Title

Games are continuation models!

On Blass phenomenon, duality and polarized linear logic

(work in progress)

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Goal

Explain Blass phenomenon
Understand the duality of control
Linearize Hyland-Ong games

How? by bridging three lines of research:

★ Game models.

— $\lambda\mu$-categories by L. Ong,

— the family construction by S. Abramsky and G. McCusker.

★ Continuation models.

— continuation categories by Y. Lafont, B. Reus and T. Streicher.

— $\otimes\neg$-categories by H. Thielecke,

— control categories by P. Selinger.

★ Linear logic.

— LC by J-Y. Girard,

— Polarized linear logic by O. Laurent.

Polarized linear logic = logic of continuations

I. Polarized games.

Polarization (Blass, Girard, Laurent)

Separate games in two dual classes.

(1) positive games starting by a Player move (=values),

(2) negative games starting by an Opponent move (=functions).

The connectives $\oplus$ and $\otimes$ and units 0 and 1 are positive.

Dually, the connectives $\&$ and $⅋$ and units $\top$ and $\bot$ are negative.

The modality ! transforms negative into positive.

Dually, the modality ? transforms positive into negative.

Eg. $$!(A \,\&\, B) \quad \cong \quad !A \otimes !B$$

Hyland-Ong games (arenas)

An arena is a triple $(M_A, \vdash_A, \lambda_A)$ where

— M_A is a set of moves,

— λ_A is a polarity function $M_A \longrightarrow \{+, -\}$,

— $\vdash_A$ is a justification relation defining a forest of alternated moves.

Hyland-Ong games (plays)

★ A justification sequence s over an arena A is a couple (f, j) such that

-1- f is a finite sequence of moves,

-2- j is a partial decreasing function (for all i in the domain of j: $j(i) < i$) over the domain of f, what is called the justification.

The i^{th} move of a justification sequence s is justified when,

— $j(i) = \perp$ and $f(i)$ is a root of the arena A,

— $j(i) = k$ and $f(k) \vdash_A f(i)$;

★ A justification sequence s is

— justified when all its moves are justified.

— well-opened when the first move is its only root move.

— player when its last move is of polarity $+$,

— opponent when its last move is of polarity $-$.

Linear Hyland-Ong games (definition)

A linear HO-game is a 4-tuple

$$A = (M_A, \vdash_A, \lambda_A, P_A)$$

where

— $(M_A, \lambda_A, \vdash_A)$ is a positive arena,

— P_A is a prefix-closed set of plays, ie. justified alternated and well-opened justification sequences.

A linear HO-game is

— positive when its arena has all roots positive,

— negative when its arena has all roots negative.

Strategies (definition)

A strategy is a set σ of plays such that:

— all plays of σ are player,

— σ is closed by "player" prefix:

$$\forall s \cdot a \cdot b \in P_A, \qquad s \cdot a \cdot b \in \sigma \implies s \in \sigma$$

— σ is deterministic:

$$\forall s \cdot a \cdot b \in P_A, s \cdot a \cdot c \in P_A, \qquad s \cdot a \cdot b \in \sigma \text{ et } s \cdot a \cdot c \in \sigma \implies b = c$$

Sum (of positive games)

Given two positive games A and B, the game $A \oplus B$ has

— the sum of forests $(M_A, \vdash_A, \lambda_A)$ and $(M_B, \vdash_B, \lambda_B)$ as arena,

— the sum of forests P_A and P_B as set of plays,

The unit 0 is the empty positive arena.

Tensor (of positive games) by distributivity

A game is simple when its arena has a root at most.

Here, we restrict to semi-simple games, that is finite sums of simple games:

$$A = \bigoplus_{i=0}^{n} B_i$$

Defining $A \otimes B$ over semi-simple games reduces to defining $A \otimes B$ over simple games, and applying the distributivity equality (**??**).

$$A \otimes (B \oplus C) = (A \otimes B) \otimes (A \otimes C)$$

The tensor product of two simple games A and B has

— the "coalesced sum" of trees $(M_A, \vdash_A, \lambda_A)$ and $(M_B, \vdash_B, \lambda_B)$ as arena,

— the "interleaved product" of P_A and P_B as set of plays $P_{A \otimes B}$.

