A functorial bridge between proofs and knots

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January 24, 2012

Abstract

In this paper, we investigate notions of cyclic and braided dialogue categories. In particular, we establish that the functor from the free balanced dialogue category to the free ribbon category is faithful.

Contents

1 Introduction

Gottlob Frege reinvented logic at the end of the 19th century by declaring that a mathematical proof could be understood and studied like any other mathematical object. This revolutionary insight was substantiated by his recent discovery of the *Begriffschrift*, a symbolic calculus where *formal proofs* are constructed mechanically, as sequences of elementary transformations on mathematical formulas. However, because of the novelty of his ideas, Frege's description of formal proofs remained somewhat rudimentary. This initiated a secular quest for a more appropriate notation. Several alternative descriptions of proofs were formulated along the years, most notably among them:

- the sequent calculus, introduced by Gentzen in the 1930s in order to establish the consistency of Peano Arithmetic,
- natural deduction, also introduced by Gentzen in the 1930s and revived by Prawitz in the mid-1960s,
- proof nets, introduced by Girard in the mid-1980s as a graphical notation for proofs of linear logic, standing at the mediating point between sequent calculus and natural deduction.

Each of these notations is designed to provide a convenient framework to manipulate formal proofs, and to enlighten some of their cardinal properties:

- the cut-elimination theorem and the subformula property in the case of the sequent calculus,

- the symmetry between the introduction and the elimination rules in the case of natural deduction,
- the correction criterion and the sequentialization theorem in the case of proof nets.

From that point of view, proof theory remains a science in construction, still looking for a satisfactory definition of its object of investigation: the very notion of logical proof. There lies one amusing paradox of proof theory. By way of principle, an intrinsic notion of "logical proof" does not depend on the symbolic notation where it is formulated. On the other hand, every symbolic notation designed to capture this intrinsic notion of proof... describes it through the lenses of its own syntax. This explains the accumulation of dedicated formalisms as well as the never-ending quest for an intrinsic and syntax-free notion of "logical proof". The situation should be contrasted with what one finds in algebra, which is more satisfactory: the same group (take for instance the group B_n of braids with n strands) may admit several algebraic presentations by generators and relations... but it is still the same group!

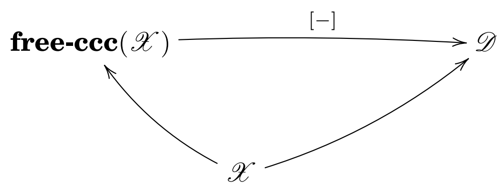
Proof invariants. This intrinsic point of view is fruitful in algebra because it enables to compare the various syntactic presentations of a given group using a sophisticated toolbox, typically including the homological notion of *syzygy*. In particular, the intrinsic notion of group is complementary to its syntactic presentations. Hence, in order to define an intrinsic notion of logical proof, it is tempting to adapt to proof theory what has been achieved for groups and for other algebraic structures. This line of research was initiated by Lambek [**?**] and promoted in his book with Scott [**?**]. The key idea is to think of a particular proof system as a presentation of a *category with structure*. Typically, Lambek established that the free cartesian closed category **free-ccc**$(\mathscr{X})$ on a given category $\mathscr{X}$ has:

- the formulas of minimal logic (constructed with the binary constructors $\times$ and $\Rightarrow$ together with the conjunctive unit 1) with atoms given by the objects of the category $\mathscr{X}$,
- the simply-typed λ-terms of type $A \Rightarrow B$ as morphisms from the formula A to the formula B, considered modulo a suitable notion of equivalence between λ-terms (that is, β and η conversion and composition of maps between atoms in $\mathscr{X}$).

This result enables to construct denotational models of the simply-typed λ-calculus in a very nice and conceptual way. Consider the full and faithful functor $\mathscr{X} \longrightarrow$ **free-ccc**$(\mathscr{X})$ which transports every object of $\mathscr{X}$ to the corresponding atomic formula in **free-ccc**$(\mathscr{X})$. The theorem established by Lambek means that every functor $\mathscr{X} \longrightarrow \mathscr{D}$ to a cartesian-closed category $\mathscr{D}$ lifts to a structure-preserving functor

$$[-] \quad : \quad \textbf{free-ccc}(\mathscr{X}) \quad \longrightarrow \quad \mathscr{D}$$

making the diagram commute:

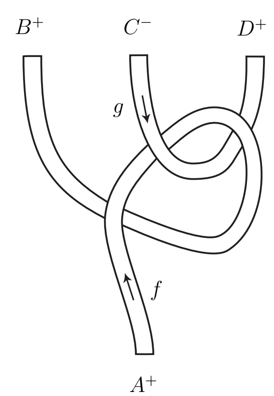


Moreover, the structure-preserving functor $[-]$ is unique modulo natural isomorphism. The benefit of the construction is that the functor $[-]$ transports very simply-typed λ-term P to a denotation $[P]$ in the cartesian-closed category $\mathscr{D}$ – this providing an invariant of the λ-term P modulo $\beta\eta$-conversion. A typical illustration of the method is to define $\mathscr{D}$ as the category of sets and functions. This enables to interpret every λ-term M of type $A \Rightarrow B$ as a function $[M] : [A] \to [B]$ between the set-theoretic interpretations of the types A and B.

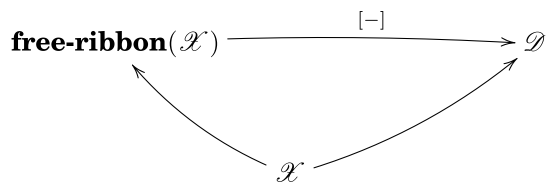
Ribbon categories. It appears that a similar story occurs at the interface of knot theory and of representation theory for quantum groups. There, one defines a *ribbon category* as a monoidal category equipped with combinators for braiding and U-turns, satisfying a series of expected equations, see Sections **??** for a brief survey. Then, one establishes a coherence theorem, which states that the free ribbon category on a category $\mathscr{X}$ has

- as objects: sequences $(A_1^{\epsilon_1}, \ldots, A_n^{\epsilon_n})$ of signed objects of $\mathscr{X}$ where each A_i is an object of the category $\mathscr{X}$, and each ϵ_i is either $+$ or $-$,
- as morphisms: oriented ribbon tangles considered modulo topological deformation, where every open strand is colored by a morphism of $\mathscr{X}$, and every closed strand is colored by an equivalence class of morphisms of $\mathscr{X}$, modulo the equality $g \circ f \sim f \circ g$ for every pair of morphisms of the form $f : A \to B$ and $g : B \to A$.

So, a typical morphism from (A^+) to (B^+, C^-, D^+) in the category **free-ribbon**$(\mathscr{X})$ looks like this



where $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are morphisms in the category $\mathscr{X}$. Now, consider the full and faithful functor $\mathscr{X} \longrightarrow$ **free-ribbon**$(\mathscr{X})$ which transports every object A of $\mathscr{X}$ to the corresponding signed sequence (A^+). Then, just as in the case of the free cartesian closed category, every functor from the category $\mathscr{X}$ to a ribbon category $\mathscr{D}$ lifts as a structure-preserving functor $[-]$ which makes the diagram below commute:



Once properly oriented and colored, every topological ribbon knot P defines a morphism $P : I \longrightarrow I$ from the tensorial unit $I = ()$ to itself in the category **free-ribbon**$(\mathscr{X})$. Hence, its image $[P]$ defines an invariant of the ribbon knot P modulo topological deformation. This well-known method enables for instance to establish that the Jones polynomial $[P]$ associated to a ribbon knot P defines a topological invariant, see **[?]** for details.

Dialogue categories. The discussion so far reveals an analogy between the functorial approaches to proof theory and to knot theory. The analogy does not imply in itself that proofs and knots are related in any deeper sense: after all, the same hammer (in that case, the notion of free category with structure) may be used to knock in different kinds of nails. So, our purpose here will be to substantiate this emerging analogy by designing an intermediate notion of *category with structure* at the gravity center between cartesian closed categories and ribbon categories. As we will see in the course of the paper, this topo–logical crossbreeding is provided by the notion of *balanced dialogue category*, whose intended position is summarized in the diagram below.

proof theory				**knot theory**
cartesian closed categories	$\leftrightarrow$	balanced dialogue categories	$\leftrightarrow$	ribbon categories

The notion of dialogue category has been already used by the author in order to reflect the *dialogical interpretations* of proofs as interactive strategies. It is defined as a monoidal category equipped with a primitive notion of duality.

Definition 1 (Dialogue categories) *A dialogue category is a monoidal category $\mathscr{C}$ equipped with an object $\bot$ together with two functors*

$$x \mapsto (x \multimap \bot) \;:\; \mathscr{C}^{op} \longrightarrow \mathscr{C} \qquad\qquad x \mapsto (\bot \multimapinv x) \;:\; \mathscr{C}^{op} \longrightarrow \mathscr{C}$$

and two families of isomorphisms

$$\begin{array}{rcrcl} \varphi_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \cong & \mathscr{C}(y, x \multimap \bot) \\ \psi_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \cong & \mathscr{C}(x, \bot \multimapinv y) \end{array}$$

natural in x and y.

A balanced dialogue category is then simply defined as a dialogue category whose underlying monoidal category $\mathscr{C}$ is balanced in the sense of Joyal and Street [**?**]. This means that the category $\mathscr{C}$ is equipped with a braiding and a twist, satisfying a series of coherence diagrams reflecting topological equalities of ribbon tangles. Interestingly, no additional coherence property is required between the dialogue structure and the balanced structure.

The proof-theoretic nature of dialogue categories is witnessed by the fact that it comes together with a logic — a braided and twisted logic of tensor and negation called *ribbon logic*. The logic is introduced in §.**??** and formulated in the traditional style of proof theory — that is, as a sequent calculus whose derivation trees are identified modulo a notion of proof equality. Just as in the case of cartesian closed categories, one establishes that the free balanced dialogue category generated by a category $\mathscr{X}$ has

- as objects: the formulas of ribbon logic (constructed with the binary tensor product $\otimes$ and its unit I together with the left negation $A \mapsto A \multimap \bot$ and the right negation $A \mapsto \bot \multimapinv A$) with atoms provided by the objects of the category $\mathscr{X}$,
- as morphisms from A to B: the derivation trees of the sequent $A \vdash B$ in ribbon logic, modulo the proof equalities of the logic.

