A micrological study of negation

Paul-André Melliès*

Abstract

The purpose of this paper is to develop a combinatorial presentation of negation in dialogue categories, and to illustrate the fundamental symmetry between proofs and anti-proofs in logic. The micrological analysis reveals that negation may be either taken as primitive, or decomposed into a series of more elementary components studied in the paper. The resulting formulation provides an algebraic presentation by generators and relations of the traditional notion of innocent strategy between dialogue games.

1 Introduction

One primary purpose of our work in tensorial logic is to transfer the ideas and constructions of linear logic to this more primitive logic of tensor and negation, where negation $A \mapsto \neg A$ is not required to be involutive anymore. Despite its conceptual relevance, the research programme bumps against the ideological objection that tensorial logic is an "intuitionistic" logic and thus lacks the nice "classical" symmetries of linear logic [5, 6]. In this objection lies an insidious but prevailing misunderstanding about the status of the "classical" symmetries of linear logic. This misunderstanding is so deeply rooted in our logical traditions that it cannot be overtaken without a genuine act of refoundation — accomplished here by showing that the "classical" symmetries of linear logic are derived from a more primitive symmetry which governs both tensorial logic and linear logic. The existence of this common symmetry for tensorial logic and linear logic challenges the dogma that the "classical" symmetries of linear logic arise from the fact that its linear negation $A \mapsto \neg A$ is involutive. An important methodological contribution of the paper is precisely to refute this dogma and to explain that an "intuitionistic" logic like tensorial logic — whose negation is not involutive — is in fact regulated by the same symmetries as a "classical" logic like linear logic.

A more harmonious and unified picture of logic emerges, properly founded on the dialogical and interactive nature of proofs underlying game semantics. In particular, the effect of the primitive symmetry which govern both tensorial logic and linear logic is simply to interchange the two sides *Prover* and *Denier* of a logical dispute. This permutation Prover ↔ Denier should be understood as an *internal symmetry* of logic itself. An analogy with Galilean Mechanics emerges here, which we find useful to mention since it guided our work. The idea is that the symmetry Prover ↔ Denier plays the same role as a uniform and rectilinear change of frame of reference in Galilean Mechanics. This invariance property expresses a *principle of relativity* which tells that Prover is given exactly the same argumental power as Denier in tensorial logic or in linear logic. The principle was originally detected as a *chirality principle* between proofs and anti-proofs in the diagrammatic language of tensorial proof-nets [10] and then reunderstood as a more general microcosm principle in logic [12].

*CNRS, Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France. This work has been partly supported by the ANR Project RECRE.

As expected, the invariance principle is not limited to tensorial logic or to linear logic and holds in essentially the same way in any logical system, either intuitionistic or classical. We like to think of it as a principle of *free discussion* discovered in the Athenian democracy and concealed today in the core of formal logic. Once this alternative and symmetric account of logic has been spelled out, the historical separation between classical logic and intuitionistic logic inherited from the 1930's gradually vanishes — and our research program on tensorial logic becomes meaningful again...

In this introduction, we will explain how this alternative and unified picture of logic gradually emerged from our attempts to transfer the "classical" symmetries of linear logic to an "intuitionistic" logic like tensorial logic. Because our approach to proof theory is essentially algebraic in the present paper, we will start from the notion of $*$-autonomous category which plays an important role here, because it provides a categorical counterpart to linear logic. Recall that a $*$-autonomous category is traditionally defined as a monoidal biclosed category $\mathscr{C}$ equipped with a dualizing object $\bot$, see [2] for details. An object $\bot$ of a biclosed monoidal category $\mathscr{C}$ is *dualizing* when the two canonical morphisms

$$x \longrightarrow \bot \multimapinv (x \multimap \bot) \qquad\qquad x \longrightarrow (\bot \multimapinv x) \multimap \bot$$

transporting x into its double negation are isomorphisms, for every object x of the dialogue category. The terminology of "dualizing object" comes from the fact that the two negation functors

$$\begin{array}{ccccccccc} (-)^{\bot} & : & x & \mapsto & x \multimap \bot & : & \mathscr{C} & \longrightarrow & \mathscr{C}^{\,op} \\ {}^{\bot}(-) & : & x & \mapsto & \bot \multimapinv x & : & \mathscr{C}^{\,op} & \longrightarrow & \mathscr{C} \end{array}$$

define in that case an equivalence

$$\mathscr{C} \underset{{}^{\bot}(-)}{\overset{(-)^{\bot}}{\rightleftarrows}} \mathscr{C}^{\,op} \quad (\text{equivalence}) \tag{1}$$

between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{\,op}$. This establishes that every $*$-autonomous category $\mathscr{C}$ is self-dual, in the technical sense that it is equivalent to its opposite category.

Although this specific definition of $*$-autonomous category is prevailing today, one is entitled to complain about the fact that it is not sufficiently "symmetric" in the sense that it starts from the conjunction $\otimes$ and from the two implications $\multimap$ and $\multimapinv$ provided by the biclosed monoidal category, rather than from the conjunction $\otimes$ and from the disjunction $\parr$ provided by the traditional presentation of linear logic. From a purely aesthetic point of view, this non-symmetric presentation of a perfectly self-dual logic like linear logic looks awkward, and one thus wonders whether the formulation may be replaced by a nicely symmetric one. A preliminary difficulty is that there are two different canonical ways to define the disjunction in a biclosed monoidal category:

$$\begin{array}{ccccc} x \parr_1 y & := & \bot \multimapinv ((y \multimap \bot) \otimes (x \multimap \bot)) & := & {}^{\bot}(y^{\bot} \otimes x^{\bot}) \\ x \parr_2 y & := & ((\bot \multimapinv y) \otimes (\bot \multimapinv x)) \multimap \bot & := & ({}^{\bot}y \otimes {}^{\bot}x)^{\bot} \end{array}$$

Despite their difference, the series of monoidal equivalences

$$(\mathscr{C}, \parr_1, \bot) \underset{{}^{\bot}(-)}{\overset{(-)^{\bot}}{\rightleftarrows}} (\mathscr{C}, \otimes, \mathbf{1})^{op(0,1)} \underset{{}^{\bot}(-)}{\overset{(-)^{\bot}}{\rightleftarrows}} (\mathscr{C}, \parr_2, \bot) \quad (\text{monoidal equivalence, monoidal equivalence})$$

establishes that the two monoidal structures $⅋_1$ and $⅋_2$ on the category $\mathscr{C}$ are related by the monoidal equivalence $x \mapsto x^{\perp\perp} = (x \multimap \perp) \multimap \perp$ in the technical sense that there exists isomorphisms

$$(x \mathbin{⅋_1} y)^{\perp\perp} \cong x^{\perp\perp} \mathbin{⅋_2} y^{\perp\perp} \qquad\qquad \perp^{\perp\perp} \cong \perp$$

natural in x and y and compatible in the expected sense with the two disjunctions $⅋_1$ and $⅋_2$ and with their common unit $\perp$. For that reason, it often makes sense to restrict oneself to the case of a symmetric or pivotal $*$-autonomous category, where the monoidal equivalence $x \mapsto x^{\perp\perp}$ is isomorphic to the identity. In that case, the left and the right implication functors coincide modulo isomorphism, and the disjunction $⅋$ may be thus defined in a non equivocal way as

$$x \mathbin{⅋} y \quad := \quad ((y \multimap \perp) \otimes (x \multimap \perp)) \multimap \perp.$$

This preliminary discussion leads to the question of presenting such a $*$-autonomous category starting from the symmetric pair of connectives $\otimes$ and $⅋$ instead of the asymmetric pair $\otimes$ and $\multimap$. The matter was resolved in a precise and elegant way by Cockett and Seely with their notion of *linearly distributive category*. Recall from [4, 3] that a linearly distributive category is defined as a category $\mathscr{C}$ equipped with two monoidal structures $(\mathscr{C}, \otimes, 1)$ and $(\mathscr{C}, ⅋, \perp)$ together with a pair of distributivity laws

$$\begin{array}{lcccc} \kappa^R_{x,y,z} & : & (x \mathbin{⅋} y) \otimes z & \longrightarrow & x \mathbin{⅋} (y \otimes z) \\ \kappa^L_{x,y,z} & : & x \otimes (y \mathbin{⅋} z) & \longrightarrow & (x \otimes y) \mathbin{⅋} z \end{array} \tag{2}$$

between the two tensor products, which satisfy a series of coherence diagrams recalled in the paper. A nice theorem established in [4] states (in particular) that a symmetric $*$-autonomous category is the same thing as a symmetric linearly distributive category where every object m comes equipped with a dual object m^* together with two maps

$$\begin{array}{lcccc} \mathbf{AX}[m] & : & \mathbf{1} & \longrightarrow & m^* \mathbin{⅋} m \\ \mathbf{CUT}[m] & : & m \otimes m^* & \longrightarrow & \perp \end{array} \tag{3}$$

satisfying a series of well-chosen coherence diagrams, see §2 for details. This alternative presentation of $*$-autonomous categories provides the expected symmetric formulation based on the conjunction $\otimes$ and on the disjunction $⅋$ of linear logic.

A fascinating aspect of the approach developed by Cockett and Seely is that the notion of linearly distributive category does not require the category $\mathscr{C}$ to be self-dual. Indeed, in this particular presentation of $*$-autonomous categories, the self-duality of linear logic is entirely decoupled from the connectives $\otimes$ and $⅋$ and comes only at the very last stage, when one equips every object x of the linearly distributive category $\mathscr{C}$ with a dual object x^*. As just explained, this very last step enforces the category $\mathscr{C}$ to be $*$-autonomous, and thus self-dual, in the technical sense that $\mathscr{C}$ is equivalent to its opposite category $\mathscr{C}^{op}$.

Now, we are ready to turn ourselves to tensorial logic. Looking at it from the functorial point of view, the notion of *dialogue category* provides a categorical counterpart to tensorial logic in exactly the same way as the notion of $*$-autonomous category does for linear logic. So, if one really believes in this apparently extravagant idea mentioned in the introduction that tensorial logic is a *refinement* of linear logic — rather than simply a *fragment* of it — then one should be able to present dialogue categories in a similarly symmetric fashion. As we have already pointed out, the project does not seem to make a lot of sense when one looks at it with the familiar but deforming spectacles of linear logic. Recall that a dialogue category is a monoidal category equipped with an object $\perp$ and two natural isomorphisms

$$\begin{array}{lcccc} \varphi_{x,y} & : & \mathscr{C}(x \otimes y, \perp) & \cong & \mathscr{C}(y, x \multimap \perp) \\ \psi_{x,y} & : & \mathscr{C}(x \otimes y, \perp) & \cong & \mathscr{C}(x, \perp \multimapinv y) \end{array}$$

So, if one wants to adapt the notion of linearly distributive category to the case of dialogue categories, one needs

a. to understand how and where the conjunction $\otimes$ and the disjunction $⅋$ should be interpreted in a dialogue category,

b. to decorrelate very carefully the self-duality (for what this means, because we do not see any self-duality arising in a dialogue category at this point) from the conjunction $\otimes$ and the disjunction $⅋$ of the dialogue category, in the same way as the self-duality $m \mapsto m^*$ is decoupled from the connectives $\otimes$ and $⅋$ in order to get a linearly distributive category from a $*$-autonomous category.

Alas... all this does not make much sense at this point, at least in the way it is currently formulated. In particular, one does not see (1) how to interpret the disjunction $⅋$ in a dialogue category and (2) what self-duality means since a dialogue category $\mathscr{C}$ has no reason to be equivalent to its opposite category $\mathscr{C}^{op}$.

In order to resolve these difficulties, one needs to look at the situation in a radically different way, following the guideline offered by 2-dimensional algebra. The first observation is that every dialogue category comes with an adjunction

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}^{op} \quad (L \dashv R) \tag{4}$$

between the two negation functors

$$L \;:\; x \;\mapsto\; \bot \multimapinv x \qquad\qquad R \;:\; x \;\mapsto\; x \multimap \bot$$

This adjunction induces a monad in $\mathscr{C}$

$$T \quad : \quad x \quad \mapsto \quad (\bot \multimapinv x) \multimap \bot \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and a comonad in $\mathscr{C}^{op}$

$$T' \quad : \quad x \quad \mapsto \quad \bot \multimapinv (x \multimap \bot) \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad \mathscr{C}^{op}$$

which may be also seen as a monad in $\mathscr{C}$. It is worth mentioning here that a $*$-autonomous category may be alternatively defined as a dialogue category where the adjunction (4) is an equivalence. By equivalence, we mean that the units η and η' of the two double negation monads T and T'

$$\eta_x \;:\; x \;\longrightarrow\; (\bot \multimapinv x) \multimap \bot \qquad\qquad \eta'_x \;:\; x \;\longrightarrow\; \bot \multimapinv (x \multimap \bot)$$

are invertible for every object x of the category, and thus, that the object $\bot$ is dualizing. This basic observation demonstrates that the primitive notion of dialogue category is sufficient in order to define the notion of $*$-autonomous category, and that the more sophisticated notion of biclosed monoidal category may be thus avoided for that purpose.

At this stage, a tentative solution for problem a. emerges from the contemplation of the adjunction (4) between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. The idea is to interpret the conjunction $\otimes$ as the tensor product taken in the dialogue category $\mathscr{C}$ and the disjunction $⅋$ as the tensor product taken in the opposite dialogue category $\mathscr{C}^{op(0,1)}$. Here, we write $\mathscr{C}^{op(0,1)}$ for the monoidal category $\mathscr{C}$ where the orientation of tensors (dimension 0) and of morphisms (dimension 1) have been reversed. After this nicely symmetric but also partly speculative resolution of problem a. we are facing two additional and quite serious difficulties:

c. on the one hand, in contrast to what happened in the case of linear logic, the conjunction $\otimes$ and the disjunction $⅋$ do not live in the same category, since the conjunction $\otimes$ lives in the category $\mathscr{C}$ and the disjunction $⅋$ lives in its opposite category,

d. on the other hand, it is difficult to understand in what sense the conjunction $\otimes$ is really different from the disjunction $⅋$ since they are both defined as the tensor product of the category $\mathscr{C}$, and consequently, they only differ modulo the apparently conventional difference between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$.

We will resolve the three difficulties b. c. and d. in three steps. The first step is fundamental, although it appears as purely notational at first sight. It consists in decoupling the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op(0,1)}$ by writing

$$\mathscr{A} \quad = \quad (\mathscr{A}, \owedge, \mathit{true})$$

for the monoidal category $\mathscr{C}$ and

$$\mathscr{B} \quad = \quad (\mathscr{B}, \ovee, \mathit{false})$$

for its opposite category $\mathscr{C}^{op(0,1)}$. This resolves problem d. or at least overcomes it, since the two categories $\mathscr{A}$ and $\mathscr{B}$ are considered from now on as intrinsically different — although they are *secretly* related by the identity

$$\mathscr{B}^{op(0,1)} \quad = \quad \mathscr{A}. \tag{5}$$

Accordingly, and for clarity's sake, we choose to write conjunction as $\owedge$ in the category $\mathscr{A}$ and disjunction as $\ovee$ in the category $\mathscr{B}$.

A guiding intuition to keep in mind is that $\mathscr{A}$ denotes a category of proofs or strategies implemented by one side of the logical dispute (typically called Prover) whereas $\mathscr{B}$ denotes a category of anti-proofs or counter-strategies implemented by the other side (typically called Denier). We find convenient to call "polarization" this decoupling of the original dialogue category $\mathscr{C}$ and of its opposite category $\mathscr{C}^{op(0,1)}$ in two independent components $\mathscr{A}$ and $\mathscr{B}$. The terminology pays tribute to what this decorrelation owes to the notion of polarity introduced by Jean-Yves Girard in his seminal work on classical logic [7]. Girard's key idea was that the formulas of classical logic should be polarized in order to interpret every positive formula as a commutative $\otimes$-comonoid and every negative formula as a commutative $⅋$-monoid in the category of coherence spaces. Our own decision of decoupling the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op(0,1)}$ is directly inspired by this construction. The novelty of our work compared to the traditional view of polarities is (*i*) to reverse the point of view and to take polarities as more primitive than linear logic itself (*ii*) to reveal the 2-categorical rather than 1-categorical nature of these polarities and thanks to this new insight (*iii*) to generalize polarities to a large variety of proof systems besides the original system devised by Girard for classical logic. Typically, decoupling a cartesian closed category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op}$ enables one to write the same "classical" decomposition of implication as in linear logic:

$$A \Rightarrow B \quad = \quad A^{*} \quad \text{or} \quad B \tag{6}$$

with a notion of polarity intrinsically different from Girard's table of polarities for classical logic, see [11] for an elaboration of the table and of the formula. The fact that the formula (6) holds in an "intuitionistic" framework like cartesian closed categories establishes it as a fundamental principle of logic itself, not circumscribed to classical logic or to linear logic as it is generally believed.

Coming back to our discussion on dialogue categories, the decorrelation of $\mathscr{C}$ and $\mathscr{C}^{op(0,1)}$ enables us to resolve problem c. by recasting the original adjunction (4) as the adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathscr{B} \tag{7}$$

between the categories $\mathscr{A}$ and $\mathscr{B}$. In that way, the conjunction $\owedge$ and the disjunction $\ovee$ live in *different* categories $\mathscr{A}$ and $\mathscr{B}$ related by the functors L and R of the adjunction (7).

The analysis is sufficiently advanced at this point in order to provide the ground for a polarized or two-sided refinement of "linearly distributive categories". The key idea is that the linearly distributive category $\mathscr{C}$ together with its two tensor products $\otimes$ and $\parr$ should be replaced by *a pair* of monoidal categories

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false) \tag{8}$$

equipped with an adjunction (7) relating them. In addition, and there lies a key contribution of the paper, the original pair of distributivity laws (2) should be replaced by the following pair of distributivity laws

$$\begin{array}{ccccc} \kappa^{\owedge} & : & R(b \ovee L(a)) \owedge m & \longrightarrow & R(b \ovee L(a \owedge m)) \\ \kappa^{\ovee} & : & L(a \owedge R(b \ovee n)) & \longrightarrow & L(a \owedge R(b)) \ovee n \end{array} \tag{9}$$

Although these distributivity laws may appear intimidating at first sight, we claim that they convey one fundamental principle of logic. In particular, both distributivity laws hold in *every* dialogue category. As such, they regulate every sufficiently compositional proof system admitting a notion of negation.

