A micrological study of tortile negation

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Abstract

In this paper, we describe an algebraic presentation of the notion of helical dialogue chirality. In particular, the helix structure enables us to decompose the dual of the left negation as the right negation of the dual.

1 Motivations

Among the several equivalent definitions of *dialogue chirality* formulated in our companion paper [6], one finds a definition directly inspired by proof-theory and based on the following combinators:

$$\begin{array}{rcrcl} \mathbf{axiom}[m] & : & L(a) & \longrightarrow & m^* \varovee L(m \varowedge a) \\ \mathbf{cut}[m] & : & m \varowedge R(m^* \varovee b) & \longrightarrow & R(b) \end{array}$$

Besides the requirement that these combinators define a transjunction accross the adjunction $L \dashv R$ for every object m of the category $\mathscr{A}$, one requires that they satisfy a monoidality property, expressed by the following coherence diagram for the $\mathbf{axiom}[-]$ combinator:

$$\begin{array}{ccc} n^* \varovee L(n \varowedge a) & \xrightarrow{\mathbf{axiom}[m]} & n^* \varovee (m^* \varovee L(m \varowedge (n \varowedge a))) \\ & & \downarrow{\scriptstyle associativity} \\ \uparrow{\scriptstyle \mathbf{axiom}[n]} & & (n^* \varovee m^*) \varovee L((m \varowedge n) \varowedge a) \\ & & \downarrow{\scriptstyle monoidality} \\ La & \xrightarrow{\mathbf{axiom}[m \varowedge n]} & (m \varowedge n)^* \varovee L((m \varowedge n) \varowedge a) \end{array} \qquad (1)$$

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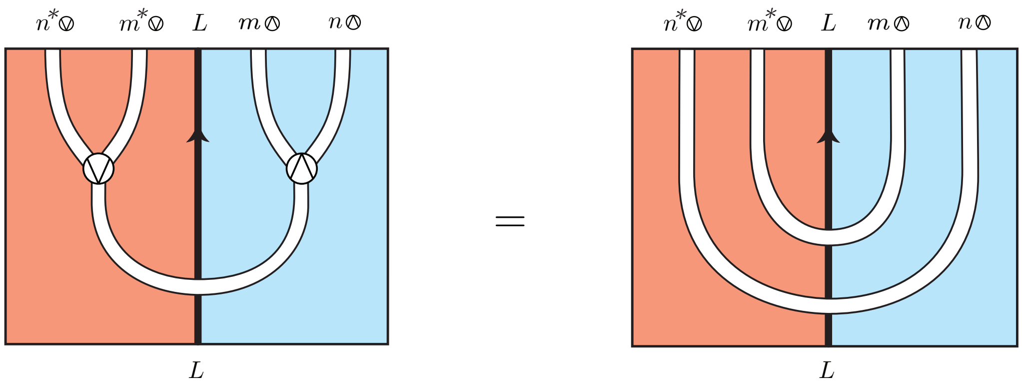
As expected, this diagram is required to commute for all objects a, m, n of the category $\mathscr{A}$. Looking at it with linear and tensorial proof-nets in mind, this coherence diagram may be understood as an η-expansion law for the axiom link. This point becomes even clearer when one translates (1) in the graphical language of string diagrams:



As should be clear from the picture, the purpose of the η-expansion law is to decompose the axiom$[m \owedge n]$ link into the pair of more elementary axiom$[m]$ and axiom$[n]$ links. A natural question is whether there exists a similar η-expansion law for the axiom link

$$La \xrightarrow{\quad \mathbf{left.axiom}[Rm] \quad} (Rm)^* \ovee L((Rm) \owedge a) \tag{2}$$

associated to the negation Rm of an object m of the category $\mathscr{B}$. The idea would be to deduce it from the axiom link of the object m itself. However, the object m lives in the category $\mathscr{B}$ and thus needs to be translated back to the object ${}^*m$ which lives in the other side $\mathscr{A}$ of the dialogue chirality. Recall that the two monoidal categories $(\mathscr{A}, \owedge, \mathbf{true})$ and $(\mathscr{B}, \ovee, \mathsf{false})$ of a dialogue chirality are related by a monoidal equivalence

$$\mathscr{A} \; \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \; \mathscr{B}^{op(0,1)} \qquad (\textit{monoidal equivalence})$$

See [5] for details. Note that we are careful here to indicate in (2) that we consider the *left* axiom link. The reason is that, in order to define an η-expansion for Rm, one needs to be careful about orientations and to start from the *right* axiom link

$$L(\mathbf{true}) \xrightarrow{\quad \mathbf{right.axiom}[m] \quad} L({}^*m) \ovee m$$

associated to the object m in the category $\mathscr{B}$. This morphism induces in turn the morphism

$$La \xrightarrow{\eta} L(RL(\mathbf{true}) \oslash a) \xrightarrow{map} L(R(L({}^*m) \otimes m) \oslash a)$$

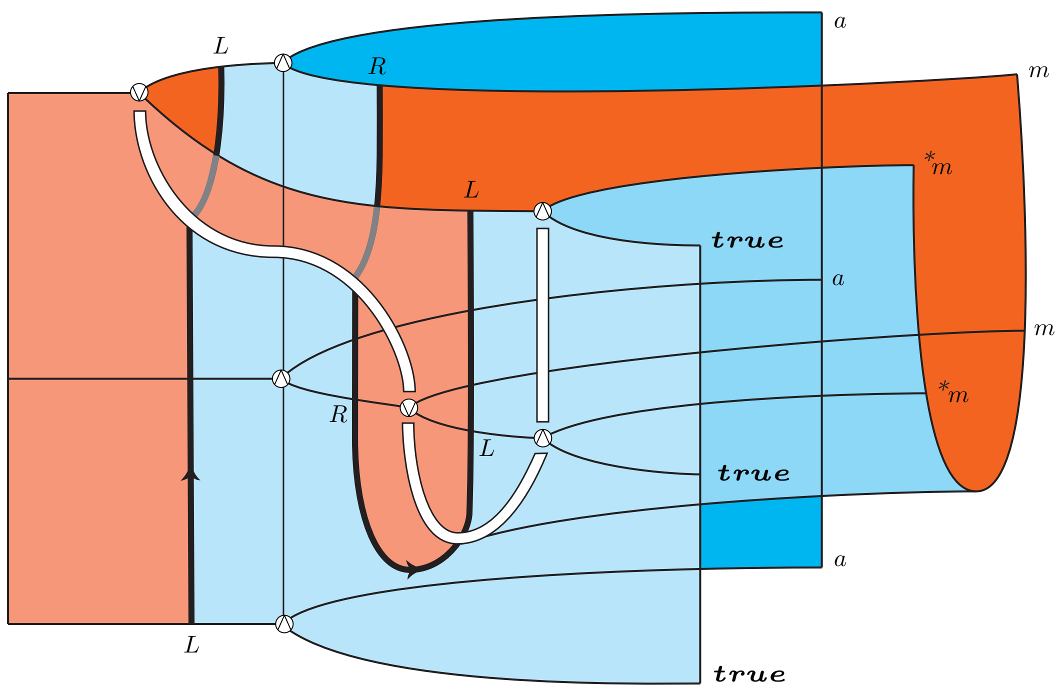
which may be composed with the distributivity law

$$L(R(L({}^*m) \otimes m) \oslash a) \xrightarrow{distributivity} L({}^*m) \otimes L((Rm) \oslash a)$$

The resulting morphism

$$La \longrightarrow L({}^*m) \otimes L((Rm) \oslash a) \tag{3}$$

may be depicted in the following way:



The different shades of blue and red are here to separate the different instances of categories $\mathscr{A}$ and $\mathscr{B}$ appearing in the 2-categorical diagram. Note that for graphical convenience, we depict a variant

$$L(\mathbf{true} \oslash a) \longrightarrow L(R(L(\mathbf{true} \oslash {}^*m) \otimes m) \oslash a) \longrightarrow L(\mathbf{true} \oslash {}^*m) \otimes L((Rm) \oslash a)$$

of the morphism (3). The last step in order to complete the construction is to require the existence of a family of isomorphisms

$$L({}^*m) \xrightarrow{isomorphism} (Rm)^* \tag{4}$$

natural in m. In this way, by composing (3) with (4), one obtains a morphism

$$La \longrightarrow (Rm)^* \varovee L((Rm) \varowedge a) \qquad (5)$$

with the expected source and target. There remains to understand whether the resulting morphism coincides with the original axiom link (2). Although the answer to this question is positive in many situations of interest, it appears that nothing compels the two morphisms to coincide in general. This technical point is subtle but fundamental because it reveals that logical negation may be *twisted* in exactly the same way as a topological ribbon. We will see that the two morphisms (2) and (5) are equal modulo a twist of angle 2π of the negation strand. This observation puts topology at the heart of logic.