The proof-as-tangle theorem. Once the proof-theoretic nature of balanced dialogue categories clarified, there remains to understand how they are related to topology. This aspect is captured by a coherence theorem established in the course of the paper. The theorem is called the *proof-as-tangle* theorem because it enables to identify the proofs of ribbon logic as ribbon tangles modulo topological deformation. Let us briefly explain how it is formulated. A pointed category $(\mathscr{C}, \bot)$ is defined as a category $\mathscr{C}$ equipped with an object $\bot$ singled out in the category. A pointed category may be alternatively defined as an S-algebra for the monad S which transports every category $\mathscr{X}$ to the category $\mathscr{X} + \mathbf{1}$ defined as the disjoint sum of $\mathscr{X}$ with the terminal category $\mathbf{1}$. The unique object of $\mathbf{1}$ is noted $\bot$ and provides the singled-out object of the pointed category $(\mathscr{X} + \mathbf{1}, \bot)$. Every category $\mathscr{X}$ induces a free ribbon category $\mathbf{free\text{-}ribbon}(\mathscr{X} + \mathbf{1})$ generated by the category $\mathscr{X} + \mathbf{1}$. The category $\mathbf{free\text{-}ribbon}(\mathscr{X} + \mathbf{1})$ is monoidal and balanced by construction. It is also a dialogue category where the left and right negation functors are defined as

$$x \multimap \bot \quad \stackrel{def}{=} \quad x^* \otimes \bot \qquad\qquad \bot \multimapinv x \quad \stackrel{def}{=} \quad \bot \otimes x^*.$$

Note that the balanced dialogue category $\mathbf{free\text{-}ribbon}(\mathscr{X} + \mathbf{1})$ is somewhat degenerate, since the canonical morphism

$$(\bot \multimapinv (x \multimap \bot)) \otimes y \quad \longrightarrow \quad \bot \multimapinv ((x \otimes y) \multimap \bot)$$

is an isomorphism. Now, the unit of the monad S instantiated at the category $\mathscr{X}$

$$\eta \quad : \quad \mathscr{X} \quad \longrightarrow \quad \mathscr{X} + \mathbf{1}$$

induces a functor

$$\mathscr{X} \quad \longrightarrow \quad \mathscr{X} + \mathbf{1} \quad \longrightarrow \quad \textbf{free-ribbon}(\mathscr{X} + \mathbf{1})$$

from $\mathscr{X}$ to the balanced dialogue category $\textbf{free-ribbon}(\mathscr{X}_\perp)$. From this follows that there exists a structure-preserving functor between balanced dialogue categories

$$[-] \quad : \quad \textbf{free-dialogue}(\mathscr{X}) \quad \longrightarrow \quad \textbf{free-ribbon}(\mathscr{X} + \mathbf{1})$$

which makes the diagram below commute:

$$\begin{array}{ccc}
\textbf{free-dialogue}(\mathscr{X}) & \xrightarrow{\quad [-] \quad} & \textbf{free-ribbon}(\mathscr{X} + \mathbf{1}) \\
\uparrow & & \uparrow \\
\mathscr{X} & \xrightarrow{\quad \eta \quad} & \mathscr{X} + \mathbf{1}
\end{array}$$

This functor $[-]$ transports:

- the formulas of ribbon logic into signed sequences of $\perp$'s and of objects of the underlying category $\mathscr{X}$,
- the proofs of ribbon logic modulo proof equality into ribbon tangles modulo topological deformation.

Note in particular that the double negation of a formula A is transported to

$$[\perp \multimapinv (A \multimap \perp)] \;=\; \perp \otimes \perp^{*} \otimes [A] \qquad\qquad [(\perp \multimapinv A) \multimap \perp] \;=\; [A] \otimes \perp^{*} \otimes \perp$$

We establish in §. the following proof-as-tangle theorem:

Theorem. The functor $[-]$ is faithful.

This statement means that two derivation trees π_1 and π_2 of the same sequent $A_1, \cdots, A_n \vdash B$ in ribbon logic are equal in the proof-theoretical sense precisely when the associated ribbon tangles $[\pi_1]$ and $[\pi_2]$ are equal in the topological sense. The proof-as-tangle theorem provides in that way a tentative solution to the problem of defining an intrinsic notion of "logical proof" in ribbon logic. More specifically, the theorem tells that a proof π of a formula A is an entangled network of ribbon wires, whose strands connect the negations and the atoms of the formula A... Although the theorem is limited here to a small fragment of logic, it should help to design similarly intrinsic notions of "logical proofs" in larger systems — typically including first-order and second-order quantification, the structural rules of contraction and weakening, as well as dependent types.

One distinctive aspect of the theorem is that the ribbon tangle *exists* independently of the way it is constructed by the proof system. This leads to something like a copernician turn, already germinating in the notion of proof-net of linear logic discussed below. Indeed, once recognized the *intrinsic* nature of the logical proof, the sequent calculus appears as a specific *procedure* in order to construct (or to deconstruct) it. This procedure may be then compared with other construction (or deconstruction) methods — typically natural deduction, or Frege-like systems based on rewriting[1].

Two main precursors. Besides the mentioned analogy between the functorial approach to proof theory and knot theory, the proof-as-tangle theorem is inspired by two different but related lines of work in proof theory:

- the dialogical interpretation of intuitionnistic and classical logic where formulas are interpreted as dialogue games, and proofs are interpreted as interactive strategies between two players: the Ego playing the proof and the Alter playing the refutation,
- the multiplicative proof-nets of linear logic where a cut-free proof of a formula is described as an involution on the atoms of the formula, satisfying an additional " long trip " correctness criterion.

Each line of work tries to articulate an intrinsic notion of logical proof — interactive in the case of dialogue games, graphical in the case of multiplicative proof-nets. Our work is an attempt of unification based on the observation that the strategy $[\pi]$ associated to a proof π may be alternatively seen as a proof-net for *tensorial logic* — a variant of linear logic where negation is not necessarily involutive. It appears that this tensorial bridge between dialogue games and linear logic magnifies the two fields, and " repairs " a series of deficiencies encountered on each side. Let us briefly describe in turn how the proof-as-tangle theorem and its connection to knot theory revisits and unifies these two well-established fields of proof theory.

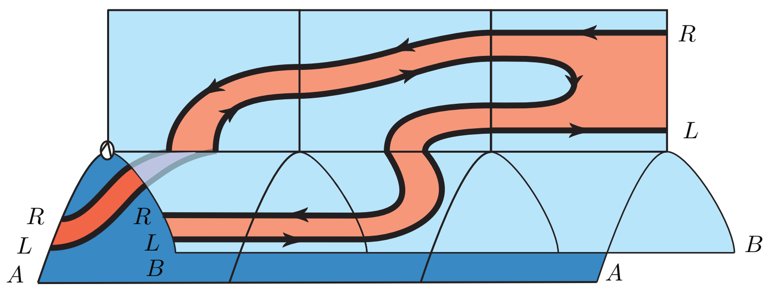
Dialogue games. As already mentioned, ribbon logic is a topological refinement of tensorial logic — a primitive logic of tensor and negation designed to reflect the *dialogical interpretations* of proofs as interactive strategies playing on dialogue games. The connection between tensorial logic and dialogue games is established in [**?**]. It is provided by the notion of *symmetric* (rather than balanced) dialogue category: a coherence theorem states that the free symmetric dialogue category generated by a category $\mathscr{X}$ is the category with dialogue games as objects, and total innocent strategies as morphisms. The theorem also ensures that two proofs π_1 and π_2 of a formula A are equal in tensorial logic precisely when their interpretation as interactive strategies $[\pi_1]$ and $[\pi_2]$ are equal. This proof-as-strategy theorem established in [**?**] should be seen as a symmetric version of the proof-as-tangle theorem, where symmetric tensorial logic replaces ribbon logic. In that sense, the proof-as-tangle theorem refines

[1]This point of view relies on the implicit assumption that every proof system of ribbon logic should recover in its own way the proofs *as well as* the proof equalities of the original sequent calculus. Although this methodological assumption is arguably strong (weaker notions of proof equality may be considered in some situations) we believe that it is quite reasonable — after all, changing the proof equality should mean changing the logic.

traditional game semantics, and provides it with new topological foundations. The main observation of [**?**] is that a proof of tensorial logic like

$$
\begin{array}{ll}
\dfrac{\overline{A \vdash A} \qquad \overline{B \vdash B}}{A , B \vdash A \otimes B} & \text{Right } \otimes \\[6pt]
\hline
A \,,\, B \,,\, (A \otimes B) \multimap \bot \;\vdash & \text{Left } \multimap \\
\hline
B \,,\, (A \otimes B) \multimap \bot \,,\, A \;\vdash & \text{Exchange} \\
\hline
(A \otimes B) \multimap \bot \,,\, A \;\vdash\; B \multimap \bot & \text{Right } \multimap \\
\hline
\bot \circ\!\!- (B \multimap \bot) \,,\, (A \otimes B) \multimap \bot \,,\, A \;\vdash & \text{Left } \circ\!\!- \\
\hline
A \,,\, \bot \circ\!\!- (B \multimap \bot) \,,\, (A \otimes B) \multimap \bot \;\vdash & \text{Exchange} \\
\hline
\bot \circ\!\!- (B \multimap \bot) \,,\, (A \otimes B) \multimap \bot \;\vdash\; A \multimap \bot & \text{Right } \multimap \\
\hline
\bot \circ\!\!- (A \multimap \bot) \,,\, \bot \circ\!\!- (B \multimap \bot) \,,\, (A \otimes B) \multimap \bot \;\vdash & \text{Left } \circ\!\!- \\
\hline
\bot \circ\!\!- (A \multimap \bot) \,,\, \bot \circ\!\!- (B \multimap \bot) \;\vdash\; \bot \circ\!\!- ((A \otimes B) \multimap \bot) & \text{Right } \circ\!\!- \\
\hline
(\bot \circ\!\!- (A \multimap \bot)) \otimes (\bot \circ\!\!- (B \multimap \bot)) \;\vdash\; \bot \circ\!\!- ((A \otimes B) \multimap \bot) & \text{Left } \otimes
\end{array}
\qquad (1)
$$

may be depicted in string diagrams as a formula tree in motion:



where $L : \mathscr{C} \to \mathscr{C}^{op}$ and $R : \mathscr{C}^{op} \to \mathscr{C}$ denote the functors

$$L \;:\; A \;\mapsto\; A \multimap \bot \qquad\qquad R \;:\; A \;\mapsto\; \bot \circ\!\!- A$$

Moreover, the flow of negations induced by the formula tree in motion coincides with the game-theoretic interpretation of the proof. As a matter of fact, it is not difficult to see that ribbon logic is conservative over tensorial logic. In particular, there exists a canonical functor

$$\textbf{free-dialogue}(\mathscr{X}) \quad\longrightarrow\quad \textbf{free-sym-dialogue}(\mathscr{X})$$

from the free *balanced* dialogue category to the free *symmetric* dialogue category generated by the same category $\mathscr{X}$. This functor is one-to-one on objects, and full on morphisms. This means that the ribbon logic has the same formulas as tensorial logic, and that every proof of tensorial logic comes from one proof (or more) of ribbon logic. In particular, several proofs of ribbon logic correspond to the proof (1) of tensorial logic. All of them are translated to a ribbon tangle defining a map

$$\bot_2 \otimes \bot_3^* \otimes [A] \otimes \bot_4 \otimes \bot_5^* \otimes [B] \quad\longrightarrow\quad \bot_1 \otimes \bot_6^* \otimes [A] \otimes [B]$$

in the free ribbon category generated by the category $\mathscr{X}_\bot$. The tags $i \in \{1, ..., 6\}$ labeling the objects $\bot$ and $\bot^*$ are here to indicate that the ribbon tangle contains three strands labelled with $j \in \{1, 3, 5\}$ each of them connecting the "input" object labelled j to the "output" object labelled $j + 1$.

Linear logic. Although it often remains implicit, the idea of looking for an intrinsic notion of "logical proof" is prominent in Girard's work in proof theory. After all, one of his main contributions to our field is the notion of *proof-net* which was introduced in his seminal article on linear logic. In its most elementary form designed for multiplicative linear logic, a proof-net is defined as a graph whose nodes are the connectives of the logic: the conjunction $\otimes$, the disjunction $\wp$, the axiom and the cut. In particular, a cut-free proof-net π of a multiplicative formula A is entirely described by the involution $[\![\pi]\!]$ defined by its axiom links on the atoms of the formula A. Girard had this revolutionary insight that any such involution on the atoms of a formula A should be understood as an attempt (possibly failed) to build a proof of multiplicative linear logic — what he coined a *proof-structure*. A purely graphical correctness criterion called the "long trip" criterion enables then to test whether a proof-structure is a proof-net — that is, whether it comes from a derivation tree of multiplicative linear logic.

