{\textit{RefChi}}\,(U(\mathscr{A}_1, \mathscr{B}_1), U(\mathscr{A}_2, \mathscr{B}_2))$$

are category isomorphisms for all dialogue chiralities $(\mathscr{A}_1, \mathscr{B}_1)$ and $(\mathscr{A}_2, \mathscr{B}_2)$.

6 The coherence theorem

In this section, we construct a 2-dimensional equivalence between the 2-category ***DiaCat*** of dialogue categories and the 2-category ***DiaChi*** of dialogue chiralities. From this follows that every dialogue chirality is equivalent to the image of a dialogue category in the 2-category ***DiaChi*** – this establishing the coherence theorem claimed in the introduction, as well as a recipe to strictify a dialogue chirality into a dialogue category.

6.1 From dialogue chiralities to dialogue categories

We start by constructing a 2-functor

$$\mathcal{F} \quad : \quad \textbf{\textit{DiaChi}} \quad \longrightarrow \quad \textbf{\textit{DiaCat}}$$

from the 2-category ***DiaChi*** of dialogue chiralities to the 2-category ***DiaCat*** of dialogue categories.

The 0-dimensional cells. The 2-functor transports every dialogue chirality $(\mathscr{A}, \mathscr{B})$ to the dialogue category defined as

$$(\mathscr{C}, \otimes, I) \quad := \quad (\mathscr{A}, \oslash, \mathbf{true})$$

equipped with the tensorial pole

$$\bot \quad := \quad R(\mathbf{false}).$$

together with the functors:

$$x \multimap \bot \;=\; (L(x))^{*} \qquad\qquad \bot \multimapinv x \;=\; R({}^{*}x).$$

The natural bijections φ and ψ are then defined by composing the series of natural bijections

$$\begin{array}{rclr}
\mathscr{C}(x \otimes y, \bot) & = & \mathscr{A}(x \oslash y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \bot, \\
& \cong & \mathscr{A}(x, R(\mathbf{false} \varobar {}^{*}y)) & \text{by applying } \chi_{y,x,\mathbf{false}}, \\
& \cong & \mathscr{A}(x, R({}^{*}y)) & \text{by applying the unit law in } \mathscr{B}, \\
& \cong & \mathscr{B}(L(x), {}^{*}y) & \text{by the adjunction } L \dashv R, \\
& \cong & \mathscr{A}(y, (L(x))^{*}) & \text{by the adjunction } {}^{*}(-) \dashv (-)^{*}, \\
& = & \mathscr{C}(y, (L(x))^{*}) & \text{by definition of } \mathscr{C}.
\end{array}$$

$$\begin{array}{rclr}
\mathscr{C}(x \otimes y, \bot) & = & \mathscr{A}(x \oslash y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \bot, \\
& \cong & \mathscr{A}(x, R(\mathbf{false} \varobar {}^{*}y)) & \text{by applying } \chi_{y,x,\mathbf{false}}, \\
& \cong & \mathscr{A}(x, R({}^{*}y)) & \text{by applying the unit law in } \mathscr{B}, \\
& = & \mathscr{C}(x, R({}^{*}y)) & \text{by definition of } \mathscr{C}.
\end{array}$$

The 1-dimensional cells. Every 1-dimensional cell

$$F = (F_{\bullet}, F_{\circ}, \widetilde{F}, \overline{F}) \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is transported to the dialogue functor $(F_{\bullet}, \bot_F)$ consisting of the functor

$$F_{\bullet} \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2.$$

and the morphism

$$\bot_F \quad : \quad F_\bullet(\bot_{\mathscr{A}_1}) \quad \longrightarrow \quad \bot_{\mathscr{A}_2}$$

defined as the composite

$$F_\bullet \circ R\,(\mathbf{false}) \xrightarrow{\overline{F}_{\mathbf{false}}} R \circ F_\circ\,(\mathbf{false}) \xrightarrow{\;\;monoidality\;\;} R\,(\mathbf{false})$$

The 2-dimensional cells. Every 2-dimensional cell $\theta = (\theta_\bullet, \theta_\circ)$ is transported to the dialogue transformation $\theta_\bullet$. One easily checks that the monoidal natural transformation $\theta_\bullet$ makes Diagram (11) commute.

6.2 From dialogue categories to dialogue chiralities

We construct a 2-functor

$$\mathcal{G} \quad : \quad \boldsymbol{DiaCat} \quad \longrightarrow \quad \boldsymbol{DiaChi}$$

from the 2-category ***DiaCat*** of dialogue categories to the 2-category ***DiaChi*** of dialogue chiralities.