Our primary purpose in this work is to shed light on these topological aspects of the logical negation by transferring ideas coming from functorial knot theory and representation theory of quantum groups. Recall that the notion of ribbon category axiomatizes the properties of the category $Mod(H)$ of finite dimensional representations of a quantum group H, defined here as a quasi-triangular and twisted Hopf algebra. Every object $\bot$ in such a ribbon category $\mathscr{C}$ defines a pair of negation functors

$$x \multimap \bot := x^{\wedge} \otimes \bot \qquad\qquad \bot \multimapinv x := \bot \otimes x^{\vee}$$

where $x^{\wedge}$ denotes the right dual and $x^{\vee}$ denotes the left dual of the object x in the ribbon category. These negation functors equip the ribbon category $\mathscr{C}$ with the structure of a dialogue category. The associated dialogue chirality $(\mathscr{A}, \mathscr{B})$ has monoidal sides defined as

$$(\mathscr{A}, \varowedge, \mathbf{true}) = (\mathscr{C}, \otimes, \mathbf{1}) \qquad\qquad (\mathscr{B}, \varovee, \mathbf{false}) = (\mathscr{C}, \otimes, \mathbf{1})^{op(0,1)}$$

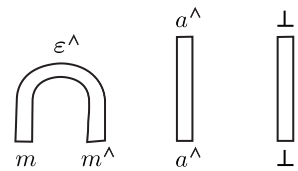
with monoidal equivalence $(-)^*$ and ${}^*(-)$ defined as the identity on $\mathscr{C}$. We wish to give a little precedence to the right axiom combinator and thus define L and R as the negation functors

$$Lx = x \multimap \bot = x^{\wedge} \otimes \bot \qquad\qquad Rx = \bot \multimapinv x = \bot \otimes x^{\vee}.$$

The right axiom associated to the object m in the category $\mathscr{C}$

$$La = a^{\wedge} \otimes \bot \xrightarrow{\textbf{left.axiom}[m]} L(a \varowedge {}^*m) \varovee m \quad \cong \quad m \otimes m^{\wedge} \otimes a^{\wedge} \otimes \bot$$

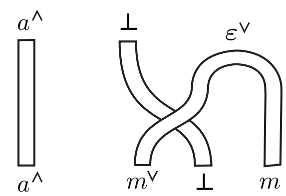
lives in the category $\mathscr{B} = \mathscr{C}^{op}$ and is thus defined as the counit of the dual pair $x \dashv x^{\wedge}$ and depicted as follows in the category $\mathscr{C}$:

right.**axiom**$[m]$ $=$ 

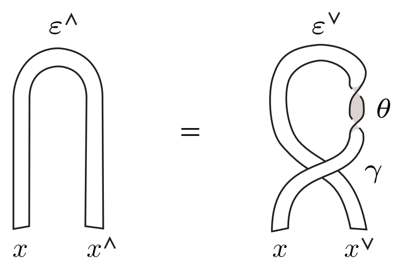
The left axiom associated an object m

$$La = a^{\wedge} \otimes \bot \xrightarrow{\textbf{left.axiom}[m]} m^{*} \varovee L(m \varowedge a) \quad \cong \quad a^{\wedge} \otimes m^{\wedge} \otimes \bot \otimes m$$

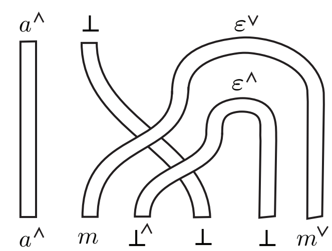
requires a braiding:

left.**axiom**$[m]$ $=$ 

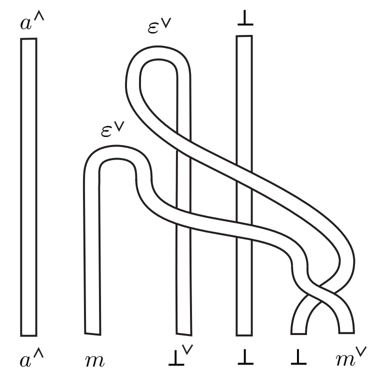
where we may for simplicity identify the left dual $x^{\vee}$ and the right dual $x^{\wedge}$ of the object x at the condition of adding the equality:



The two morphisms are not equal... but equal modulo a twist.

left.**axiom**$[Rm]$ $=$ 

On the other hand, the composite map is equal to:

composite morphism (5) $=$ 

This shows that the two morphisms (2) and (5) are different in the particular case of ribbon categories. They differ of a twist on the object $\bot$.

$$La = a^\wedge \otimes \bot \quad \longrightarrow \quad (Rm)^* \oslash L((Rm) \obslash a) \quad \cong \quad (a^\wedge \otimes \bot \otimes m^\vee) \otimes \bot \otimes \bot \otimes m^\vee$$

The twist may be obtained in any dialogue category !!!

As we will see, this additional combinator has a topological content, and may be interpreted as an *helical* structure on the tensorial negation of the underlying dialogue category $\mathscr{C}$. This helical structure generalizes the familiar notion of pivotal or cyclic monoidal category. This requires to integrate this helical map in the general theory of distributivity laws developed in our companion paper [6]. We will see that the two morphisms are not equal, that one needs to twist the negation functors R and L.

2 Helical dialogue categories and chiralities

The notion of helical dialogue category was introduced in the companion paper [7] among a series alternative notions of dialogue categories motivated by topology. The notion of helical dialogue category is important because it appears (at least in our current understanding) as the most primitive instance of these topological notions of dialogue categories. In particular, every cyclic or ribbon dialogue category is helical in a canonical way. Our main purpose in this section is to introduce and to justify the corresponding notion of *helical dialogue chirality*. In order to validate this 2-sided account of helical dialogue categories, we proceed in exactly the same way as for the notion of dialogue chirality, see [5] for details. We thus construct a 2-category ***HeliCat*** of helical dialogue categories in §2.2 and a 2-category ***HeliChir*** of helical dialogue chiralities in §2.3 and §2.4. Finally, we exhibit a 2-dimensional equivalence between the pair of 2-categories in §3.

2.1 Helical dialogue categories

A dialogue category is defined as a monoidal category $(\mathscr{C}, \otimes, I)$ equipped with an object $\bot$ coming together with a representation

$$\varphi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(y, x \multimap \bot)$$

of the functor

$$y \mapsto \mathscr{C}(x \otimes y, \bot) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object x, and with a representation

$$\psi_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \cong \quad \mathscr{C}(x, \bot \multimapinv y)$$

of the functor

$$x \mapsto \mathscr{C}(x \otimes y, \bot) \quad : \quad \mathscr{C}^{op} \longrightarrow Set$$

for each object y. The following notion of dialogue category is introduced in [6].

Definition 1 (helical dialogue category) *A helical dialogue category is a dialogue category $\mathscr{C}$ equipped with a family of bijections*

$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \longrightarrow \quad \mathscr{C}(y \otimes x, \bot)$$

natural in x and y and required to make the diagram

$$\begin{array}{ccc}
\mathscr{C}((y \otimes z) \otimes x, \bot) & \xrightarrow{\ associativity\ } & \mathscr{C}(y \otimes (z \otimes x), \bot) \\
{\scriptstyle wheel_{x, y \otimes z}} \uparrow & & \downarrow {\scriptstyle wheel_{y, z \otimes x}} \\
\mathscr{C}(x \otimes (y \otimes z), \bot) & & \mathscr{C}((z \otimes x) \otimes y, \bot) \\
{\scriptstyle associativity} \downarrow & & \uparrow {\scriptstyle associativity} \\
\mathscr{C}((x \otimes y) \otimes z, \bot) & \xrightarrow{\ wheel_{x \otimes y, z}\ } & \mathscr{C}(z \otimes (x \otimes y), \bot)
\end{array} \qquad (6)$$

commute for all objects x, y, z of the category $\mathscr{C}$.