The $\parr$-product (of negative games) and lift operators

Given two negative games A and B, define

$$A \parr B \quad = \quad (A^{\perp} \otimes B^{\perp})^{\perp}$$

and given a positive game A and a negative game B, define:

$$A \multimap B \quad = \quad A^{\perp} \parr B$$

Remark: the first move of $A \multimap B$ is a pair of a move in A and a move in B.

A lift $\downarrow$ from $-$ to $+$ and a lift $\uparrow$ from $+$ to $-$.

Remark: the positive game $\downarrow A$ and negative game $\uparrow A$ are simple.

II. Polar categories

Two dual categories: $\mathcal{G}[n]$ and $\mathcal{G}[v]$

Negative games form a category $\mathcal{G}[n]$:

— its morphisms $A \longrightarrow B$ are the strategies of $A \multimap_{n} B$,

where the negative game $A \multimap_{n} B$ is defined as:

$$\boxed{A \multimap_{n} B \quad = \quad \downarrow A \multimap B}$$

for negative games A and B.

Positive games form a category $\mathcal{G}[v]$:

— its morphisms $A \longrightarrow B$ are the strategies of $A \multimap_{v} B$.

where the negative game $A \multimap_{v} B$ is defined as:

$$\boxed{A \multimap_{v} B \quad = \quad A \multimap \uparrow B}$$

for positive games A and B.

Property: The categories $\mathcal{G}[v]$ and $\mathcal{G}[n]$ are dual.

Two dual categories: $\mathfrak{P}$ and $\mathfrak{N}$ of transverse strategies.

In order to have a better grip on control...

Definition: A strategy of $A \multimap\uparrow B$ is transverse (from A to B) if for every root a in A, there exists a root b in B such that $a \cdot b$ is a play of σ.

★ The category $\mathfrak{P}$ has
- positive games as objects,
- transverse strategies of $\mathcal{G}[v]$ as maps $A \xrightarrow{+} B$.

★ The category $\mathfrak{N}$ has
- negative games as objects,
- transverse strategies of $\mathcal{G}[n]$ as maps $A \xrightarrow{-} B$.

Two remarks:

— The categories $\mathfrak{P}$ and $\mathfrak{N}$ are dual.

— The category $\mathfrak{P}$ is monoidal $(\otimes, 1)$ and has sums $(\oplus, 0)$.

A remarkable adjunction

Two functors

$$\uparrow : \mathfrak{P} \longrightarrow \mathfrak{N} \qquad \downarrow : \mathfrak{N} \longrightarrow \mathfrak{P}$$

and an adjunction

$$\frac{\uparrow A \quad \stackrel{-}{\longrightarrow} \quad B}{A \quad \stackrel{+}{\longrightarrow} \quad \downarrow B}$$

Intuitively, the two hom-sets

$$\uparrow A \stackrel{-}{\longrightarrow} B \qquad\qquad A \stackrel{+}{\longrightarrow} \downarrow B$$

describe, each one in its own paradigm (call-by-name or call-by-value) the strategies of

$$A \multimap B$$

which wait for a simultaneous move of Opponent in $A^{\perp}$ and in B.

The meridian category of an adjunction $\uparrow \dashv \downarrow$

To every adjunction $\uparrow \dashv \downarrow$ between functors:

$$\uparrow : \mathfrak{P} \longrightarrow \mathfrak{N} \qquad \downarrow : \mathfrak{N} \longrightarrow \mathfrak{P}$$

one associates a meridian category $[\mathfrak{P}, \mathfrak{N}]$

— whose objects are the objects of $\mathfrak{P}$ and of $\mathfrak{N}$,

— whose morphisms $A \longrightarrow B$, are

- — the morphisms of $\mathfrak{P}$ between positive objects,
- — the morphisms of $\mathfrak{N}$ between negative objects,
- — the morphisms $A \longrightarrow \downarrow B$ in $\mathfrak{P}$ or $\uparrow A \longrightarrow B$ in $\mathfrak{N}$, from a positive object A^+ to a negative object B^-.