The 0-dimensional cells. To every dialogue category $\mathscr{C}$, the 2-functor $\mathcal{G}$ associates the dialogue chirality defined as follows:

$$(\mathscr{A}, \oslash, \mathbf{true}) \;:=\; (\mathscr{C}, \otimes, e) \qquad\qquad (\mathscr{B}, \obslash, \mathbf{false}) \;:=\; (\mathscr{C}, \otimes, e)^{op(0,1)}.$$

The monoidal equivalence between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$ is defined by the identity functors on the category $\mathscr{C}$. between $L(x) = x \multimap \bot$ and $R(x) = \bot \multimapinv x$ is witnessed by the series of bijections

$$\begin{array}{rcl} \mathscr{A}(x, R(y)) & = & \mathscr{C}(x, \bot \multimapinv y) \\ & \cong & \mathscr{C}(x \otimes y, \bot) \\ & \cong & \mathscr{C}(y, x \multimap \bot) \\ & = & \mathscr{B}(L(x), y) \end{array}$$

natural in x and y. The natural bijection $\chi_{m,x,y}$ is defined as follows:

$$\begin{array}{ccc} \mathscr{C}(x \otimes m, \bot \multimapinv y) & & \mathscr{C}(x, \bot \multimapinv (m \otimes y)) \\ \Big\downarrow {\scriptstyle \psi^{-1}_{x \otimes m, y}} & & \Big\uparrow {\scriptstyle \psi_{x, m \otimes y}} \\ \mathscr{C}((x \otimes m) \otimes y, \bot) & \xrightarrow{\;\;associativity\;\;} & \mathscr{C}(x \otimes (m \otimes y), \bot) \end{array}$$

It follows easily from the naturality of χ that the family χ satisfies Equation (16) required by the notion of reflection chirality.

The 1-dimensional cells. To every dialogue functor

$$(F, \bot_F) \quad : \quad (\mathscr{C}, \bot_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \bot_{\mathscr{D}})$$

the 2-functor $\mathcal{G}$ associates the 1-dimensional cell $\mathcal{G}(F)$ defined as the quadruple consisting of the lax monoidal functor

$$\mathcal{G}(F)_{\bullet} \ : \ \mathscr{C} \ \xrightarrow{F} \ \mathscr{D}$$

the oplax monoidal functor

$$\mathcal{G}(F)_{\circ} \ : \ \mathscr{C}^{\,op(0,1)} \ \xrightarrow{F^{\,op(0,1)}} \ \mathscr{D}^{\,op(0,1)}$$

the monoidal isomorphism $\widetilde{\mathcal{G}(F)}$ defined as the identity on the functor F, and the natural transformation

$$\overline{\mathcal{G}(F)} \quad : \quad R \circ F \quad \longrightarrow \quad F \circ R$$

whose components

$$F(\bot_{\mathscr{C}} \multimapinv x) \quad \longrightarrow \quad \bot_{\mathscr{D}} \multimapinv F(x)$$

is associated by $\chi_{F(x)}$ to the morphism

$$F(\bot_{\mathscr{C}} \multimapinv x) \otimes F(x) \ \longrightarrow \ F((\bot_{\mathscr{C}} \multimapinv x) \otimes x) \ \longrightarrow \ F(\bot_{\mathscr{C}}) \ \longrightarrow \ \bot_{\mathscr{D}}.$$

The monoidality of the functor F implies that this definition of the quadruple $\mathcal{G}(F)$ satisfies the equation (17) required of a 1-cell between dialogue chiralities.

The 2-dimensional cells. To every dialogue transformation

$$\mathscr{C} \ \underset{G}{\overset{F}{\Downarrow \theta}} \ \mathscr{D}$$

the 2-functor $\mathcal{G}$ associates the 2-dimensional cell $\mathcal{G}(\theta)$ defined as the pair of monoidal natural transformations

$$\mathcal{G}(\theta)_{\bullet} \quad = \quad \mathscr{A}_1 = \mathscr{C} \ \underset{G}{\overset{F}{\Downarrow \theta}} \ \mathscr{D} = \mathscr{A}_2$$

$$\mathcal{G}(\theta)_{\circ} \quad = \quad \mathscr{B}_1 = \mathscr{C}^{\,op(0,1)} \ \underset{G}{\overset{F}{\Uparrow \theta^{\,op(0,1)}}} \ \mathscr{D}^{\,op(0,1)} = \mathscr{B}_2$$

One then checks that the two equations required by the definition of a 2-cell in ***DiaChi*** are satisfied: Equation (4.2) is immediate while Equation (18) follows easily from the naturality of χ.