2.2 A 2-category of helical dialogue categories

We define a 2-category ***HeliCat*** with

- helical dialogue categories as 0-cells,
- helical functors as 1-cells,
- dialogue natural transformations as 2-cells.

The 1-dimensional cells. A helical functor between two helical dialogue categories is defined as a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a morphism

$$\bot_F \quad : \quad F(\bot) \quad \longrightarrow \quad \bot$$

such that the diagram

$$\begin{array}{ccccc}
\mathscr{C}(x \otimes y, \bot) & \xrightarrow{F} & \mathscr{D}(F(x \otimes y), F(\bot)) & \xrightarrow{coercion} & \mathscr{D}(F(x) \otimes F(y), \bot) \\
\downarrow{\scriptstyle wheel_{x,y}} & & & & \downarrow{\scriptstyle wheel_{F(x),F(y)}} \\
\mathscr{C}(y \otimes x, \bot) & \xrightarrow{F} & \mathscr{D}(F(y \otimes x), F(\bot)) & \xrightarrow{coercion} & \mathscr{D}(F(y) \otimes F(x), \bot)
\end{array} \tag{7}$$

commutes for all objects x, y of the category $\mathscr{C}$. In this diagram, the two coercion maps are deduced by precomposing with the lax monoidal structure of the functor F

$$m_{x,y} \quad : \quad F(x) \otimes F(y) \quad \longrightarrow \quad F(x \otimes y)$$

and by postcomposing with the map $\bot_F$.

The 2-dimensional cells. The 2-cells are defined in the same way as in the case of dialogue categories, see [5] for details. A dialogue natural transformation

$$\theta \quad : \quad (F, \bot_F) \quad \Rightarrow \quad (G, \bot_G)$$

is defined there as a natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

making the diagram

$$\begin{array}{ccc}
F(\bot) & & \\
 & \searrow{\scriptstyle \bot_F} & \\
\downarrow{\scriptstyle \theta_\bot} & & \bot \\
 & \nearrow{\scriptstyle \bot_G} & \\
G(\bot) & &
\end{array}$$

commute. We leave the reader check that the expected notions of (horizontal and vertical) identity and composition define a 2-category ***HeliCat*** together with a forgetful 2-functor

$$U \quad : \quad \textbf{\textit{HeliCat}} \quad \longrightarrow \quad \textbf{\textit{DiaCat}}$$

to the 2-category ***DiaCat*** of dialogue categories constructed in [5]. Note that by construction, the 2-functor U is fully faithful on 2-dimensional cells.

2.3 Helical dialogue chiralities

Recall from [5, 6] that a dialogue chirality is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with an adjunction $L \dashv R$ and with a monoidal equivalence

$$\mathscr{A} \;\underset{R}{\overset{L}{\rightleftarrows}}\; \mathscr{B} \quad (\perp) \qquad\qquad \mathscr{A} \;\underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}}\; \mathscr{B}^{op(0,1)} \quad (\textit{monoidal equivalence})$$

together with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \ovee b \,\rangle \tag{8}$$

natural in a and b, where $\langle\, a \,|\, b \,\rangle$ is defined as

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R\, b\,).$$

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad\chi_{m \owedge n}\quad} & & \langle\, a \,|\, (m \owedge n)^* \ovee b \,\rangle \\
\downarrow \scriptstyle{associativity} & & & & \uparrow \scriptstyle{associativity,\ monoidality\ of\ negation} \\
\langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi_m} & \langle\, n \owedge a \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b) \,\rangle
\end{array} \tag{9}$$

commute. A helical dialogue chirality is then defined as a dialogue category equipped with a natural family of bijections *helix* permuting the two sides of the evaluation bracket $\langle - | - \rangle$.

Definition 2 (helical chirality) *A helical structure in a dialogue chirality is defined as a family of bijections*

$$helix_{a,b} \quad : \quad \langle\, a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, {}^*b \,|\, a^* \,\rangle$$

natural in a and b, and making the diagram below commute:

$$\begin{array}{ccccc}
\langle\, a_1 \owedge a_2 \,|\, b \,\rangle & \xrightarrow{helix} & \langle\, {}^*b \,|\, (a_1 \owedge a_2)^* \,\rangle & = & \langle\, a_2 \,|\, a_1^* \ovee b \,\rangle \\
\downarrow \scriptstyle{\chi_{a_1,a_2,b}} & & & & \uparrow \scriptstyle{\chi_{a_2,{}^*b,a_1^*}} \\
\langle\, a_2 \,|\, a_1^* \ovee b \,\rangle & & & & \langle\, a_2 \owedge {}^*b \,|\, a_1^* \,\rangle \\
\downarrow \scriptstyle{helix} & & & & \| \\
\langle\, {}^*(a_1^* \ovee b) \,|\, a_2^* \,\rangle & & & & \langle\, {}^*(b \ovee a_2^*) \,|\, a_1^* \,\rangle \\
\| & & & & \uparrow \scriptstyle{helix} \\
\langle\, {}^*b \owedge a_1 \,|\, a_2^* \,\rangle & \xrightarrow{\chi_{{}^*b,a_1,a_2^*}} & \langle\, a_1 \,|\, ({}^*b)^* \ovee a_2^* \,\rangle & = & \langle\, a_1 \,|\, b \ovee a_2^* \,\rangle
\end{array} \tag{10}$$

where every double edge is meant to describe a canonical coercion isomorphism induced by the monoidal equivalence between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$. A helical dialogue category is then defined as a dialogue category equipped with a helical structure.

The family $helix$ induces a series of bijections

$$\mathscr{A}(a, Rb) \xrightarrow{helix} \mathscr{A}(a, {}^*(L({}^*b))) \xrightarrow{equivalence} \mathscr{B}(L({}^*b), a^*) \xrightarrow{adjunction} \mathscr{A}({}^*b, R(a^*))$$

each of them, and thus their composite, natural in a and b. From this follows by the usual Yoneda argument that the natural family of bijections $helix$ may be alternatively formulated as a family of isomorphisms ${}^*(L({}^*b)) \to Rb$ natural in b, or equivalently, as a family of isomorphisms

$$helix_a \quad : \quad {}^*(La) \quad \longrightarrow \quad R(a^*) \tag{11}$$

natural in a.

Remark. It should be noted that a dialogue chirality is typically obtained from a dialogue category by defining the negation functors L and R as $Lx = \bot \multimapinv x$ and $Rx = x \multimap \bot$. In that case, the family (11) amounts to a family of isomorphisms

$$helix_x \quad : \quad \bot \multimapinv x \quad \longrightarrow \quad x \multimap \bot. \tag{12}$$

This choice of orientation is essentially arbitrary, so much arbitrary in fact that we take the reverse one in our companion paper [7]. However, there is a reason for making the particular choice (12) when one starts from the notion of dialogue chirality (10) considered here, with currification (8) acting on the left. Thanks to that particular choice, we are able to keep "positive" all the instances of the combinators $helix$ and χ in the coherence diagram (10). Another choice would have been to reverse the orientation of $helix_{a,b}$, to take decurrification (the reverse of currification) as primitive and to start from $\langle\, a \,|\, b_1 \otimes b_2 \,\rangle$ in the coherence diagram (10).

2.4 A 2-category of helical dialogue chiralities

We define a 2-category ***HeliChir*** with helical dialogue chiralities as objects (or 0-dimensional cells). This construction will be compared in §3 with the 2-category ***HeliCat*** of helical categories just constructed in §2.4.