— composition is defined using the adjunction $\uparrow \dashv \downarrow$.

Remark: No morphism from a negative to a positive object.

Definition: a morphism from $\mathfrak{P}$ to $\mathfrak{N}$ is called meridian,
a morphism of $\mathfrak{P}$ or $\mathfrak{N}$ is called polar.

When A is positive and B is negative, one may write:

$$\mathbf{Hom}_{[\mathfrak{P},\mathfrak{N}]}(A, B) \cong \mathbf{Hom}_{\mathfrak{P}}(A, \downarrow B) \cong \mathbf{Hom}_{\mathfrak{N}}(\uparrow A, B)$$

The meridian category(2)

From the point of view of data flows...

Polar categories (definition)

A polar category is a:

— two categories $\mathfrak{P}$ and $\mathfrak{N} \cong \mathfrak{P}^{op}$

— the category $\mathfrak{P}$ is symmetric monoidal $(\otimes, 1)$ and has sums $(\oplus, 0)$,

— the tensor distributes over the sum,

— a functor $\uparrow : \mathfrak{P} \longrightarrow \mathfrak{N}$ and its dual functor $\downarrow : \mathfrak{N} \longrightarrow \mathfrak{P}$,

— for every positive object A, an adjunction $\uparrow(- \otimes B) \dashv \downarrow(B^{\perp} \parr -)$

$$\frac{\dfrac{\dfrac{\uparrow(A \otimes B) \xrightarrow{-} C}{A \otimes B \multimap C}}{A \multimap B^{\perp} \parr C}}{A \xrightarrow{+} \downarrow(B^{\perp} \parr C)}$$

where $(-)^{\perp} : \mathfrak{P} \longrightarrow \mathfrak{N}^{op}$ is the negation functor, and $\parr$ is the dual of $\otimes$.

The "usual" categories $\mathcal{G}[n]$ and $\mathcal{G}[v]$ as kleisli constructions

The adjunction $\uparrow\ \dashv\ \downarrow$ between $\uparrow\colon \mathfrak{P} \longrightarrow \mathfrak{N}$ and $\downarrow\colon \mathfrak{N} \longrightarrow \mathfrak{P}$ induces

— a comonad $\uparrow\downarrow$ on the category $\mathfrak{N}$.

— a monad $\downarrow\uparrow$ on the category $\mathfrak{N}$.

Fact:

$\mathcal{G}[n]$ is the cokleisli category over $\mathfrak{N}$ induced by the comonad $\uparrow\downarrow\colon \mathfrak{N} \longrightarrow \mathfrak{N}$.

Dually, $\mathcal{G}[v]$ is the kleisli category over $\mathfrak{P}$ induced by the monad $\downarrow\uparrow\colon \mathfrak{P} \longrightarrow \mathfrak{P}$.

III. Blass phenomenon revisited

Blass non associativity phenomenon.

$$A^+ = \begin{pmatrix} a' : - \\ \uparrow \\ a : + \end{pmatrix} \qquad B^+ = \begin{pmatrix} b' : - \\ \uparrow \\ b : + \end{pmatrix} \qquad C^- = \begin{pmatrix} c' : + \\ \uparrow \\ c : - \end{pmatrix} \qquad D^- = \begin{pmatrix} d' : + \\ \uparrow \\ d : - \end{pmatrix}$$

One defines

$$\begin{array}{llll} \sigma : & A^+ & \longrightarrow & B^+ \qquad \sigma = \{\epsilon, a \cdot a'\} \\ \tau : & B^+ & \longrightarrow & C^- \qquad \tau = \{\epsilon\} \\ \nu : & C^- & \longrightarrow & D^- \qquad \nu = \{\epsilon, d \cdot d'\} \end{array}$$