6.3 The pseudo-natural transformation Φ

As in the introductory case of categories and reflection chiralities investigated in Section 2, the composite 2-functor

$$\textbf{\textit{DiaCat}} \xrightarrow{\mathcal{G}} \textbf{\textit{DiaChi}} \xrightarrow{\mathcal{F}} \textbf{\textit{DiaCat}}$$

coincides with the identity on the 2-category ***DiaCat*** of dialogue categories. In order to establish that ***DiaCat*** and ***DiaChi*** are biequivalent, we proceed as in the proof of Theorem 1 and construct a pair of pseudo-natural transformations

$$\Phi \ : \ Id \longrightarrow \mathcal{G}\circ\mathcal{F} \qquad\qquad \Psi \ : \ \mathcal{G}\circ\mathcal{F} \longrightarrow Id$$

between the identity 2-functor on ***Chi*** and the 2-functor $\mathcal{G}\circ\mathcal{F}$, and show that their components $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ define an equivalence in the 2-category ***DiaChi***, for every dialogue chirality $(\mathscr{A},\mathscr{B})$. To that purpose, it is important to describe precisely the structure of the dialogue chiralities $(\mathscr{A},\mathscr{A}^{\,op(0,1)})$ obtained by applying the 2-functor $\mathcal{G}\circ\mathcal{F}$ to a dialogue chirality $(\mathscr{A},\mathscr{B})$. The dialogue chirality $(\mathscr{A},\mathscr{A}^{\,op(0,1)})$ is equipped with the trivial monoidal equivalence:

$$\mathscr{A} \ \underset{id}{\overset{id}{\rightleftarrows}}\ \left(\mathscr{A}^{\,op(0,1)}\right)^{op(0,1)} \qquad (\textit{monoidal equivalence})$$

with the adjunction

$$\mathscr{A} \ \underset{R}{\overset{L}{\rightleftarrows}}\ \mathscr{B} \ \underset{({}^{*}(-))^{\,op(0,1)}}{\overset{((-)^{*})^{\,op(0,1)}}{\rightleftarrows}}\ \mathscr{A}^{\,op(0,1)} \qquad (L \dashv R, \ \perp)$$

From this follows that

$$\langle\, a_1 \,|\, a_2 \,\rangle_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})} \quad = \quad \mathscr{A}(a_1, R\,({}^{*}a_2)) \quad = \quad \langle\, a_1 \,|\, {}^{*}a_2 \,\rangle_{(\mathscr{A},\mathscr{B})}$$

Moreover, the natural transformation $\chi_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})}$ at instance (a,m,b) is defined as the composite function

$$
\begin{array}{ccc}
\langle\, a_1 \owedge m \,|\, {}^*a_2 \,\rangle & & \langle\, a_1 \,|\, {}^*(m \owedge a_2) \,\rangle \\
\big\downarrow & & \big\uparrow \\
\langle\, a_1 \owedge m \,|\, \mathbf{false} \ovee {}^*a_2 \,\rangle & & \langle\, a_1 \,|\, \mathbf{false} \ovee {}^*(m \owedge a_2) \,\rangle \\
{\scriptstyle (\chi_{(\mathscr{A},\mathscr{B})})^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle \chi_{(\mathscr{A},\mathscr{B})}} \\
\langle\, (a_1 \owedge m) \owedge a_2 \,|\, \mathbf{false} \,\rangle & \longrightarrow & \langle\, a_1 \owedge (m \owedge a_2) \,|\, \mathbf{false} \,\rangle
\end{array}
$$

At this point, it is also helpful to notice that the coherence diagram (16) satisfied by the reflection chirality $(\mathscr{A},\mathscr{B})$ ensures that the diagram

$$
\begin{array}{ccc}
 & \langle\, a_1 \owedge m \,|\, {}^*a_2 \,\rangle & \\
{\scriptstyle \chi_{(\mathscr{A},\mathscr{B})}}\swarrow & & \searrow{\scriptstyle \chi_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})}} \\
\langle\, a_1 \,|\, {}^*a_2 \ovee {}^*m \,\rangle & \xrightarrow{\ \ monoidality\ \ } & \langle\, a_1 \,|\, {}^*(m \owedge a_2) \,\rangle
\end{array}
\tag{19}
$$

commutes.