Putting (7-8-9) together leads to the notion of *linearly distributive chirality* introduced in the technical core of the paper, see §6 for details. This two-sided notion of chirality refines the one-sided notion of linearly distributive category in just the same way as the notion of dialogue category refines the notion of $*$-autonomous category. It is worth mentioning that the notion of chirality we have in mind here is *mixed* rather than *pure* in the sense of [11]. Recall that a mixed chirality is a pair of categories in which the side $\mathscr{B}$ is not required to be equivalent to the opposite of the side $\mathscr{A}$ in contrast to the case of a pure chirality where $\mathscr{B} \cong \mathscr{A}^{op}$. The resulting notion of linearly distributive chirality satisfies two cardinal properties:

- every dialogue category $\mathscr{C}$ defines a linearly distributive chirality whose two sides $\mathscr{A}$ and $\mathscr{B}$ are defined as $\mathscr{A} = \mathscr{C}$ and $\mathscr{B} = \mathscr{C}^{op(0,1)}$ and whose functors L and R are defined as negation functors,
- a linearly distributive category is the same thing as a linearly distributive chirality where the two sides $\mathscr{A}$ and $\mathscr{B}$ happen to coincide with the same category $\mathscr{C}$ and where the two functors L and R are defined as the identity functor between the category $\mathscr{C}$ and itself.

In the same way as we like to call "polarization" the process of decoupling $\mathscr{C}$ and its opposite category $\mathscr{C}^{op(0,1)}$, we find convenient to call "depolarization" the operation of restricting the definition of a linearly distributive chirality to the specific case when:

- the two sides $\mathscr{A}$ and $\mathscr{B}$ coincide
- and the two functors L and R are defined as identity functors.

The philosophy of tensorial logic is that every concept and construction of linear logic should have a counterpart which refines it in tensorial logic — and from which it may be then recovered by depolarization. The notion of *linearly distributive category* offers a typical illustration of this principle, since as we have just mentioned, the notion may be recovered by depolarization from the notion of *linearly distributive chirality* introduced in the paper.

At this stage, it is worth stressing the perfect symmetry between the two distributivity laws $\kappa^{\owedge}$ and $\kappa^{\ovee}$. This symmetry is an instance of the general *chirality principle* between proofs and anti-proofs observed in [10]. This symmetry is witnessed in this case by the fact that each of the two laws may be obtained from the other one by applying the involution

$$R \leftrightarrow L \qquad x \owedge y \leftrightarrow x \ovee y \qquad a \leftrightarrow b \qquad m \leftrightarrow n$$

and by reversing the orientation of the map. After depolarization, this fundamental symmetry between the two combinators $\kappa^{\owedge}$ and $\kappa^{\ovee}$ boils down to the *self-symmetry* between the two sides κ^R and κ^L of the original distributivity law (2) for linearly distributive categories. It should be mentioned that the discovery of this unexpected degeneracy of linear logic was the starting point of tensorial logic. The distinction between $\kappa^{\owedge}$ and $\kappa^{\ovee}$ in tensorial logic reflects the distinction between Opponent views (generated by $\kappa^{\owedge}$) and Proponent views (generated by $\kappa^{\ovee}$) in game semantics, see [10] for details. The discovery that the two combinators $\kappa^{\owedge}$ and $\kappa^{\ovee}$ of tensorial logic are identified as $\kappa^R = \kappa^L$ after depolarization into linear logic was a clear challenge to our belief of the time that linear logic should be taken as an ultimate horizon of proof-theory.

Now that we have designed a tensorial counterpart to the notion of linearly distributive category, we are very close to the resolution of problem b. For memories, our original problem was to understand how the self-duality of $*$-autonomous categories may be adapted to dialogue categories. The key insight is to take the identity (5) very seriously, and to relax it to a monoidal equivalence

$$\mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \qquad \text{(monoidal equivalence)} \tag{10}$$

between the monoidal categories $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$. Our decision of relaxing the identity (5) into a monoidal equivalence (10) is motivated by the desire to encapsulate the original notion of *duality* introduced by Cockett and Seely [4, 3] for linearly distributive categories, see §2.2 for details. The technical point to remember here is that the negation functor of a $*$-autonomous category is not strictly involutive in general, but only involutive up to equivalence. By shifting from an equality to an equivalence (10) we thus ensure that the notion of duality on a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ encapsulates the two cases we want to capture:

- the case of a dialogue category $\mathscr{C}$ where the equivalence (10) is defined by taking the identity functor $\mathscr{C} \to \mathscr{C}$ between the category $\mathscr{C}$ and itself,
- the case of a $*$-autonomous category $\mathscr{C}$ where the equivalence (10) is defined by taking the negation functors $\mathscr{C} \to \mathscr{C}^{op(0,1)}$ and $\mathscr{C}^{op(0,1)} \to \mathscr{C}$ between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op(0,1)}$.

At this stage, we have resolved the four obstructions a. b. c. and d. In order to complete our program, there still remains to refine (3) and to formulate a variant of duality between an object m in the category $\mathscr{A}$ and an object m^* in the category $\mathscr{B}$. We will see in the course of the paper that this two-sided notion of duality consists of the two combinators

$$\begin{array}{rcc} \mathbf{AX}[m] & : & true \longrightarrow R(m^* \ovee L(m)) \\ \mathbf{CUT}[m] & : & L(m \owedge R(m^*)) \longrightarrow false. \end{array}$$

The reader will easily check that one recovers the two combinators (3) of a right duality in a linearly distributive category $\mathscr{C}$ after depolarization to a situation where the two sides $\mathscr{A}$ and $\mathscr{B}$ of the linearly distributive chirality coincide with the category $\mathscr{C}$, and where the functors L and R coincide with the identity. Conversely, we will see that the existence of such a family of duality combinators $\mathbf{AX}[m]$ and $\mathbf{CUT}[m]$ implies that the category $\mathscr{A}$ defines a dialogue category — at least when $\mathscr{A}$ and $\mathscr{B}$ are *symmetric* monoidal categories, see §7 for details. This last point justifies our claim that linearly distributive chiralities provide a tensorial refinement of the notion of linearly distributive category.

A brief summary. Let us summarize what we have learned from this short excursion:

- that every category $\mathscr{A} = \mathscr{C}$ of proofs comes with an opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$ of anti-proofs (or refutations) which may be entirely decoupled from the original category $\mathscr{C}$,
- that every formula x of the logic may be either seen from the point of view of Prover or from the point of view of Denier, and that these two points of view are related by a change $x \mapsto x^*$ of reference frame between Prover in $\mathscr{A}$ and Denier in $\mathscr{B}$.

We advocate that this new scenography of logic is more harmonious than the traditional picture with its rigid and somewhat conventional separation between classical and intuitionistic logic. One main benefit of the approach is that the change of reference frame $x \mapsto x^*$ remains involutive in the case of an "intuitionistic" system like tensorial logic.

Besides our debt to the logical notion of polarity introduced by Girard, we would like to stress how much this alternative picture of logic owes to the technical shift from 1-dimensional to 2-dimensional categories. The higher-dimensional point of view is indeed necessary in order to understand that the primitive symmetry of logic between Prover and Denier lives at a 2-dimensional level, rather than at the 1-dimensional (that is, categorical) level. This idea is supported by a microcosm principle adapted by the author [11] from a similar microcosm principle originally formulated by Baez and Dolan [1] for higher dimensional algebra. In the case of monoidal categories, the microcosm principle tells that any contravariant operation — like negation or implication — relies in the end on the symmetry

$$op(0,1) \quad : \quad \mathscr{C} \mapsto \mathscr{C}^{op(0,1)} \quad : \quad LaxMonCat \quad \longrightarrow \quad OpLaxMonCat^{op(0,2)}$$

between the cartesian 2-category *LaxMonCat* of monoidal categories and lax monoidal functors and the cartesian 2-category *OpLaxMonCat* constructed with oplax monoidal functors. In other words, the primary duality of logic is provided by the 2-dimensional operation of "reversing" a category $\mathscr{C}$ of proofs into its opposite category $\mathscr{C}^{op(0,1)}$ of anti-proofs. We like to think that this purely formal and algebraic observation reflects the phylogenetic principle that *symmetry comes before logic* [1].

It is worth looking in retrospect at the discovery of linear logic in the 1980's with this unified and perfectly symmetric picture of logic in mind. Linear logic appears then as the historical encounter of this internal symmetry $\mathscr{C} \mapsto \mathscr{C}^{op(0,1)}$ which lives at dimension 2 and the notion of $*$-autonomous category which lives at dimension 1. This foundational quiproquo between dimensions 1 and 2 induced the long-standing misconception that linear logic benefits from the "classical" symmetries of classical logic because its linear negation is involutive. One main ambition of the present paper is to correct this foundational mistake and to take this opportunity to dissect the micro-structure of tensorial negation in such a way as to make the flow of time explicit in logic, in the same way as in game semantics — see §4 for a discussion.

Plan of the paper. We start by recalling in §2 the notion of linearly distributive category together with the notion of right duality we will be specifically interested in. We recall in §3 the notion of dialogue chirality introduced by the author in [11] as a two-sided and deformed notion of dialogue category. Follows in §4 a discussion on the integration of time in the traditional proof-net syntax of linear logic. The notion of bimodulation between modulations (also called lax actions or parametric monads) is then recalled in §5. We reach in §6 the technical core of the paper, where we introduce the notion of linearly distributive chirality, and formulate the corresponding notion of right duality. We establish at the end of the section that the notion of dialogue chirality coincides with the notion of linearly distributive chirality with a right

[1] An intellectual challenge for logicians today is to develop a sufficiently fine-grained understanding (or anatomy) of formal logic in order to articulate it with the investigations on the evolutionary mechanisms which lead to the emergence of logical or prelogical reasoning in the human societies.

duality when the underlying monoidal categories $\mathscr{A}$ and $\mathscr{B}$ are symmetric. We conclude in §7 by observing that a mismatch remains between the two notions of dialogue chirality and linearly distributive chirality in the case of non-symmetric monoidal categories. We leave the task of resolving this matter to the companion paper devoted to helical negation [13].

2 Linearly distributive categories

The purpose of the section is to recall (and sometimes refine) the main definitions and results appearing in the original work by Cockett and Seely [4]. We start by recalling the notions of linearly distributive category in §2.1 and of right duality in §2.2. We establish in §2.3 that every linearly distributive category with a right duality is monoidal closed on the left. We introduce the symmetric notion of left duality in §2.4 and show that every linearly distributive category equipped with a left *and* a right duality defines a $*$-autonomous category. Finally, we establish in §2.5 that every right duality defines a monoidal functor between the underlying linearly distributive category $\mathscr{C}$ equipped with $\otimes$ and its opposite category equipped with $\parr$.

2.1 Definition

A (planar) linearly distributive category $\mathscr{C}$ is defined in [4, 3] as a category equipped with two monoidal structures, the first one called "tensor product" given by the bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ with unit $\mathbf{1}$ and natural isomorphisms

$$\alpha^{\otimes}_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

$$\lambda^{\otimes}_{x} : \mathbf{1} \otimes x \longrightarrow x \qquad \rho^{\otimes}_{x} : x \longrightarrow x \otimes \mathbf{1}$$

and the second one called "cotensor product" given by the bifunctor $\parr : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ with unit $\perp$ and natural isomorphisms

$$\alpha^{\parr}_{x,y,z} : x \parr (y \parr z) \longrightarrow (x \parr y) \parr z$$

$$\lambda^{\parr}_{x} : x \longrightarrow \perp \parr x \qquad \rho^{\parr}_{x} : x \parr \perp \longrightarrow x.$$

Linearly distributive categories are moreover equipped with two natural morphisms

$$\kappa^{L}_{x,y,z} : x \otimes (y \parr z) \longrightarrow (x \otimes y) \parr z$$

$$\kappa^{R}_{x,y,z} : (x \parr y) \otimes z \longrightarrow x \parr (y \otimes z)$$

required to satisfy a series of commutativity axioms, consisting of six pentagons and four triangles. The six pentagons may be separated into three series of two pentagons. The first series describes how the distributive law κ^R interacts with the associativity laws:

$$\begin{array}{ccccc}
((w \parr x) \otimes y) \otimes z & \xrightarrow{\kappa^R} & (w \parr (x \otimes y)) \otimes z & \xrightarrow{\kappa^R} & w \parr ((x \otimes y) \otimes z) \\
\downarrow{\scriptstyle \alpha^{\otimes}} & & & & \downarrow{\scriptstyle \alpha^{\otimes}} \\
(w \parr x) \otimes (y \otimes z) & & \xrightarrow{\qquad\kappa^R\qquad} & & w \parr (x \otimes (y \otimes z))
\end{array}$$

$$\begin{array}{ccccc}
(w \parr (x \parr y)) \otimes z & \xrightarrow{\kappa^R} & w \parr ((x \parr y) \otimes z) & \xrightarrow{\kappa^R} & w \parr (x \parr (y \otimes z)) \\
\downarrow{\scriptstyle \alpha^{\parr}} & & & & \downarrow{\scriptstyle \alpha^{\parr}} \\
((w \parr x) \parr y) \otimes z & & \xrightarrow{\qquad\kappa^R\qquad} & & (w \parr x) \parr (y \otimes z)
\end{array}$$

the second series describes how the distributive law κ^L interacts with the associativity laws:

$$\begin{array}{ccccc}
(w \otimes x) \otimes (y \parr z) & & \xrightarrow{\kappa^L} & & ((w \otimes x) \otimes y) \parr z \\
\downarrow{\scriptstyle \alpha^{\otimes}} & & & & \downarrow{\scriptstyle \alpha^{\otimes}} \\
w \otimes (x \otimes (y \parr z)) & \xrightarrow{\kappa^L} & w \otimes ((x \otimes y) \parr z) & \xrightarrow{\kappa^L} & (w \otimes (x \otimes y)) \parr z
\end{array}$$

$$\begin{array}{ccccc}
w \otimes (x \parr (y \parr z)) & & \xrightarrow{\kappa^L} & & (w \otimes x) \parr (y \parr z) \\
\downarrow{\scriptstyle \alpha^{\parr}} & & & & \downarrow{\scriptstyle \alpha^{\parr}} \\
w \otimes ((x \parr y) \parr z) & \xrightarrow{\kappa^L} & (w \otimes (x \parr y)) \parr z & \xrightarrow{\kappa^L} & ((w \otimes x) \parr y) \parr z
\end{array}$$

and the last series describes how κ^L and κ^R interact together with the associativity laws:

$$\begin{array}{ccccc}
(w \otimes (x \parr y)) \otimes z & \xrightarrow{\alpha^{\otimes}} & w \otimes ((x \parr y) \otimes z) & \xrightarrow{\kappa^R} & w \otimes (x \parr (y \otimes z)) \\
\downarrow{\scriptstyle \kappa^L} & & & & \downarrow{\scriptstyle \kappa^L} \\
((w \otimes x) \parr y) \otimes z & & \xrightarrow{\kappa^R} & & (w \otimes x) \parr (y \otimes z)
\end{array}$$

$$\begin{array}{ccccc}
(w \parr x) \otimes (y \parr z) & & \xrightarrow{\kappa^L} & & ((w \parr x) \otimes y) \parr z \\
\downarrow{\scriptstyle \kappa^R} & & & & \downarrow{\scriptstyle \kappa^R} \\
w \parr (x \otimes (y \parr z)) & \xrightarrow{\kappa^L} & w \parr ((x \otimes y) \parr z) & \xrightarrow{\alpha^{\parr}} & (w \parr (x \otimes y)) \parr z
\end{array}$$

Similarly, the four coherence triangles may be separated in two series of two triangles. The first series describes how the distributive law κ^R interacts with the units:

$$\begin{array}{ccc}
& x \otimes y & \\
{\scriptstyle \lambda^{\parr}} \swarrow & & \searrow {\scriptstyle \lambda^{\parr}} \\
(\bot \parr x) \otimes y & \xrightarrow{\kappa^R} & \bot \parr (x \otimes y)
\end{array}
\qquad
\begin{array}{ccc}
& x \parr y & \\
{\scriptstyle \rho^{\otimes}} \swarrow & & \searrow {\scriptstyle \rho^{\otimes}} \\
(x \parr y) \otimes \mathbf{1} & \xrightarrow{\kappa^R} & x \parr (y \otimes \mathbf{1})
\end{array}$$

and the second series describes how the distributive law κ^L interacts with the units:

$$\begin{array}{ccc}
\mathbf{1} \otimes (x \parr y) & \xrightarrow{\kappa^L} & (\mathbf{1} \otimes x) \parr y \\
{\scriptstyle \lambda^{\otimes}} \searrow & & \swarrow {\scriptstyle \lambda^{\otimes}} \\
& x \parr y &
\end{array}
\qquad
\begin{array}{ccc}
x \otimes (y \parr \bot) & \xrightarrow{\kappa^L} & (x \otimes y) \parr \bot \\
{\scriptstyle \rho^{\parr}} \searrow & & \swarrow {\scriptstyle \rho^{\parr}} \\
& x \otimes y &
\end{array}$$

One requires as expected that these diagrams commute for all objects w, x, y, z of the linearly distributive category $\mathscr{C}$.