The 1-dimensional cells. A 1-dimensional cell in ***HeliChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined in essentially the same way as a 1-dimensional cell in the 2-category ***DiaChir*** of dialogue chiralities constructed in [5]. Hence, it is a quadruple $F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor $F_\bullet : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_\circ : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\ {\scriptstyle (-)^*}\big\downarrow & \widetilde{F} \;\; \Swarrow & \big\downarrow{\scriptstyle (-)^*} \\ \mathscr{B}_1^{op(0,1)} & \xrightarrow[\quad F_\circ^{op(0,1)} \quad]{} & \mathscr{B}_2^{op(0,1)} \end{array} \tag{13}$$

together with a natural transformation:

$$\begin{array}{ccc} \mathscr{A}_1 & \xrightarrow{\quad F_\bullet \quad} & \mathscr{A}_2 \\ {\scriptstyle R}\big\uparrow & \overline{F} \;\; \Searrow & \big\uparrow{\scriptstyle R} \\ \mathscr{B}_1 & \xrightarrow[\quad F_\circ \quad]{} & \mathscr{B}_2 \end{array} \tag{14}$$

making the diagram

$$\begin{array}{ccc} \langle\, m \owedge a \,|\, b \,\rangle & \xrightarrow{\qquad \chi_m \qquad} & \langle\, a \,|\, m^* \ovee b \,\rangle \\ {\scriptstyle F_{m \owedge a, b}}\big\downarrow & & \big\downarrow{\scriptstyle F_{a, m^* \ovee b}} \\ \langle\, F_\bullet(m \owedge a) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(m^* \ovee b) \,\rangle \\ \big\downarrow{\scriptstyle \textit{monoidality of } F_\bullet} & & \big\downarrow{\scriptstyle \textit{monoidality of } F_\circ} \\ & & \langle\, F_\bullet(a) \,|\, F_\circ(m^*) \ovee F_\circ(b) \,\rangle \\ & & \big\downarrow{\scriptstyle \widetilde{F}} \\ \langle\, F_\bullet(m) \owedge F_\bullet(a) \,|\, F_\circ(b) \,\rangle & \xrightarrow{\; \chi_{F_\bullet(m)} \;} & \langle\, F_\bullet(a) \,|\, F_\bullet(m)^* \ovee F_\circ(b) \,\rangle \end{array} \tag{15}$$

commute for all objects a, m in $\mathscr{A}_1$ and b in $\mathscr{B}_1$. Here, the map

$$F_{a,b} \quad : \quad \langle\, a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle$$

is defined as the composite

$$\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\quad F_{a,b} \quad} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle \\
\| & & & & \| \\
\mathscr{A}_1(a, Rb) & \xrightarrow{F_\bullet} & \mathscr{A}_2(F_\bullet(a), F_\bullet(Rb)) & \xrightarrow{\overline{F}} & \mathscr{A}_2(F_\bullet(a), RF_\circ(b))
\end{array}$$

The only additional requirement compared to a 1-dimensional cell in ***DiaChir*** is that the diagram below commutes:

$$\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\quad F_{a,b} \quad} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle \\
\downarrow{\scriptstyle helix_{a,b}} & & & & \downarrow{\scriptstyle helix_{F_\bullet(a),F_\circ(b)}} \\
\langle\, {}^*b \,|\, a^* \,\rangle & \xrightarrow{F_{{}^*b,a^*}} & \langle\, F_\bullet({}^*b) \,|\, F_\circ(a^*) \,\rangle & \xrightarrow{\widetilde{F}} & \langle\, {}^*(F_\circ(b)) \,|\, (F_\bullet(a))^* \,\rangle
\end{array} \tag{16}$$

Here, the map $\widetilde{F}_{b,a}$ is defined by applying the natural isomorphism $\widetilde{F}$ on the object $F_\circ(a^*)$ in order to get the object $(F_\bullet(a))^*$ and *at the same time* its mate

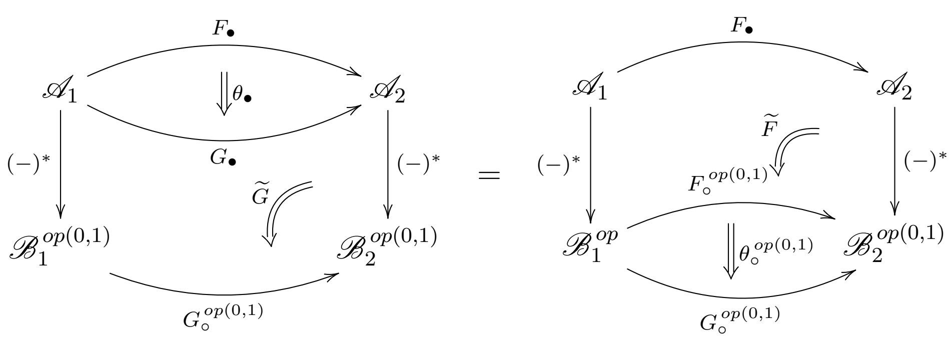
$$\widetilde{F}_{mate} \quad : \quad F_\bullet \circ {}^*(-) \quad \Rightarrow \quad {}^*(-) \circ F_\circ \quad : \quad \mathscr{B}_1^{op(0,1)} \quad \longrightarrow \quad \mathscr{A}_2$$

on the object $F_\bullet({}^*b)$ in order to get the object ${}^*(F_\circ(b))$.

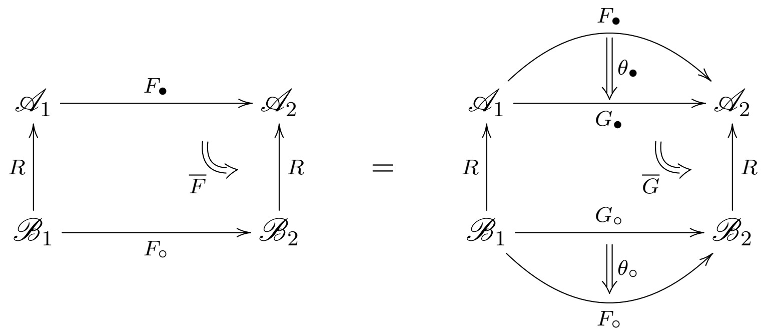
The 2-dimensional cells. A 2-dimensional cell in ***HeliChir***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined in exactly the same way as a 2-dimensional cell in the 2-category ***DiaChir*** of dialogue chiralities constructed in [5]. It is a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations $\theta_\bullet : F_\bullet \Rightarrow G_\bullet$ and $\theta_\circ : G_\circ \Rightarrow F_\circ$ satisfying the two equations below:



(17)



(18)

As in the case of the 2-category ***HeliCat*** in §2.2, we let the reader check that the expected notions of (horizontal and vertical) identity and composition define a 2-category ***HeliChir*** together with a forgetful 2-functor

$$U' \quad : \quad \textbf{\textit{HeliChir}} \quad \longrightarrow \quad \textbf{\textit{DiaChir}}$$

to the 2-category ***DiaChir*** of dialogue chiralities constructed in [5]. As in the case of the forgetful 2-functor from ***HeliCat*** in §2.2, the forgetful 2-functor U is fully faithful on 2-dimensional cells.

3 Construction of the 2-categorical equivalence

In this section, we show that the 2-categories ***HeliCat*** and ***HeliChir*** are equivalent in the appropriate 2-dimensional sense. The construction is a direct adaptation of the 2-dimensional equivalence between ***DiaCat*** and ***DiaChir*** exhibited in our companion paper [5]. So, it is not particularly difficult, but it should done with great care. In the same way as in the original case of dialogue categories, the equivalence may be understood as a coherence theorem for helical dialogue chiralities. In particular, it provides a general recipe to strictify a helical dialogue chirality into an equivalent helical dialogue category. Recall that a dialogue chirality $(\mathscr{A}, \mathscr{B})$ is called *strict* when $\mathscr{B} = \mathscr{A}^{\,op(0,1)}$ and moreover, the two functors ${}^*(-)$ and $(-)^*$ and their monoidal equivalence are trivial — that is, equal to the identity on $\mathscr{A}$.

3.1 From dialogue categories to dialogue chiralities

We start by constructing a 2-functor

$$\mathcal{F} \quad : \quad \textbf{\textit{HeliCat}} \quad \longrightarrow \quad \textbf{\textit{HeliChir}}$$

from the 2-category ***HeliCat*** of helical dialogue categories to the 2-category ***DiaChir*** of helical dialogue chiralities.