The notion of proof-net is nice, simple and elegant. It would provide a perfect candidate for an intrinsic concept of logical proof... Unfortunately, it appeared very soon in the short history of linear logic that something was simply wrong with the definition of a multiplicative proof-net. The defect appears when one takes the units of linear logic seriously. In order to detect when a proof-structure is a proof-net, the correctness criterion needs to know when the disjunctive unit $\bot$ of multiplicative linear logic has been introduced by the rule

$$\frac{\vdash \quad A_1, \ldots, A_n}{\vdash \quad A_1, \ldots, A_n, \bot} \quad \text{introduction of } \bot$$

in the derivation tree. In order to deal with this problem, Girard suggested to equip every disjunctive unit $\bot$ of the proof-net with a dedicated link called a "jump". This jump connects the unit $\bot$ to any of the connectives appearing in the context $A_1, \ldots, A_n$ of the introduction rule. The technical device enables to reconstruct the derivation tree from the proof-net with jumps... but it breaks at the same time the correspondence between proof-nets and derivation trees modulo proof equality.

A purely categorical account of this defect of linear logic is possible. Just like tensorial logic is underlined by dialogue categories, linear logic is underlined by $*$-autonomous categories. The notion of symmetric $*$-autonomous category may be defined in several ways, for instance as a dialogue category $\mathscr{C}$ with the additional requirement that the object $\bot$ is *dualizing* — this meaning that the canonical morphisms

$$x \longrightarrow \bot \multimapinv (x \multimap \bot) \qquad\qquad x \longrightarrow (\bot \multimapinv x) \multimap \bot$$

are isomorphisms for every object x of the category $\mathscr{C}$. Such a $*$-autonomous category is called symmetric when the underlying dialogue category is symmetric. Recall that a symmetric monoidal category is a balanced monoidal category whose twist is the identity. Besides, a compact closed category is defined as a ribbon category whose twist is an identity. Every compact-closed category is symmetric $*$-autonomous. From this follows that given a category $\mathscr{X}$, there exists a structure-preserving functor

$$[\![-]\!] \quad : \quad \mathbf{free\text{-}sym\text{-}*\text{-}autonomous}(\mathscr{X}) \quad \longrightarrow \quad \mathbf{free\text{-}compact\text{-}closed}(\mathscr{X})$$

between symmetric $*$-autonomous categories. It appears that the functor transports every proof π of a formula A of multiplicative linear logic to an involution $[\![\pi]\!]$ on the atoms in the category $\mathscr{X}$ of the formula. The functor $[\![-]\!]$ is not faithful. This means that two proofs A typical example is

...

On the other hand, the proof-as-strategy theorem states that the functor

$$[-] \quad : \quad \mathbf{free\text{-}sym\text{-}dialogue}(\mathscr{X}) \quad \longrightarrow \quad \mathbf{free\text{-}compact\text{-}closed}(\mathscr{X}_{\perp})$$

is faithful. This clarifies in what sense the shift to dialogue games and tensorial logic provides a tentative solution to the traditional problem of proof equality in linear logic. By removing the flow of negations, and keeping only the flow of negation, linear logic does not reflect the intrinsic dynamic of proofs. This is fine when formulas have a space. But the unit $\perp$ has no space. This explains why one needs to add a technical device like jumps.

Representation theory. Much work has been devoted to understand how the algebraic dualities of quantum groups are related to the topological invariants of knots. These investigations have been generally performed in categories of finite dimensional representations where the duality is involutive, in the sense that the canonical map $A \to A^{**}$ transporting a space A to its bidual space A^{**} is invertible. One primary task of this work is to recast this well-established connection inside categories where the duality is not necessarily involutive. To that purpose, we start from the primitive notion of duality provided by dialogue categories. A typical illustration is provided by the category $\mathscr{C}$ of representations (of arbitrary dimension) of a Hopf algebra with an invertible antipode, where the object $\perp$ is defined as the base field k. More generally, any object $\perp$ picked in a monoidal closed category $\mathscr{C}$ (closed on the left and on the right) defines a dialogue category. As it stands, the notion of dialogue category reflects a primitive (and pervasive) notion of duality, and it is thus natural to look for a satisfactory definition of cyclic dialogue and ribbon dialogue category. In particular, we characterize the Hopf algebras H whose category H-Mod of left representations is cyclic dialogue or ribbon dialogue.

Related works. Speak of non commutative linear logic, balanced star-autonomous...

this topological flow of negations and atoms is precisely what every formal syntax of ribbon logic is describing. Either Hilbert-style, natural deduction or sequent calculus –

This work in proof theory is part of a wider research program, whose purpose is to refine to tensorial logic the components of linear logic, where negation is necessarily involutive. From an algebraic point of view, this means extending to dialogue categories the body of tools and concepts developed for $*$-autonomous categories. It should be noted that a $*$-autonomous is the same thing as a dialogue category where the object $\bot$ is dualizing, this meaning that the canonical morphisms

$$x \longrightarrow \bot \multimapinv (x \multimap \bot) \qquad\qquad x \longrightarrow (\bot \multimapinv x) \multimap \bot$$

are isomorphisms for every object x. In particular, we will be careful to check that the notions of cyclic and braided dialogue categories match with the existing notions of cyclic and braided $*$-autonomous categories. The notion of dialogue category is connected to game semantics. This work is part of a plan to develop cyclic as well as braided notions of game semantics, with a clear topological status. Again, strategies should collapse to proof-nets in cyclic and in braided linear logic when the flow of negation is removed.

the sequent calculus, in natural deduction one symbolic framework or another:

The theorem has far-reaching proof-theoretic consequences which go beyond its conceptual formulation. The whole point of the coherence theorem is to provide an intrinsic notion of proof, independent of its formal construction. The theorem establishes that proof equality boils down to the topology of the network of links connecting negations and atoms flowing inside the proof. From that point of view, it makes sense to think of the proof as this network of negations and atoms

Linear logic – proof-nets. Tensorial logic – game semantics.

This theorem is the latest

safer topological foundations to the dialogical

The notion of dialogue category was introduced by the author in order to reflect the *dialogical interpretations* of proofs as interactive strategies playing on dialogue games. In particular, we have established that the free *symmetric* dialogue category generated by a category $\mathscr{X}$ has dialogue games as objects, and total innocent strategies as morphisms.

As such, the category We have already established in a companion paper that the free *symmetric* dialogue category generated by a category $\mathscr{X}$ has dialogue games as objects, and total innocent strategies as morphisms. In this framework, a proof $A \vdash B$ of the associated tensorial logic may be seen as a family of links connecting the negations of A and the negations of B.

It appears that the free *balanced* dialogue category has the same objects, and its morphisms are a refinement of innocent strategies where the flow of negation (as well as the morphisms of $\mathscr{X}$) behave like topological strands in a ribbon tangle.

The point is that these notations are not supported by any *intrinsic* notion of formal proof. From that point of view, The situation is a bit similar as if there were various notions of presentations of a group, but no *intrinsic* notion of proof.

n proof theory, which is the search for the intrinsic meaning of logical proofs, independently of the formal syntax in which they are traditionally expressed. This is an old question in logic, which has never found a satisfactory answer. Semantics is the study of invariants of proofs modulo execution. A difficult question in the field is to decide when two proofs should be considered as equal. It appears that the notion of braided dialogue category enables to recast this traditional problem to the topology of ribbon tangles. This is achieved in two steps. First of all, one shows that the free braided dialogue category $\mathbf{free\text{-}dialogue}(\mathscr{X})$ generated by a category $\mathscr{X}$ is a category of formulas and proofs (modulo execution) of a braided variant of tensorial logic. Tensorial logic is a primitive logic of tensor and negation, where negation is not necessarily involutive. Then, one shows that there exists a faithful functor from $\mathbf{free\text{-}dialogue}(\mathscr{X})$ to the free ribbon category $\mathbf{free\text{-}ribbon}(\mathscr{X}_*)$ over the category $\mathscr{X}$ extended with one object $\bot$. The fact that the functor is *faithful* shows that equality of proofs modulo execution reduces in this case to equality of ribbon tangles modulo deformation.

Plan of the paper.

2 Balanced dialogue categories

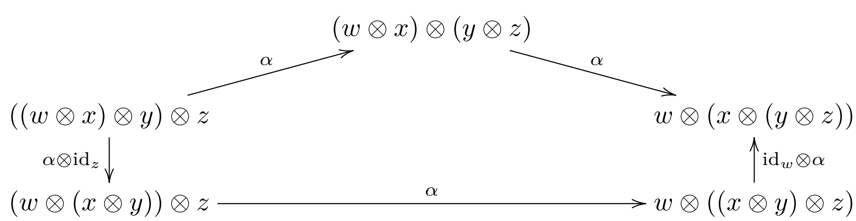
2.1 Monoidal categories

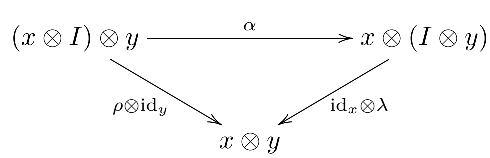
In order to fix notations, we recall that a monoidal category $\mathscr{C}$ is a category equipped with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and an object I and three natural isomorphisms

$$\alpha_{x,y,z} \quad : \quad (x \otimes y) \otimes z \quad \longrightarrow \quad x \otimes (y \otimes z)$$

$$\lambda_x \quad : \quad I \otimes x \quad \longrightarrow \quad x \qquad\qquad \rho_x \quad : \quad x \otimes I \quad \longrightarrow \quad x$$

making the two coherence diagrams below commute.





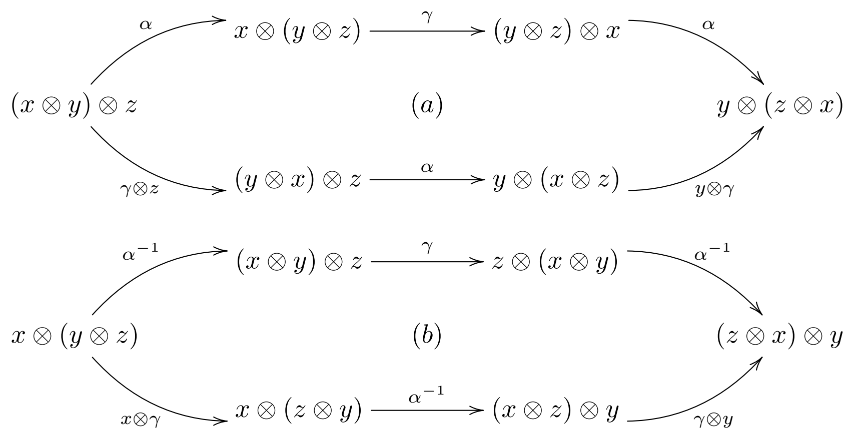
2.2 Braided categories

The notion of braided monoidal category $\mathscr{C}$ is introduced in ....

Definition 2 (braiding) *A braiding in a monoidal category $\mathscr{C}$ is a family of isomorphisms*

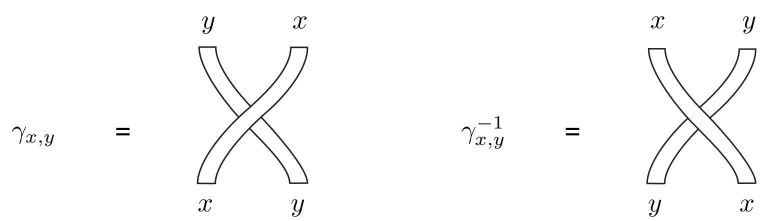
$$\gamma_{x,y} \quad : \quad x \otimes y \quad \longrightarrow \quad y \otimes x$$

natural in x and y such that the two diagrams

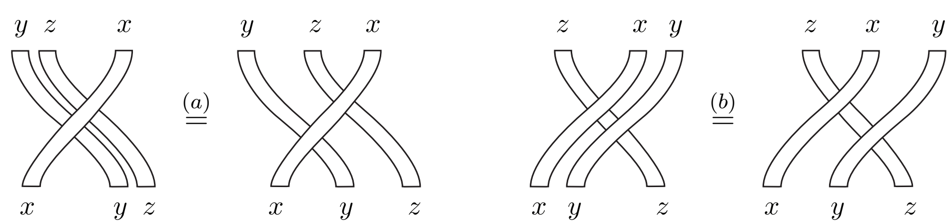


commute.