The 1-dimensional cells $\Phi_{(\mathscr{A},\mathscr{B})}$. To every reflection chirality $(\mathscr{A},\mathscr{B})$ one associates the 1-cell

$$
\Phi_{(\mathscr{A},\mathscr{B})} \quad : \quad (\mathscr{A},\mathscr{B}) \quad \longrightarrow \quad \left(\mathscr{A},\mathscr{A}^{\,op(0,1)}\right)
$$

defined as the pair of monoidal functors

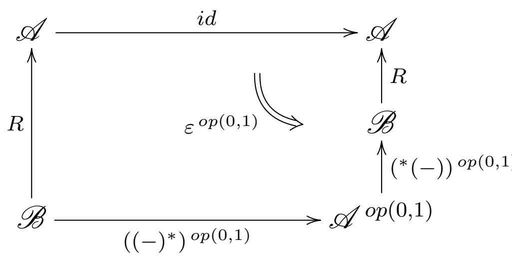
$$
\left(\Phi_{(\mathscr{A},\mathscr{B})}\right)_{\bullet} \ : \ \mathscr{A} \xrightarrow{\ id\ } \mathscr{A} \qquad\qquad \left(\Phi_{(\mathscr{A},\mathscr{B})}\right)_{\circ} \ : \ \mathscr{B} \xrightarrow{\ ((-)^*)^{\,op(0,1)}\ } \mathscr{A}^{\,op(0,1)}
$$

together with the monoidal natural isomorphism

$$
\widetilde{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = \quad
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\qquad id \qquad} & \mathscr{A} \\
{\scriptstyle {}^*(-)}\big\downarrow & \overset{\eta}{\Swarrow} & \big\downarrow{\scriptstyle id} \\
\mathscr{B}^{\,op(0,1)} & \xrightarrow[\ (-)^*\ ]{} & \left(\mathscr{A}^{\,op(0,1)}\right)^{op(0,1)}
\end{array}
$$

and the natural transformation

$$\overline{\Phi_{(\mathscr{A},\mathscr{B})}} \quad = $$



where η (resp. ε) denotes the unit (resp. counit) of the adjunction $^*(-) \dashv (-)^*$. Once commutativity of (19) established, one easily checks that this definition of $\Phi_{(\mathscr{A},\mathscr{B})}$ makes the diagram (17) commute, and thus provides a valid definition of a 1-cell in the 2-category ***DiaChi***.

The 2-dimensional cells Φ_F. To every 1-dimensional cell in ***DiaChi***

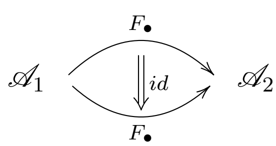
$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

one associates the 2-cell in ***DiaChi***

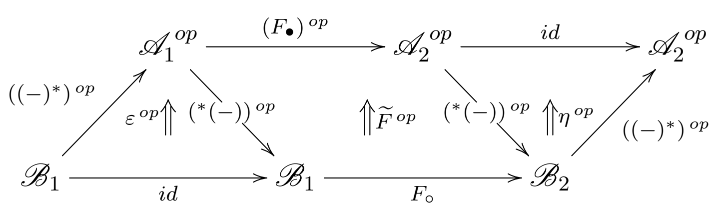
$$\Phi_F \quad : \quad \Phi_{(\mathscr{A}_2,\mathscr{B}_2)} \circ F \quad \Rightarrow \quad \mathcal{GF}(F) \circ \Phi_{(\mathscr{A}_1,\mathscr{B}_1)}$$

defined as the pair of monoidal natural transformations

$$(\Phi_F)_\bullet \quad = $$



$$(\Phi_F)_\circ \quad = $$



Note that the 2-cell Φ_F is defined in exactly the same way as in the proof of Theorem 1. One easily checks that the definition makes the diagrams (15) and (18) commute, and thus defines a 2-cell in the 2-category ***DiaChi***. Moreover, the family Φ defines a pseudo-natural transformation.

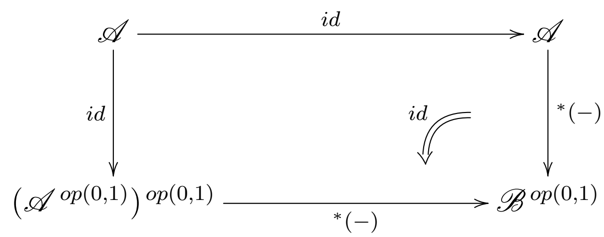
6.4 The pseudo-natural transformation Ψ

The 1-dimensional cells $\Psi_{(\mathscr{A},\mathscr{B})}$. To every dialogue chirality $(\mathscr{A},\mathscr{B})$, one associates the 1-cell $\Psi_{(\mathscr{A},\mathscr{B})}$ defined as the pair of functors

$$(\Psi_{(\mathscr{A},\mathscr{B})})_\bullet \;:\; \mathscr{A} \xrightarrow{id} \mathscr{A} \qquad\qquad (\Psi_{(\mathscr{A},\mathscr{B})})_\circ \;:\; \mathscr{A}^{\,op(0,1)} \xrightarrow{(^*(-))^{\,op(0,1)}} \mathscr{B}$$

