An algebraic presentation of dialogue categories

Paul-André Melliès*

August 22, 2012

Abstract

In this paper, we describe an algebraic presentation of the notion of helical dialogue chirality. In particular, the helix structure enables us to decompose the dual of the left negation as the right negation of the dual.

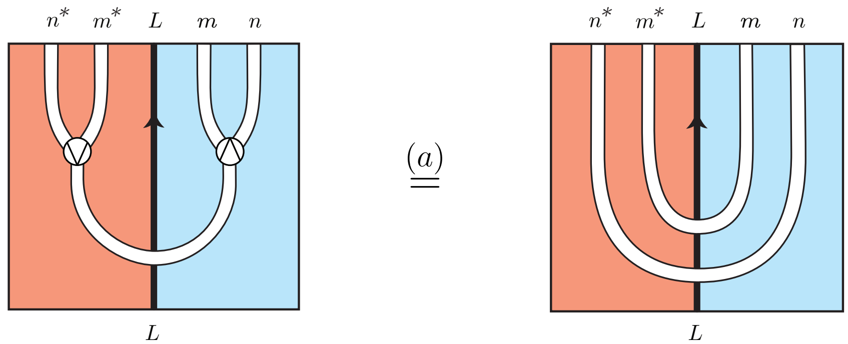
1 Motivations

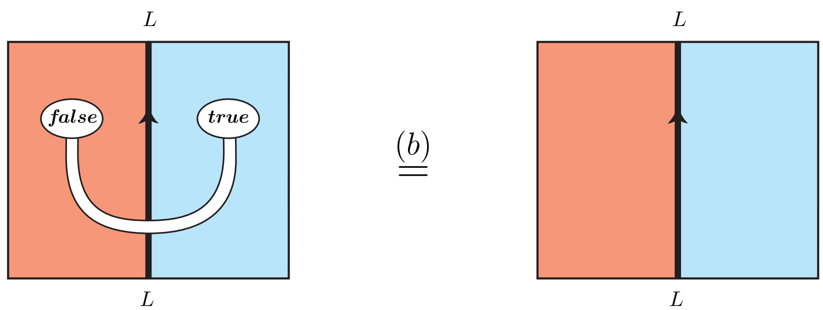
The study of dialogue categories and chiralities leads to the following coherence diagrams for the axiom combinator:

$$\begin{array}{ccc}
L(a \owedge m) \ovee m^* & \xrightarrow{\;\mathbf{axiom}[n]\;} & (L((a \owedge m) \owedge n) \ovee n^*) \ovee m^* \\
 & & \big\downarrow \scriptstyle{associativity} \\
\mathbf{axiom}[m] \big\uparrow & & L(a \owedge (m \owedge n)) \ovee (n^* \ovee m^*) \\
 & & \big\downarrow \scriptstyle{monoidality} \\
La & \xrightarrow{\;\mathbf{axiom}[m \owedge n]\;} & L(a \owedge (m \owedge n)) \ovee (m \owedge n)^*
\end{array}$$

commutes for all objects a, m, n and morphisms $f : m \to n$ of the category $\mathscr{A}$. In string diagrams:

*CNRS, Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France. This work has been partly supported by the ANR Project RECRE.





From the logical point of view, the two coherence diagrams **(??)** and **(??)** should be understood as η-expansion laws for the axiom link. Typically, the purpose of the η-expansion law **(??)** is to decompose the link **axiom**$[m \owedge n]$ into the more elementary links **axiom**$[m]$ and **axiom**$[n]$. A natural question is whether there exists a similar η-expansion law which decomposes the axiom link

$$La \xrightarrow{\quad \mathbf{right.axiom}[Rm] \quad} L(a \owedge Rm) \ovee (Rm)^*$$

associated to the negation of m into the axiom link associated to the object m of the category $\mathscr{B}$. To that purpose, it appears necessary to start from the *left* axiom link

$$L(\mathbf{true}) \xrightarrow{\quad \mathbf{left.axiom}[{}^*m] \quad} ({}^*m)^* \ovee L({}^*m) \xrightarrow{\quad equivalence \quad} m \ovee L({}^*m)$$

associated to the object ${}^*m$ living this time in the category $\mathscr{A}$. Then, the helical structure

$$L({}^*m) \xrightarrow{\quad isomorphism \quad} (Rm)^*$$

This defines a morphism

$$La \xrightarrow{\ \eta\ } L(a \owedge RL(\mathbf{true})) \xrightarrow{\quad map \quad} L(a \owedge R(m \ovee (Rm)^*))$$