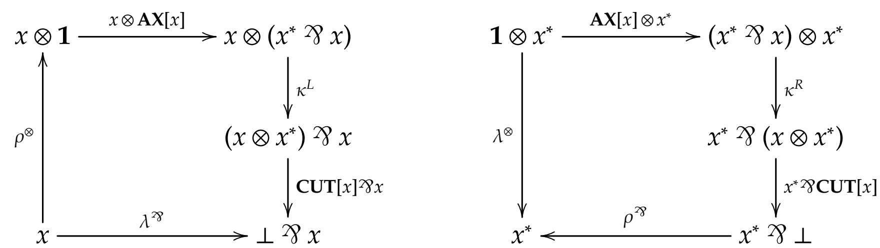
2.2 Right dualities in linearly distributive categories

A general notion of negation in a linearly distributive category $\mathscr{C}$ is introduced in [4]. Here, we find convenient to keep only half of it. This leads us to the notion of *right duality* in a linearly distributive category $\mathscr{C}$, defined below. The complementary notion of *left duality* is discussed in §2.4, at the end of the section.

Definition 1 *A right duality in a linearly distributive category $\mathscr{C}$ consists of the following data:*

- *an object x^*,*
- *two morphisms $\mathbf{AX}[x] : \mathbf{1} \longrightarrow x^* ⅋ x$ and $\mathbf{CUT}[x] : x \otimes x^* \longrightarrow \bot$*

for every object x of the category $\mathscr{C}$. The morphisms are moreover required to make the diagrams



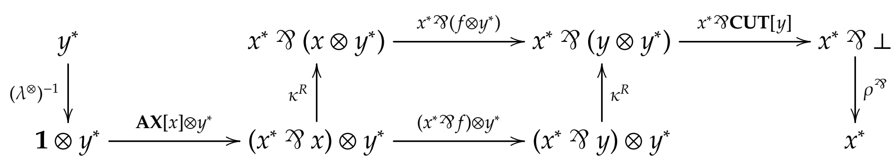
(11)

commute.

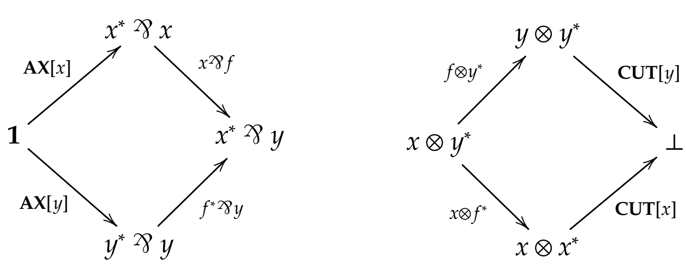
Note that every such right duality defines a contravariant functor

$$x \mapsto x^* \quad : \quad \mathscr{C} \longrightarrow \mathscr{C}^{op}$$

which transports every morphism $f : x \longrightarrow y$ of the linearly distributive category $\mathscr{C}$ to the morphism $f^* : y^* \longrightarrow x^*$ defined in the following way:



The functoriality of $x \mapsto x^*$ follows easily from the coherence diagrams. The construction of f^* also ensures that the two families $\mathbf{AX}[-]$ and $\mathbf{CUT}[-]$ are dinatural in the sense that the two diagrams



commute for every morphism $f : x \to y$.

2.3 Main statement

One main purpose of the notion of linearly distributive category is to isolate in a $*$-autonomous category the properties of the connectives$\otimes$ and⅋ from the properties of the duality. In particular, the notion of right duality is designed in order to satisfy the following property, which states that every linearly distributive category $\mathscr{C}$ equipped with a right duality is monoidal closed on the left.

Proposition 1 *In any linearly distributive category $\mathscr{C}$ with a right duality,*

- *the functor $(x \otimes -)$ is left adjoint to the functor $(x^* ⅋ -)$*

- *the functor* $(- \parr y)$ *is right adjoint to the functor* $(- \otimes y^*)$

for all objects x, y *of the category. In particular, every such category* $\mathscr{C}$ *is monoidal closed on the left, with implication defined as*

$$x \multimap y \quad = \quad x^* \parr y.$$

The proof is essentially immediate. One associates to every morphism

$$f \quad : \quad x \otimes y \quad \longrightarrow \quad z$$

the morphism $\varphi_{x,y,z}(f)$ defined as the composite

$$y \xrightarrow{(\lambda^{\otimes})^{-1}} \mathbf{1} \otimes y \xrightarrow{\mathbf{AX}[x]} (x^* \parr x) \otimes y \xrightarrow{\kappa^R} x^* \parr (x \otimes y) \xrightarrow{f} x^* \parr z$$

Conversely, one associates to every morphism

$$g \quad : \quad y \quad \longrightarrow \quad x^* \parr z$$

the morphism $\psi_{x,y,z}(g)$ defined as the composite

$$x \otimes y \xrightarrow{g} x \otimes (x^* \parr z) \xrightarrow{\kappa^L} (x \otimes x^*) \parr z \xrightarrow{\mathbf{CUT}[x]} \bot \parr z \xrightarrow{\rho^{\parr}} z$$

The coherence properties of the linearly distributive category $\mathscr{C}$ and of the right duality $x \mapsto x^*$ imply together that the correspondence is one-to-one and natural in y and z. This establishes the property. It is worth mentioning that the unit η_x and counit ε_x of the adjunction are equal to the composite morphisms below:

$$\begin{array}{lcl} \eta_x & : & y \xrightarrow{(\lambda^{\otimes})^{-1}} \mathbf{1} \otimes y \xrightarrow{\mathbf{AX}[x]} (x^* \parr x) \otimes y \xrightarrow{\kappa^R} x^* \parr (x \otimes y) \\ \varepsilon_x & : & x \otimes (x^* \parr y) \xrightarrow{\kappa^L} (x \otimes x^*) \parr y \xrightarrow{\mathbf{CUT}[x]} \bot \parr y \xrightarrow{\rho^{\parr}} y \end{array}$$

2.4 Left dualities in linearly distributive categories

A basic but important feature of the notion of linearly distributive category is that it is invariant under the following symmetry: given a linearly distributive category $\mathscr{C}$, the category $\mathscr{C}^{\,op(0)}$ obtained by reversing the orientation of the tensor $\otimes$ and cotensor $\parr$ products is also linearly distributive. This enables to define a left duality $x \mapsto {}^*x$ in a linearly distributive category $\mathscr{C}$ as a right duality in the linearly distributive category $\mathscr{C}^{\,op(0)}$. The definition may be expanded in the following way. A left duality consists of a family of morphisms

$$\overline{\mathbf{AX}}[x] : \mathbf{1} \longrightarrow x \parr {}^*x \qquad \overline{\mathbf{CUT}}[x] : {}^*x \otimes x \longrightarrow \bot$$

parametrized by the objects x of the category $\mathscr{C}$, and making the two diagrams below commute:

$$\begin{array}{ccc} \mathbf{1} \otimes x & \xrightarrow{\overline{\mathbf{AX}}[x] \otimes x} & (x \parr {}^*x) \otimes x \\ & & \downarrow \kappa^R \\ \lambda^{\otimes} \downarrow & & x \parr ({}^*x \otimes x) \\ & & \downarrow x \parr \overline{\mathbf{CUT}}[x] \\ x & \xleftarrow{\rho^{\parr}} & x \parr \bot \end{array} \qquad \begin{array}{ccc} {}^*x \otimes \mathbf{1} & \xrightarrow{{}^*x \otimes \overline{\mathbf{AX}}[x]} & {}^*x \otimes (x \parr {}^*x) \\ & & \downarrow \kappa^L \\ \rho^{\otimes} \uparrow & & ({}^*x \otimes x) \parr {}^*x \\ & & \downarrow \overline{\mathbf{CUT}}[x] \parr {}^*x \\ {}^*x & \xrightarrow{\lambda^{\parr}} & \bot \parr {}^*x \end{array} \tag{12}$$

The following statement is a direct consequence of Proposition 1 when applied to the right duality in the companion linearly distributive category $\mathscr{C}^{\,op(0)}$.

Proposition 2 *In any linearly distributive category $\mathscr{C}$ with a left duality,*

- *the functor $(- \otimes x)$ is left adjoint to the functor $(- \parr {}^*x)$*
- *the functor $(y \parr -)$ is right adjoint to the functor $({}^*y \otimes -)$*

for all objects x, y of the category. In particular, every such category $\mathscr{C}$ is monoidal closed on the right, with implication $y \multimapinv x$ defined as $y \parr {}^*x$.

This leads to the following statement which justifies the very definition of linearly distributive category:

Proposition 3 (Cockett-Seely) *Every linearly distributive category $\mathscr{C}$ equipped with a right duality and a left duality is $*$-autonomous with negation functors defined as*

$$x \multimap \bot = x^* \parr \bot \qquad\qquad \bot \multimapinv x = \bot \parr {}^*x$$

Shifting to symmetric monoidal categories, it may be established that a symmetric $*$-autonomous category is essentially the same thing as a symmetric linearly distributive category equipped with a right duality, see §7 for a discussion.

2.5 Monoidality of right duality

This section is mainly technical: it is essentially here to prepare the later comparison with right dualities in linearly distributive chiralities in §6.4. We establish that every right duality defines a monoidal functor (and not just a functor) from the underlying linearly distributive category $\mathscr{C}$ equipped with $\otimes$ to its opposite category $\mathscr{C}^{op(0,1)}$ equipped with $\parr$. Suppose given such a right duality in a linearly distributive category $\mathscr{C}$. We know by Proposition 1 that the right duality associates to every pair of objects x and y a pair of adjunctions

$$\mathscr{C} \underset{y^* \parr -}{\overset{y \otimes -}{\rightleftarrows}} \mathscr{C} \underset{x^* \parr -}{\overset{x \otimes -}{\rightleftarrows}} \mathscr{C} \qquad (y \otimes - \dashv y^* \parr -, \quad x \otimes - \dashv x^* \parr -)$$

which may be then composed into an adjunction

$$x \otimes (y \otimes -) \quad \dashv \quad y^* \parr (x^* \parr -)$$

Composing with associativity on both sides induces the adjunction

$$(x \otimes y) \otimes - \quad \dashv \quad (y^* \parr x^*) \parr -$$

which may be then compared with the adjunction associated by the right duality to the object $x \otimes y$:

$$(x \otimes y) \otimes - \quad \dashv \quad (x \otimes y)^* \parr -$$

by Proposition 1 again. The uniqueness of a right adjoint modulo isomorphism implies the existence of a family of isomorphisms

$$\theta_{x,y,z} \quad : \quad (y^* \parr x^*) \parr z \longrightarrow (x \otimes y)^* \parr z$$

natural in z, defined as the composite morphism

$$\begin{array}{ccc}
(x \otimes y)^* \parr ((x \otimes y) \otimes ((y^* \parr x^*) \parr z)) & \xrightarrow{\alpha^{\otimes}, \alpha^{\parr}} & (x \otimes y)^* \parr (x \otimes (y \otimes (y^* \parr (x^* \parr z)))) \\
 & & \downarrow \varepsilon_y \\
\uparrow \eta_{x \otimes y} & & (x \otimes y)^* \parr (x \otimes (x^* \parr z)) \\
 & & \downarrow \varepsilon_x \\
(y^* \parr x^*) \parr z & \xrightarrow{\theta_{x,y,z}} & (x \otimes y)^* \parr z
\end{array}$$

In the same way, one deduces from the adjunction $(\mathbf{1}\otimes - \dashv \mathbf{1}^* \parr -)$ the existence of a natural isomorphism

$$\theta_z \quad : \quad z \longrightarrow \mathbf{1}^* \parr z$$

defined as the composite

$$z \xrightarrow{\eta_{\mathbf{1}}} \mathbf{1}^* \parr (\mathbf{1}\otimes z) \xrightarrow{\lambda^{\otimes}} \mathbf{1}^* \parr z.$$

From this, we deduce the existence of two families of isomorphisms

$$\tilde{\theta}_{x,y} : y^* \parr x^* \longrightarrow (x\otimes y)^* \qquad\qquad \tilde{\theta}_{\mathbf{1}} : \bot \longrightarrow \mathbf{1}^* \tag{13}$$

simply defined by instantiating z as $\bot$. It is not difficult to check that

Proposition 4 *In a linearly distributive category $\mathscr{C}$, the right duality functor equipped with the coercion maps (13) defines a monoidal functor*

$$(-)^* \quad : \quad (\mathscr{C},\otimes,\mathbf{1}) \quad \longrightarrow \quad (\mathscr{C},\parr,\bot)^{op(0,1)}.$$

One checks moreover that the two coercion maps $\tilde{\theta}$ make the four diagrams

$$\begin{array}{ccccc}
y^* \parr (\mathbf{1}\otimes y) & \xrightarrow{\mathbf{AX}[x]} & y^* \parr ((x^* \parr x)\otimes y) & \xrightarrow{\kappa^R} & y^* \parr (x^* \parr (x\otimes y)) \\
\uparrow{\scriptstyle (\lambda^{\otimes})^{-1}} & & & & \downarrow{\scriptstyle \alpha^{\parr}} \\
y^* \parr y & & & & (y^* \parr x^*) \parr (x\otimes y) \\
\uparrow{\scriptstyle \mathbf{AX}[y]} & & & & \downarrow{\scriptstyle \tilde{\theta}_{x,y}} \\
\mathbf{1} & & \xrightarrow{\mathbf{AX}[x\otimes y]} & & (x\otimes y)^* \parr (x\otimes y)
\end{array}$$

$$\begin{array}{ccccc}
x\otimes (y\otimes (y^* \parr x^*)) & \xrightarrow{\kappa^L} & x\otimes ((y\otimes y^*) \parr x^*) & \xrightarrow{\mathbf{CUT}[y]} & x\otimes (\bot \parr x^*) \\
\uparrow{\scriptstyle \alpha^{\otimes}} & & & & \downarrow{\scriptstyle (\lambda^{\parr})^{-1}} \\
(x\otimes y)\otimes (y^* \parr x^*) & & & & x\otimes x^* \\
\uparrow{\scriptstyle \tilde{\theta}^{-1}_{x,y}} & & & & \downarrow{\scriptstyle \mathbf{CUT}[x]} \\
(x\otimes y)\otimes (x\otimes y)^* & & \xrightarrow{\mathbf{CUT}[x\otimes y]} & & \bot
\end{array}$$

$$\begin{array}{ccc}
\mathbf{1}^* \parr \mathbf{1} & \xrightarrow{\tilde{\theta}_{\mathbf{1}}} & \bot \parr \mathbf{1} \\
\uparrow{\scriptstyle \mathbf{AX}[\mathbf{1}]} & & \downarrow{\scriptstyle (\lambda^{\parr})^{-1}} \\
\mathbf{1} & \xrightarrow{id} & \mathbf{1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{1}\otimes \bot & \xrightarrow{\tilde{\theta}_{\mathbf{1}}} & \mathbf{1}\otimes \mathbf{1}^* \\
\uparrow{\scriptstyle (\lambda^{\otimes})^{-1}} & & \downarrow{\scriptstyle \mathbf{CUT}[\mathbf{1}]} \\
\bot & \xrightarrow{id} & \bot
\end{array}$$

commute for all objects x, y of the linearly distributive category $\mathscr{C}$. We will see these four diagrams reappear in §6.4 when we compare the corresponding notion of right duality in a linearly distributive chirality.

3 Dialogue chiralities

The purpose of the section is to recall (and also slightly adapt) the notion of *dialogue chirality* originally introduced in [11]. As explained in the introduction, the notion of dialogue chirality was designed in order to provide a two-sided formulation of the familiar but one-sided notion of dialogue category. We start the section in §3.1 by slightly relaxing the original definition, in order to adapt it to the specific purposes of the paper. We then investigate a series of equivalent formulations of this notion of dialogue chirality, either based on a family of adjunctions in §3.2 or on a family of transjunctions in §3.3. We complete the section by recalling in §3.4 the notion of transjunction introduced in [12] together with its pictorial representation in string diagrams.

3.1 Dialogue chiralities on the left

We start from the original definition of dialogue chiralities given in [11] where we replace the monoidal equivalence (10) by the appropriate monoidal functor (15). The change is essentially innocuous. It is mainly justified by the slick formulation of our main result (Proposition 9) established in the second part of the paper, see §6.3.

Definition 2 (dialogue chiralities) *A dialogue chirality on the left is defined as a pair of monoidal categories*

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \quad (L \dashv R) \tag{14}$$

whose unit and counit are denoted as

$$\eta \;:\; Id \longrightarrow R \circ L \qquad\qquad \varepsilon \;:\; L \circ R \longrightarrow Id$$

together with a monoidal functor

$$(-)^{*} \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}^{op(0,1)} \tag{15}$$

and a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^{*} \ovee b \,\rangle$$

natural in m, a, b, which we call currification *in honor of the logician Haskell Curry. Here, the bracket* $\langle\, a \,|\, b \,\rangle$ *denotes the set of morphisms from a to* $R(b)$ *in the category* $\mathscr{A}$*:*

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R(b)\,).$$

The family χ *is moreover required to make the diagram*

$$\begin{array}{ccccc}
\langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad\chi_{m \owedge n}\quad} & & \langle\, a \,|\, (m \owedge n)^{*} \ovee b \,\rangle \\
\Big\downarrow{\scriptstyle associativity} & & & & \Big\uparrow{\scriptstyle associativity,\ monoidality\ of\ negation} \\
\langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi_{m}} & \langle\, n \owedge a \,|\, m^{*} \ovee b \,\rangle & \xrightarrow{\chi_{n}} & \langle\, a \,|\, n^{*} \ovee (m^{*} \ovee b) \,\rangle
\end{array} \tag{16}$$

commute for all objects a, m, n *of the category* $\mathscr{A}$*, and all objects b of the category* $\mathscr{B}$*.*

3.2 A formulation based on adjunctions

A preliminary step towards the algebraic presentation of dialogue chiralities performed later in the paper (see §6) is to replace the currification isomorphism χ by a family of adjunctions, in the following way.