The braiding map $\gamma_{x,y}$ is depicted in string diagrams as a positive braiding of the ribbon strands x and y where its inverse is depicted as the negative braiding:



The two coherence diagrams (a) and (b) are then depicted as equalities between string diagrams:



2.3 Balanced categories

The notion of balanced category is introduced in...

Definition 3 (balanced category) *A balanced category $\mathscr{C}$ is a braided monoidal category equipped with a family of morphisms*

$$\theta_x \quad : \quad x \quad \longrightarrow \quad x$$

natural in x, satisfying the equality

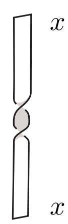
$$\theta_I \quad = \quad \mathrm{id}_I$$

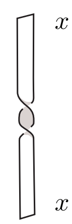
and making the diagram

$$\begin{array}{ccc} x \otimes y & \xrightarrow{\gamma_{x,y}} & y \otimes x \\ \downarrow{\scriptstyle \theta_{x \otimes y}} & & \downarrow{\scriptstyle \theta_y \otimes \theta_x} \\ x \otimes y & \xleftarrow[\gamma_{y,x}]{} & y \otimes x \end{array} \qquad (2)$$

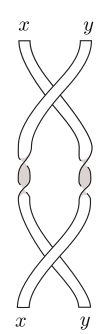
commute for all objects x and y of the category $\mathscr{C}$.

The twist θ_x is depicted as the ribbon x twisted positively in the trigonometric direction with an angle 2π whereas its inverse θ_x^{-1} is depicted as the same ribbon x twisted this time negatively with an angle -2π:

θ_x = 

θ_x^{-1} = 

This notation enables us to give a topological motivation to the axioms of a balanced category. The first requirement that θ_I is the identity means that the ribbon strand I should be thought as ultra thin. The second requirement that the coherence Diagram 2 commutes says that topological equality between string diagrams:

$\theta_{x \otimes y}$ = 

2.4 Balanced dialogue categories

At this stage, we are ready to introduce the notion of *balanced dialogue category* which provides a functorial bridge between proof theory and knot topology.

Definition 4 (balanced dialogue category) *A balanced dialogue category is a dialogue category $\mathscr{C}$ moreover equipped with a braiding and a twist defining a balanced category.*

3 Ribbon logic

We introduce below the sequent calculus of ribbon logic, this including the equalities associated by the cut-elimination procedure.

3.1 The ribbon groups

Recall that the braid group $\mathbf{B}_n$ on n strands is presented by the generators σ_i for $1 \leq i \leq n-1$ and the equations

$$\begin{array}{rcl} \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1} & & \\ \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i & & \text{when } |j-i| \geq 2. \end{array} \tag{3}$$

There is an obvious left action

$$\triangleright \quad : \quad \mathbf{B}_n \times [n] \quad \longrightarrow \quad [n] \tag{4}$$

of the group $\mathbf{B}_n$ on the set $[n] = \{1, \cdots, n\}$ of strands. This action enables to define a wreath product of $\mathbf{B}_n$ on the additive group $(\mathbb{Z}, +, 0)$. The resulting group $\mathbf{G}_n$ is called the *ribbon group* on n strands. The group is presented by the generators σ_i for $1 \leq i \leq n-1$ and θ_i for $1 \leq i \leq n$, together with the equations (3) of the braid group $\mathbf{B}_n$ and the equations below:

$$\begin{array}{ll} \sigma_i \circ \theta_i = \theta_{i+1} \circ \sigma_i & \\ \sigma_i \circ \theta_{i+1} = \theta_i \circ \sigma_i & \\ \sigma_i \circ \theta_j = \theta_j \circ \sigma_i & \text{when } j < i \text{ or when } j \geq j+2. \end{array}$$

Each group $\mathbf{G}_n$ may be alternatively seen as a groupoid noted $S\,\mathbf{G}_n$, with a unique object $*$ and $S\,\mathbf{G}_n(*,*) = \mathbf{G}_n$. There is a nice and conceptual definition of the ribbon groups $\mathbf{G}_n$ which goes as follows. The groupoid $\mathscr{G}$ defined as the disjoint sum of the groupoids $S\,\mathbf{G}_n$ coincides with the free balanced category generated by the terminal category $\mathbf{1}$. Recall that the category $\mathbf{1}$ has a unique object $*$ and a unique map. Hence, the group $\mathbf{G}_n$ may be alternatively defined as $\mathscr{G}(n,n)$ where $n = 1 \otimes \cdots \otimes 1$ is the n-fold tensor product of the generator 1 of the category $\mathscr{G}$. This is just the ribbon-theoretic counterpart to the well-known fact that the free braided monoidal category $\mathscr{B}$ generated by the category $\mathbf{1}$ coincides with the disjoint sum of the groupoids $S\,\mathbf{B}_n$. From this observation follows a family of group homomorphisms

$$\otimes \quad : \quad \mathbf{G}_p \times \mathbf{G}_q \quad \longrightarrow \quad \mathbf{G}_{p+q}$$

which reflects the monoidal structure of the category $\mathscr{G}$. It is also useful to observe that the action (4) extends to a left action

$$\triangleright \quad : \quad \mathbf{G}_n \times [n] \quad \longrightarrow \quad [n]$$

where each generator θ_i acts trivially, in the sense that $\theta_i \triangleright k = k$ for all $k \in [n]$.

3.2 The sequent calculus

The formulas of ribbon logic are finite trees generated by the grammar

$$A, B ::= A \otimes B \ | \ I \ | \ A \multimap \bot \ | \ \bot \multimapinv A \ | \ \bot.$$

The sequence are two-sided

$$A_1, \ldots, A_m \ \vdash \ B$$

with a *sequence* of formulas $A_1, ..., A_m$ on the left-hand side, and a *unique* formula B on the right-hand side. The proof of ribbon logic are then defined as derivation trees in a sequent calculus. The sequent calculus is defined as the usual sequent calculus of tensorial logic, recalled in Figure 1, together with a family of exchange rules

$$\frac{A_1, \ldots, A_n \vdash B}{A_{g \triangleright 1}, \ldots, A_{g \triangleright n} \vdash B} \text{ Exchange}[g]$$

parametrized by the elements g of the ribbon group $\mathbf{G}_n$.

Axiom	$\dfrac{}{A \vdash A}$
Cut	$\dfrac{\Gamma \vdash A \qquad \Upsilon_1, A, \Upsilon_2 \vdash B}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}$
Right $\otimes$	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$
Left $\otimes$	$\dfrac{\Upsilon_1, A, B, \Upsilon_2 \vdash C}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}$
Right I	$\dfrac{}{\vdash I}$
Left I	$\dfrac{\Upsilon_1, \Upsilon_2 \vdash A}{\Upsilon_1, I, \Upsilon_2 \vdash A}$
Right $\multimapinv$	$\dfrac{\Gamma, A \vdash \bot}{\Gamma \vdash \bot \multimapinv A}$
Left $\multimapinv$	$\dfrac{\Gamma \vdash A}{\bot \multimapinv A, \Gamma \vdash \bot}$
Right $\multimap$	$\dfrac{A, \Gamma \vdash \bot}{\Gamma \vdash A \multimap \bot}$
Left $\multimap$	$\dfrac{\Gamma \vdash A}{\Gamma, A \multimap \bot \vdash \bot}$

Figure 1: Sequent calculus of tensorial logic.

3.3 The proof equalities

Ribbon logic comes equipped with a notion of equality between proofs. The equality is refined as a rewriting system modulo equations. A typical example of such a rewriting rule is provided by the exchange rule.

3.3.1 The group-theoretic equations

Exchange (composition) —

$$
\text{Exch}[g]\;\dfrac{\text{Exch}[h]\;\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A_1,\ldots,A_n \vdash B}}{A_{h\rhd 1},\ldots,A_{h\rhd n} \vdash B}}{A_{g\rhd(h\rhd 1)},\ldots,A_{g\rhd(h\rhd n)} \vdash B}
\quad\leftrightsquigarrow\quad
\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A_1,\ldots,A_n \vdash B}}{A_{g\circ h\rhd 1},\ldots,A_{g\circ h\rhd n} \vdash B}\;\text{Exch}[g\circ h]
$$

Exchange (unit) —

$$
\text{Exchange}[e]\;\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A_1,\ldots,A_n \vdash B}}{A_{e\rhd 1},\ldots,A_{e\rhd n} \vdash B}
\quad\leftrightsquigarrow\quad
\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A_1,\ldots,A_n \vdash B}
$$

where e denotes the unit of the ribbon group $\mathbf{G}_n$.

3.3.2 The braid and twist equations

We start by a series of three basic rewriting steps which reflect the coherence diagrams required of the braiding and of the twist in a balanced monoidal category.

The braid equations

$$
[p\otimes\sigma\otimes q]\;\dfrac{\text{Left}\otimes\;\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Upsilon_1,A,B,C,\Upsilon_2\vdash D}}{\Upsilon_1,A\otimes B,C,\Upsilon_2\vdash D}}{\Upsilon_1,C,A\otimes B,\Upsilon_2\vdash D}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Upsilon_1,A,B,C,\Upsilon_2\vdash D}}{\Upsilon_1,A,C,B,\Upsilon_2\vdash C}\;[p\otimes 1\otimes\sigma\otimes q]}{\Upsilon_1,C,A,B,\Upsilon_2\vdash C}\;[p\otimes\sigma\otimes 1\otimes q]}{\Upsilon_1,C,A\otimes B,\Upsilon_2\vdash C}\;\text{Left}\otimes
$$

$$
[p\otimes\sigma\otimes q]\;\dfrac{\text{Left}\otimes\;\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Upsilon_1,A,B,C,\Upsilon_2\vdash D}}{\Upsilon_1,A,B\otimes C,\Upsilon_2\vdash D}}{\Upsilon_1,B\otimes C,A,\Upsilon_2\vdash D}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Upsilon_1,A,B,C,\Upsilon_2\vdash D}}{\Upsilon_1,B,A,C,\Upsilon_2\vdash C}\;[p\otimes\sigma\otimes 1\otimes q]}{\Upsilon_1,B,C,A,\Upsilon_2\vdash C}\;[p\otimes 1\otimes\sigma\otimes q]}{\Upsilon_1,B\otimes C,A,\Upsilon_2\vdash C}\;\text{Left}\otimes
$$

where p and q denote the respective lengths of Υ_1 and Υ_2.

The twist equation

$$
[p\otimes\theta\otimes q]\;\dfrac{\text{Left}\otimes\;\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A\otimes B, \Upsilon_2 \vdash C}}{\Upsilon_1, A\otimes B, \Upsilon_2 \vdash C}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}\;[p\otimes\theta\otimes\theta\otimes q]}{\Upsilon_1, B, A, \Upsilon_2 \vdash C}\;[p\otimes\sigma\otimes q]}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}\;[p\otimes\sigma\otimes q]}{\Upsilon_1, A\otimes B, \Upsilon_2 \vdash C}\;\text{Left}\otimes
$$

where p and q denote the respective lengths of Υ_1 and Υ_2.

3.3.3 Exchange vs. the left introduction rules

We describe the proof transformations induced by the interaction between an exchange rule and a left introduction rule. In each case, the transformation has the effect of permuting the exchange rule before the left introduction rule.