Definition of $\mathcal{F}$: the 0-dimensional cells. To every helical dialogue category $\mathscr{C}$, the 2-functor $\mathcal{F}$ associates the helical dialogue chirality defined in exactly the same way as in the case of basic dialogue categories:

$$(\mathscr{A}, \otimes, \mathbf{true}) \;:=\; (\mathscr{C}, \otimes, e) \qquad\qquad (\mathscr{B}, \otimes, \mathbf{false}) \;:=\; (\mathscr{C}, \otimes, e)^{op(0,1)}.$$

The monoidal equivalence between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$ is defined as the identity functor on the monoidal category $\mathscr{C}$. The two adjoint functors L and R are defined as

$$L : x \mapsto \bot \multimapinv x \qquad\qquad R : x \mapsto x \multimap \bot$$

with the adjunction $L \dashv R$ witnessed by the series of bijections

$$\begin{array}{rcl} \mathscr{A}(x, R(y)) & = & \mathscr{C}(x, y \multimap \bot) \\ & \cong & \mathscr{C}(y \otimes x, \bot) \\ & \cong & \mathscr{C}(y, \bot \multimapinv x) \\ & = & \mathscr{B}(L(x), y) \end{array}$$

natural in x and y. The bijection $\chi_{m,x,y}$ is defined as the composite

$$\mathscr{C}(m \otimes x, y \multimap \bot) \xrightarrow{\varphi^{-1}_{x \otimes m, y}} \mathscr{C}(y \otimes m \otimes x, \bot) \xrightarrow{\varphi_{y \otimes m, x}} \mathscr{C}(x, (y \otimes m) \multimap \bot)$$

where for simplicity, we forget the associativity map between $y \otimes (m \otimes x)$ and $(y \otimes m) \otimes x$. Up to that stage, the 2-functor $\mathcal{F}$ is defined in exactly the same way in the original case of dialogue categories, see [5] for details. This already ensures that the data introduced for $(\mathscr{A}, \mathscr{B})$ define a dialogue chirality. The only novelty is the definition of the helical structure

$$\langle\, a \,|\, b \,\rangle = \mathscr{A}(a, Rb) = \mathscr{C}(a, b \multimap \bot) \quad \xrightarrow{helix_{a,b}} \quad \mathscr{C}(b, a \multimap \bot) = \mathscr{A}({}^*b, Ra^*) = \langle\, {}^*b \,|\, a^* \,\rangle$$

as the natural family of isomorphisms

$$\mathscr{C}(a, b \multimap \bot) \xrightarrow{\varphi^{-1}_{b,a}} \mathscr{C}(b \otimes a, \bot) \xrightarrow{wheel^{-1}_{a,b}} \mathscr{C}(a \otimes b, \bot) \xrightarrow{\varphi_{a,b}} \mathscr{C}(a, b \multimap \bot)$$

It is not difficult to check that this defines a helical dialogue chirality $(\mathscr{A}, \mathscr{B})$. The point is that the two coherence diagrams (6) for dialogue categories and (10) for dialogue chiralities essentially coincide.

Definition of $\mathcal{F}$: the 1-dimensional cells. To every dialogue functor

$$(F, \bot_F) \quad : \quad (\mathscr{C}, \bot_{\mathscr{C}}) \quad \longrightarrow \quad (\mathscr{D}, \bot_{\mathscr{D}})$$

the 2-functor $\mathcal{F}$ associates the 1-dimensional cell $\mathcal{F}(F)$ defined as the quadruple consisting of the lax monoidal functor

$$\mathcal{F}(F)_\bullet \ : \ \mathscr{C} \xrightarrow{F} \mathscr{D}$$

the oplax monoidal functor

$$\mathcal{F}(F)_\circ \ : \ \mathscr{C}^{op(0,1)} \xrightarrow{F^{op(0,1)}} \mathscr{D}^{op(0,1)}$$

the monoidal isomorphism $\widetilde{\mathcal{F}(F)}$ defined as the identity on the functor F, and the natural transformation

$$\overline{\mathcal{F}(F)} \quad : \quad R \circ F \quad \longrightarrow \quad F \circ R$$

whose components

$$F(x \multimap \bot_{\mathscr{C}}) \quad \longrightarrow \quad F(x) \multimap \bot_{\mathscr{D}}$$

is associated by currification $\varphi_{F(x),F(x \multimap \bot_{\mathscr{C}})}$ to the morphism

$$F(x) \otimes F(x \multimap \bot_{\mathscr{C}}) \ \longrightarrow \ F(x \otimes (x \multimap \bot_{\mathscr{C}})) \ \longrightarrow \ F(\bot_{\mathscr{C}}) \ \longrightarrow \ \bot_{\mathscr{D}}.$$

We know from [5] that this defines a 1-dimensional cell between dialogue chiralities. There remains to show that $\mathcal{F}(F)$ is compatible with the helical structure, in the technical sense that diagram (16) commutes. This fact is essentially immediate to deduce from the fact that F is compatible with $wheel$ in the technical sense that diagram (7) commutes.

Definition of $\mathcal{F}$: the 2-dimensional cells. The 2-functor $\mathcal{F}$ acts on 2-cells in exactly the same way as in the original case of dialogue categories considered in [5], see that paper for details.

3.2 From dialogue chiralities to dialogue categories

Now that the 2-functor $\mathcal{F}$ has been constructed, we complement it with a 2-functor in the reverse direction:

$$\mathcal{G} \quad : \quad \textbf{\textit{HeliChir}} \quad \longrightarrow \quad \textbf{\textit{HeliCat}}$$

from the 2-category of helical dialogue chiralities to the 2-category of helical dialogue categories.

Definition of $\mathcal{G}$: the 0-dimensional cells. The 2-functor transports every helical dialogue chirality $(\mathscr{A}, \mathscr{B})$ to the helical dialogue category defined as

$$(\mathscr{C}, \otimes, I) \quad := \quad (\mathscr{A}, \owedge, \mathbf{true})$$

equipped with the tensorial pole

$$\bot \quad := \quad R(\mathbf{false}).$$

together with the functors:

$$\bot \multimapinv x \ = \ {}^*(L(x)) \qquad\qquad x \multimap \bot \ = \ R(x^*).$$

The natural bijections φ and ψ are then defined by composing the series of natural bijections

$$\begin{array}{rcll}
\mathscr{C}(x \otimes y, \bot) & = & \mathscr{A}(x \owedge y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \bot, \\
& \cong & \mathscr{A}(y, R(x^* \ovee \mathbf{false})) & \text{by applying } \chi_{x,y,\mathbf{false}}, \\
& \cong & \mathscr{A}(y, R(x^*)) & \text{by applying the unit law in } \mathscr{B}, \\
& \cong & \mathscr{B}(L(y), x^*) & \text{by the adjunction } L \dashv R, \\
& \cong & \mathscr{A}(x, {}^*(L(y))) & \text{by the adjunction } (-)^* \dashv {}^*(-), \\
& = & \mathscr{C}(x, {}^*(L(y))) & \text{by definition of } \mathscr{C}.
\end{array}$$

$$\begin{array}{rcll}
\mathscr{C}(x \otimes y, \bot) & = & \mathscr{A}(x \owedge y, R(\mathbf{false})) & \text{by definition of } \mathscr{C} \text{ and of } \bot, \\
& \cong & \mathscr{A}(y, R(x^* \ovee \mathbf{false})) & \text{by applying } \chi_{x,y,\mathbf{false}}, \\
& \cong & \mathscr{A}(y, R(x^*)) & \text{by applying the unit law in } \mathscr{B}, \\
& = & \mathscr{C}(y, R(x^*)) & \text{by definition of } \mathscr{C}.
\end{array}$$

The helical structure *wheel* on the dialogue category $\mathscr{C}$ is defined as the composite of natural bijections:

$$\begin{array}{ccccccc}
\mathscr{C}(x \otimes y, \bot) & & & \xrightarrow{\quad wheel_{x,y} \quad} & & & \mathscr{C}(y \otimes x, \bot) \\
\Big\downarrow \varphi_{x,y} & & & & & & \Big\uparrow \varphi^{-1}_{y,x} \\
\mathscr{C}(y, x \multimap \bot) & = & \langle\, {}^*y \mid x^* \,\rangle & \xrightarrow{helix^{-1}_{x,y}} & \langle\, x \mid y \,\rangle & = & \mathscr{C}(x, y \multimap \bot)
\end{array}$$

The fact that this defines a helical structure is essentially immediate: the reason already mentioned is that the two coherence diagrams (6) for dialogue categories and (10) for dialogue chiralities essentially coincide modulo translation of one into the other.