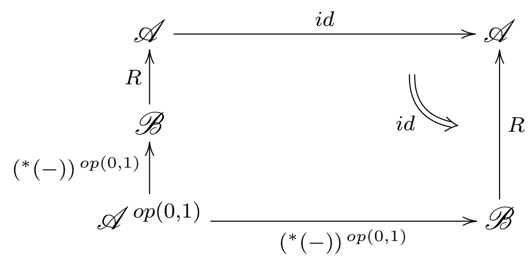
equipped with the trivial monoidal natural isomorphism

$$\widetilde{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = $$



and with the trivial natural transformation

$$\overline{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = $$



Just as in the case of $\Phi_{(\mathscr{A},\mathscr{B})}$, it is easy to check that this definition of $\Psi_{(\mathscr{A},\mathscr{B})}$ makes the diagram (17) commute once commutativity of (19) is established. As such, it provides a valid definition of a 1-cell in the 2-category ***DiaChi***.

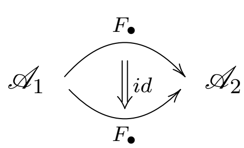
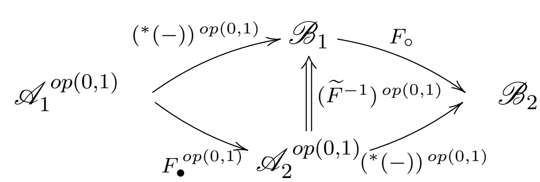
The 2-dimensional cells Ψ_F. To every 1-dimensional cell F in ***DiaChi***

$$F \quad : \quad (\mathscr{A}_1,\mathscr{B}_1) \quad\longrightarrow\quad (\mathscr{A}_2,\mathscr{B}_2)$$

one associates the reversible 2-cell in ***DiaChi***

$$\Psi_F \quad : \quad F\circ\Psi_{(\mathscr{A}_1,\mathscr{B}_1)} \quad\Rightarrow\quad \Psi_{(\mathscr{A}_2,\mathscr{B}_2)}\,\circ\,\mathcal{GF}(F)$$

defined as the pair of monoidal natural transformations

$(\Psi_F)_\bullet =$  $(\Psi_F)_\circ =$ 

One checks easily that the natural transformations $(\Psi_F)_\circ$ and $(\Psi_F)_\bullet$ satisfy the equation (15) of §4.2. as well as the equation (18) of §5. It is not difficult either to see that Ψ defines a pseudo-natural transformation, in the same way as in the proof of Theorem 1 in §2.

6.5 Main theorem

We proceed just as in the proof of Theorem 1 in §2. and establish that the pair of 1-cells $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ defines an equivalence in the 2-category ***DiaChi***, for every dialogue chirality $(\mathscr{A},\mathscr{B})$. This fact is essentially immediate to check, this leading us to the main result of the article:

Theorem 2 (Coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence between the 2-categories* ***DiaCat*** *and* ***DiaChi****.*

7 Reflection categories

We are now in the position of doing some reverse analysis, and pto define the notion of reflection category which underlies the unbiased notion of reflection chirality.

7.1 Reflection categories

A reflection category is defined as a pair $(\mathscr{C},\bot_{\mathscr{C}})$ consisting of a monoidal category $\mathscr{C}$ together with a functor

$$\bot_{\mathscr{C}} \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad \boldsymbol{Set}.$$

The 2-category ***RefCat*** is defined as the 2-category with reflection categories as objects, reflection functors as 1-cells, and reflection transformations as 2-cells.

The 1-dimensional cells. A reflection functor

$$(F,\bot_F) \quad : \quad (\mathscr{C},\bot_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D},\bot_{\mathscr{D}})$$

between reflection categories is defined as a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a natural transformation

$$\bot_F \quad : \quad \bot_{\mathscr{C}} \quad \Longrightarrow \quad \bot_{\mathscr{D}} \circ F.$$

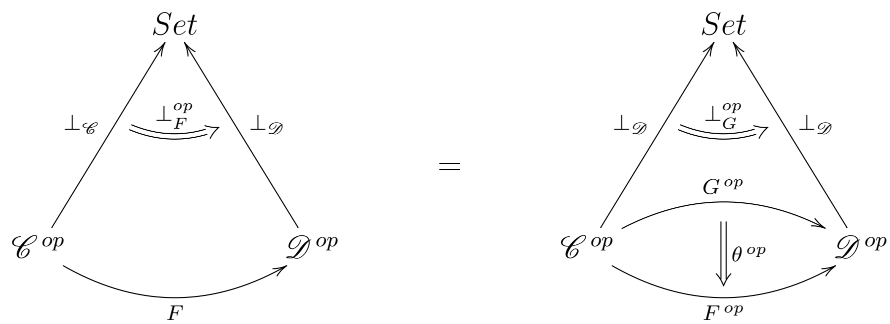
The 2-dimensional cells. A reflection transformation

$$\theta \quad : \quad (F, \bot_F) \quad \Rightarrow \quad (G, \bot_G)$$

is defined as a monoidal natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

satisfying the equality:



It is not difficult to see that the resulting 2-category ***RefCat*** is biequivalent to the 2-category ***RefChi*** of reflection chiralities.