At this point, we want a map

$$L(a \owedge R(m \ovee (Rm)^*)) \xrightarrow{\quad distributivity \quad} L(a \owedge Rm) \ovee (Rm)^*$$

This requires to develop a general theory of these distributivity laws, describing the coherence diagrams.

2 Helical dialogue categories and chiralities

We construct a 2-category of helical dialogue categories, helical functors and helical natural transformations.

2.1 A 2-category of helical dialogue categories

We define a 2-category ***HelCat*** with

- helical dialogue categories as 0-cells,
- helical functors as 1-cells,
- dialogue natural transformations as 2-cells.

The 0-dimensional cells. Recall from [6] that a helical dialogue category $\mathscr{C}$ is defined as a dialogue category equipped with a family of bijections

$$wheel_{x,y} \quad : \quad \mathscr{C}(x \otimes y, \bot) \quad \longrightarrow \quad \mathscr{C}(y \otimes x, \bot)$$

natural in x and y and required to make the diagram

$$\begin{array}{ccc}
\mathscr{C}((y \otimes z) \otimes x, \bot) & \xrightarrow{\ associativity\ } & \mathscr{C}(y \otimes (z \otimes x), \bot) \\
\uparrow {\scriptstyle wheel_{x,y\otimes z}} & & \downarrow {\scriptstyle wheel_{y,z\otimes x}} \\
\mathscr{C}(x \otimes (y \otimes z), \bot) & & \mathscr{C}((z \otimes x) \otimes y, \bot) \\
\downarrow {\scriptstyle associativity} & & \uparrow {\scriptstyle associativity} \\
\mathscr{C}((x \otimes y) \otimes z, \bot) & \xrightarrow{\ wheel_{x\otimes y,z}\ } & \mathscr{C}(z \otimes (x \otimes y), \bot)
\end{array} \tag{1}$$

commute for all objects x, y, z of the category $\mathscr{C}$.

The 1-dimensional cells. A helical functor between two helical dialogue categories is a lax monoidal functor

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

equipped with a morphism

$$\bot_F \quad : \quad F(\bot) \quad \longrightarrow \quad \bot$$

such that the diagram

$$\begin{array}{ccccc}
\mathscr{C}(x \otimes y, \bot) & \xrightarrow{F} & \mathscr{D}(F(x \otimes y), F(\bot)) & \xrightarrow{coercion} & \mathscr{D}(F(x) \otimes F(y), \bot) \\
\Big\downarrow{\scriptstyle wheel_{x,y}} & & & & \Big\downarrow{\scriptstyle wheel_{F(x),F(y)}} \\
\mathscr{C}(y \otimes x, \bot) & \xrightarrow{F} & \mathscr{D}(F(y \otimes x), F(\bot)) & \xrightarrow{coercion} & \mathscr{D}(F(y) \otimes F(x), \bot)
\end{array}$$

commutes for all objects x, y of the category $\mathscr{C}$. In this diagram, the two coercion maps are deduced by precomposing with the lax monoidal structure of the functor F

$$m_{x,y} \quad : \quad F(x) \otimes F(y) \quad \longrightarrow \quad F(x \otimes y)$$

and by postcomposing with the map $\bot_F$.

The 2-dimensional cells. A dialogue natural transformation

$$\theta \quad : \quad (F, \bot_F) \quad \Rightarrow \quad (G, \bot_G)$$

is defined as a natural transformation

$$\theta \quad : \quad F \quad \Rightarrow \quad G$$

making the diagram

$$\begin{array}{ccc}
F(\bot) & & \\
\Big\downarrow{\scriptstyle \theta_\bot} & \searrow^{\bot_F} & \\
 & & \bot \\
 & \nearrow_{\bot_G} & \\
G(\bot) & &
\end{array}$$

commute. The composition law and identities of the 2-category ***HelCat*** are defined as expected.