Left introduction of the tensor product — The proof

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_p, A, B, B_1,\ldots,B_q \vdash C}}{A_1,\ldots,A_p, A\otimes B, B_1,\ldots,B_q \vdash C}\;\text{Left}\otimes}{A_{g\rhd 1},\ldots,A_{g\rhd p}, A\otimes B, B_{h\rhd 1},\ldots,B_{h\rhd q} \vdash C}\;[g\otimes 2\otimes h]
$$

is transformed into the proof

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_p, A, B, B_1,\ldots,B_q \vdash C}}{A_{g\rhd 1},\ldots,A_{g\rhd p}, A, B, B_{h\rhd 1},\ldots,B_{h\rhd q} \vdash C}\;[g\otimes 1\otimes h]}{A_{g\rhd 1},\ldots,A_{g\rhd p}, A\otimes B, B_{h\rhd 1},\ldots,B_{h\rhd q} \vdash C}\;\text{Left}\otimes
$$

Left introduction of the left negation —

$$
\text{Left}\multimap\;\dfrac{[g]\;\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_n \vdash B}}{A_{g\rhd 1},\ldots,A_{g\rhd n} \vdash B}}{A_{g\rhd 1},\ldots,A_{g\rhd n}, B\multimap\bot \vdash \bot}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_n \vdash B}}{A_1,\ldots,A_n, B\multimap\bot \vdash \bot}\;\text{Left}\multimap}{A_{g\rhd 1},\ldots,A_{g\rhd n}, B\multimap\bot \vdash \bot}\;[g\otimes 1]
$$

Left introduction of the right negation —

$$
\text{Left}\circ\!\!-\;\dfrac{[g]\;\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_n \vdash B}}{B_{g\rhd 1},\ldots,B_{g\rhd n} \vdash A}}{\bot\circ\!\!- A, B_{g\rhd 1},\ldots,B_{g\rhd n} \vdash \bot}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{B_1,\ldots,B_n \vdash A}}{\bot\circ\!\!- A, B_1,\ldots,B_n \vdash \bot}\;\text{Left}\circ\!\!-}{\bot\circ\!\!- A, B_{g\rhd 1},\ldots,B_{g\rhd n} \vdash \bot}\;[1\otimes g]
$$

3.3.4 Exchange vs. the right introduction rules

We describe now the proof transformations induced by the interaction between an exchange rule and a right introduction rule.

Right introduction of the tensor product — The proof

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1,\cdots,A_p \vdash C} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1,\ldots,B_q \vdash D}}{E_1,\ldots,E_{p+q} \vdash C\otimes D}\ \text{Right}\otimes}{E_{g\otimes h\rhd 1},\cdots,E_{g\otimes h\rhd p+q} \vdash C\otimes D}\ \text{Exchange}[g\otimes h]
$$

where $E_i = A_i$ for $1 \le i \le p$ and $E_{p+i} = B_i$ for $1 \le i \le q$, is transformed into

$$
\dfrac{\text{Exchange}[g]\ \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1,\cdots,A_p \vdash C}}{A_{g\rhd 1},\ldots,A_{g\rhd p} \vdash C} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1,\ldots,B_q \vdash D}}{B_{h\rhd 1},\ldots,B_{h\rhd p} \vdash D}\ \text{Exchange}[h]}{A_{g\rhd 1},\ldots,A_{g\rhd p},B_{h\rhd 1},\cdots,B_{h\rhd q} \vdash C\otimes D}\ \text{Right}\otimes
$$

Right introduction of the left negation —

$$
[g]\ \dfrac{\text{Right}\multimap\ \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A,B_1,\ldots,B_n \vdash \bot}}{B_1,\ldots,B_n \vdash A\multimap\bot}}{B_{g\rhd 1},\ldots,B_{g\rhd n} \vdash A\multimap\bot}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A,B_1,\ldots,B_n \vdash \bot}}{A,B_{g\rhd 1},\ldots,B_{g\rhd n} \vdash \bot}\ [1\otimes g]}{B_{g\rhd 1},\ldots,B_{g\rhd n} \vdash A\multimap\bot}\ \text{Right}\multimap
$$

Right introduction of the right negation —

$$
[g]\ \dfrac{\text{Right}\multimapinv\ \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_n,B \vdash \bot}}{A_1,\ldots,A_n \vdash \bot\multimapinv B}}{A_{g\rhd 1},\ldots,A_{g\rhd n} \vdash \bot\multimapinv B}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A_1,\ldots,A_n,B \vdash \bot}}{A_{g\rhd 1},\ldots,A_{g\rhd n},B \vdash \bot}\ [g\otimes 1]}{A_{g\rhd 1},\ldots,A_{g\rhd n} \vdash \bot\multimapinv B}\ \text{Right}\multimapinv
$$

3.4 A normal form

There is a cut-elimination procedure: cuts can be removed. Once cuts have been removed, then one may apply the proof transformations above, as well as the proof transformations in the second appendix, in order to get a cut-free proof in normal form. What does that mean? First we do

We can start by Left introduction of $\otimes$ and of I until we reach a negation or an atom everywhere. Then a series of exchange rules. Then we apply the right introduction of the tensor product in order to separate the subproofs which were tonsured. We get sequents Gamma |- A where A is a negation or an atom. We can then remove a right introduction of a left or right negation if we picked properly the exchange laws. We start again...

3.5 Soundness theorem

Suppose given a proof π of ribbon logic over the category $\mathscr{C}$. Then, Every formal proof of ribbon logic may be interpreted in a ribbon dialogue category

Proposition 1 *Every derivation tree of ribbon logic may be interpreted as a morphism in a ribbon dialogue category.*

4 The free balanced dialogue category

We give a categorical account of the soundness theorem.

4.1 The free balanced dialogue category

Suppose given a category $\mathscr{X}$. Let $\mathscr{C}$ denote the category

- with the formulas of ribbon logic as objects,
- with the proofs of $A \vdash B$ modulo proof equality as morphisms $A \longrightarrow B$.

Composition is defined using the cut-rule.

Proposition 2 *The category $\mathscr{C}$ defines a ribbon dialogue category.*

Proof. there is a one-to-one relationship between the proofs of $A_1, \cdots, A_n \vdash B$ and the proofs of $A_1 \otimes \cdots \otimes A_n \vdash B$

The category with formulas as objects, proofs modulo equality as morphisms.

5 Ribbon categories

To that purpose, we start by recalling the definition of monoidal categories in § 2.1, of braided monoidal categories in § 2.2, of balanced monoidal categories in § 2.3 and of ribbon categories in § 5.1. We finally introduce the notion of balanced dialogue category in § 2.4. We show in § **??** that every balanced dialogue category induces two cyclic structures on the tensorial pivot $\perp$. We conclude the section by illustrating in § **??** the notion of balanced dialogue category with an example coming from representation theory of quantum groups: the category of (finite and infinite dimensional) H-modules associated to a ribbon Hopf algebra H.

5.1 Ribbon categories

Definition 5 (dual pairs) *A dual pair in a monoidal category $\mathscr{C}$ is a quadruple $(x, y, \eta, \varepsilon)$ consisting of two objects x and y and two morphisms*

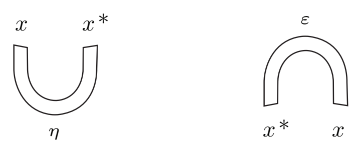
$$\eta : I \longrightarrow x \otimes y \qquad\qquad \varepsilon : y \otimes x \longrightarrow I$$

making the two diagrams below commute:

$$\begin{array}{ccc} (x \otimes y) \otimes x & \xrightarrow{\alpha^{-1}} & x \otimes (y \otimes x) \\ \uparrow{\scriptstyle \eta\otimes x} & & \downarrow{\scriptstyle x\otimes\varepsilon} \\ x & \xrightarrow{\mathrm{id}_x} & x \end{array} \qquad\qquad \begin{array}{ccc} y \otimes (x \otimes y) & \xrightarrow{\alpha^{-1}} & (y \otimes x) \otimes y \\ \uparrow{\scriptstyle y\otimes\eta} & & \downarrow{\scriptstyle \varepsilon\otimes y} \\ y & \xrightarrow{\mathrm{id}_y} & y \end{array}$$

One often writes $x \dashv y$ in that case, and says that y is a left dual of x, and that x is a right dual of y.

The unit η and count ε are depicted as U-turns:



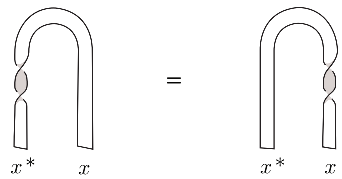
The two coherence diagrams express how a U-turn combines with a U-turn in the other direction:

Definition 6 (ribbon category) *A ribbon category $\mathscr{C}$ is a balanced category where every object x is equipped with a duality $(x, x^*, \eta_x, \varepsilon_x)$ and such that the diagram*

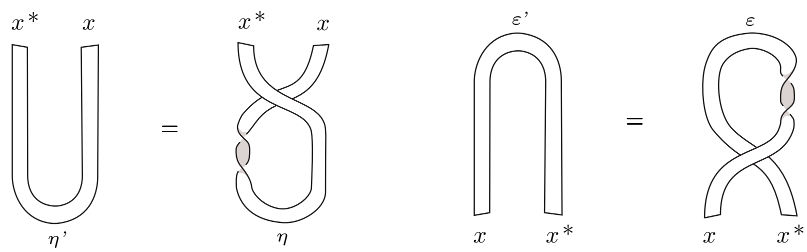
$$\begin{array}{ccc} x^* \otimes x & \xrightarrow{x^* \otimes \theta_x} & x^* \otimes x \\ \downarrow{\scriptstyle \theta_{x^*}\otimes x} & & \downarrow{\scriptstyle \varepsilon} \\ x^* \otimes x & \xrightarrow{\varepsilon} & I \end{array}$$

commutes for every object x of the category $\mathscr{C}$.

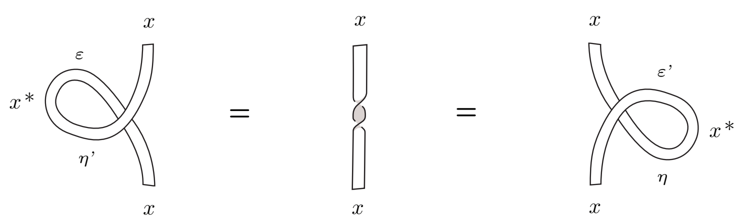
The coherence diagram of ribbon categories is depicted as



Note that in a ribbon category, every object x^* is also right dual to the object x thanks to the dual pair $x \dashv x^*$ with unit η' and counit ε' defined as



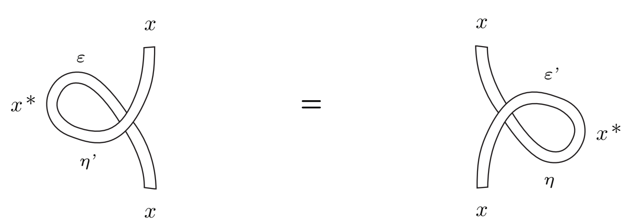
This implies in particular that the following equality holds in every ribbon category:



(5)

This leads to a concise definition of ribbon category, which does not mention the balanced structure:

Proposition 3 *A ribbon category is the same thing as a braided category where every object x is equipped with a dual pair $(x, x^*, \eta, \varepsilon)$ and a dual pair $(x^*, x, \eta', \varepsilon')$ satisfying the equality:*



Note that the object x^* is at the same time a left dual and a right dual of the object x.

5.2 The free ribbon category

Shum – framed ribbon tangles

5.3 The proof-as-tangle functor

Every ribbon category $\mathscr{C}$ equipped with a distinguished object $\bot$ defines a balanced dialogue category where the two negation functors are defined as

$$x \multimap \bot \quad = \quad x^* \otimes \bot \qquad\qquad \bot \multimapinv x \quad = \quad \bot \otimes x^*$$

As explained in § 7.2, the balanced structure on the dialogue category $\mathscr{C}$ induces a cyclic structure *wheel* defined in Equation (8). The twist in Equation (8) ensures then that the associated *turn* described in § **??** coincides with the negative braiding

$$turn_x \quad = \quad \gamma^{-1}_{\bot, x^*} \quad : \quad x^* \otimes \bot \quad \longrightarrow \quad \bot \otimes x^*$$

permuting the object x^* under the tensorial pivot $\bot$. This also justifies the informal topological explanation for the definition of *wheel* in § 7.2. Indeed, the topological equality of Equation (9) makes sense in any balanced dialogue category coming from a ribbon category with a distinguished object $\bot$ — although the diagrams are meaningless in a general balanced dialogue category.