Proposition 5 *A dialogue chirality on the left is the same thing as a pair of monoidal categories* $(\mathscr{A}, \owedge, true)$ *and* $(\mathscr{B}, \ovee, false)$ *equipped with an adjunction (14) and a monoidal functor (15) together with an adjunction*

$$L(m \owedge -) \quad \dashv \quad R(m^{*} \ovee -) \tag{17}$$

for every object m of the category $\mathscr{A}$, whose unit and counit are denoted

$$\eta[m] \;:\; a \longrightarrow R(m^* \ovee L(m \owedge a)) \qquad\qquad \varepsilon[m] \;:\; L(m \owedge R(m^* \ovee b)) \longrightarrow b$$

The family $\eta[-]$ is moreover required to be natural and monoidal, this meaning that the diagrams below

$$\begin{array}{ccccc} & \overset{\eta[m]}{\nearrow} & R(m^* \ovee L(m \owedge a)) & \overset{f}{\searrow} & \\ a & & & & R(m^* \ovee L(n \owedge a)) \\ & \underset{\eta[n]}{\searrow} & R(n^* \ovee L(n \owedge a)) & \underset{f^*}{\nearrow} & \end{array} \tag{18}$$

$$\begin{array}{ccc} a & \xrightarrow{\eta[m \owedge n]} & R((m \owedge n)^* \ovee L((m \owedge n) \owedge a)) \\ \downarrow{\scriptstyle \eta[n]} & & \\ R(n^* \ovee L(n \owedge a)) & & \downarrow{\scriptstyle \text{associativity \& monoidality of negation}} \\ \downarrow{\scriptstyle \eta[m]} & & \\ R(n^* \ovee LR(m^* \ovee L(m \owedge (n \owedge a)))) & \xrightarrow{\varepsilon} & R(n^* \ovee (m^* \ovee L(m \owedge (n \owedge a)))) \end{array} \tag{19}$$

should commute for all objects a, m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

Remark. The family of adjunctions (17) instantiated at the unit *true* induces an adjunction $L \dashv R$ between the functors L and R, whose unit and counit are defined as the expected families of morphisms

$$\begin{array}{lllllll} \eta'_a & : & a & \xrightarrow{\eta[true]} & R(true^* \ovee L(true \owedge a)) & \xrightarrow{\text{associativity \& monoidality}} & RL(a) \\ \varepsilon'_b & : & LR(b) & \xrightarrow{\text{associativity \& monoidality}} & L(true \owedge R(true^* \ovee b)) & \xrightarrow{\varepsilon[true]} & b \end{array}$$

An important question is to understand whether this adjunction necessarily coincides with the original adjunction (4) between L and R. The answer is positive. Indeed, it is not difficult to see that the coherence diagram (19) implies that the two adjunctions coincide in the sense that $\eta = \eta'$ and $\varepsilon = \varepsilon'$. The main idea is to instantiate the coherence diagram (19) with $m = n = e$, and to apply the coherence laws of the monoidal categories $\mathscr{A}$ and $\mathscr{B}$ in order to show that the diagram

$$\begin{array}{ccc} a & \xrightarrow{\eta'} & RLa \\ \downarrow{\scriptstyle \eta'} & & \| \\ RLa & & \| \\ \downarrow{\scriptstyle \eta'} & & \| \\ RLRLa & \xrightarrow{\varepsilon} & RLa \end{array}$$

commutes. It easily follows from this and from the properties of adjunctions that $\eta = \eta'$ and $\varepsilon' = \varepsilon$. This means in particular that the coherence diagram

$$\begin{array}{ccc} R(true^* \ovee L(true \owedge a)) & \xrightarrow{\text{monoidality}} & R(false \ovee L(true \owedge a)) \\ \uparrow{\scriptstyle \eta[true]} & & \downarrow{\scriptstyle \text{associativity}} \\ a & \xrightarrow{\eta} & RL(a) \end{array} \tag{20}$$

is a consequence of the two coherence diagrams (18) and (19) formulated in the statement of Proposition 5.

Remark. Once established that the adjunction $(L, R, \eta', \varepsilon')$ coincides with the adjunction $(L, R, \eta, \varepsilon)$, it is not difficult to deduce from their companion diagrams (18) and (19) that the two coherence diagrams

$$\begin{array}{ccccc}
 & & L(n \obslash R(n^* \oslash b)) & & \\
 & \nearrow^{f} & & \searrow^{\varepsilon[n]} & \\
L(m \obslash R(n^* \oslash b)) & & & & b \\
 & \searrow_{f^*} & & \nearrow_{\varepsilon[m]} & \\
 & & L(m \obslash R(m^* \oslash b)) & &
\end{array}$$

$$\begin{array}{ccc}
L(m \obslash (n \obslash R(n^* \oslash (m^* \oslash b)))) & \xrightarrow{\eta} & L(m \obslash RL(n \obslash R(n^* \oslash (m^* \oslash b)))) \\
 & & \downarrow{\scriptstyle \varepsilon[n]} \\
\downarrow{\scriptstyle \text{associativity \& monoidality of negation}} & & L(m \obslash R(m^* \oslash b)) \\
 & & \downarrow{\scriptstyle \varepsilon[m]} \\
L((m \obslash n) \obslash R((m \obslash n)^* \oslash b)) & \xrightarrow{\varepsilon[m \obslash n]} & b
\end{array}$$

commute for all object b of the category $\mathscr{B}$ and all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$. At this point, it is worth mentioning that there is an element of choice in the definition of dialogue chiralities formulated in Proposition 5 since the two coherence diagrams for $\varepsilon[-]$ may very well replace the two corresponding diagrams (18) and (19) for $\eta[-]$.

3.3 A formulation based on transjunctions

The formulation of dialogue chiralities described in §3.2 is fine... but it may be marginally improved. The idea is to replace the original pair of combinators $\eta[-]$ and $\varepsilon[-]$ presenting the adjunctions by a pair of somewhat simpler combinators:

$$\begin{array}{rcl}
\textbf{axiom}[m] & : & L(a) \longrightarrow m^* \oslash L(m \obslash a) \\
\textbf{cut}[m] & : & m \obslash R(m^* \oslash b) \longrightarrow R(b)
\end{array}$$

directly inspired by proof-theory. The two combinators are defined from $\eta[-]$ and $\varepsilon[-]$ in the following way:

$$\begin{array}{rcl}
\textbf{axiom}[m] & : & L(a) \xrightarrow{\eta[m]} LR(m^* \oslash L(m \obslash a)) \xrightarrow{\varepsilon} m^* \oslash L(m \obslash a) \\
\textbf{cut}[m] & : & m \obslash R(m^* \oslash b) \xrightarrow{\eta} RL(m \obslash R(m^* \oslash b)) \xrightarrow{\varepsilon[m]} R(b)
\end{array}$$

Conversely, the combinators $\eta[-]$ and $\varepsilon[-]$ may be recovered from the combinators **axiom**$[-]$ and **cut**$[-]$ in the following way:

$$\begin{array}{rcl}
\eta[m] & : & a \xrightarrow{\eta} RL(a) \xrightarrow{\textbf{axiom}[m]} R(m^* \oslash L(m \obslash a)) \\
\varepsilon[m] & : & L(m \obslash R(m^* \oslash b)) \xrightarrow{\textbf{cut}[m]} LR(b) \xrightarrow{\varepsilon} b
\end{array}$$

It is easy to show that this back-and-forth translation between the pair of combinators $(\eta[-], \varepsilon[-])$ and the pair of combinators $(\textbf{axiom}[-], \textbf{cut}[-])$ defines a one-to-one relationship. This observation leads to an alternative formulation of dialogue chiralities, based this time on the *transjunction*

$$(m \obslash -) \quad \multimapinv \quad (m^* \oslash -) \tag{21}$$

between the two functors

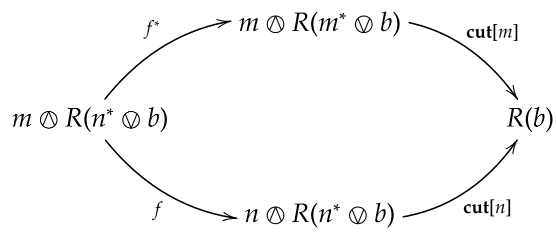
$$m \oslash - \;:\; \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad m^* \obslash - \;:\; \mathscr{B} \longrightarrow \mathscr{B}$$

accross the adjunction $L \dashv R$ between $\mathscr{A}$ and $\mathscr{B}$. The reader unaware of the notion of transjunction will find the notion recalled in §3.4. One main reason for introducing this notion of transjunction is that it enables us to replace the original combinators $\eta[-]$ and $\varepsilon[-]$ by the logically flavoured combinators **axiom**$[-]$ and **cut**$[-]$.

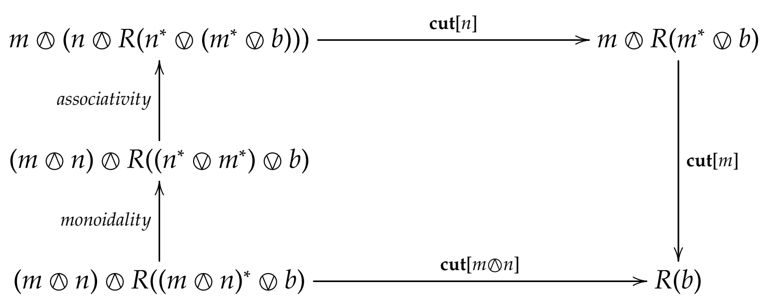
Proposition 6 *A dialogue chirality on the left may be alternatively defined as a pair of categories* $(\mathscr{A}, \oslash, true)$ *and* $(\mathscr{B}, \obslash, false)$ *equipped with an adjunction (14) and a monoidal functor (15) together with a family of transjunctions*

$$\textbf{axiom}[m] \;:\; L(a) \longrightarrow m^* \obslash L(m \oslash a) \qquad\qquad \textbf{cut}[m] \;:\; m \oslash R(m^* \obslash b) \longrightarrow R(b)$$

natural in b and m. The family **cut**$[-]$ *is moreover required to be natural and monoidal, in the sense that the two diagrams*



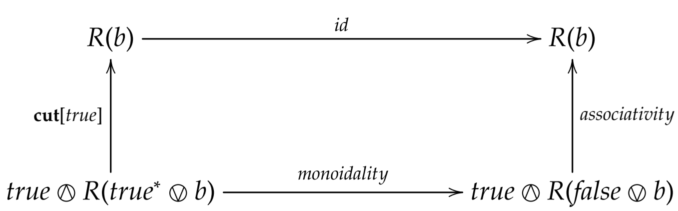
(22)



(23)

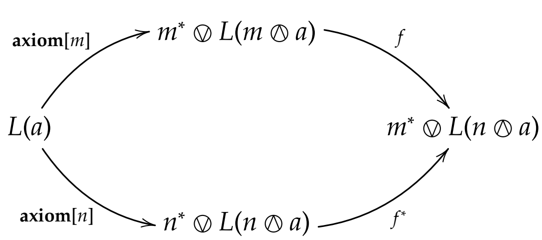
commute for all objects a, m, n *and morphisms* $f : m \to n$ *of the category* $\mathscr{A}$.

Remark. In the same way as in §3.2, the coherence diagram



(24)

follows from the coherence diagrams (22) and (23). In the same way, one can easily check that the two coherence diagrams for the combinator **cut**$[-]$ are equivalent to the corresponding coherence diagrams for **axiom**$[-]$ below:



$$
\begin{array}{ccc}
n^* \varovee L(n \varowedge a) & \xrightarrow{\textbf{axiom}[m]} & n^* \varovee (m^* \varovee L(m \varowedge (n \varowedge a))) \\
\uparrow{\scriptstyle \textbf{axiom}[n]} & & \downarrow{\scriptstyle associativity} \\
 & & (n^* \varovee m^*) \varovee L((m \varowedge n) \varowedge a) \\
 & & \downarrow{\scriptstyle monoidality} \\
La & \xrightarrow{\textbf{axiom}[m \varowedge n]} & (m \varowedge n)^* \varovee L((m \varowedge n) \varowedge a)
\end{array}
\qquad (25)
$$

and the coherence diagram

$$
\begin{array}{ccc}
true^* \varovee L(true \varowedge a) & \xrightarrow{monoidality} & false \varovee L(true \varowedge a) \\
\uparrow{\scriptstyle \textbf{axiom}[true]} & & \downarrow{\scriptstyle associativity} \\
L(a) & \xrightarrow{id} & L(a)
\end{array}
\qquad (26)
$$

follows from the coherence diagrams (22) and (23).

3.4 Transjunctions

It is probably worth recalling here the notion of transjunction introduced in [12].

Definition 3 (transjunction) *Suppose given a pair of adjunctions*

$$
\mathscr{A}_1 \underset{R_1}{\overset{L_1}{\rightleftarrows}} \mathscr{B}_1 \quad (L_1 \dashv R_1) \qquad\qquad \mathscr{A}_2 \underset{R_2}{\overset{L_2}{\rightleftarrows}} \mathscr{B}_2 \quad (L_2 \dashv R_2)
$$

whose units and counits are denoted η_1, η_2 and $\varepsilon_1, \varepsilon_2$ respectively. A transjunction $F \dashv\!\!\!\!\exists\; G$ between a pair of functors

$$F : \mathscr{A}_1 \to \mathscr{A}_2 \qquad G : \mathscr{B}_2 \to \mathscr{B}_1$$

accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

$$\textbf{axiom} : L_1 \Rightarrow G \circ L_2 \circ F \qquad \textbf{cut} : F \circ R_1 \circ G \Rightarrow R_2$$

making the two diagrams

$$
\begin{array}{ccc}
F \circ R_1 \circ L_1 & \xRightarrow{\textbf{axiom}} & F \circ R_1 \circ G \circ L_2 \circ F \\
\Uparrow{\scriptstyle \eta_1} & (a) & \Downarrow{\scriptstyle \textbf{cut}} \\
F & \xRightarrow{\eta_2} & R_2 \circ L_2 \circ F
\end{array}
\qquad
\begin{array}{ccc}
G \circ L_2 \circ F \circ R_1 \circ G & \xRightarrow{\textbf{cut}} & G \circ L_2 \circ R_2 \\
\Uparrow{\scriptstyle \textbf{axiom}} & (b) & \Downarrow{\scriptstyle \varepsilon_2} \\
L_1 \circ R_1 \circ G & \xRightarrow{\varepsilon_1} & G
\end{array}
\qquad (27)
$$

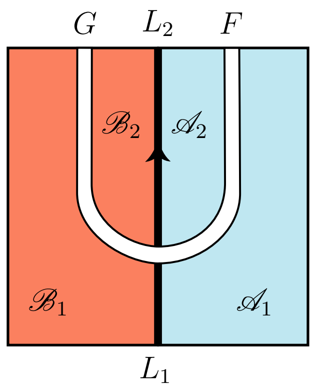
commute.

Although the notion of transjunction may appear complicated at first sight, it is ultimately justified by the following observation.

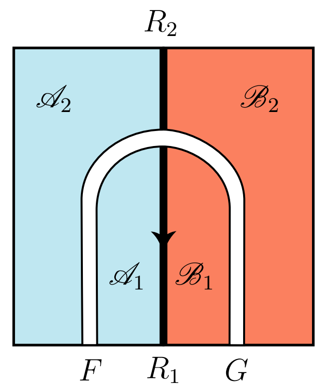
Proposition 7 *A transjunction $F \dashv\!\!\!\!\exists\; G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is the same thing as an adjunction $F \circ L_1 \dashv R_2 \circ G$.*

One important benefit of shifting from the notion of adjunction $F \circ L_1 \dashv R_2 \circ G$ to the notion of transjunction is that it is much easier to depict them in the language of string diagrams. Remember that we work in the 2-category of categories, functors and natural transformations. The generators **axiom** and **cut** of a transjunction $F \dashv\!\!\!\exists\, G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ are thus depicted in the following way:

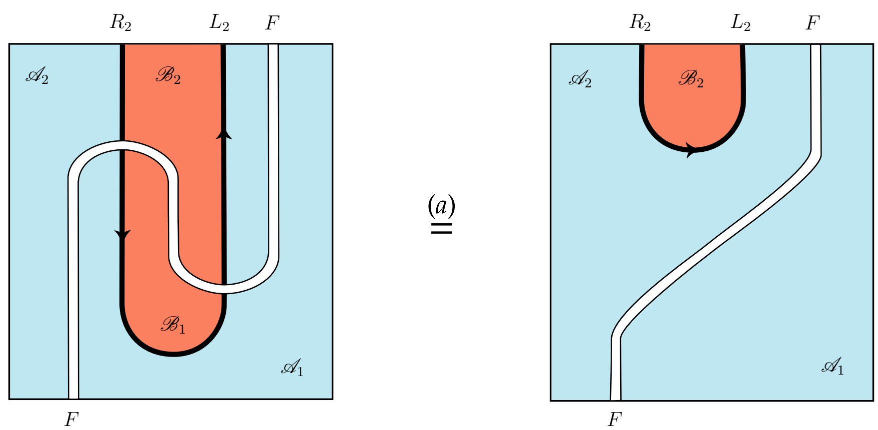
axiom =

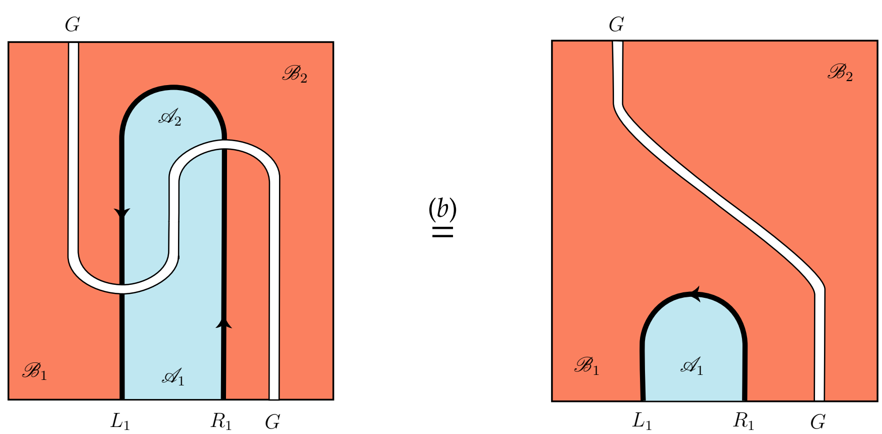


cut =



The two commutative diagrams (a) and (b) of Definition 3 are then depicted as:



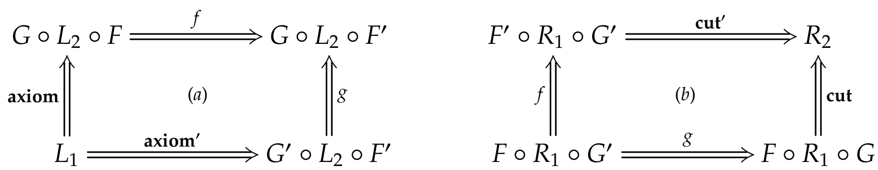


A notion of homorphism between transjunctions may be also introduced, this giving rise to a category of transjunctions.