2.2 A 2-category of helical dialogue chiralities

We construct a 2-category ***HelChir*** with helical dialogue chiralities as 0-cells.

The 0-dimensional cells.

Definition 1 (Helical chirality) *A helical chirality is a dialogue chirality equipped with a family of bijections*

$$\sigma_{a,b} \quad : \quad \langle\, a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, {}^{*}b \,|\, a^{*} \,\rangle$$

natural in a, b*, and making the diagram below commute:*

$$\begin{array}{ccccccc}
\langle\, a_1 \owedge a_2 \,|\, b \,\rangle & \xrightarrow{\chi^{R}_{a_2,a_1,b}} & \langle\, a_1 \,|\, b \ovee a_2^{*} \,\rangle & \xrightarrow{\sigma} & \langle\, {}^{*}(b \ovee a_2^{*}) \,|\, a_1^{*} \,\rangle & = & \langle\, a_2 \owedge {}^{*}b \,|\, a_1^{*} \,\rangle \\
\Big\downarrow{\scriptstyle \sigma} & & & & & & \Big\downarrow{\scriptstyle \chi^{R}_{{}^{*}b,a_2,a_1}} \\
\langle\, {}^{*}b \,|\, (a_1 \owedge a_2)^{*} \,\rangle & & & & & & \langle\, a_2 \,|\, a_1^{*} \ovee ({}^{*}b)^{*} \,\rangle \\
\| & & & & & & \| \\
\langle\, {}^{*}b \,|\, a_2^{*} \ovee a_1^{*} \,\rangle & \xleftarrow{\chi^{R}_{a_1,{}^{*}b,a_2^{*}}} & \langle\, {}^{*}b \owedge a_1 \,|\, a_2^{*} \,\rangle & = & \langle\, {}^{*}(a_1^{*} \ovee b) \,|\, a_2^{*} \,\rangle & \xleftarrow{\sigma} & \langle\, a_2 \,|\, a_1^{*} \ovee b \,\rangle
\end{array} \tag{2}$$

The 1-dimensional cells. A 1-dimensional cell in ***HelChir***

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a quadruple $F = (F_{\bullet}, F_{\circ}, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor $F_{\bullet} : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_{\circ} : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism

$$\begin{array}{ccc}
\mathscr{A}_1 & \xrightarrow{\quad F_{\bullet} \quad} & \mathscr{A}_2 \\
\Big\downarrow{\scriptstyle (-)^{*}} & \widetilde{F} \;\Swarrow & \Big\downarrow{\scriptstyle (-)^{*}} \\
\mathscr{B}_1^{op(0,1)} & \xrightarrow[\quad F_{\circ}^{op(0,1)} \quad]{} & \mathscr{B}_2^{op(0,1)}
\end{array} \tag{3}$$

together with a natural transformation:

$$
\begin{array}{ccc}
\mathscr{A}_1 & \xrightarrow{\;F_\bullet\;} & \mathscr{A}_2 \\
{\scriptstyle R}\uparrow & \overset{\overline{F}}{\Rightarrow} & \uparrow{\scriptstyle R} \\
\mathscr{B}_1 & \xrightarrow[\;F_\circ\;]{} & \mathscr{B}_2
\end{array}
\tag{4}
$$