6 The proof-as-tangle theorem

Every object $\bot$ picked in a ribbon category $\mathscr{D}$ defines a balanced dialogue category with left and right negations defined as:

$$A \multimap \bot = A^* \otimes \bot \qquad\qquad \bot \multimapinv A = \bot \otimes A^*$$

Moreover, the canonical definition of turn induced from the balanced structure of the category $\mathscr{D}$ coincides with the braiding map:

$$\gamma_{A^*, \bot} : A^* \otimes \bot \longrightarrow \bot \otimes A^*.$$

Now, suppose given a category $\mathscr{C}$, and define the category $\mathscr{C}_\bot$ as the category $\mathscr{C}$ extended with an object $\bot$. Note in particular that $\mathscr{C}_\bot$ is isomorphic to the category $\mathscr{C} + 1$ where 1 is the singleton category. By definition, the free ribbon category **free-ribbon**($\mathscr{C}_\bot$) is a ribbon category, and thus a balanced dialogue category with $\bot$ defined as tensorial pole. From this follows that there exists, up to isomorphism, a unique dialogue functor

$$[-] \quad : \quad \textbf{free-dialogue}(\mathscr{C}) \quad \longrightarrow \quad \textbf{free-ribbon}(\mathscr{C}_\bot)$$

making the diagram commute:

$$\begin{array}{ccc} \textbf{free-dialogue}(\mathscr{C}) & \longrightarrow & \textbf{free-ribbon}(\mathscr{C}_\bot) \\ \uparrow & & \uparrow \\ \mathscr{C} & \longrightarrow & \mathscr{C}_\bot \end{array}$$

This leads to the main theorem of the article.

Theorem 1 *The functor* $[-]$ *is faithful.*

Proof. Suppose that two cut-free derivation trees

$$\frac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{A \vdash B} \qquad \frac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{A \vdash B}$$

of ribbon logic induce the same tangle $[\pi_1] = [\pi_2]$ modulo topological deformation in the category **free-ribbon**$(\mathscr{C}_\perp)$. We show that $\pi_1 \leftrightsquigarrow \pi_2$ and conclude. We proceed by induction on the number of links in the tangle. By the normal form theorem, we know that the proofs π_1 and π_2 are equal modulo logical equality to:

$$\frac{\begin{array}{c}\pi_1'\\ \vdots\end{array}}{A_1, \ldots, A_n \vdash B} \qquad \frac{\begin{array}{c}\pi_2'\\ \vdots\end{array}}{A_1, \ldots, A_n \vdash B}$$

where each A_i is either a negation or an atom, followed by the same sequence of left introduction of tensor and left introduction of unit. Suppose that $B = \perp$. In that case, the formula $\perp$ was either introduced by:

- an axiom rule,
- the left introduction of a left negation,
- or the left introduction of a right negation.

The case of the axiom is easy to treat, and left to the reader.. We may suppose without loss of generality that it is a left negation rule. In that case, the proof π_1' is equal to

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\pi_1''\\ \vdots\end{array}}{X_1, \ldots, X_{n-1} \vdash X}}{X_1, \ldots, X_{n-1}, X \multimap \perp \vdash \perp}\ \text{Left} \multimap}{A_1, \ldots, A_n \vdash \perp}\ \text{Exchange}[g]$$

The topological equality of $[\pi_1]$ and $[pi_2]$ implies that $\perp$ is also introduced by the left introduction of a left negation. The proof π_2' factors as

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\pi_2''\\ \vdots\end{array}}{Y_1, \ldots, Y_{n-1} \vdash Y}}{Y_1, \ldots, Y_{n-1}, Y \multimap \perp \vdash \perp}\ \text{Left} \multimap}{A_1, \ldots, A_n \vdash \perp}\ \text{Exchange}[h]$$

From this, we conclude that the proof

$$\cfrac{\cfrac{\begin{array}{c}\pi_1''\\ \vdots\end{array}}{X_1, \ldots, X_{n-1} \vdash X}}{X_1, \ldots, X_{n-1}, X \multimap \perp \vdash \perp}\ \text{Left} \multimap$$

has the same topological tangle as

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_2'' \atop \vdots}{Y_1, \ldots, Y_{n-1} \vdash Y}}{Y_1, \ldots, Y_{n-1}, Y \multimap \bot \vdash \bot} \text{ Left } \multimap}{A_1, \ldots, A_n \vdash \bot} \text{ Exchange}[h]}{X_1, \ldots, X_{n-1}, X \multimap \bot \vdash \bot} \text{ Exchange}[g^{-1}]$$

From this we deduce that

$$g^{-1} \circ h \in \mathbf{G}_n$$

is of the form

$$f \otimes 1.$$

From this follows that the second proof is equal to

$$\dfrac{\pi_1'' \atop \vdots}{X_1, \ldots, X_{n-1} \vdash X}$$

Moreover, the two proofs

$$\dfrac{\pi_1'' \atop \vdots}{X_1, \ldots, X_{n-1} \vdash X}$$

and

$$\dfrac{\dfrac{\pi_2'' \atop \vdots}{Y_1, \ldots, Y_{n-1} \vdash Y}}{Y_{f \rhd 1}, \ldots, Y_{f \rhd n-1} \vdash Y} \text{ Exchange[f]}$$

have the same tangle, and thus are equal in the logical equality by induction hypothesis.

Now, suppose that $B = B_1 \otimes B_2$. In that case, the last rule is a tensor. We get a tensor product of proofs, and we carry on.

7 Cyclic dialogue categories and cyclic logic

7.1 Cyclic dialogue categories

The notion of *cyclic* dialogue category is recalled in this section. The notion may be defined in two different but equivalent ways. On the one hand, it is defined a dialogue category equipped with a family of bijections

$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \longrightarrow \quad \mathscr{C}(y \otimes x, \bot)$$

natural in x and y, required to make the diagram below

$$\begin{array}{ccc}
\mathscr{C}((y \otimes z) \otimes x, \bot) & \xrightarrow{\textit{associativity}} & \mathscr{C}(y \otimes (z \otimes x), \bot) \\
\uparrow {\scriptstyle wheel_{x,y\otimes z}} & & \downarrow {\scriptstyle wheel_{y,z\otimes x}} \\
\mathscr{C}(x \otimes (y \otimes z), \bot) & & \mathscr{C}((z \otimes x) \otimes y, \bot) \\
\downarrow {\scriptstyle associativity} & & \uparrow {\scriptstyle associativity} \\
\mathscr{C}((x \otimes y) \otimes z, \bot) & \xrightarrow{wheel_{x\otimes y, z}} & \mathscr{C}(z \otimes (x \otimes y), \bot)
\end{array} \tag{6}$$

commute for all objects x, y, z. A cyclic dialogue category may be alternatively defined as a dialogue category $\mathscr{C}$ equipped with a natural isomorphism

$$turn_x \quad : \quad x \multimap \bot \quad \longrightarrow \quad \bot \mathrel{\circ\!\!-} x$$

making the coherence diagram

$$\begin{array}{ccccc}
 & & \bot & & \\
 & \nearrow {\scriptstyle reval} & & \nwarrow {\scriptstyle leval} & \\
(\bot \mathrel{\circ\!\!-} x) \otimes x & & & & y \otimes (y \multimap \bot) \\
\uparrow {\scriptstyle turn_x} & & & & \uparrow {\scriptstyle turn_y^{-1}} \\
(x \multimap \bot) \otimes x & & & & y \otimes (\bot \mathrel{\circ\!\!-} y) \\
\uparrow {\scriptstyle lev} & & & & \uparrow {\scriptstyle rev} \\
y \otimes ((x \otimes y) \multimap \bot) \otimes x & & \xrightarrow{turn_{x\otimes y}} & & y \otimes (\bot \mathrel{\circ\!\!-} (x \otimes y)) \otimes x
\end{array} \tag{7}$$

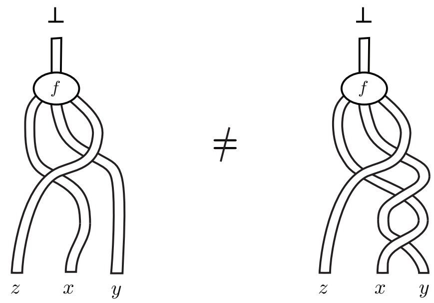
commute for all objects x, y of the category $\mathscr{C}$.

7.2 Every balanced dialogue category is cyclic

Every dialogue category $\mathscr{C}$ whose underlying monoidal category is braided comes equipped with a natural bijection

$$\begin{array}{rcccc}
wheel_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \longrightarrow & \mathscr{C}(y \otimes x, \bot) \\
 & & f & \longmapsto & f \circ \gamma_{y,x}
\end{array}$$

Unfortunately, the bijection does not satisfy the coherence diagram (6) required of a cyclic structure in § **??**. The trouble comes from the fact that the two diagrams below are not necessarily equal because the category $\mathscr{C}$ is braided, rather than symmetric:



So, in order to obtain the desired equality

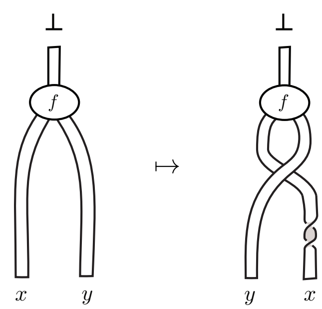
$$wheel_{\,y,z\otimes x} \quad = \quad wheel_{\,y,z\otimes x} \quad \circ \quad wheel_{\,x,y\otimes z}$$

one needs to define *wheel* in a slightly different way. However, braided categories are not sufficient to that purpose. This is precisely the reason for shifting to balanced categories, since they provide us with a satisfactory solution based on the ability to twist ribbon strands. Indeed, every balanced dialogue category $\mathscr{C}$ comes equipped with a natural bijection *wheel* defined this time as:

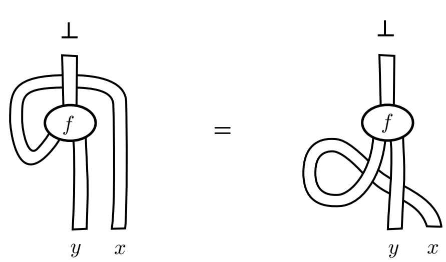
$$\begin{array}{rclcl} wheel_{\,x,y} & : & \mathscr{C}(x\otimes y,\bot) & \longrightarrow & \mathscr{C}(y\otimes x,\bot) \\ & & f & \longmapsto & f\circ\gamma_{y,x}\circ(\mathrm{id}_y\otimes\theta_x) \end{array} \tag{8}$$

This bijection satisfies the coherence diagram (6) and thus defines a cyclic structure on the balanced dialogue category $\mathscr{C}$. Pictorially:

$wheel_{\,x,y}$:



This definition is also supported (at least informally) by the topological equality which relates the pictorial notation for *wheel* on the one hand, and the topological reformulation in (5) of the twist map on the other hand.



(9)

Although this diagram does not make sense in the general case of a balanced dialogue category, we will see in § **??** that it becomes meaningful in the particular case of a balanced dialogue category coming from a ribbon category.