Definition of $\mathcal{G}$: the 1-dimensional cells. Every 1-dimensional cell

$$F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F}) \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is transported to the dialogue functor $(F_\bullet, \bot_F)$ consisting of the functor

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2$$

and of the morphism

$$\bot_F \quad : \quad F_\bullet(\bot_{\mathscr{A}_1}) \quad \longrightarrow \quad \bot_{\mathscr{A}_2}$$

defined as the composite

$$F_\bullet \circ R\,(\mathbf{false}) \xrightarrow{\overline{F}_{\mathbf{false}}} R \circ F_\circ\,(\mathbf{false}) \xrightarrow{\textit{monoidality}} R\,(\mathbf{false})$$

There remains to show that $\mathcal{G}(F) = F_\bullet$ is compatible with the wheel structure, in the technical sense that diagram (7) commutes. This fact is essentially immediate to deduce from the diagram chase below:

$$\begin{array}{ccccc}
\mathscr{C}(x \otimes y, \bot) & \xrightarrow{F} & \mathscr{D}(F(x \otimes y), \bot) & \xrightarrow{\textit{coercion}} & \mathscr{D}(F(x) \otimes F(y), \bot) \\
\downarrow{\scriptstyle \psi_{x,y}} & & (a) & & \downarrow{\scriptstyle \psi_{Fx,Fy}} \\
\mathscr{C}(x, \bot \multimapinv y) & \xrightarrow{F} & \mathscr{D}(Fx, F(\bot \multimapinv y)) & \longrightarrow & \mathscr{D}(Fx, \bot \multimapinv Fy) \\
\downarrow{\scriptstyle \textit{helix}^{-1}_{x,y}} & & (b) & & \downarrow{\scriptstyle \textit{helix}^{-1}_{Fx,Fy}} \\
\mathscr{C}(y, \bot \multimapinv x) & \xrightarrow{F} & \mathscr{D}(Fy, F(\bot \multimapinv x)) & \longrightarrow & \mathscr{D}(Fy, \bot \multimapinv Fx) \\
\downarrow{\scriptstyle \psi^{-1}_{y,x}} & & (c) & & \downarrow{\scriptstyle \psi^{-1}_{Fy,Fx}} \\
\mathscr{C}(y \otimes x, \bot) & \xrightarrow{F} & \mathscr{D}(F(y \otimes x), \bot) & \xrightarrow{\textit{coercion}} & \mathscr{D}(F(y) \otimes F(x), \bot)
\end{array}$$

whose hexagons (a) and (c) commute in every dialogue category, and whose inner hexagon (b) commutes because diagram (16) commutes.

Definition of $\mathcal{G}$: the 2-dimensional cells. The 2-functor $\mathcal{G}$ acts on 2-dimensional cells in exactly the same way as in the original case of dialogue categories considered in [5]. We refer the reader to that paper for details. We simply recall here that every 2-dimensional cell $\theta = (\theta_\bullet, \theta_\circ)$ is transported to the dialogue transformation $\theta_\bullet$.

3.3 The pseudo-natural transformation Φ

The composite 2-functor

$$\textbf{\textit{HeliCat}} \quad \overset{\mathcal{F}}{\longrightarrow} \quad \textbf{\textit{HeliChir}} \quad \overset{\mathcal{G}}{\longrightarrow} \quad \textbf{\textit{HeliCat}}$$

coincides with the identity on the 2-category ***HeliCat*** of helical dialogue categories. In order to establish that ***HeliCat*** and ***HeliChir*** are biequivalent, we construct a pair of pseudo-natural transformations

$$\Phi \;:\; Id \;\longrightarrow\; \mathcal{F}\circ\mathcal{G} \qquad\qquad \Psi \;:\; \mathcal{F}\circ\mathcal{G} \;\longrightarrow Id$$

between the identity 2-functor on ***HeliChir*** and the 2-functor $\mathcal{F}\circ\mathcal{G}$. We then show that their components $\Phi_{(\mathscr{A},\mathscr{B})}$ and $\Psi_{(\mathscr{A},\mathscr{B})}$ define an equivalence in the 2-category ***DiaChir***, for every helical dialogue chirality $(\mathscr{A},\mathscr{B})$. Before proceeding further, we find convenient to give a detailed account of the helical dialogue chirality $(\mathscr{A},\mathscr{A}^{\,op(0,1)})$ obtained by applying the 2-functor $\mathcal{F}\circ\mathcal{G}$ to a given dialogue chirality $(\mathscr{A},\mathscr{B})$. The dialogue chirality $(\mathscr{A},\mathscr{A}^{\,op(0,1)})$ is equipped with the trivial monoidal equivalence:

$$\mathscr{A} \quad \underset{id}{\overset{id}{\rightleftarrows}}\;{\scriptstyle monoidal\ equivalence} \quad \left(\mathscr{A}^{\,op(0,1)}\right)^{op(0,1)}$$

with the adjunction

$$\mathscr{A} \quad \underset{R}{\overset{L}{\rightleftarrows}}\;\bot \quad \mathscr{B} \quad \underset{((-)^{*})^{\,op(0,1)}}{\overset{({}^{*}(-))^{\,op(0,1)}}{\rightleftarrows}}\;\bot \quad \mathscr{A}^{\,op(0,1)}$$

From this follows that

$$\langle\, a_1 \,|\, a_2 \,\rangle_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})} \quad = \quad \mathscr{A}(a_1, R\,(a_2^{*})) \quad = \quad \langle\, a_1 \,|\, a_2^{*} \,\rangle_{(\mathscr{A},\mathscr{B})}$$

The natural transformation $\chi_{(\mathscr{A},\mathscr{A}^{\,op(0,1)})}$ at instance (m,a,b) is defined as the composite function

$$\begin{array}{ccc}
\langle\, m \oslash a_1 \,|\, a_2^{*} \,\rangle & & \langle\, a_1 \,|\, (a_2 \oslash m)^{*} \,\rangle \\
\downarrow & & \uparrow \\
\langle\, m \oslash a_1 \,|\, a_2^{*} \obslash \mathbf{false} \,\rangle & & \langle\, a_1 \,|\, (a_2 \oslash m)^{*} \obslash \mathbf{false} \,\rangle \\
{\scriptstyle (\chi_{(\mathscr{A},\mathscr{B})})^{-1}}\downarrow & & \uparrow{\scriptstyle \chi_{(\mathscr{A},\mathscr{B})}} \\
\langle\, a_2 \oslash (m \oslash a_1) \,|\, \mathbf{false} \,\rangle & \longrightarrow & \langle\, (a_2 \oslash m) \oslash a_1 \,|\, \mathbf{false} \,\rangle
\end{array}$$

Similarly, the dialogue chirality $\mathcal{G} \circ \mathcal{F}(\mathscr{A}, \mathscr{B})$ has helical structure defined as

$$\langle\, a_1 \mid a_2^* \,\rangle_{(\mathscr{A},\mathscr{B})} \xrightarrow{helix_{a_1,a_2^*}} \langle\, {}^*(a_2^*) \mid a_1^* \,\rangle_{(\mathscr{A},\mathscr{B})} \xrightarrow{equivalence} \langle\, a_2 \mid a_1^* \,\rangle_{(\mathscr{A},\mathscr{B})}$$

where $helix$ is the helical structure of the original dialogue chirality $(\mathscr{A}, \mathscr{B})$.

After this detailed description of the "strictified" version $\mathcal{G} \circ \mathcal{F}(\mathscr{A}, \mathscr{B})$ obtained from the helical dialogue chirality $(\mathscr{A}, \mathscr{B})$, we are ready to introduce the pseudo-natural transformations Φ and Ψ. The construction is exactly the same as for dialogue chiralities in [5]. Our main concern is thus to check that the constructions are compatible with the helical structures of the original dialogue chirality $(\mathscr{A}, \mathscr{B})$ and of its strictified version.