Definition 4 (homomorphism) *A homomorphism between two transjunctions $F \dashv\!\!\!\exists\, G$ and $F' \dashv\!\!\!\exists\, G'$ accross the same pair of adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations*

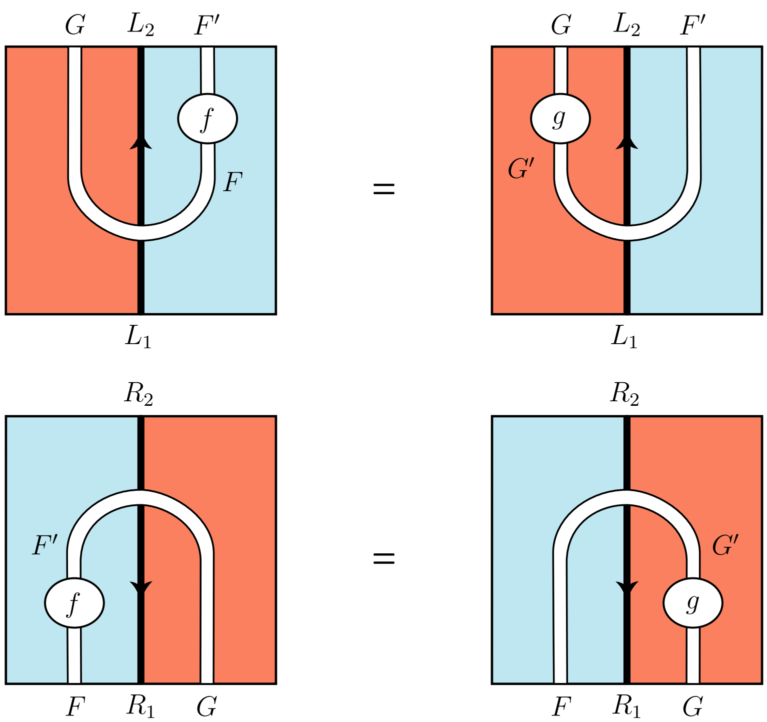
$$f \ : \ F \ \Rightarrow \ F' \qquad\qquad g \ : \ G' \ \Rightarrow \ G$$

making the two diagrams



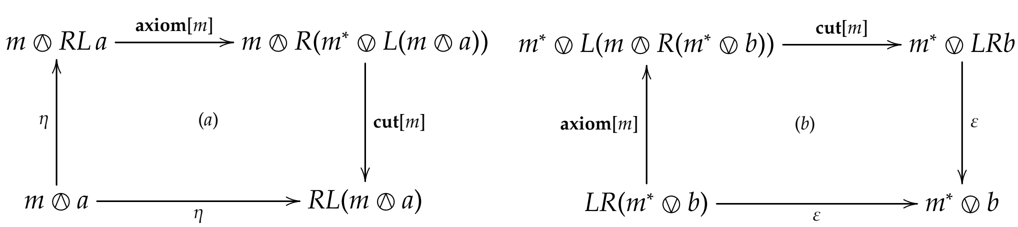
commute.

Pictorially, such a homomorphism (f, g) is defined as a pair of natural transformations $f : F \Rightarrow F'$ and $g : G' \Rightarrow G$ satisfying the diagrammatic equalities below:

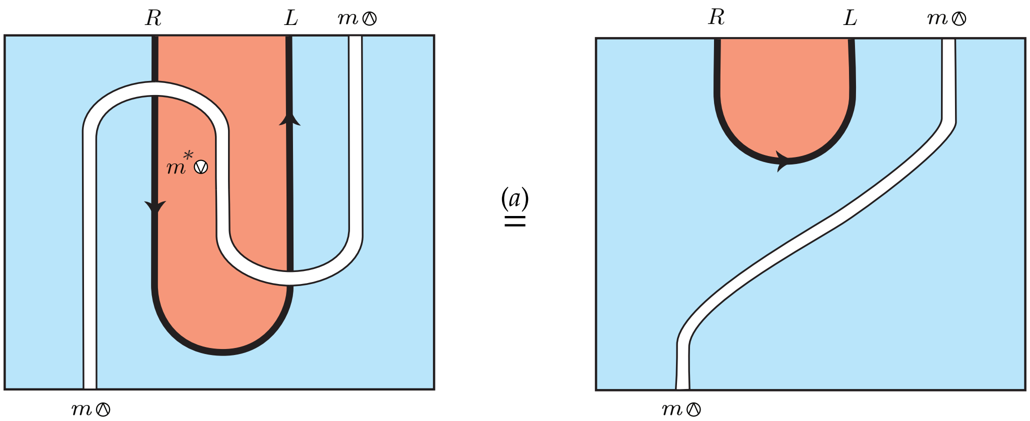


4 Tensorial proof-nets vs. multiplicative proof-nets

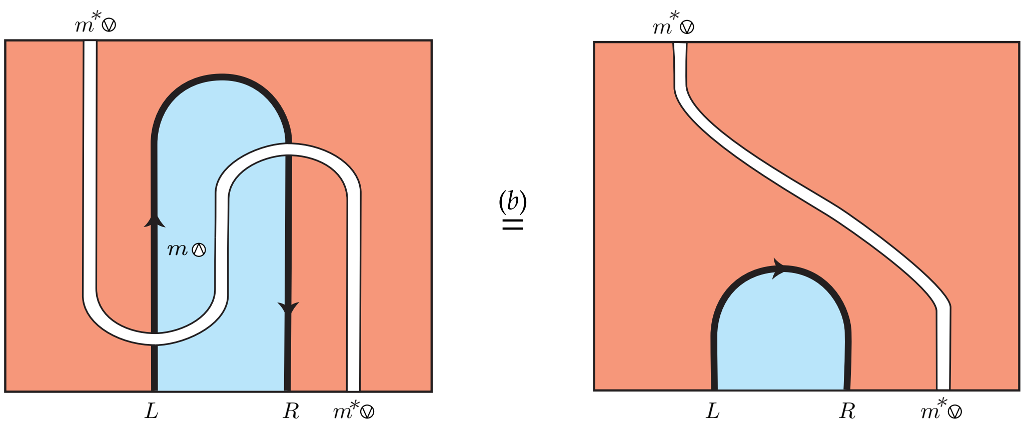
As already mentioned in the introduction, the philosophy guiding our work on tensorial logic is to lift the constructions of linear logic to a situation where negation is not required to be involutive anymore. A typical question is to refine the axiom-cut annihilation of linear logic — as it appears in multiplicative proof-nets — in order to apply it to tensorial logic. One benefit of our algebraic approach is that the problem has been (although somewhat secretly) resolved in the two previous sections §3.3 and §3.4. Indeed, recall that Definition 3 and Proposition 6 express together that the two coherence diagrams (27) required of a transjunction should be instantiated as follows in the definition of a dialogue chirality:



These diagrams are required to commute for all objects m, a of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$. Once translated in string diagrams, the coherence diagram (a) is depicted as



while its companion diagram (b) is depicted as its mirror image:



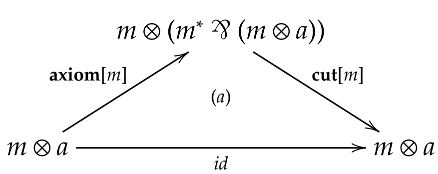
An important observation here is that the two equations (a) and (b) refine the traditional axiom-cut annihilation of linear logic, by integrating the non involutive negation functors L and R in it. Indeed, specialized to the particular case of a $*$-autonomous category $\mathscr{C}$, where the two sides $\mathscr{A}$ and $\mathscr{B}$ of the dialogue chirality coincide with $\mathscr{C}$, and where the two functors R and L are defined as the identity functor $\mathscr{C} \to \mathscr{C}$, the family of transjunctions defining the $*$-autonomous category in Proposition 6 boils down to a family of adjunctions

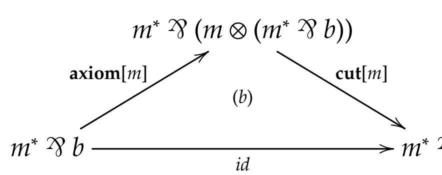
$$m \otimes - \quad \dashv \quad m^* \parr - \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \tag{28}$$

with unit and counit defined as

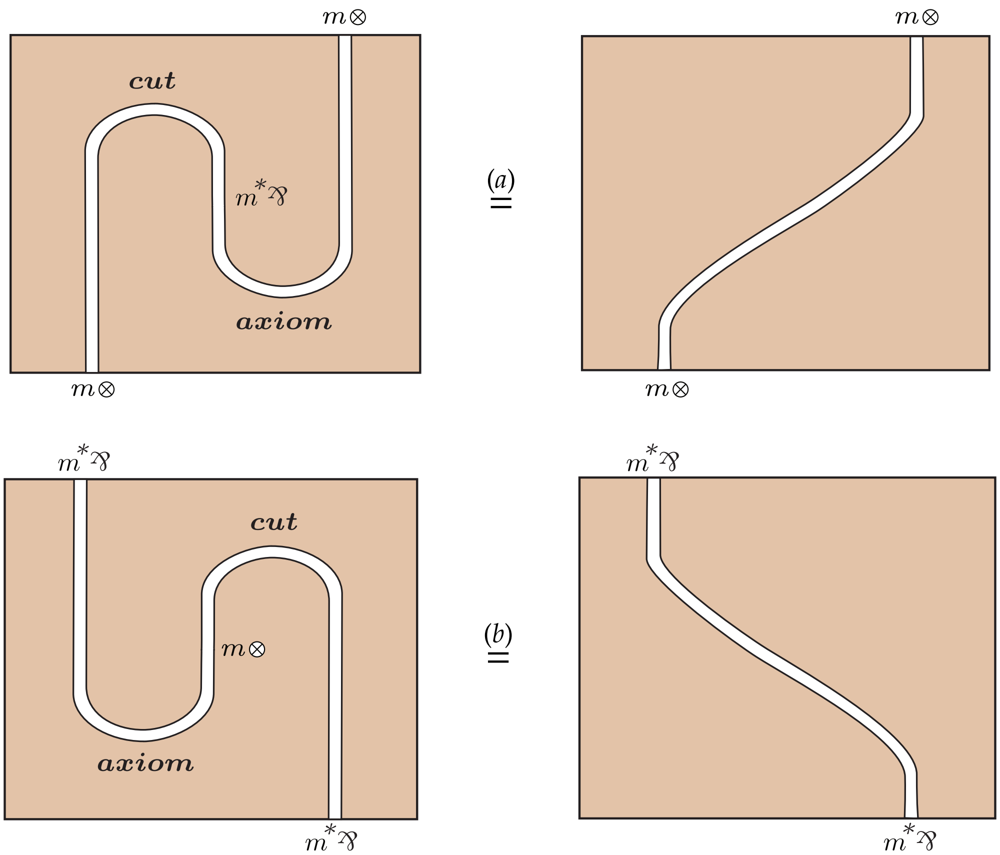
$$\mathbf{axiom}[m] \ : \ a \ \longrightarrow \ m^* \parr (m \otimes a) \qquad\qquad \mathbf{cut}[m] \ : \ m \otimes (m^* \parr b) \ \longrightarrow \ b.$$

Accordingly, the equations (a) and (b) are replaced by the familiar triangular laws of the adjunction (28):





which are required to commute for all objects m, a of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$. One recovers in this way the familiar axiom-cut annihilation of multiplicative proof-nets in linear logic, turned upside-down and formulated in the graphical language of string diagrams:



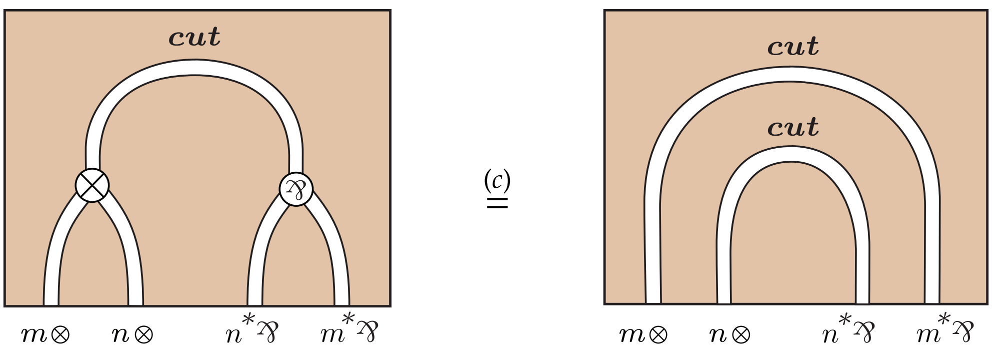
At this stage, it should be clear that the purpose of each transjunction (21) mentioned in Proposition 6 is to refine the corresponding adjunction (28) of $*$-autonomous categories by taking care of the explicit negation functors L and R of dialogue chiralities. Conversely, and just as expected, the axiom-cut annihilation of $*$-autonomous categories is recovered by "depolarizing" the dialogue chirality into a situation where its two sides $\mathscr{A}$ and $\mathscr{B}$ are equal, and where the negation functors L and R disappear.

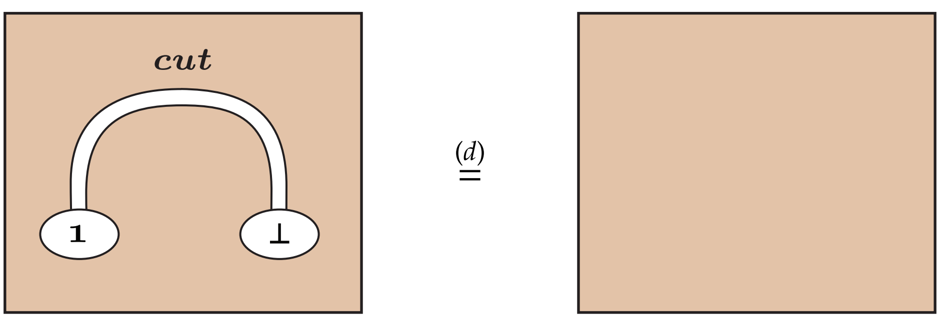
As we have just seen, the refinement of linear logic into tensorial logic works surprisingly well in the case of the axiom-cut annihilation. One may thus wonder whether the two other fundamental equations of multiplicative proof nets — namely the tensor-par and the **1**-$\perp$ annihilation rules — may be upgraded to tensorial logic in the same way. The two rules of linear logic are recalled below as commutative diagrams (c) and (d) in a generic $*$-autonomous category. The reader will easily check that they express together that the family of adjunctions (28) is monoidal in its parameter m.

$$
\begin{array}{ccc}
m \otimes (n \otimes (n^* \parr (m^* \parr b))) & \xrightarrow{\ \mathbf{cut}[n]\ } & m \otimes (m^* \parr b) \\
\uparrow {\scriptstyle associativity} & & \\
(m \otimes n) \otimes ((n^* \parr m^*) \parr b) & (c) & \downarrow {\scriptstyle \mathbf{cut}[m]} \\
\uparrow {\scriptstyle monoidality} & & \\
(m \otimes n) \otimes ((m \otimes n)^* \parr b) & \xrightarrow[\ \mathbf{cut}[m \otimes n]\ ]{} & b
\end{array}
$$

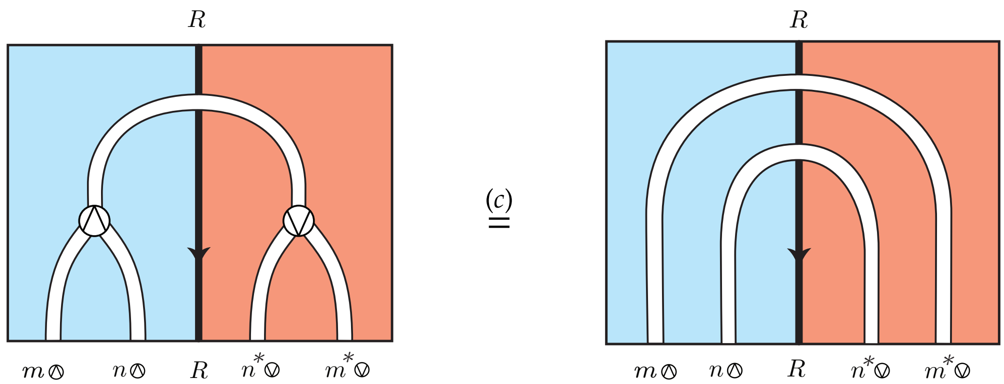
$$
\begin{array}{ccc}
b & \xrightarrow{\ id\ } & b \\
 & & \uparrow {\scriptstyle unit} \\
\uparrow {\scriptstyle \mathbf{cut}[\mathbf{1}]} & (d) & \mathbf{1} \otimes b \\
 & & \uparrow {\scriptstyle unit} \\
\mathbf{1} \otimes (\mathbf{1}^* \parr b) & \xrightarrow[\ monoidality\ ]{} & \mathbf{1} \otimes (\bot \parr b)
\end{array}
$$

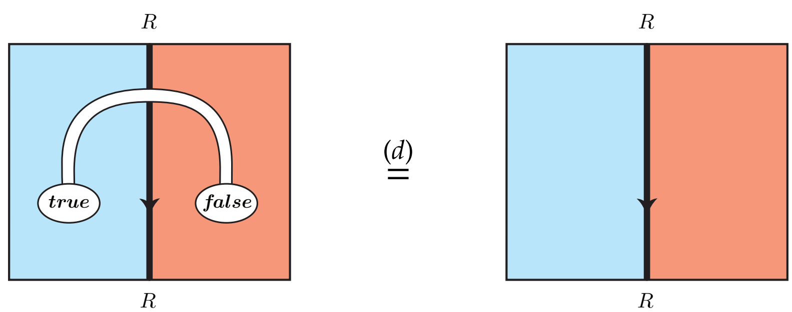
These two equations (c) and (d) of linear logic are depicted below in the appropriate language of string diagrams.