Then

$$\begin{array}{rcl} (\sigma;\tau) & = & \{\epsilon, (a,c) \cdot a'\} \\ (\sigma;\tau);\nu & = & \{\epsilon, (a,d) \cdot d'\} \end{array}$$

and

$$\begin{array}{rcl} (\tau;\nu) & = & \{\epsilon, (b,d) \cdot d'\} \\ \sigma;(\tau;\nu) & = & \{\epsilon, (a,d) \cdot a'\} \end{array}$$

Blass phenomenon: a clash between the kleisli and co-kleisli constructions

$$\sigma : A \xrightarrow{+} \downarrow\uparrow B \quad \text{or} \quad \sigma : \uparrow A \xrightarrow{-} \uparrow B$$

$$\tau : B \xrightarrow{+} \downarrow C \quad \text{or} \quad \tau : \uparrow B \xrightarrow{-} C$$

$$\nu : \downarrow C \xrightarrow{+} \downarrow D \quad \text{or} \quad \nu : \uparrow\downarrow C \xrightarrow{-} D$$

Thus, to compose $\sigma;\tau$ with ν (in $\mathfrak{P}$):

$$A \xrightarrow{\sigma} \downarrow\uparrow B \xrightarrow{\downarrow\tau} \downarrow C \xrightarrow{\nu} \downarrow D$$

and to compose σ with $\tau;\nu$ (in $\mathfrak{N}$):

$$\uparrow A \xrightarrow{\sigma} \uparrow B \xrightarrow{\uparrow\tau} \uparrow\downarrow C \xrightarrow{\nu} D$$

and applying the adjunction, does not commute generally.

Remark: this is equivalent to premonoidality of $\parr$ in $\mathcal{G}[n]$ (as in Peter's control categories) or that the monad $\downarrow\uparrow$ is strong, but not commutatively so.

Consequence: the Blass phenomenon is not limited to games!

$$\mathfrak{P} = \mathbf{Ens} \qquad\qquad \mathfrak{N} = \mathbf{Ens}^{op}$$

$$\uparrow : X \mapsto R^X \qquad\qquad \downarrow : X \mapsto R^X$$

Again, a strong monad $X \mapsto R^{R^X}$ in $\mathbf{Ens}$,

$$X \xrightarrow{x \mapsto \lambda f. f(x)} R^{R^X} \xleftarrow{h \mapsto \lambda h'. h(\lambda f. f(h'))} R^{R^{R^{R^X}}}$$

but not commutatively so.

The Blass phenomenon: Let us compose

$$\sigma : R^B \longrightarrow R^A \qquad \tau : B \times C \longrightarrow R \qquad \nu : R^C \longrightarrow R^D$$

To compose σ et τ, one transforms τ as $C \longrightarrow R^B$,

$$C \xrightarrow{\tau} R^B \xrightarrow{\sigma} R^A$$

then transforms the composite $\sigma ; \tau$ as $A \longrightarrow R^C$:

$$(\sigma ; \tau) ; \nu : A \longrightarrow R^C \xrightarrow{\nu} R^D$$

The functions $(\sigma ; \tau) ; \nu$ and $\sigma ; (\tau ; \nu)$ are generally different.

IV. Polar categories

as (monoidal) continuation categories

Polar categories = "monoidal" continuation category

After some thought... a polar category boils down to

— a symmetric monoidal category $(\mathfrak{P}, \otimes, 1)$ with sums $(\oplus, 0)$,

— where the tensor distributes over the sum,

— an object R with all linear exponentials R^A

(1) a contravariant functor $R^- : \mathfrak{P} \longrightarrow \mathfrak{P}^{op}$ and

(2) a bijection

$$\frac{A \otimes B \quad \longrightarrow \quad R}{A \quad \multimap \quad R^B}$$

natural in A and B.