7.2 The companion of a dialogue category

Every dialogue category $\mathscr{C}$ defines an adjunction between the left adjoint functor

$$L \quad : \quad x \mapsto (x \multimap \bot) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{\,op(0,1)}$$

and the right adjoint functor

$$R \quad : \quad x \mapsto (\bot \multimapinv x) \quad : \quad \mathscr{C}^{\,op(0,1)} \quad \longrightarrow \quad \mathscr{C}.$$

We have seen in §3.3 that this choice of adjunction is somewhat arbitrary, since the functor $R^{op(1)}$ is left adjoint to the functor $L^{op(1)}$. This symmetry between $L \dashv R$ and $R^{op(1)} \dashv L^{op(1)}$ implies that every dialogue category $\mathscr{C}$ induces another dialogue category $\mathcal{H}(\mathscr{C}) = \mathscr{C}^{\,op(0)}$, called its companion. Moreover, this operation $\mathcal{H}$ defines a 2-functor

$$\mathcal{H} \quad : \quad \textbf{\textit{DiaCat}} \quad \longrightarrow \quad \textbf{\textit{DiaCat}}$$

By the coherence theorem, there exists a bifunctor (in fact it is a 2-functor)

$$\mathcal{H} \quad : \quad \textbf{\textit{DiaChi}} \quad \longrightarrow \quad \textbf{\textit{DiaChi}}$$

which transports every dialogue chirality $(\mathscr{A}, \mathscr{B})$ to its companion dialogue chirality. This 2-functor is defined as follows:

$$\mathcal{H} \quad : \quad (\mathscr{A}, \mathscr{B}) \quad \mapsto \quad (\mathscr{A}^{op(0)}, \mathscr{B}^{op(0)})$$

where the adjunction on $\mathcal{H}(\mathscr{A}, \mathscr{B})$ is defined as the composite of the three adjunctions below:

$$\mathscr{A} \underset{(-)^*}{\overset{{}^*(-)}{\rightleftarrows}}_{\perp} \mathscr{B}^{op} \underset{L^{op}}{\overset{R^{op}}{\rightleftarrows}}_{\perp} \mathscr{A}^{op} \underset{(-)^*}{\overset{{}^*(-)}{\rightleftarrows}}_{\perp} \mathscr{B}$$

Note also the existence of an isomorphism

$$R(\mathbf{false}) \quad \cong \quad L(\mathbf{true})^* \tag{20}$$

in every dialogue chirality which reflects the isomorphism between $(\perp \multimapinv I)$ and $(I \multimap \perp)$ in every dialogue category. This isomorphism follows from the series of bijections:

$$\begin{array}{rclc}
\mathscr{A}(a, R(\mathbf{false})) & \cong & \mathscr{A}(\mathbf{true} \owedge a, R(\mathbf{false})) & \text{by the unit law,} \\
 & \cong & \mathscr{A}(\mathbf{true}, R(\mathbf{false} \ovee {}^*a)) & \text{by currification } \chi_a, \\
 & \cong & \mathscr{A}(\mathbf{true}, R({}^*a)) & \text{by the unit law,} \\
 & \cong & \mathscr{B}(L(\mathbf{true}), {}^*a) & \text{by the adjunction } L \dashv R\,, \\
 & \cong & \mathscr{B}^{op}({}^*a, L(\mathbf{true})) & \text{by definition of } \mathscr{B}^{op}\,, \\
 & \cong & \mathscr{A}(a, L(\mathbf{true})^*) & \text{by the equivalence } {}^*(-) \dashv (-)^*
\end{array}$$

natural in a, together with the Yoneda lemma.