making the two diagrams

$$
\begin{array}{ccc}
\langle\, a \owedge m \,|\, b \,\rangle & \xrightarrow{\;\chi_m\;} & \langle\, a \,|\, b \ovee m^* \,\rangle \\
\downarrow{\scriptstyle F_{a\owedge m,b}} & & \downarrow{\scriptstyle F_{a,b\ovee m^*}} \\
\langle\, F_\bullet(a \owedge m) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(b \ovee m^*) \,\rangle \\
 & & \downarrow{\scriptstyle \textit{monoidality of } F_\circ} \\
\downarrow{\scriptstyle \textit{monoidality of } F_\bullet} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \ovee F_\circ(m^*) \,\rangle \\
 & & \downarrow{\scriptstyle \widetilde{F}} \\
\langle\, F_\bullet(a) \owedge F_\bullet(m) \,|\, F_\circ(b) \,\rangle & \xrightarrow{\;\chi_{F_\bullet(m)}\;} & \langle\, F_\bullet(a) \,|\, F_\circ(b) \ovee F_\bullet(m)^* \,\rangle
\end{array}
\tag{5}
$$

$$
\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\;F_{a,b}\;} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle \\
\downarrow{\scriptstyle \sigma_{a,b}} & & & & \downarrow{\scriptstyle \sigma_{F_\bullet(a),F_\circ(b)}} \\
\langle\, {}^*b \,|\, a^* \,\rangle & \xrightarrow{\;F_{{}^*b,a^*}\;} & \langle\, F_\bullet({}^*b) \,|\, F_\circ(a^*) \,\rangle & \xrightarrow{\;\widetilde{F}\;} & \langle\, {}^*(F_\circ(b)) \,|\, (F_\bullet(a))^* \,\rangle
\end{array}
\tag{6}
$$

commute for all objects a, m in $\mathscr{A}_1$ and b in $\mathscr{B}_1$. Here, the map

$$F_{a,b} \quad : \quad \langle\, a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle$$

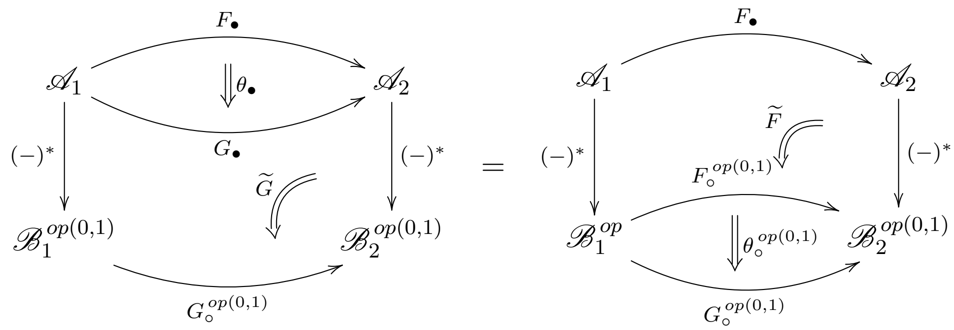
is defined as the composite

$$
\begin{array}{ccccc}
\langle\, a \,|\, b \,\rangle & & \xrightarrow{\;F_{a,b}\;} & & \langle\, F_\bullet(a) \,|\, F_\circ(b) \,\rangle \\
\| & & & & \| \\
\mathscr{A}_1(a, Rb) & \xrightarrow{\;F_\bullet\;} & \mathscr{A}_2(F_\bullet(a), F_\bullet(Rb)) & \xrightarrow{\;\overline{F}\;} & \mathscr{A}_2(F_\bullet(a), RF_\circ(b))
\end{array}
$$

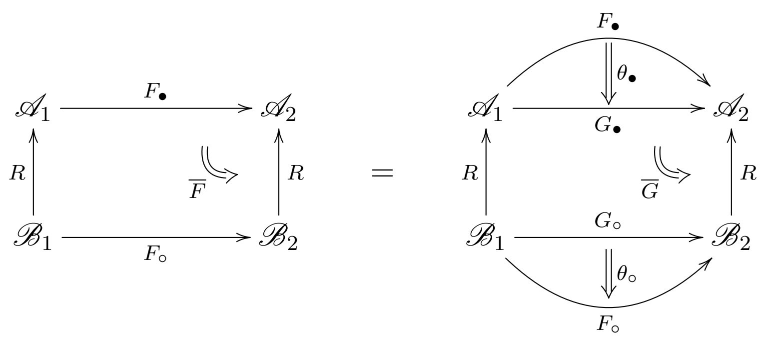
The 2-dimensional cells. A 2-dimensional cell in ***HelChir***

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations $\theta_\bullet : F_\bullet \Rightarrow G_\bullet$ and $\theta_\circ : G_\circ \Rightarrow F_\circ$ satisfying the two equations below:



(7)



(8)

2.3 Equivalence

The 2-functor induces a biequivalence between the 2-categories ***HelCat*** and ***HelChir***.