In the case of Hyland-Ong games, R is the Sierpinski game $\begin{pmatrix} *_O \\ \uparrow \\ *_P \end{pmatrix}$

Remark: a polar category with cartesian tensor is called a continuation category (Lafont, Reus, Streicher)

Polarized linear logic = a logic of continuations

$$A, B ::= A \oplus B \mid A \otimes B \mid !(A^{\perp}) = R^A \mid 0 \mid \mathbf{1} \mid \alpha$$

right $\oplus$	$\dfrac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}$ $\quad$ $\dfrac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$	left $\oplus$	$\dfrac{\Gamma, A \vdash \Pi \qquad \Gamma, B \vdash \Pi}{\Gamma, A \oplus B \vdash \Pi}$
right $\otimes$	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$	left $\otimes$	$\dfrac{\Gamma, A, B \vdash \Pi}{\Gamma, A \otimes B \vdash \Pi}$
left 0	$\dfrac{}{\Gamma, 0 \vdash}$	right 0	no rule
left 1	$\dfrac{\Gamma \vdash A}{\Gamma, 1 \vdash A}$	right 1	$\dfrac{}{\vdash \mathbf{1}}$
right !	$\dfrac{\Gamma, A \vdash}{\Gamma \vdash R^A}$	left !	$\dfrac{\Gamma \vdash A}{\Gamma, R^A \vdash}$
contraction	$\dfrac{\Gamma, A, A \vdash \Pi}{\Gamma, A \vdash \Pi}$	weakening	$\dfrac{\Gamma \vdash \Pi}{\Gamma, A \vdash \Pi}$

+ axiom, cut and permutation rules.

Interpretation of LLP in a continuation category

a proof of $\Gamma \vdash A$	is interpreted as	a morphism $\Gamma \longrightarrow A$
a proof of $\Gamma \vdash$	is interpreted as	a morphism $\Gamma \longrightarrow R$

Thus, the right !-introduction rule is reversible

$$\frac{\Gamma \otimes A \longrightarrow R}{\Gamma \longrightarrow R^A}$$

while the left !-introduction rule is non-reversible

$$\frac{\Gamma \longrightarrow A}{\Gamma \otimes R^A \longrightarrow R}$$

Continuations, continuations, continuations...

This observation not reductive at all:

— a cut-elimination theorem for continuations, with a nice proof-theoretic characterization of proofs (O. Laurent's correctness criterion for LLP)

— after all, every control category is equivalent to a response category (a structure theorem by Peter).

V. One step further: linearizing Hyland-Ong games

The family construction (Abramsky and McCusker)

The polar category of games enjoys two important properties:

-1- every R^A is π-atomic (Joyal)

$$\frac{\downarrow A \xrightarrow{+} B \oplus C}{(\downarrow A \xrightarrow{+} B) + (\downarrow A \xrightarrow{+} C)}$$

-2- every game is a (finite) sum of games of the form R^A.

Thus,

$$\begin{array}{ccc} \bigoplus_i \downarrow A_i & \xrightarrow{+} & \bigoplus_j \downarrow B_j \\ \hline \Pi_i(\downarrow A_i & \xrightarrow{+} & \bigoplus_j \downarrow B_j) \\ \hline \Pi_i \Sigma_j(\downarrow A_i & \xrightarrow{+} & \downarrow B_j) \end{array}$$

But $\downarrow A_i \xrightarrow{+} \downarrow B_j$ in $\mathfrak{P}$ is simply $A_i \longrightarrow B_j$ in $\mathcal{G}[n]$.

Thus, $\mathfrak{P}$ is (equivalent to) the free co-complete category over $\mathcal{G}[n]$, also called the family construction (by Samson and Guy).

Better: the family construction (Abramsky, McCusker)

To obtain a polar category, start from a symmetric monoidal closed category $(\mathcal{C}, \odot, I)$ with products $\&, \top$.

Define the category $\mathfrak{P}$ as the free co-cartesian category over $\mathcal{C}$:

— its objects are $\{A_i \mid i \in I\}$

— its morphisms $\{A_i \mid i \in I\} \longrightarrow \{B_i \mid i \in I\}$ is a reindexing function $\phi : I \longrightarrow J$ and a morphism $A_i \longrightarrow B_{\phi(i)}$ for every $i \in I$.