The 1-dimensional cells $\Phi_{(\mathscr{A},\mathscr{B})}$. Recall from [5] that to every dialogue chirality $(\mathscr{A}, \mathscr{B})$ one associates the 1-cell

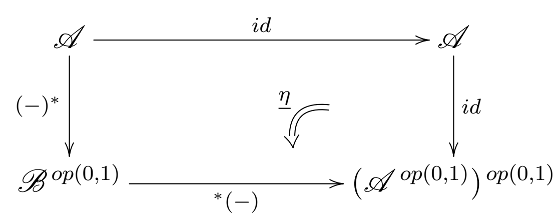
$$\Phi_{(\mathscr{A},\mathscr{B})} \quad : \quad (\mathscr{A}, \mathscr{B}) \quad \longrightarrow \quad \left(\mathscr{A}, \mathscr{A}^{\,op(0,1)}\right)$$

defined as the pair of monoidal functors

$$\left(\Phi_{(\mathscr{A},\mathscr{B})}\right)_{\bullet} \;:\; \mathscr{A} \xrightarrow{id} \mathscr{A} \qquad\qquad \left(\Phi_{(\mathscr{A},\mathscr{B})}\right)_{\circ} \;:\; \mathscr{B} \xrightarrow{({}^*(-))^{\,op(0,1)}} \mathscr{A}^{\,op(0,1)}$$

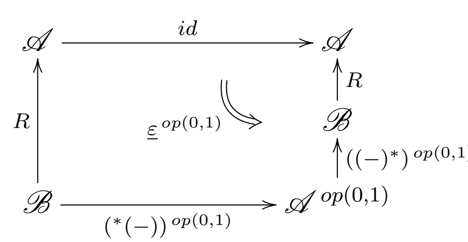
together with the monoidal natural isomorphism

$\widetilde{\Phi_{(\mathscr{A},\mathscr{B})}} \quad =$



and the natural transformation

$\overline{\Phi_{(\mathscr{A},\mathscr{B})}} \quad =$



where $\underline{\eta}$ and $\underline{\varepsilon}$ denote the unit and counit of the adjunction $(-)^* \dashv {}^*(-)$. We know from [5] that Φ defines a 1-dimensional cell between the dialogue chirality $(\mathscr{A}, \mathscr{B})$ and its strictified version. There remains to check that this 1-dimensional cell is compatible with the helical structures of the two dialogue chiralities. Technically speaking, this simply means that the coherence diagram (16) commutes when instantiated as follows:

$$\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\qquad\quad \Phi_{a,b} \quad\qquad} & & \langle\, a \,|\, ({}^*b)^* \,\rangle \\
 & & & & \Big\downarrow {\scriptstyle helix_{a,({}^*b)^*}} \\
{\scriptstyle helix_{a,b}} \Big\downarrow & & & & \langle\, {}^*(({}^*b)^*) \,|\, a^* \,\rangle \\
 & & & & \Big\downarrow {\scriptstyle equivalence} \\
\langle\, {}^*b \,|\, a^* \,\rangle & \xrightarrow{\ \Phi_{{}^*b,a^*}\ } & \langle\, {}^*b \,|\, ({}^*(a^*))^* \,\rangle & \xrightarrow{\ \widetilde{\Phi}_{{}^*b,a^*}\ } & \langle\, {}^*b \,|\, a \,\rangle
\end{array}$$

We leave the reader check that this diagram commutes by naturality of $helix$ and because all the bijections (except for the two instantiations of $helix$) involved in it are deduced from the equivalence $(-)^* \dashv {}^*(-)$.

The 2-dimensional cells Φ_F. Are constructed just as the 2-dimensional cells Φ_F in [5]. From this follows that each of them defines a 2-cell in the 2-category ***HeliChir*** and that the family Φ itself defines a pseudo-natural transformation.

3.4 The pseudo-natural transformation Ψ

The 1-dimensional cells $\Psi_{(\mathscr{A},\mathscr{B})}$. To every dialogue chirality $(\mathscr{A},\mathscr{B})$, one associates the 1-cell $\Psi_{(\mathscr{A},\mathscr{B})}$ defined as the pair of functors

$$(\Psi_{(\mathscr{A},\mathscr{B})})_{\bullet} \;:\; \mathscr{A} \xrightarrow{\ id\ } \mathscr{A} \qquad\qquad (\Psi_{(\mathscr{A},\mathscr{B})})_{\circ} \;:\; \mathscr{A}^{\,op(0,1)} \xrightarrow{\ ((-)^*)^{\,op(0,1)}\ } \mathscr{B}$$

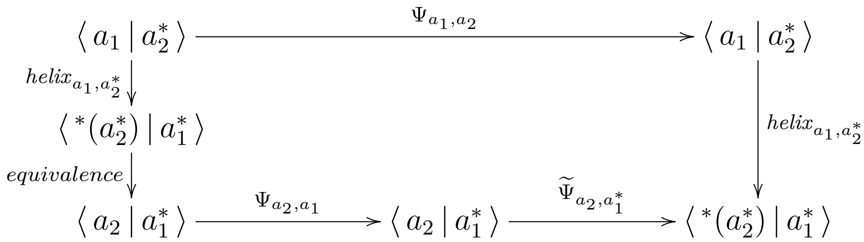
equipped with the trivial monoidal natural isomorphism

$$\widetilde{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{array}{ccc}
\mathscr{A} & \xrightarrow{\qquad id \qquad} & \mathscr{A} \\
{\scriptstyle id}\Big\downarrow & \overset{id}{\Swarrow} & \Big\downarrow{\scriptstyle (-)^*} \\
(\mathscr{A}^{\,op(0,1)})^{\,op(0,1)} & \xrightarrow[\ (-)^*\ ]{} & \mathscr{B}^{\,op(0,1)}
\end{array}$$

and with the trivial natural transformation

$$\overline{\Psi_{(\mathscr{A},\mathscr{B})}} \quad = \quad \begin{array}{ccc}
\mathscr{A} & \xrightarrow{\qquad id \qquad} & \mathscr{A} \\
{\scriptstyle R}\Big\uparrow & & \\
\mathscr{B} & \overset{id}{\Searrow} & \Big\uparrow{\scriptstyle R} \\
{\scriptstyle ((-)^*)^{\,op(0,1)}}\Big\uparrow & & \\
\mathscr{A}^{\,op(0,1)} & \xrightarrow[\ ((-)^*)^{\,op(0,1)}\ ]{} & \mathscr{B}
\end{array}$$

We know from [5] that Ψ defines a 1-dimensional cell between the dialogue chirality $(\mathscr{A}, \mathscr{B})$ and its strictified version. So, just as in the case of the 1-dimensional cell Φ in §3.4, we only need to check that this 1-dimensional cell Ψ is compatible with the helical structures of the two dialogue chiralities. The coherence diagram (16) is instantiated as follows in that case:



where we write $\langle a \mid b \rangle$ for the evaluation bracket $\langle a \mid b \rangle_{(\mathscr{A},\mathscr{B})}$ of the original dialogue chirality. We leave the reader check that this diagram commutes because Ψ_{a_1,a_2} and Ψ_{a_2,a_1} are equal to the identity, and because $\widetilde{\Psi}_{a_2,a_1^*}$ is obtained by applying the unit $\underline{\eta}$ of the equivalence $a_2 \to {}^*(a_2^*)$ inside the evaluation bracket.

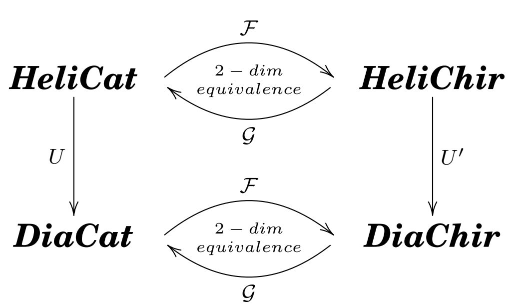
The 2-dimensional cells Ψ_F. Are constructed just as the 2-dimensional cells Ψ_F in [5]. From this follows that each of them defines a 2-cell in the 2-category ***HeliCat*** and that the family Ψ itself defines a pseudo-natural transformation.

3.5 Coherence theorem for helical dialogue chiralities

We have just established that

Theorem 1 (coherence theorem) *The pair of 2-functors $\mathcal{F}$ and $\mathcal{G}$ defines a biequivalence between the 2-categories* ***HeliCat*** *and* ***HeliChir***.

Note that the pair of forgetful 2-functors U and U' defines a homomorphism in the appropriate 2-dimensional sense between the biequivalences:



This homomorphism reflects the fact that strictification of a helical dialogue chirality $(\mathscr{A}, \mathscr{B})$ is performed in the same way in ***HeliChir*** and in ***DiaChir***.