One shows moreover that

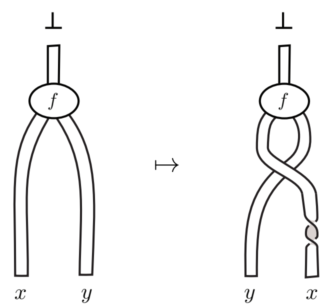
Proposition 4 *In every balanced dialogue category $\mathscr{C}$, the dialogical twist associated to the cyclic structure (8) in Proposition* **??** *coincides with the twist map $\theta_\perp$ associated to the tensorial focus.*

In every balanced dialogue category, there exists another cyclic structure defined as:

$$\overline{wheel}_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \perp) \longrightarrow \mathscr{C}(y \otimes x, \perp) \qquad f \longmapsto f \circ \gamma^{-1}_{x,y} \circ (\mathrm{id}_y \otimes \theta_x^{-1}) \tag{10}$$

Pictorially,

$\overline{wheel}_{x,y}$:



Note that the induced dialogical twist is equal to $\theta_\perp^{-1}$.

7.3 Cyclic logic

In the case of cyclic logic, the exchange rule is limited to

$$\frac{A_1, \ldots, A_n \vdash \perp}{A_{g \triangleright 1}, \ldots, A_{g \triangleright n} \vdash \perp} \text{Exchange}[g]$$

where g is an element of the abelian group $(\mathbb{Z}, +, 0)$.

Exchange vs. cut. Note that such a h does not exist in the case of cyclic logic, when $B = \perp$ and one of the two contexts Υ_1 and Υ_2 is nonempty. We will see that this rewrite rule is not necessary in the case of cyclic logic. This has to do with the status of $\perp$ in a context

$$\Upsilon_1, \perp, \Upsilon_2 \vdash A$$

which has to introduce $\perp$ on the left with an axiom.

7.4 The free cyclic dialogue category

....

8 Appendix: the cut-elimination procedure

The cut-elimination procedure is described as a series of symbolic transformations on proofs in Sections 8.2 – **??**.

8.1 Interaction with the exchange rule

Before starting the comprehensive list of rewriting steps, we complete the series of interactions coming from the exchange rule.

Exchange vs. cut — The proof

$$\dfrac{\text{Exchange}[g]\ \dfrac{\dfrac{\pi_1 \atop \vdots}{A_1,\cdots,A_n \vdash B}}{A_{g\rhd 1},\ldots,A_{g\rhd n}\vdash B} \qquad \dfrac{\pi_2 \atop \vdots}{\Upsilon_1, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A_{g\rhd 1},\ldots,A_{g\rhd n},\Upsilon_2 \vdash C}\ \text{Cut}$$

is transformed into

$$\dfrac{\dfrac{\pi_1 \atop \vdots}{A_1,\cdots,A_n \vdash B} \qquad \dfrac{\dfrac{\pi_2 \atop \vdots}{\Upsilon_1, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A_1,\cdots,A_n,\Upsilon_2 \vdash C}\ \text{Cut}}{\Upsilon_1, A_{g\rhd 1},\ldots,A_{g\rhd n},\Upsilon_2 \vdash C}\ \text{Exchange}[h]$$

where $h = p \otimes h \otimes q$ is deduced from g and the size p and q of the two contexts Υ_1 and Υ_2.

Cut vs. exchange — The proof

$$\dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A_i} \qquad \dfrac{\dfrac{\pi_2 \atop \vdots}{A_1,\ldots,A_n \vdash B}}{A_{g\rhd 1},\ldots,A_{g\rhd n}\vdash B}\ \text{Exchange}[g]}{A_{g\rhd 1},\ldots,A_{j-1},\Gamma,A_{j+1},\ldots,A_{g\rhd n}\vdash B}\ \text{Cut}$$

where $j = g^{-1} \rhd i$ is the unique index such that $g \rhd j = i$, is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A_i} \qquad \dfrac{\pi_2 \atop \vdots}{A_1,\ldots,A_n \vdash B}}{A_1,\ldots,A_{i-1},\Gamma,A_{i+1},A_n \vdash B}\ \text{Cut}}{A_{g\rhd 1},\ldots,A_{j-1},\Gamma,A_{j+1},\ldots,A_{g\rhd n}\vdash B}\ \text{Exchange}[h]$$

where h is deduced from g and the size of the context Γ.

8.2 Commuting conversions

The two proofs below are equivalent from the point of view of cut-elimination:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma\vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_2, A, \Upsilon_3\vdash B\end{array}\qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1,B,\Upsilon_4\vdash C\end{array}}{\Upsilon_1,\Upsilon_2,A,\Upsilon_3,\Upsilon_4\vdash C}\,\text{Cut}}{\Upsilon_1,\Upsilon_2,\Gamma,\Upsilon_3,\Upsilon_4\vdash C}\,\text{Cut}$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma\vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_2, A, \Upsilon_3\vdash B\end{array}}{\Upsilon_2,\Gamma,\Upsilon_3\vdash B}\,\text{Cut}\qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1,B,\Upsilon_4\vdash C\end{array}}{\Upsilon_1,\Upsilon_2,\Gamma,\Upsilon_3,\Upsilon_4\vdash C}\,\text{Cut}$$

In particular, the cut-elimination procedure is allowed to transform the first proof into the second one, and conversely. The two proofs below are also equivalent from the point of view of cut-elimination:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma\vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta\vdash B\end{array}\qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1,A,\Upsilon_2,B,\Upsilon_3\vdash C\end{array}}{\Upsilon_1,A,\Upsilon_2,\Delta,\Upsilon_3\vdash C}\,\text{Cut}}{\Upsilon_1,\Gamma,\Upsilon_2,\Delta,\Upsilon_3\vdash C}\,\text{Cut}$$

$$\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta\vdash B\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma\vdash A\end{array}\qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1,A,\Upsilon_2,B,\Upsilon_3\vdash C\end{array}}{\Upsilon_1,\Gamma,\Upsilon_2,B,\Upsilon_3\vdash C}\,\text{Cut}}{\Upsilon_1,\Gamma,\Upsilon_2,\Delta,\Upsilon_3\vdash C}\,\text{Cut}$$

8.3 The axiom steps

Axiom vs. cut

$$\dfrac{\text{Axiom}\,\dfrac{}{A\vdash A}\qquad \begin{array}{c}\pi\\ \vdots\\ \hline \Upsilon_1,A,\Upsilon_2\vdash B\end{array}}{\Upsilon_1,A,\Upsilon_2\vdash B}\,\text{Cut} \quad\rightsquigarrow\quad \begin{array}{c}\pi\\ \vdots\\ \hline \Upsilon_1,A,\Upsilon_2\vdash B\end{array}$$

Cut vs. axiom

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \hline \Gamma\vdash A\end{array}\qquad \dfrac{}{A\vdash A}\,\text{Axiom}}{\Gamma\vdash A}\,\text{Cut} \quad\rightsquigarrow\quad \begin{array}{c}\pi\\ \vdots\\ \hline \Gamma\vdash A\end{array}$$

8.4 Principal formulas

The tensor product The proof

$$\dfrac{\text{Right}\otimes\ \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}\ \text{Left}\otimes}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B} \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}$$

A choice has been made here: indeed, the cut rule on the formula $A \otimes B$ is replaced by a cut rule on the formula B, followed by a cut rule on the formula A. The other order could have been followed instead, with the cut rule on A applied before the cut rule on B. However, this choice is innocuous, because the two derivations resulting from this choice are equivalent modulo the conversion rule.

Tensor unit The proof

$$\dfrac{\text{Right } I\ \dfrac{}{\vdash I} \qquad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Upsilon_1, \Upsilon_2 \vdash A}}{\Upsilon_1, I, \Upsilon_2 \vdash A}\ \text{Left } I}{\Upsilon_1, \Upsilon_2 \vdash A}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Upsilon_1, \Upsilon_2 \vdash A}$$

Left negation The proof

$$\dfrac{\text{Right} \multimap\ \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A, \Delta \vdash \bot}}{\Delta \vdash A \multimap \bot} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma \vdash A}}{\Gamma, A \multimap \bot \vdash \bot}\ \text{Left} \multimap}{\Gamma, \Delta \vdash \bot}\ \text{Cut}$$

is transformed into the proof

$$
\dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_1}{\vdots}}{A, \Delta \vdash \bot}}{\Gamma, \Delta \vdash \bot}\,\text{Cut}
$$

Right negation The proof

$$
\dfrac{\text{Right}\multimapinv\ \dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma, A \vdash \bot}}{\Gamma \vdash \bot \multimapinv A} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Delta \vdash A}}{\bot \multimapinv A, \Delta \vdash \bot}\,\text{Left}\multimapinv}{\Gamma, \Delta \vdash \bot}\,\text{Cut}
$$

is transformed into the proof

$$
\dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Delta \vdash A} \qquad \dfrac{\overset{\pi_1}{\vdots}}{\Gamma, A \vdash \bot}}{\Gamma, \Delta \vdash \bot}\,\text{Cut}
$$

8.5 Expansion steps

Tensor product

$$
\dfrac{}{A \otimes B \vdash A \otimes B}\,\text{Axiom} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \qquad \dfrac{}{B \vdash B}\,\text{Axiom}}{A, B \vdash A \otimes B}\,\text{Right}\otimes}{A \otimes B \vdash A \otimes B}\,\text{Left}\otimes
$$

Left negation

$$
\dfrac{}{A \multimap \bot \vdash A \multimap \bot}\,\text{Axiom} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{A, A \multimap \bot \vdash \bot}\,\text{Left}\multimap}{A \multimap \bot \vdash A \multimap \bot}\,\text{Right}\multimap
$$

Right negation

$$
\dfrac{}{\bot \multimapinv A \vdash \bot \multimapinv A}\,\text{Axiom} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{\bot \multimapinv A, A \vdash \bot}\,\text{Left}\multimapinv}{\bot \multimapinv A \vdash \bot \multimapinv A}\,\text{Right}\multimapinv
$$

Tensor unit

$$
\dfrac{}{I \vdash I}\,\text{Axiom} \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\vdash I}\,\text{Right}\ I}{I \vdash I}\,\text{Left}\ I
$$

8.6 Secondary hypothesis

Tensor product (right introduction) The proof

$$\dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\pi_2 \atop \vdots}{\Upsilon_1, A, \Upsilon_2 \vdash B} \quad \dfrac{\pi_3 \atop \vdots}{\Delta \vdash C}}{\Upsilon_1, A, \Upsilon_2, \Delta \vdash B \otimes C}\ \text{Right} \otimes}{\Upsilon_1, \Gamma, \Upsilon_2, \Delta \vdash B \otimes C}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\text{Cut}\ \dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A} \quad \dfrac{\pi_2 \atop \vdots}{\Upsilon_1, A, \Upsilon_2 \vdash B}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B} \qquad \dfrac{\pi_3 \atop \vdots}{\Delta \vdash C}}{\Upsilon_1, \Gamma, \Upsilon_2, \Delta \vdash B \otimes C}\ \text{Right} \otimes$$

Similarly, the proof

$$\dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\pi_2 \atop \vdots}{\Delta \vdash B} \quad \dfrac{\pi_3 \atop \vdots}{\Upsilon_1, A, \Upsilon_2 \vdash C}}{\Delta, \Upsilon_1, A, \Upsilon_2 \vdash B \otimes C}\ \text{Right} \otimes}{\Delta, \Upsilon_1, \Gamma, \Upsilon_2 \vdash B \otimes C}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\pi_2 \atop \vdots}{\Delta \vdash B} \qquad \dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A} \quad \dfrac{\pi_3 \atop \vdots}{\Upsilon_1, A, \Upsilon_2 \vdash C}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash C}\ \text{Cut}}{\Delta, \Upsilon_1, \Gamma, \Upsilon_2 \vdash B \otimes C}\ \text{Right} \otimes$$