Property (adapted from Abramsky,McCusker)

The category $\mathfrak{P}$ is symmetric monoidal closed, with tensor product distributive over sums. Thus it is a polar category.

Relaxing the polarity constraints

Define $\otimes$ as follows:

— $P \otimes Q$

— $[P_1, ..., P_m] \otimes Q = [..., P_i \otimes Q, ...]$

— $[P_1, ..., P_m] \otimes [Q_1, ..., Q_n] = [..., P_i \otimes Q_j, ...]$

where $[P_1, ..., P_n]$ is the negative game $\&_i \uparrow P_i$.

Tentative: this defines a monoidal structure on the meridian category.

Linearizing the threaded model of Idealized Algol (Abramsky, McCusker)

Fact: There exists a functor ! from $\mathfrak{N}$ to its category of comonoids such that

$$\begin{array}{ccc} !A & \xrightarrow{\quad\sigma\quad} & !B \\ \big\downarrow{\scriptstyle \delta_A} & & \big\downarrow{\scriptstyle \delta_B} \\ !A \otimes !A & \xrightarrow{\ \sigma\otimes\sigma\ } & !B \otimes !B \end{array}$$

threaded strategies from A to B $\quad = \quad$ comonoidal maps from $!A$ to $!B$

Besides, there is an adjunction:

$$\frac{!A \xrightarrow{-} B}{!A \xrightarrow{m} !B}$$

This makes the subcategory **Threaded** of objects

$$[!A_1, ..., !A_n] \quad = \quad \oplus_i \downarrow !A_i$$

in $\mathfrak{P}$ a continuation category (that is: cartesian polar) with $!\bot = !(1^\bot)$ as response object. Indeed:

$$\begin{array}{ccc}
[!A_1, ..., !A_m] \otimes [!B_1, ..., !B_n] & \xrightarrow{m} & [!\bot] \\ \hline
\sqcap_i \Sigma_j (!A_i \otimes !B_j & \xrightarrow{m} & !\bot) \\ \hline
\sqcap_i \Sigma_j (!A_i \otimes !B_j & \xrightarrow{-} & \bot) \\ \hline
\sqcap_i \Sigma_j (!A_i & \xrightarrow{-} & (!B_j \multimap \bot)) \\ \hline
([!A_1, ..., !A_n] & \xrightarrow{-} & [\&_j (!B_j \multimap \bot))] \\ \hline
([!A_1, ..., !A_n] & \xrightarrow{m} & [!\&_j (!B_j \multimap \bot))]
\end{array}$$

All in all: a cokleisli category.

The interpretation of `new` in ALGOL

Consider the non-comonoidal non-threaded map

$$\mathtt{cell} : 1 \longrightarrow !\mathtt{var}$$

and the threaded map interpretation of the ALGOL term M:

$$!\mathtt{var} \xrightarrow{m} !B$$

The two maps may be composed as follows: derelict the second one

$$!\mathtt{var} \xrightarrow{-} B$$

to compose it in $\mathfrak{N}$ with the first one

$$1 \xrightarrow{-} !\mathtt{var} \xrightarrow{-} B$$

then exponentiate the whole map

$$1 \xrightarrow{m} !B$$

to obtain the interpretation of the term $\mathtt{new}\ x\ \mathtt{in}\ M$.

Innocent strategies

The same kind of comonoidal characterization is possible for innocent strategies... but more complicated.

V. Conclusion

Conclusion

— games are continuation models,

— polarized linear logic is a logic of continuations,

— Blass problem (first noticed by Abramsky) becomes a general phenomenon of continuation models, akin to premonoidality.

— polar categories make the positive and negative worlds interact: positive objects = values, past and negative objects= functions, future,

— finally, a linearization of threaded as well as innocent Hyland-Ong games. The secret: alter the arenas.