7.3 The companion of a reflection category

The previous construction generalizes to an equivalence between reflection categories and reflection chiralities. Indeed, there exists a 2-functor

$$\mathcal{H}' \quad : \quad \boldsymbol{RefCat} \quad \longrightarrow \quad \boldsymbol{RefCat}$$

which transports every reflection category $(\mathscr{C}, \perp_{\mathscr{C}})$ to the reflection category $(\mathscr{C}^{op(0)}, \perp_{\mathscr{C}})$. As in the case of dialogue categories, the coherence theorem induces the existence of a bifunctor (in fact a 2-functor)

$$\mathcal{H}' \quad : \quad \boldsymbol{RefChi} \quad \longrightarrow \quad \boldsymbol{RefChi}$$

which transports every reflection chirality $(\mathscr{A}, \mathscr{B})$ to the reflection chirality $(\mathscr{A}^{op(0)}, \mathscr{B}^{op(0)})$ whose underlying monoidal equivalence

$$\mathscr{A}^{op(0)} \underset{(-)^{\circledast}}{\overset{^{\circledast}(-)}{\rightleftarrows}} \mathscr{B}^{op(1)} \qquad \text{(monoidal equivalence)}$$

is defined as the opposite of degree 0 of the original equivalence $(-)^* \dashv {}^*(-)$ between the monoidal categories $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$, whose distributor

$$\langle\!\langle - \,|\, - \rangle\!\rangle \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

is defined as the composite

$$\mathscr{A}^{op} \times \mathscr{B} \xrightarrow{\ {}^*(-)\times(-)^*\ } \mathscr{B} \times \mathscr{A}^{op} \xrightarrow{\ permute\ } \mathscr{A}^{op} \times \mathscr{B} \xrightarrow{\ \langle - | - \rangle\ } Set$$

and whose natural bijection

$$\overline{\chi}_{m,a,b} \quad : \quad \langle\!\langle\, a \owedge^{op} m \,|\, b \,\rangle\!\rangle \quad \longrightarrow \quad \langle\!\langle\, a \,|\, b \ovee^{op} {}^*m \,\rangle\!\rangle$$

is defined as the unique function making the diagram below commute:

$$\begin{array}{ccc}
\langle\!\langle\, a \owedge^{op} m \,|\, b \,\rangle\!\rangle & \xrightarrow{\ \overline{\chi}_{m,a,b}\ } & \langle\!\langle\, a \,|\, b \ovee^{op} {}^*m \,\rangle\!\rangle \\
\Big| \, \scriptstyle{definition} & & \Big| \, \scriptstyle{definition} \\
\langle\!\langle\, m \owedge a \,|\, b \,\rangle\!\rangle & & \langle\!\langle\, a \,|\, {}^*m \ovee b \,\rangle\!\rangle \\
\Big| \, \scriptstyle{definition} & & \Big| \, \scriptstyle{definition} \\
\langle\, b^* \,|\, {}^*(m \owedge a) \,\rangle & & \langle\, ({}^*m \ovee b)^* \,|\, {}^*a \,\rangle \\
\Big| \, \scriptstyle{monoidality} & & \Big| \, \scriptstyle{monoidality} \\
\langle\, b^* \,|\, {}^*a \ovee {}^*m \,\rangle & \xleftarrow[\ \chi_{({}^*m)}\ ]{} & \langle\, b^* \owedge m \,|\, {}^*a \,\rangle
\end{array}$$

One easily checks that $\overline{\chi}_m$ satisfies the two coherence axioms required of a reflection chirality. Note that there exists a family of isomorphisms

$$\langle\, a \,|\, \mathbf{false} \,\rangle \quad \cong \quad \langle\!\langle\, a \,|\, \mathbf{false} \,\rangle\!\rangle \tag{21}$$

natural in a, defined as the composite:

$$
\begin{array}{cccc}
\langle\!\langle\, a \,|\, \mathbf{false} \,\rangle\!\rangle & = & \langle\, \mathbf{false}^{*} \,|\, {}^{*}a \,\rangle & \text{by definition of } \langle\!\langle\, a \,|\, b \,\rangle\!\rangle \\
& \cong & \langle\, \mathbf{true} \,|\, {}^{*}a \,\rangle & \text{by monoidality of } (-)^{*}, \\
& \cong & \langle\, \mathbf{true} \,|\, \mathbf{false} \oslash {}^{*}a \,\rangle & \text{by associativity,} \\
& \cong & \langle\, \mathbf{true} \owedge a \,|\, \mathbf{false} \,\rangle & \text{by applying the isomorphism } \chi_a, \\
& \cong & \langle\, a \,|\, \mathbf{false} \,\rangle & \text{by associativity.}
\end{array}
$$

This provides a counterpart to the isomorphism (20). Indeed, in the particular case of a reflection chirality induced from a dialogue chirality $(\mathscr{A}, \mathscr{B})$, the series of natural bijections

$$
\begin{array}{cccl}
\langle\!\langle\, a \,|\, b \,\rangle\!\rangle & = & \langle\, b^{*} \,|\, {}^{*}a \,\rangle & \text{by definition of } \langle\!\langle\, - \,|\, - \,\rangle\!\rangle \\
& \cong & \mathscr{A}(b^{*}, R({}^{*}a)) & \text{by definition of a special distributor} \\
& \cong & \mathscr{B}(L(b^{*}), {}^{*}a) & \text{by adjunction } L \dashv R \\
& \cong & \mathscr{A}(a, (L(b^{*}))^{*}) & \text{by adjunction } (-)^{*} \dashv {}^{*}(-)
\end{array}
$$

ensures that the two notions of companionship coincide for dialogue chiralities and for reflection chiralities.