4 Ambidextrous chiralities

In this section, we introduce the notion of *ambidextruous chirality* which provides the most primitive notion of dialogue chirality equipped with a left *as well as* a righthand side currification. The notion is introduced in §4.1. Following our general policy, we construct in §4.2 a 2-category ***AmbiChir*** of ambidextrous chiralities. We establish in §4.3 our main result of the section, which states that the 2-category of ambidextrous chiralities is *isomorphic* to the 2-category ***HeliChir*** of helical dialogue chiralities. This very strong correspondance provides a purely logical justification for the topological notion of helical dialogue category.

4.1 Definition

Definition 3 (ambidextrous chirality) *An ambidextrous chirality is a pair of monoidal categories*

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with a monoidal equivalence

$$\mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \qquad \text{(monoidal equivalence)}$$

and with two families of bijections

$$\begin{array}{ccccc} \chi^R_{m,a,b} & : & \langle\, a \owedge m \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, b \ovee m^* \,\rangle \\ \chi^L_{m,a,b} & : & \langle\, m \owedge a \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, m^* \ovee b \,\rangle \end{array}$$

natural in a, b and m, where the evaluation bracket is defined as

$$\langle\, - \,|\, - \,\rangle \quad := \quad \mathscr{A}(\, - \,,\, R(-)\,) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

The families χ^L and χ^R are required to make the three diagrams commute:

$$\begin{array}{ccccc} \langle\,(m \owedge n) \owedge a \,|\, b\,\rangle & & \xrightarrow{\quad \chi^L_{m \owedge n} \quad} & & \langle\, a \,|\, (m \owedge n)^* \ovee b\,\rangle \\ \big\downarrow {\scriptstyle associativity} & & & & \big\uparrow {\scriptstyle associativity,\ monoidality\ of\ negation} \\ \langle\, m \owedge (n \owedge a) \,|\, b\,\rangle & \xrightarrow{\chi^L_m} & \langle\, n \owedge a \,|\, m^* \ovee b\,\rangle & \xrightarrow{\chi^L_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b)\,\rangle \end{array} \qquad (19)$$

$$
\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\quad \chi^R_{m\owedge n} \quad} & & \langle\, a \,|\, b \ovee (m \owedge n)^* \,\rangle \\
\Big\downarrow {\scriptstyle associativity} & & & & \Big\uparrow {\scriptstyle associativity \atop monoidality\ of\ negation} \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \owedge m \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^R_m} & \langle\, a \,|\, (b \ovee n^*) \ovee m^* \,\rangle
\end{array}
\tag{20}
$$

$$
\begin{array}{ccccc}
\langle\, (m \owedge a) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, m \owedge a \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \,|\, m^* \ovee (b \ovee n^*) \,\rangle \\
\Big| {\scriptstyle associativity} & & & & \Big| {\scriptstyle associativity} \\
\langle\, m \owedge (a \owedge n) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \owedge n \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \,|\, (m^* \ovee b) \ovee n^* \,\rangle
\end{array}
\tag{21}
$$

for all objects a, m, n *of the category* $\mathscr{A}$ *and all objects* b *of the category* $\mathscr{B}$.

The two first coherence diagrams (19) and (20) may be seen as left and right instances of the familiar coherence diagram (9) for left currification in dialogue chiralities. The last coherence diagram (21) requires that the left and right currification are compatible in the expected sense. We will see that this last requirement has the somewhat unexpected consequence of equipping the ambidextrous chirality with a helical structure described in §4.3.

4.2 The 2-category of ambidextrous chiralities

Here, we define the 2-category ***AmbiChir*** whose 0-dimensional cells are the ambidextrous chiralities, and whose 1 and 2-dimensional cells are defined as follows.

The 1-dimensional cells. An ambidextrous homomorphism is defined as a quadruple

$$F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F})$$

consisting of a lax monoidal functor $F_\bullet : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_\circ : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism (13) and a natural isomorphism (14) making the diagram (15) commute for $\chi = \chi^L$ together with the corresponding diagram for the right currification χ^R, given below:

$$
\begin{array}{ccc}
\langle\, a \owedge m \,|\, b \,\rangle & \xrightarrow{\quad \chi^R_m \quad} & \langle\, a \,|\, b \ovee m^* \,\rangle \\
\big\downarrow {\scriptstyle F_{a \owedge m, b}} & & \big\downarrow {\scriptstyle F_{a, b \ovee m^*}} \\
\langle\, F_\bullet(a \owedge m) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(b \ovee m^*) \,\rangle \\
 & & \big\downarrow {\scriptstyle \textit{monoidality of } F_\circ} \\
\big\downarrow {\scriptstyle \textit{monoidality of } F_\bullet} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \ovee F_\circ(m^*) \,\rangle \\
 & & \big\downarrow {\scriptstyle \widetilde{F}} \\
\langle\, F_\bullet(a) \owedge F_\bullet(m) \,|\, F_\circ(b) \,\rangle & \xrightarrow{\quad \chi^R_{F_\bullet(m)} \quad} & \langle\, F_\bullet(a) \,|\, F_\circ(b) \ovee F_\bullet(m)^* \,\rangle
\end{array}
\tag{22}
$$

The 2-dimensional cells. The 2-dimensional cells are defined in exactly the same way as in the 2-categories ***DiaChir*** and ***HeliChir***.

4.3 An isomorphism between ambidextrous and helical

We establish here that

Theorem 2 *The 2-category* ***AmbiChir*** *of ambidextrous chiralities is isomorphic to the 2-category* ***HeliChir*** *of helical chiralities.*

To that purpose, we construct a pair of 2-functors

$$
\mathcal{F} \;:\; \textbf{\textit{AmbiChir}} \longrightarrow \textbf{\textit{HeliChir}} \qquad\qquad \mathcal{G} \;:\; \textbf{\textit{HeliChir}} \longrightarrow \textbf{\textit{AmbiChir}}
$$

and then show that they define an isomorphism between the 2-categories.

Ambidextrous ⇒ Helical. The 2-functor $\mathcal{F}$ transports every ambidextrous chirality $(\mathscr{A}, \mathscr{B}, \chi^L, \chi^R)$ to the underlying dialogue chirality $(\mathscr{A}, \mathscr{B}, \chi)$ with left currification $\chi = \chi^L$. The dialogue chirality is moreover equipped with the following helical structure:

$$
\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\qquad\qquad helix_{a,b} \qquad\qquad} & & \langle\, {}^*b \,|\, a^* \,\rangle \\
\Vert & & & & \Vert \\
\langle\, a \,|\, ({}^*b)^* \ovee \textbf{false} \,\rangle & \xrightarrow{(\chi^L_{({}^*b)})^{-1}} & \langle\, {}^*b \owedge a \,|\, \textbf{false} \,\rangle & \xrightarrow{\chi^R_a} & \langle\, {}^*b \,|\, \textbf{false} \ovee a^* \,\rangle
\end{array}
$$

We need to establish that (10) commutes. Then the same for ambidetrous homomorphisms.

Helical ⇒ Ambidextrous. The 2-functor $\mathcal{G}$ transports every helical chirality $(\mathscr{A}, \mathscr{B}, \chi)$ into the ambidextrous chirality $(\mathscr{A}, \mathscr{B}, \chi^L, \chi^R)$ with left currification χ^L defined as the currification χ of the original chirality, and right currification χ^R defined as the composite morphism

$$
\begin{array}{ccccc}
\langle\, a \owedge m \,|\, b \,\rangle & \xrightarrow{\chi^R_{m,a,b}} & \langle\, a \,|\, b \ovee m^* \,\rangle & = & \langle\, a \,|\, ({}^*b)^* \ovee m^* \,\rangle \\
\Big\downarrow {\scriptstyle \chi_{a,m,b}} & & & & \Big\uparrow {\scriptstyle \chi_{{}^*b,a,m^*}} \\
\langle\, m \,|\, a^* \ovee b \,\rangle & \xrightarrow{helix} & \langle\, {}^*(a^* \ovee b) \,|\, m^* \,\rangle & = & \langle\, {}^*b \owedge a \,|\, m^* \,\rangle
\end{array}
$$

We need to establish that the two coherence diagrams (20) and (21) of ambidextrous categories commute. Then the same for helical homomorphisms.

A series of chase diagrams establish that the relationship is one-to-one.

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