Tensor product (left introduction) The proof

$$\dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\pi_2 \atop \vdots}{\Upsilon_1, A, \Upsilon_2, B, C, \Upsilon_3 \vdash D}}{\Upsilon_1, A, \Upsilon_2, B \otimes C, \Upsilon_3 \vdash D}\ \text{Left} \otimes}{\Upsilon_1, \Gamma, \Upsilon_2, B \otimes C, \Upsilon_3 \vdash D}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\pi_1 \atop \vdots}{\Gamma \vdash A} \qquad \dfrac{\pi_2 \atop \vdots}{\Upsilon_1, A, \Upsilon_2, B, C, \Upsilon_3 \vdash D}}{\Upsilon_1, \Gamma, \Upsilon_2, B, C, \Upsilon_3 \vdash D}\ \text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, B \otimes C, \Upsilon_3 \vdash D}\ \text{Left} \otimes$$

Similarly, the proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash C} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, A, B, \Upsilon_2, C, \Upsilon_3 \vdash D}}{\Upsilon_1, A \otimes B, \Upsilon_2, C, \Upsilon_3 \vdash D}\ \text{Left} \otimes}{\Upsilon_1, A \otimes B, \Upsilon_2, \Gamma, \Upsilon_3 \vdash D}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash C} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, A, B, \Upsilon_2, C, \Upsilon_3 \vdash D}}{\Upsilon_1, A, B, \Upsilon_2, \Gamma, \Upsilon_3 \vdash D}\ \text{Cut}}{\Upsilon_1, A \otimes B, \Upsilon_2, \Gamma, \Upsilon_3 \vdash D}\ \text{Left} \otimes$$

Tensor unit (left introduction) The proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, A, \Upsilon_2, \Upsilon_3 \vdash D}}{\Upsilon_1, A, \Upsilon_2, I, \Upsilon_3 \vdash D}\ \text{Left } I}{\Upsilon_1, \Gamma, \Upsilon_2, I, \Upsilon_3 \vdash D}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, A, \Upsilon_2, \Upsilon_3 \vdash D}}{\Upsilon_1, \Gamma, \Upsilon_2, \Upsilon_3 \vdash D}\ \text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, I, \Upsilon_3 \vdash D}\ \text{Left } I$$

Similarly, the proof

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, \Upsilon_2, A, \Upsilon_3 \vdash D}}{\Upsilon_1, I, \Upsilon_2, A, \Upsilon_3 \vdash D}\ \text{Left } I}{\Upsilon_1, I, \Upsilon_2, \Gamma, \Upsilon_3 \vdash D}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, \Upsilon_2, A, \Upsilon_3 \vdash D}}{\Upsilon_1, \Upsilon_2, \Gamma, \Upsilon_3 \vdash D}\ \text{Cut}}{\Upsilon_1, I, \Upsilon_2, \Gamma, \Upsilon_3 \vdash D}\ \text{Left } I$$

Left negation (left introduction) The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2 \vdash B}}{\Upsilon_1, A, \Upsilon_2, B \multimap \bot \vdash \bot}\ \text{Left} \multimap}{\Upsilon_1, \Gamma, \Upsilon_2, B \multimap \bot \vdash \bot}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2 \vdash B}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}\ \text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, B \multimap \bot \vdash \bot}\ \text{Left} \multimap$$

Right negation (left introduction) The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, B, \Upsilon_2 \vdash A}}{\bot \multimapinv A, \Upsilon_1, B, \Upsilon_2 \vdash \bot}\ \text{Left} \multimapinv}{\bot \multimapinv A, \Upsilon_1, \Gamma, \Upsilon_2 \vdash \bot}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash B} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, B, \Upsilon_2 \vdash A}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash A}\ \text{Cut}}{\bot \multimapinv A, \Upsilon_1, \Gamma, \Upsilon_2 \vdash \bot}\ \text{Left} \multimapinv$$

Left negation (right introduction) The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, B \vdash \bot}}{\Upsilon_1, A, \Upsilon_2 \vdash \bot \multimapinv B}\ \text{Right} \multimapinv}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash \bot \multimapinv B}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, B \vdash \bot}}{\Upsilon_1, \Gamma, \Upsilon_2, B \vdash \bot}\ \text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash \bot \multimapinv B}\ \text{Right} \multimapinv$$

Right negation (right introduction) The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{A, \Upsilon_1, B, \Upsilon_2 \vdash \bot}}{\Upsilon_1, B, \Upsilon_2 \vdash A \multimap \bot}\,\text{Right } \multimap}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash A \multimap \bot}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash B} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{A, \Upsilon_1, B, \Upsilon_2 \vdash \bot}}{A, \Upsilon_1, \Gamma, \Upsilon_2 \vdash \bot}\,\text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash A \multimap \bot}\,\text{Right } \multimap$$

8.7 Secondary conclusion

Tensor product The proof

$$\dfrac{\text{Left } \otimes\,\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Upsilon_2, A, B, \Upsilon_3 \vdash C}}{\Upsilon_2, A \otimes B, \Upsilon_3 \vdash C} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, A \otimes B, \Upsilon_3, \Upsilon_4 \vdash D}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Upsilon_2, A, B, \Upsilon_3 \vdash C} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, A, B, \Upsilon_3, \Upsilon_4 \vdash D}\,\text{Cut}}{\Upsilon_1, \Upsilon_2, A \otimes B, \Upsilon_3, \Upsilon_4 \vdash D}\,\text{Left } \otimes$$

Tensor unit The proof

$$\dfrac{\text{Left } I\,\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Upsilon_2, \Upsilon_3 \vdash A}}{\Upsilon_2, I, \Upsilon_3 \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_4 \vdash B}}{\Upsilon_1, \Upsilon_2, I, \Upsilon_3, \Upsilon_4 \vdash B}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Upsilon_2, \Upsilon_3 \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_4 \vdash B}}{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4 \vdash B}\,\text{Cut}}{\Upsilon_1, \Upsilon_2, I, \Upsilon_3, \Upsilon_4 \vdash B}\,\text{Left } I$$

9 Appendix: the cut-elimination procedure (bis)

Permutation between the left and the right rules:

9.1 Left introduction of the tensor product

Right introduction of the tensor product

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Upsilon_1, A, B, \Upsilon_2 \vdash C\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Delta \vdash D\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2, \Delta \vdash C \otimes D}\ \text{Right } \otimes}{\Upsilon_1, A \otimes B, \Upsilon_2, \Delta \vdash C \otimes D}\ \text{Left } \otimes$$

is transformed into

$$\dfrac{\text{Left } \otimes\ \dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Upsilon_1, A, B, \Upsilon_2 \vdash C\end{matrix}}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Delta \vdash D\end{matrix}}{\Upsilon_1, A \otimes B, \Upsilon_2, \Delta \vdash C \otimes D}\ \text{Right } \otimes$$

Similarly, the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma \vdash C\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Upsilon_1, A, B, \Upsilon_2 \vdash D\end{matrix}}{\Gamma, \Upsilon_1, A, B, \Upsilon_2 \vdash C \otimes D}\ \text{Right } \otimes}{\Gamma, \Upsilon_1, A \otimes B, \Upsilon_2 \vdash C \otimes D}\ \text{Left } \otimes$$

is transformed into

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma \vdash C\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \Upsilon_1, A, B, \Upsilon_2 \vdash D\end{matrix}}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash D}\ \text{Left } \otimes}{\Gamma, \Upsilon_1, A \otimes B, \Upsilon_2 \vdash C \otimes D}\ \text{Right } \otimes$$

Right introduction of the left negation

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ A, \Upsilon_1, B, C, \Upsilon_2 \vdash \bot\end{matrix}}{\Upsilon_1, B, C, \Upsilon_2 \vdash A \multimap \bot}\ \text{Right } \multimap}{\Upsilon_1, B \otimes C, \Upsilon_2 \vdash A \multimap \bot}\ \text{Left } \otimes$$

is transformed into

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ \hline A, \Upsilon_1, B, C, \Upsilon_2 \vdash \bot\end{array}}{A, \Upsilon_1, B \otimes C, \Upsilon_2 \vdash \bot}\ \text{Left } \otimes}{\Upsilon_1, B \otimes C, \Upsilon_2 \vdash A \multimap \bot}\ \text{Right } \multimap$$

Right introduction of the right negation

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2, C \vdash \bot\end{array}}{\Upsilon_1, A, B, \Upsilon_2 \vdash \bot \multimapinv C}\ \text{Right} \multimapinv}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash \bot \multimapinv C}\ \text{Left } \otimes$$

is transformed into

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2, C \vdash \bot\end{array}}{\Upsilon_1, A \otimes B, \Upsilon_2, C \vdash \bot}\ \text{Left } \otimes}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash \bot \multimapinv C}\ \text{Right} \multimapinv$$

9.2 Left introduction of the unit

Right introduction of the unit

$$\dfrac{\dfrac{}{\vdash I}\ \text{Right } I}{I \vdash I}\ \text{Left } I$$

is transformed into

$$\dfrac{}{I \vdash I}\ \text{Axiom}$$

Right introduction of the tensor

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Upsilon_1, \Upsilon_2 \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \vdash B\end{array}}{\Upsilon_1, \Upsilon_2, \Delta \vdash A \otimes B}\ \text{Right } \otimes}{\Upsilon_1, I, \Upsilon_2, \Delta \vdash A \otimes B}\ \text{Left } I$$

is transformed into

$$\dfrac{\text{Left } I\ \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Upsilon_1, \Upsilon_2 \vdash A\end{array}}{\Upsilon_1, I, \Upsilon_2 \vdash A} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \vdash B\end{array}}{\Upsilon_1, I, \Upsilon_2, \Delta \vdash A \otimes B}\ \text{Right } \otimes$$

Similarly, the proof

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, \Upsilon_2 \vdash B}}{\Gamma, \Upsilon_1, \Upsilon_2 \vdash A \otimes B}\ \text{Right } \otimes}{\Gamma, \Upsilon_1, I, \Upsilon_2 \vdash A \otimes B}\ \text{Left } I$$

is transformed into

$$\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\overset{\pi_2}{\vdots}}{\Upsilon_1, \Upsilon_2 \vdash B}}{\Upsilon_1, I, \Upsilon_2 \vdash B}\ \text{Left } I}{\Gamma, \Upsilon_1, I, \Upsilon_2 \vdash A \otimes B}\ \text{Right } \otimes$$

Right introduction of the left negation.

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdots}}{A, \Upsilon_1, \Upsilon_2 \vdash \bot}}{\Upsilon_1, \Upsilon_2 \vdash A \multimap \bot}\ \text{Right } \multimap}{\Upsilon_1, I, \Upsilon_2 \vdash A \multimap \bot}\ \text{Left } I$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdots}}{A, \Upsilon_1, \Upsilon_2 \vdash \bot}}{A, \Upsilon_1, I, \Upsilon_2 \vdash A \multimap \bot}\ \text{Left } I}{\Upsilon_1, I, \Upsilon_2 \vdash A \multimap \bot}\ \text{Right } \multimap$$

Right introduction of the right negation.

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdots}}{\Upsilon_1, \Upsilon_2, A \vdash \bot}}{\Upsilon_1, \Upsilon_2 \vdash \bot \multimapinv A}\ \text{Right} \multimapinv}{\Upsilon_1, I, \Upsilon_2 \vdash \bot \multimapinv A}\ \text{Left } I$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdots}}{\Upsilon_1, \Upsilon_2, A \vdash \bot}}{\Upsilon_1, I, \Upsilon_2, A \vdash \bot}\ \text{Left } I}{\Upsilon_1, I, \Upsilon_2 \vdash \bot \multimapinv A}\ \text{Right } \multimap$$

9.3 Left introduction of the negation

No permutation, typically from the proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash A} \quad \dfrac{\pi_2}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\text{Right } \otimes}{\Gamma, \Delta, (A \otimes B) \multimap \bot \vdash \bot}\text{Left } \multimap$$

This means that the last rule may be a Left introduction of a negation. But in that case, the formula proved is $\bot$.