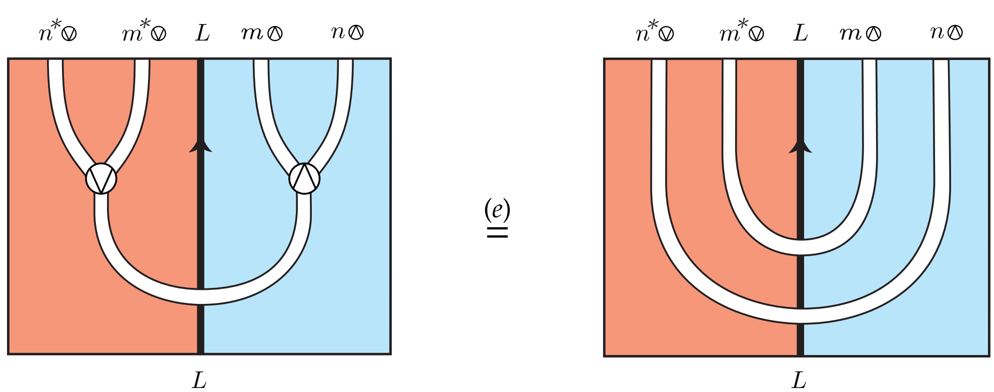
Note that they correspond to the usual equations of multiplicative proof-nets of linear logic, once appropriately turned upside-down. One benefit of our purely algebraic approach to proof-theory is to provide a simple answer to the question of upgrading (c) and (d) to tensorial logic. As we have already seen, the solution for upgrading the axiom-cut annihilation equations (a) and (b) to tensorial logic is entirely conveyed by the first part of Proposition 6. Accordingly, the second part of Proposition 6 indicates how to upgrade the two equations (c) and (d). This fact becomes apparent when one depicts the coherence diagrams (23) and (24) as string diagrams:

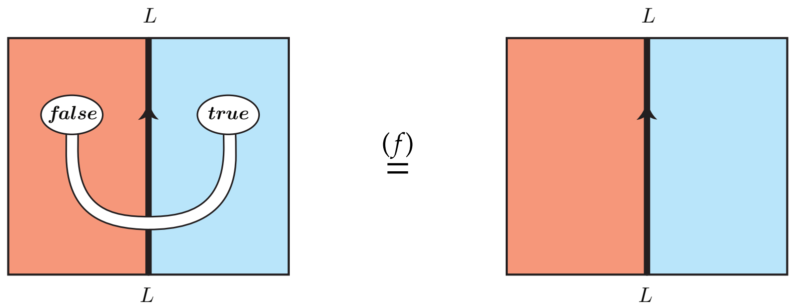




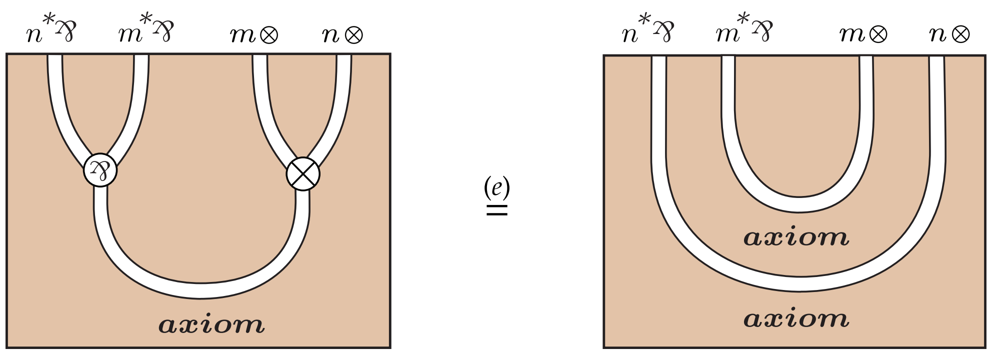
Just as in the case of the two diagrams (*a*) and (*b*), one recovers the two equations (*c*) and (*d*) of linear logic by “depolarizing” the corresponding equations of tensorial logic. Diagrammatically speaking, depolarization amounts to replacing the blue area for $\mathscr{A}$ and the red area for $\mathscr{B}$ by a unique brown area for $\mathscr{C}$, and by removing accordingly the separating frontiers L and R between $\mathscr{A}$ and $\mathscr{B}$.

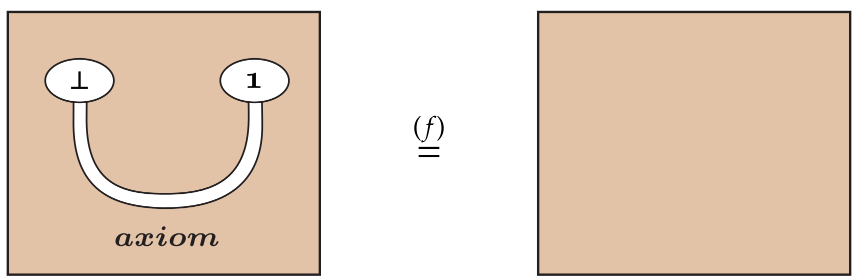
As already mentioned, a basic principle of dialogue chiralities is that every coherence diagram comes with a mirror image. A typical illustration is provided by diagram (*a*) whose mirror image is diagram (*b*). Similarly, the mirror image of the two diagrams (*c*) and (*d*) is provided by the coherence diagrams (25) and (26) depicted below as two equations (*e*) and (*f*) between string diagrams:





These diagrams refine the familiar η-expansion rule of multiplicative proof-nets in linear logic, which are deduced from their tensorial counterpart by "depolarizing" them in the expected way:





This series of elementary observations leads to the idea that the theory of proof-nets in linear logic may be lifted to a similarly rich theory of proof-nets in tensorial logic. A preliminary step in this program has been performed in [10] where a notion of tensorial proof-net at the crossroad of linear logic and game semantics has been introduced. In that respect, the process of depolarizing a tensorial proof-net into a multiplicative proof-net of linear logic deserves the alternative name of "detemporalization" or "desequentialization" since it removes the flow of execution (and thus of time) described by the functors R and L.

5 Bimodulations

The notion of *bimodulation* between two *modulations* was introduced in our companion paper [12] in order to describe the algebraic structure of dialogue categories. By modulation, we mean a *lax action* of a monoidal category $\mathscr{I}$ on a category $\mathscr{A}$, what we also like to call a *parametric monad* depending on the context of application. The purpose of a bimodulation is then to relate two such modulations. One main reason for introducing the notion of bimodulation is that it unifies the familiar notions of *monadic strength* on the one hand, and of *distributivity law* between two monads on the other hand. We start the section by recalling the two notions of modulation in §5.1 and of bimodulation in §5.2. The precise definition of a dialogue chirality on the right (rather than on the left) is then formulated in §5.3. This leads us to §5.4 where we illustrate the notion of bimodulation with the example of the continuation bimodulation associated to any dialogue chirality on the right.

5.1 Modulations

Instead of directly defining a modulation as a lax action of a monoidal category $\mathscr{I}$ on a category $\mathscr{A}$, we prefer to slightly extend our span of application and thus to work in a general 2-category $\mathscr{W}$ in the spirit of formal monad theory [14].

Definition 5 (modulation) *A modulation in a 2-category $\mathscr{W}$ is defined as a 0-cell $\mathscr{A}$ equipped with a lax monoidal functor*

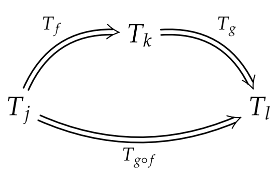
$$(T, m) \;:\; \mathscr{I} \;\longrightarrow\; \mathrm{End}(\mathscr{A}).$$

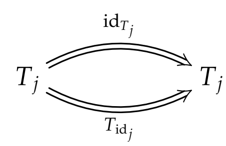
The monoidal category $(\mathscr{I}, \otimes, e)$ is called the parameter category of the $\mathscr{I}$-modulation. Accordingly, an object j of the category $\mathscr{I}$ is called a parameter of the modulation.

Hence, a $\mathscr{I}$-modulation (T, m) consists of

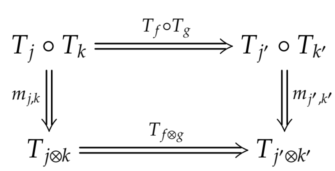
- a 1-cell $T_j : \mathscr{A} \longrightarrow \mathscr{A}$ for every parameter j and a 2-cell $T_f : T_j \Rightarrow T_k$ for every morphism $f : j \longrightarrow k$ between such parameters,
- a 2-cell $m_e : 1_A \Rightarrow T_e$ called the unit of the modulation,
- a 2-cell $m_{j,k} : T_j \circ T_k \Rightarrow T_{j \otimes k}$ called the (j,k)-component of the multiplication of the modulation, for every pair of parameters j and k.

These data are moreover required to make a series of coherence diagrams commute in the category End($\mathscr{A}$). First, the diagrams

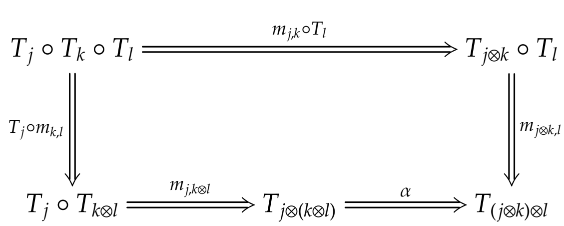


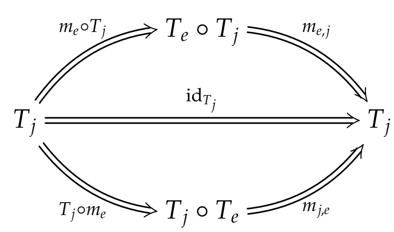


express the functoriality of T. Then, the diagrams



express the naturality of m. Finally, the two diagrams





express the monoidality of m, for all indices j, j', k, k', l and morphisms f, g, h in the parameter category $\mathscr{I}$. Note that a modulation T parametrized by the trivial monoidal category $\mathbf{1}$ with exactly one object $*$ and exactly one morphism (thus equal to the identity morphism of $*$) is the same thing as a formal monad in the 2-category $\mathscr{W}$ in the sense of Street [14].

5.2 Bimodulations between modulations

From now on, we suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathscr{W}$ equipped with a $\mathscr{I}$-modulation

$$T = \bullet \quad : \quad \mathscr{I} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

and a $\mathscr{M}^{op(0)}$-modulation

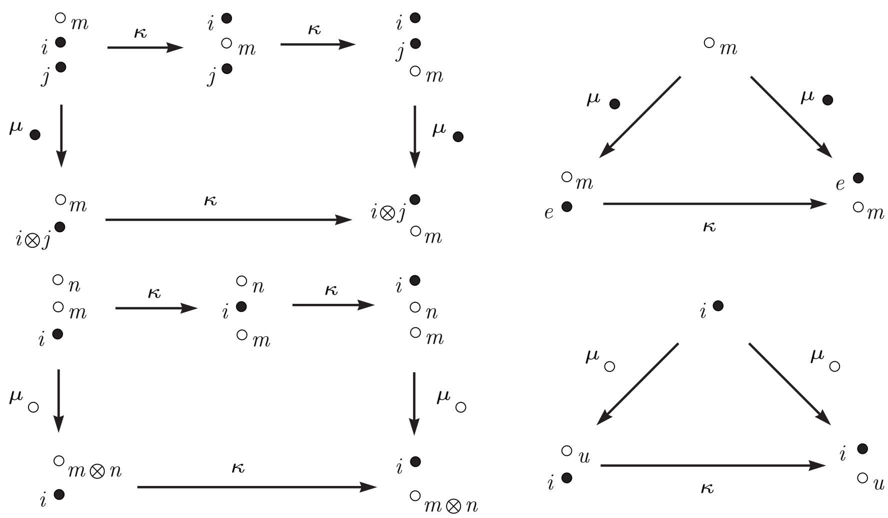
$$S = \circ \quad : \quad \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

with parameters taken in the monoidal categories $(\mathscr{I}, \otimes, e)$ and $(\mathscr{M}, \otimes, u)$.

Definition 6 (bimodulation) *A bimodulation between the two modulations $T = \bullet$ and $S = \circ$ is defined as a natural transformation*

$$\kappa \quad : \quad S\,T \quad \Rightarrow \quad T\,S \quad : \quad \mathscr{I} \times \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

making the four diagrams below commute



for all objects i, j of the category $\mathscr{I}$ and all objects m, n of the category $\mathscr{M}$.

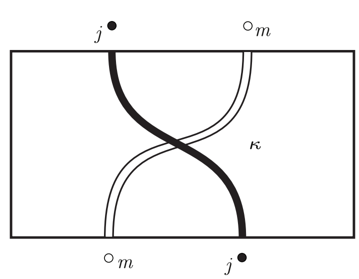
Remark. In the particular case when $\mathscr{W} = \mathbf{Cat}$, a bimodulation may be alternatively defined as a natural transformation

$$\kappa \quad : \quad (- \bullet -) \circ - \quad \Rightarrow \quad - \bullet (- \circ -) \quad : \quad \mathscr{I} \times \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

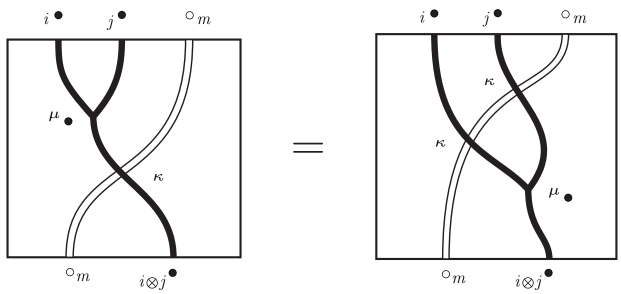
with components

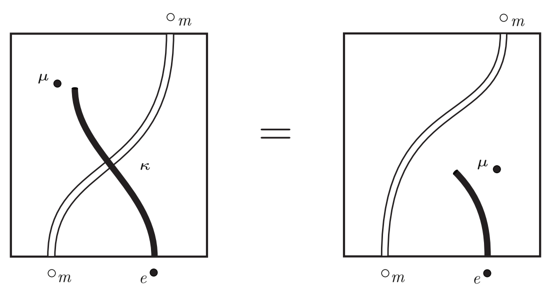
$$\kappa_{i,m,A} \quad : \quad (i \bullet A) \circ m \quad \longrightarrow \quad i \bullet (A \circ m)$$

parametrized by the objects i of the category $\mathscr{I}$, by the objects m of the category $\mathscr{M}$ and by the objects A of the category $\mathscr{C}$. We like to think of a bimodulation κ as a lax version of a $(\mathscr{I}, \mathscr{M})$-bimodule. This justifies our terminology. A bimodulation κ is represented in string diagrams as a braiding

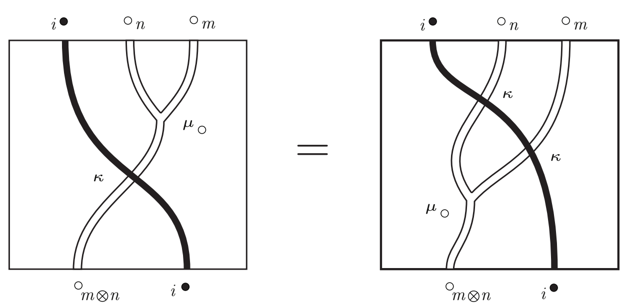


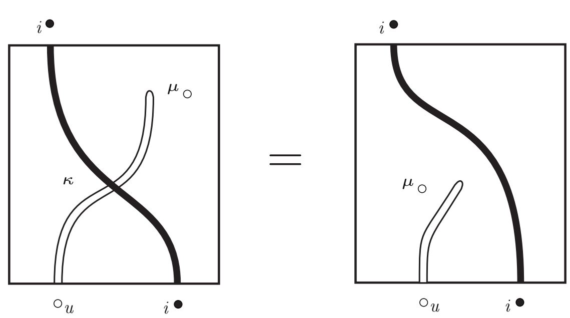
commuting the black string representing the modulation • under the white string representing the modulation ∘. The notation enables to depict the coherence diagrams of the bimodulation κ as a series of topologically intuitive equations, permuting the multiplication and unit of a modulation under or over the string representing the other modulation. Typically, the first series of equations in the definition of a bimodulation "permutes" the multiplication $\mu_\bullet$ over the modulation ∘





while the second series of equations "permutes" the multiplication $\mu_\circ$ under the modulation • in the following way:





Remark. A *comodulation* parametrized by the monoidal category $(\mathscr{I}, \otimes, e)$ in a 2-category $\mathscr{W}$ is defined as a $\mathscr{I}$-modulation in the 2-category $\mathscr{W}^{op(2)}$. Here, the notation $\mathscr{W}^{op(2)}$ is for the

2-category obtained from $\mathscr{W}$ by reversing the orientation of its 2-cells. Similarly, a *bimodulation* between two comodulations is defined as a bimodulation between the corresponding modulations in the 2-category $\mathscr{W}^{op(2)}$. A notion of bimodulation between a modulation and a comodulation may be also introduced in order to generalize the notion of distributivity law between a monad and a comonad.