8 Postliminary remarks

We conclude with two independent remarks on tentative variants of the notions of dialogue category and of chirality which we studied in the present article.

Dialogue categories. The notion of dialogue category considered in this article was chosen for its simplicity: it is designed to provide an elementary and tractable notion of a category with a duality. On the other hand, it should be noticed that a satisfactory notion of dialogue category should be also equipped with a functor

$$
\begin{array}{ccc}
\mathscr{C}^{op} \times \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \\
(x, y) & \mapsto & x \multimap \bot \multimapinv y
\end{array}
$$

and a family of bijections

$$
\mathscr{C}(x \otimes y \otimes z, \bot) \quad \cong \quad \mathscr{C}(y, x \multimap \bot \multimapinv z)
$$

natural in x, y and z. It appears that this is always the case when the object $\bot$ is cyclic, or when the underlying monoidal category $\mathscr{C}$ is balanced or symmetric, see the companion paper [12] for details. This is also the case when the monoidal category $\mathscr{C}$ is biclosed, that is, the two functors $(x \otimes -)$ and $(- \otimes x)$ have a right adjoint noted $(x \multimap -)$ and $(- \multimapinv x)$ respectively, for every object x of the category. In that case, there exists an isomorphism

$$(x \multimap \bot) \multimapinv y \quad \cong \quad x \multimap (\bot \multimapinv y)$$

natural in x and y, and the object $x \multimap \bot \multimapinv y$ may be defined as any of these two objects of the category.

Exponential ideals. Although this aspect is not explored in this article, it should be mentioned here that it is possible to relax the constraint that the category $\mathscr{B}$ is equivalent to the category $\mathscr{A}^{op}$ in the unbiased description of a category $\mathscr{C}$ with duality. This relaxation enables us to consider situation where the categories $\mathscr{A}$ and $\mathscr{B}$ are properly independent, and not opposite modulo equivalence. A typical illustration is provided by the notion of an *exponential ideal* $\mathscr{B}$ in a cartesian category $\mathscr{A}$. In order to define this notion, one observes that there exists another way to characterize a pair $(\mathscr{A}, \mathscr{B})$ where $\mathscr{A}$ is equivalent to a cartesian closed category $\mathscr{C}$ and $\mathscr{B}$ is equivalent to its opposite category $\mathscr{C}^{op}$. This leads to the definition of an *exponential chirality* which is defined just as in the case of a cartesian closed chirality, as a pair of a category $(\mathscr{A}, \wedge, \mathbf{true})$ with finite products and of a category $(\mathscr{B}, \vee, \mathrm{false})$ with finite sums, together with an equivalence $\sim$ between $\mathscr{A}$ and $\mathscr{B}^{op}$. The difference between the two notions comes from the fact that the pseudo-action (3) and the natural bijection (4) are replaced by:

- a pseudo-action
$$\wedge \quad : \quad \mathscr{A} \quad \times \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{B} \tag{22}$$
of the monoidal category $(\mathscr{A}, \wedge, \mathbf{true})$ on the category $\mathscr{B}$,
- a bijection
$$\mathscr{A}(a_1 \wedge a_2, \sim b) \quad \cong \quad \mathscr{A}(a_2, \sim (a_1 \wedge b)) \tag{23}$$
natural in a_1, a_2 and b.

Note that in every cartesian closed category $\mathscr{C}$ may be seen as the exponential chirality $(\mathscr{C}, \mathscr{C}^{op})$ where the pseudo-action (22) transports the pair (a, b) into the object $a \Rightarrow b$ of the category $\mathscr{C}^{op}$.

The definition of a cartesian category $\mathscr{A}$ equipped with an exponential ideal $\mathscr{B}$ is then recovered by relaxing the definition of exponential chirality as follows:

- remove the hypothesis that the category $\mathscr{B}$ has finite sums,
- remove the functor $\sim : \mathscr{A} \to \mathscr{B}^{op}$,
- keep the functor $\sim : \mathscr{B}^{op} \to \mathscr{A}$ but remove the hypothesis that it defines an equivalence of categories.

Everything else remains the same, in particular the natural bijection (23) as well as the two coherence diagrams corresponding to (5) and (6) in the definition of cartesian closed chirality .

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Contents