5.3 Dialogue chiralities on the right

There is an element of choice in our definition of a dialogue chirality in §3 and it thus makes sense to call it a dialogue chirality *on the left*. As we explained, every dialogue category $\mathscr{C}$ induces such a dialogue chirality on the left, obtained by defining L and R as the negation functors

$$L \;:\; x \;\mapsto\; \bot \multimapinv x \qquad\qquad R \;:\; x \;\mapsto\; x \multimap \bot.$$

The other choice of negation functors

$$L \;:\; x \;\mapsto\; x \multimap \bot \qquad\qquad R \;:\; x \;\mapsto\; \bot \multimapinv x$$

also induces a dialogue chirality, but this time *on the right*. As expected, the notion of dialogue chirality on the right defined below is a mirror image of the notion of dialogue chirality on the left.

Definition 7 *A dialogue chirality on the right is defined as a pair of monoidal categories*

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with a monoidal functor

$$\mathscr{A} \xrightarrow{\;{}^{\circledast}(-)\;} \mathscr{B}^{op(0,1)}$$

with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R)$$

and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad\longrightarrow\quad \langle\, a \,|\, b \ovee {}^{\circledast}m \,\rangle$$

natural in m, a, b. In the same way as in the case of a dialogue chirality on the left, the bracket $\langle\, a \,|\, b \,\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:

$$\langle\, a \,|\, b \,\rangle \quad=\quad \mathscr{A}(\, a\, ,\, R(b)\,).$$

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\quad \chi_{m \owedge n} \quad} & & \langle\, a \,|\, b \ovee {}^{\circledast}(m \owedge n) \,\rangle \\
\big\downarrow{\scriptstyle associativity} & & & & \big\uparrow{\scriptstyle associativity \; monoidality\; of\; negation} \\
\langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \owedge m \,|\, b \ovee {}^{\circledast}n \,\rangle & \xrightarrow{\chi_m} & \langle\, a \,|\, (b \ovee {}^{\circledast}n) \ovee {}^{\circledast}m \,\rangle
\end{array}$$

commute for all objects a, m, n of the category $\mathscr{A}$, and all objects b of the category $\mathscr{B}$.

Note that every dialogue chirality on the left whose monoidal functor $(-)^*$ is a monoidal equivalence induces a dialogue chirality on the right by applying the involution

$$\mathscr{A} \mapsto \mathscr{B}^{op(0,1)} \qquad \mathscr{B} \mapsto \mathscr{A}^{op(0,1)} \qquad L \mapsto R^{op(0,1)} \qquad R \mapsto L^{op(0,1)}.$$

The interested reader is advised to look at [11] for a discussion on this point.

5.4 The continuation bimodulation

An important reason for introducing the notion of bimodulation in [12] is that every dialogue chirality (on the right) comes together with a bimodulation

$$\kappa^{\owedge} \quad : \quad S\,T \quad \longrightarrow \quad T\,S$$

between the two modulations

$$\begin{array}{ccccccccc} T & : & (b,a) & \mapsto & R(b \ovee L(a)) & : & \mathscr{B} \times \mathscr{A} & \longrightarrow & \mathscr{A} \\ S & : & (a,m) & \mapsto & a \owedge m & : & \mathscr{A} \times \mathscr{A} & \longrightarrow & \mathscr{A} \end{array}$$

Symmetrically, every dialogue chirality (on the right) comes equipped with a bimodulation

$$\kappa^{\ovee} \quad : \quad K\,L \quad \longrightarrow \quad L\,K$$

between the two comodulations

$$\begin{array}{ccccccccc} L & : & (b,n) & \mapsto & b \ovee n & : & \mathscr{B} \times \mathscr{B} & \longrightarrow & \mathscr{B} \\ K & : & (a,b) & \mapsto & L(a \owedge R(b)) & : & \mathscr{A} \times \mathscr{B} & \longrightarrow & \mathscr{B} \end{array}$$

This pair of bimodulations will play a central role in the definition of a linearly distributive chirality in the following section.

6 Linearly distributive chiralities

At this point, we are ready to articulate the formal definition of linearly distributive chirality discussed in the introduction. The reader will find it fully exposed in §6.1. We then proceed by analogy with linearly distributive categories, and introduce in §6.2 a notion of right duality adapted to linearly distributive chiralities. The main technical result of the paper appears in §6.3. We establish there that every linearly distributive chirality equipped with a right duality defines a dialogue chirality in the sense of §3.1. Finally, we conclude in §6.4 by showing the notion of linearly distributive chirality (and of right duality) coincides with the traditional notion of linearly distributive category (and of right duality) in the "depolarized" case when the two sides $\mathscr{A}$ and $\mathscr{B}$ of the chirality coincide, and the two functors L and R are equal to the identity functor.

6.1 Definition

A *linearly distributive chirality* is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R)$$

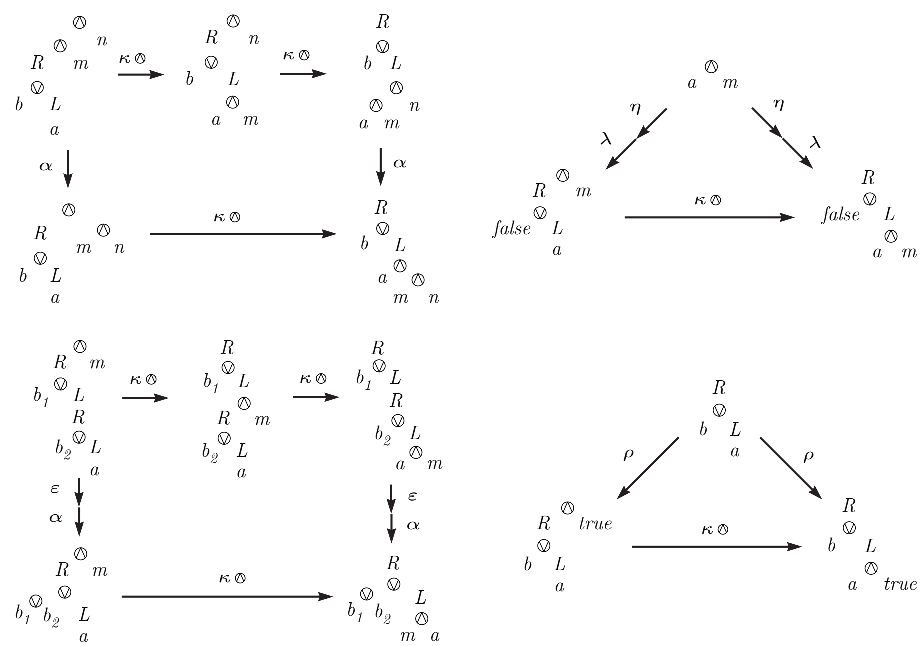
together with two bimodulations

$$\begin{array}{lcccc} \kappa^{\owedge} & : & R(b \ovee L(a)) \owedge m & \longrightarrow & R(b \ovee L(a \owedge m)) \\ \kappa^{\ovee} & : & L(a \owedge R(b \ovee n)) & \longrightarrow & L(a \owedge R(b)) \ovee n \end{array} \tag{29}$$

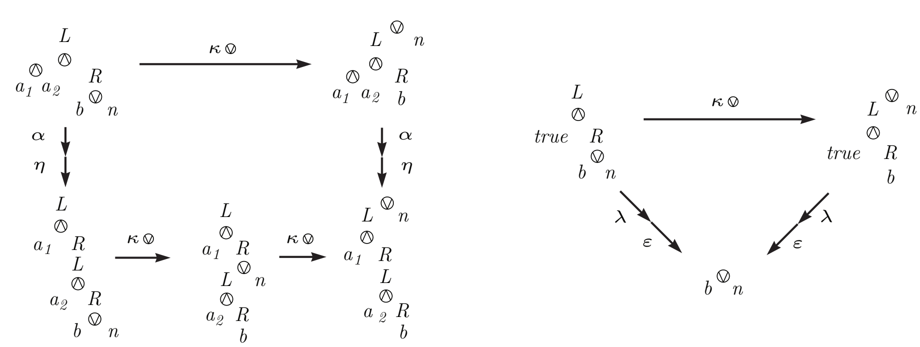
between

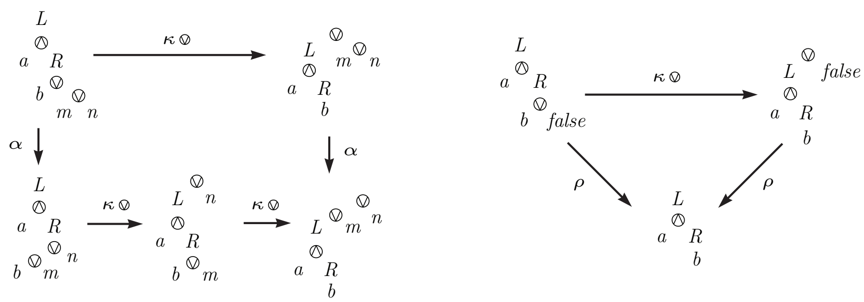
- the $\mathscr{B}$-modulation $T_b : a \mapsto R(L(b) \owedge a)$ and the $\mathscr{A}$-modulation $S_m : a \mapsto a \owedge m$ on the category $\mathscr{A}$,
- the $\mathscr{B}$-comodulation $K_n : b \mapsto b \ovee n$ and the $\mathscr{A}$-comodulation $L_a : b \mapsto L(a \ovee R(b))$ on the category $\mathscr{B}$.

Each definition of a bimodulation requires four diagrams to commute. We find useful to review the $2 \times 4 = 8$ diagrams in turn. First of all, the fact that $\kappa^{\owedge}$ defines a bimodulation means that the four diagrams

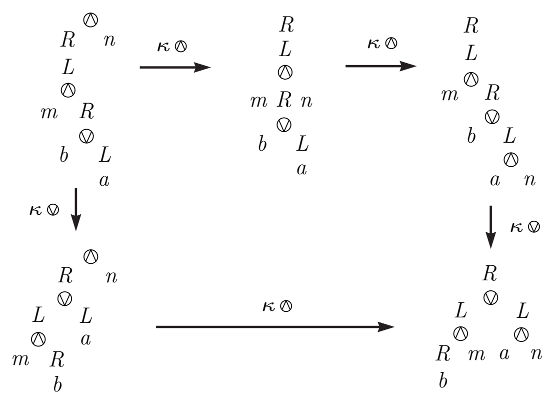


commute for all objects a, m, n in $\mathscr{A}$ and b, b_1, b_2 in $\mathscr{B}$. Symmetrically, the fact that $\kappa^{\ovee}$ defines a bimodulation means that the four diagrams

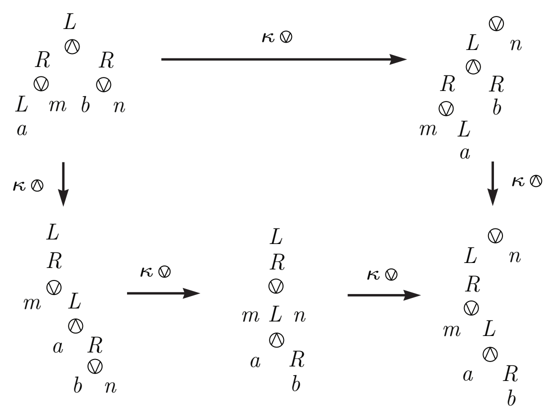




commute for all objects a, a_1, a_2 in $\mathscr{A}$ and b, m, n in $\mathscr{B}$. Besides this series of $2 \times 4 = 8$ coherence diagrams, we ask that the diagram



commutes for all objects a, m, n of the category $\mathscr{A}$ and all object b of the category $\mathscr{B}$. Symmetrically, we ask that its mirror image



commutes for any object a of the category $\mathscr{A}$ and all objects b, m, n of the category $\mathscr{B}$.

In contrast to the $2 \times 4 = 8$ coherence diagrams which are prescribed by the two bimodulations $\kappa^{\oslash}$ and $\kappa^{\otimes}$ — the algebraic nature of these last two coherence diagrams remains somewhat mysterious at the current stage. In particular, these two diagrams are not justified by the previous discussions on bimodulations since their purpose is precisely to relate the two bimodulations $\kappa^{\oslash}$ and $\kappa^{\otimes}$. We know (see §5.4) that every dialogue chirality on the right is equipped with two such bimodulations. It is not difficult to check that these last two diagrams are also satisfied by any dialogue chirality on the right. From this follows that

Proposition 8 *Every dialogue chirality on the right defines a linearly distributive chirality.*

This result leads to an important question, which is to understand what additional structure on a linearly distributive chirality turns it into a dialogue chirality on the right. The next two sections §6.2 and §6.3 are devoted to clarifying this specific point with the notion of right duality in a linearly distributive chirality.

6.2 Right duality in linearly distributive chiralities

A right duality in a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ is defined as a monoidal functor

$$\mathscr{A} \xrightarrow{\quad (-)^* \quad} \mathscr{B}^{op(0,1)}$$

as such equipped with natural isomorphisms

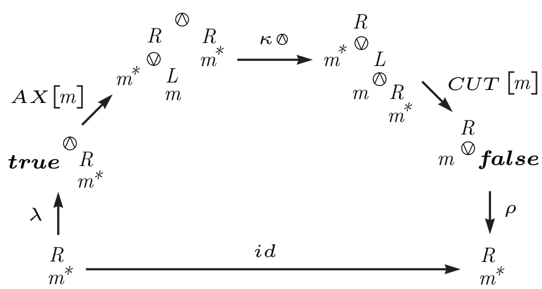
$$(a_1 \oslash a_2)^* \cong a_2^* \obslash a_1^* \qquad\qquad true^* \cong false$$

together with two families of morphisms

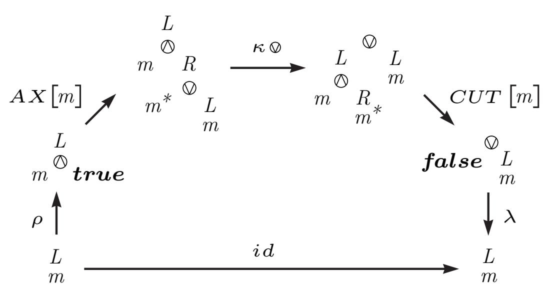
$$\mathbf{AX}[m] \quad : \quad true \quad \longrightarrow \quad R\,(\,m^* \obslash L\,(\,m\,)\,)$$

$$\mathbf{CUT}[m] \quad : \quad L\,(\,m \oslash R\,(\,m^*\,)\,) \quad \longrightarrow \quad false$$

each of them parametrized by the objects m of the category $\mathscr{A}$. These morphisms are required to make a series of 4×2 coherence diagrams commute, each of the four pairs consisting of a diagram and of its mirror image. The first pair of diagrams adapts the usual triangular axiom of adjunctions to the combinators **AX**[−] and **CUT**[−]:

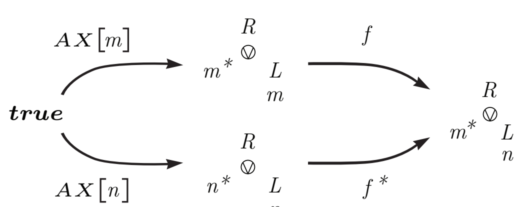


(30)

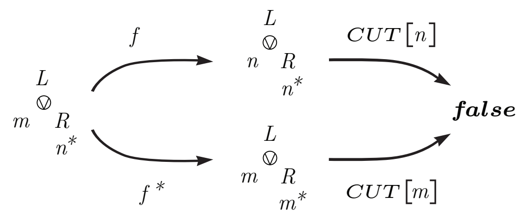


(31)

The second pair of coherence diagrams expresses that the two combinators **AX**[−] and **CUT**[−] are dinatural:

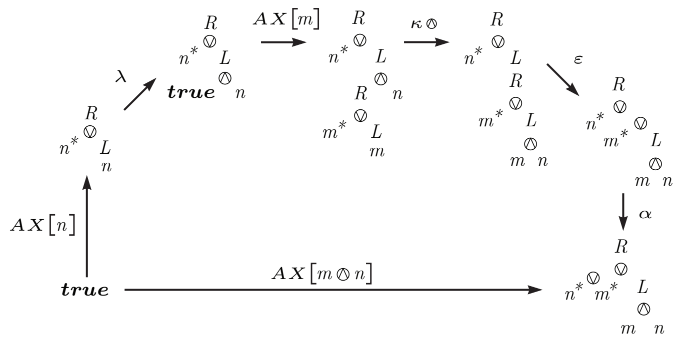


(32)

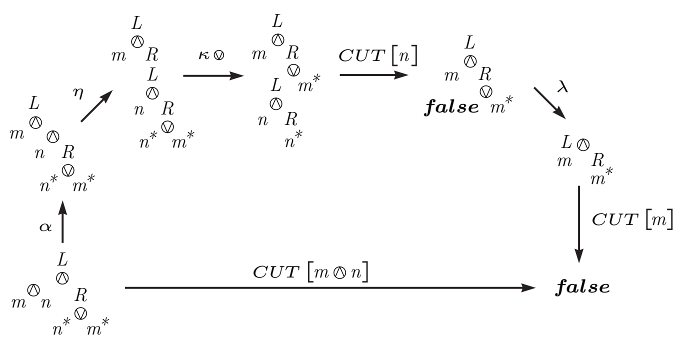


(33)

The third pair of coherence diagrams expresses that the combinators **AX**[−] and **CUT**[−] are monoidal:

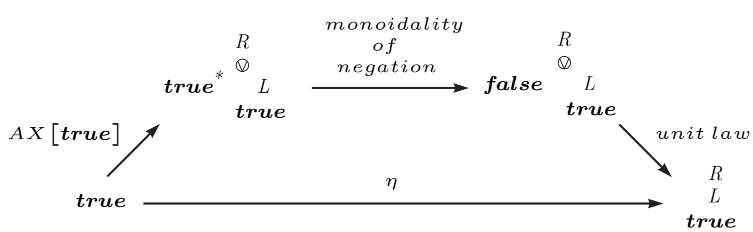


(34)

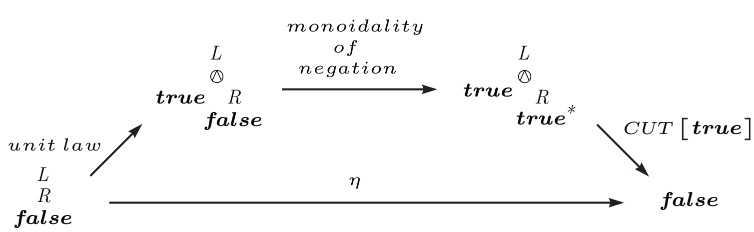


(35)

The fourth and last pair of coherence diagrams expresses that the combinators **AX**[−] and **CUT**[−] are monoidal, this time with respect to the units:



(36)



(37)

As expected, these diagrams are required to commute for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

6.3 Main statement

At this stage, we are ready to establish the main result of the paper which states that

Proposition 9 *Every linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ with a right duality defines a dialogue chirality on the left.*

This statement provides a natural justification for the two notions of linearly distributive chirality and of right duality, which have been carefully carved in order to establish the result. We start by establishing an important preliminary property:

Proposition 10 *In a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ equipped with a right duality, the functor*

$$L(m \oslash -) \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is left adjoint to the functor

$$R(m^* \obslash -) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$$

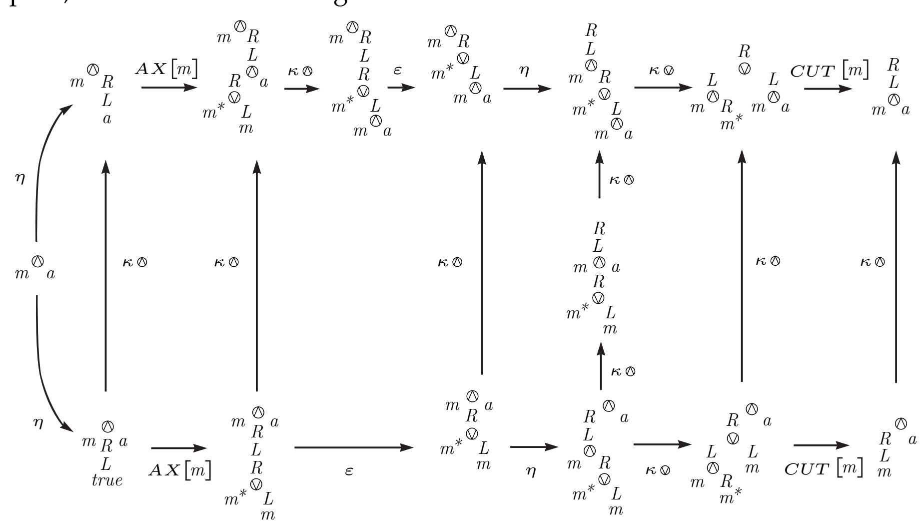
for every object m of the category $\mathscr{A}$. The family of adjunctions is moreover natural in the parameter m.

Proof. We have seen in Proposition 7 that such an adjunction may be alternatively formulated as a transjunction between $m \oslash -$ and $m^* \obslash -$ accross the adjunction $L \dashv R$. We find convenient to use this formulation here. Hence, given a linearly distributive chirality with a right duality, we introduce the two categorical combinators **axiom**[−] and **cut**[−] defined in such a way as to make the diagrams below commute:

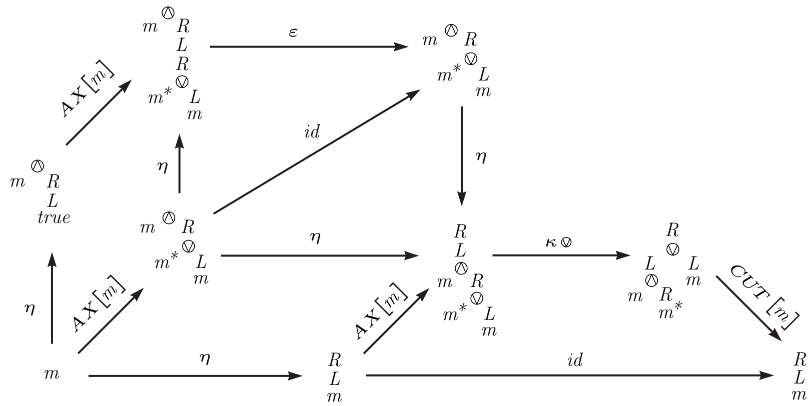
$$\begin{array}{ccc} L(R(m^* \obslash Lm) \oslash a) & \xrightarrow{\kappa^{\oslash}} & LR(m^* \obslash L(m \oslash a)) \\ \uparrow{\scriptstyle \mathbf{AX}[m]} & & \downarrow{\scriptstyle \varepsilon} \\ L(a) & \xrightarrow{\mathbf{axiom}[m]} & m^* \obslash L(m \oslash a) \end{array}$$

$$\begin{array}{ccc} RL(m \oslash R(m^* \obslash b)) & \xrightarrow{\kappa^{\obslash}} & R(L(m \oslash R(m^*)) \obslash b) \\ \uparrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \mathbf{CUT}[m]} \\ m \oslash R(m^* \obslash b) & \xrightarrow{\mathbf{cut}[m]} & R(b) \end{array}$$

Note that the very definition of the families **axiom**[−] and **cut**[−] ensures that they are natural in a and b respectively. In order to establish that this pair of combinators defines a transjunction, there simply remains to check that the two coherence diagrams (*a*) and (*b*) commute in (27). To that purpose, one constructs the diagram chase below

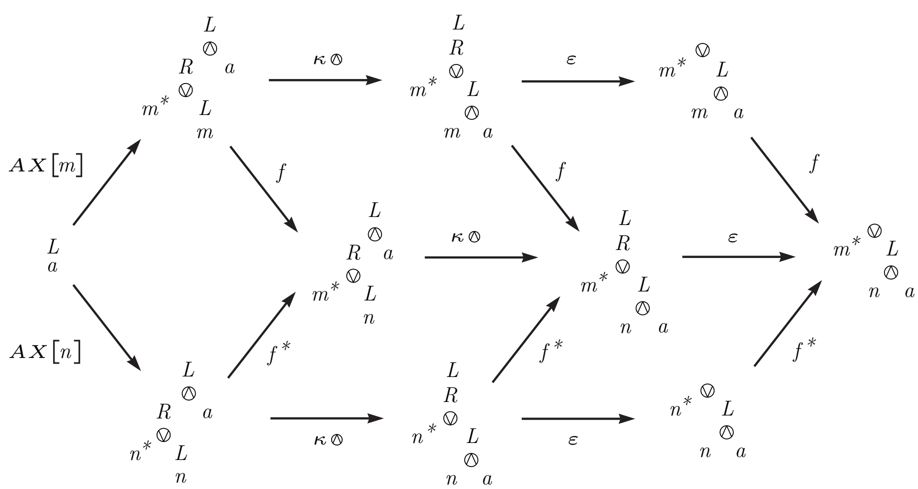


completed by the following one:

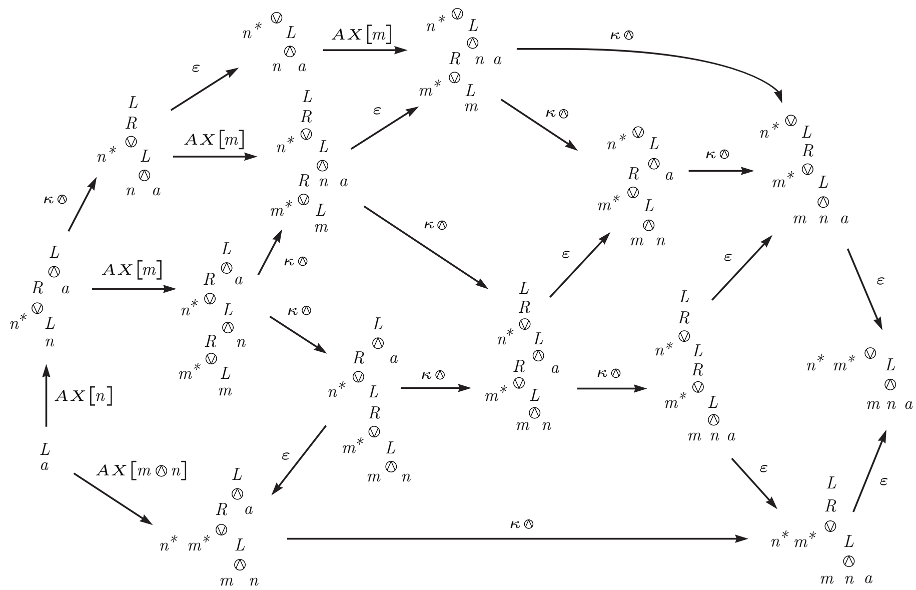


The reader will check that the two diagrams establish that the coherence diagram (*a*) commutes in (27). The fact that the other coherence diagram (*b*) commutes in (27) is proved in exactly the same way, by replacing each diagram chase by its mirror image. This proves that the pair of morphisms **axiom**[*m*] and **cut**[*m*] defines a transjunction accross the adjunction $L \dashv R$. By Proposition 7, this means that we have just constructed an adjunction between the functors $L(m \oslash -)$ and $R(m^* \varodot -)$. The fact that this family of adjunctions is natural in *m* boils down to the fact that the family of combinators **axiom**[−] and **cut**[−] are dinatural in *m*. This last assertion follows from the very definition of these combinators and from the requirement that the coherence diagrams (30) and (31) commute in the definition of a right duality. This concludes the proof of Proposition 10.

Proof of Proposition 9. At this stage, there remains to establish the two additional facts required by the alternative formulation of dialogue chiralities expressed in the statement of Proposition 6. This is achieved by the two diagram chases below, which establish that the family **axiom**[−] is dinatural in *m*:



and at the same time monoidal:



This concludes the proof that the family of transjunctions **axiom**[−] and **cut**[−] provides a dialogue chirality structure to the linearly distributive chirality $(\mathscr{A}, \mathscr{B})$.

Remark. It may come as a surprise to the careful reader that only four of the 4×2 coherence diagrams required of a right duality — namely equations (30), (31), (32) and (34) — are used in the proof of Proposition 9. We keep the four other coherence diagrams in the definition of a right duality because each of them makes perfect sense, either as the mirror image of another coherence diagram — in the case of (33) and (35) — or as the degenerate case for units of another coherence diagram — in the case of (36) and (37). On the other hand, it follows from Proposition 9 that the four coherence diagrams (33), (35), (36) and (37) are automatically valid when the coherence diagrams (30), (31), (32) and (34) hold in a linearly distributive chirality.

6.4 A comparison with linearly distributive categories

According to the general philosophy of depolarization, we should recover the notions of linearly distributive category (and of right duality) by depolarizing the corresponding notions in tensorial logic. In order to proceed with the comparison, we formally introduce the notion of *depolarized* linearly distributive chirality.

Definition 8 *A linearly distributive chirality is called* depolarized *when its two sides $\mathscr{A}$ and $\mathscr{B}$ are equal and when the two functors L and R are the identity functors.*

It is essentially immediate to check that

Proposition 11 *A linearly distributive category is the same thing as a depolarized linearly distributive chirality.*

The proof is based on a direct comparison of the coherence diagrams for linearly distributive category in §2.1 and for linearly distributive chirality in §6.1. It is easy to show that there is a one-to-one relationship between them, where each diagram §2.1 is obtained by depolarizing the corresponding diagram in §6.1. A somewhat less obvious statement is that

Proposition 12 *The notion of right duality in a linearly distributive category coincides with the notion of right duality in a depolarized linearly distributive chirality.*

Proof. Suppose given a linear distributive category $\mathscr{C}$ and the corresponding depolarized linearly distributive chirality $(\mathscr{C}, \mathscr{C})$. The easy direction of the proof is to show that every right duality in the sense of §6.2 for the chirality $(\mathscr{C}, \mathscr{C})$ defines a right duality in the sense of §2.2 for the category $\mathscr{C}$. The essential point to observe is that the two coherence diagrams (*a*) and (*b*) in Equation (11) are obtained by instantiating (or depolarizing) the coherence diagrams (30) and (31). The other direction is less obvious, but the argument has been carefully prepared in §2.2 and §2.5. Indeed, suppose given a right duality in the sense of §2.2 for the linearly distributive category $\mathscr{C}$. Then, we have established in §2.2 and §2.5 that $(-)^*$ defines a monoidal functor, and that all the coherence diagrams required by the definition of a right duality in the sense of §6.2 commute. This concludes the proof of Proposition 12.

7 Concluding remarks

Despite the work done in this paper, we are still halfway of the complete story. There is indeed a serious technical obstruction to a full resolution, which comes from the fact that we have chosen here to investigate the most primitive notion of dialogue category where the tensor product $\otimes$ is not equipped with any braiding or symmetry. This lack of structure implies that there are two *intrinsically different* choices of adjunctions $L \dashv R$. The first choice $L(x) = \bot \circ\!\!- x$ and $R(x) = x \multimap \bot$ induces a dialogue chirality on the left (see §3.1) whereas the second choice $L(x) = x \multimap \bot$ and $R(x) = \bot \circ\!\!- x$ induces a dialogue chirality on the right (see §5.3). Although this distinction between left and right dialogue chiralities is legitimate and even fundamental in tensorial logic, it induces a subtle mismatch in the present paper:

- on the one hand, we observe (see §5.4) that every dialogue chirality *on the right* defines a linearly distributive chirality *on the right*,
- then, we establish in Proposition 9 (see §6.3) that in the presence of a right duality, every linearly distributive chirality *on the right* defines a dialogue chirality *on the left* — rather than a dialogue chirality *on the right* as one would have naively expected...

This mismatch explains why we keep Proposition 9 (in §6.3) as unsatisfactory as it stands, and why we do not even attempt to establish that a dialogue chirality is *the same thing* as a linearly distributive chirality equipped with an appropriate notion of left and/or right duality. The difference of orientation between left and right dialogue chiralities is serious, and the plan would thus necessarily fail... A tentative solution would be to shift to an extended notion of dialogue category $\mathscr{C}$ with a *binary* rather than a *unary* negation, equipped with a two-sided version of currification

$$\varphi_{m,x,n} \quad : \quad \mathscr{C}(m \otimes x \otimes n, \bot) \quad \cong \quad \mathscr{C}(x, m \multimap \bot \circ\!\!- n)$$

The solution is nice and appealing, but we prefer to explore a more sophisticated direction, based on the idea that the left and the right negations $x \multimap \bot$ and $\bot \circ\!\!- x$ should coincide up to isomorphism. This track leads us to develop the notion of *helical dialogue category* where the two negations are related by a natural isomorphism

$$\mathbf{turn}_A \quad : \quad A \multimap \bot \quad \longrightarrow \quad \bot \circ\!\!- A$$

satisfying a carefully chosen coherence diagram. This helical structure leads us to a topological refinement of traditional game semantics, where the control flow of the tensorial net represented by the functors L and R may be "twisted" in the same way as a ribbon. Following the principle of depolarization, one recovers the familiar notion of cyclic proof-net when one specializes to the situation of a self-dual dialogue chirality where the two sides $\mathscr{A}$ and $\mathscr{B}$ coincide, and where the two functors L and R are the identity. These issues are treated in the companion paper [13].

However, we understand that the reader may find frustrating to have done such a long way in the paper without getting in the end a clear resolution of the questions raised in the introduction. We thus mention that a satisfactory answer exists in the specific case of a *symmetric* dialogue category. Let us start by the case of $*$-autonomous categories. Recall that a linearly distributive category $\mathscr{C}$ is called symmetric when its tensor product is equipped with a symmetry.

Proposition 13 (Cockett-Seely) *Every symmetric linearly distributive category $\mathscr{C}$ with a right duality defines a symmetric $*$-autonomous category.*

The proof is based on the observation that every right duality defines a left duality (see §2.4) in the symmetric case, with combinators defined as:

$$\overline{\mathbf{AX}}[x] \;:\; \mathbf{1} \xrightarrow{\mathbf{AX}[x]} x^* \parr x \xrightarrow{symmetry} x \parr x^* \qquad\qquad \overline{\mathbf{CUT}}[x] \;:\; x^* \otimes x \xrightarrow{symmetry} x \otimes x^* \xrightarrow{\mathbf{CUT}[x]} \bot$$

From this follows that every symmetric linearly distributive category $\mathscr{C}$ with a right duality is also equipped with a left duality — and thus defines a $*$-autonomous category by Proposition 3. In fact, one may show that the two notions of symmetric $*$-autonomous category and of symmetric linearly distributive category $\mathscr{C}$ with a duality (either left or right) are equivalent in the appropriate 2-categorical sense.

This result on $*$-autonomous categories may be adapted to the case of dialogue chiralities in the original sense of [11]. We declare that such a dialogue chirality $(\mathscr{A}, \mathscr{B})$ is *symmetric* when the monoidal categories $(\mathscr{A}, \oslash, true)$ and $(\mathscr{B}, \obslash, false)$ are both equipped with a symmetry, and when the original equivalence of monoidal categories (10) is an equivalence of *symmetric* monoidal categories:

$$\mathscr{A} \;\underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}}\; \mathscr{B}^{op(0,1)} \quad (\text{sym.monoidal equivalence}) \tag{38}$$

Similarly, we declare that a linearly distributive chirality $\mathscr{C}$ is *symmetric* when the monoidal categories $(\mathscr{A}, \oslash, true)$ and $(\mathscr{B}, \obslash, false)$ are both equipped with a symmetry. A right duality $a \mapsto a^*$ on such a symmetric linearly distributive chirality is called *complemented* when it comes with a functor $b \mapsto {}^*b$ and an equivalence (38) of symmetric monoidal categories. The key observation in that case is that:

Proposition 14 *Every symmetric linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ equipped with a complemented right duality defines a symmetric dialogue chirality.*

In fact, one could show that the notion of symmetric dialogue chirality is equivalent (in the appropriate 2-categorical sense) to the notion of symmetric linearly distributive chirality equipped with a complemented right duality. This closes the circle in the case of symmetric dialogue categories since a straightforward adaptation of [11] establishes that the notion of symmetric dialogue chirality is itself equivalent to the notion of symmetric dialogue category, in an appropriate 2-categorical sense.

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