3 Helical chiralities revisited

In this section, we study another formulation of helical chiralities.

3.1 Helical dialogue chiralities (bis)

Definition 2 (Helical chirality bis) *A helical chirality is a pair of monoidal categories*

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with a monoidal equivalence

$$\mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \qquad \text{(monoidal equivalence)}$$

with two families of bijections

$$\begin{array}{lcccc} \chi^R_{m,a,b} & : & \langle\, a \owedge m \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, b \ovee m^* \,\rangle \\ \chi^L_{m,a,b} & : & \langle\, m \owedge a \,|\, b \,\rangle & \longrightarrow & \langle\, a \,|\, m^* \ovee b \,\rangle \end{array}$$

natural in a, b and m, where

$$\langle\, - \,|\, - \,\rangle \quad = \quad \mathscr{A}(\,-\,,\; R(-)\,) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

The families χ^L and χ^R are moreover required to make the diagrams below commute:

$$\begin{array}{ccccc} \langle\, a \owedge (m \owedge n) \,|\, b \,\rangle & & \xrightarrow{\quad \chi^R_{m \owedge n} \quad} & & \langle\, a \,|\, b \ovee (m \owedge n)^* \,\rangle \\ \downarrow{\scriptstyle associativity} & & & & \uparrow{\scriptstyle associativity,\ monoidality\ of\ negation} \\ \langle\, (a \owedge m) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \owedge m \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^R_m} & \langle\, a \,|\, (b \ovee n^*) \ovee m^* \,\rangle \end{array} \tag{9}$$

$$\begin{array}{ccccc} \langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad \chi^L_{m \owedge n} \quad} & & \langle\, a \,|\, (m \owedge n)^* \ovee b \,\rangle \\ \downarrow{\scriptstyle associativity} & & & & \uparrow{\scriptstyle associativity,\ monoidality\ of\ negation} \\ \langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, n \owedge a \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^L_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b) \,\rangle \end{array} \tag{10}$$

together with the additional coherence diagram between χ^L and χ^R:

$$\begin{array}{ccccc} \langle\, (m \owedge a) \owedge n \,|\, b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, m \owedge a \,|\, b \ovee n^* \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \,|\, m^* \ovee (b \ovee n^*) \,\rangle \\ \vert{\scriptstyle associativity} & & & & \vert{\scriptstyle associativity} \\ \langle\, m \owedge (a \owedge n) \,|\, b \,\rangle & \xrightarrow{\chi^L_m} & \langle\, a \owedge n \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi^R_n} & \langle\, a \,|\, (m^* \ovee b) \ovee n^* \,\rangle \end{array} \tag{11}$$

Proposition 1 *The two notions of helical chirality formulated in Definitions 1 and 2 are equivalent.*

3.2 The 2-category of helical dialogue categories

The 1-dimensional cells. The helical functors may be reformulated in this style. A helical functor is defined as a quadruple $F = (F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor $F_\bullet : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$, an oplax monoidal functor $F_\circ : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$, a monoidal natural isomorphism (3) and a natural isomorphism (4) making the diagram (5) commute for $\chi = \chi^R$ together with the corresponding diagram for the left currification χ^L, given below:

$$
\begin{array}{ccc}
\langle\, m \owedge a \,|\, b \,\rangle & \xrightarrow{\quad \chi^L_m \quad} & \langle\, a \,|\, m^* \ovee b \,\rangle \\
\Big\downarrow{\scriptstyle F_{m \owedge a, b}} & & \Big\downarrow{\scriptstyle F_{a, m^* \ovee b}} \\
\langle\, F_\bullet(m \owedge a) \,|\, F_\circ(b) \,\rangle & & \langle\, F_\bullet(a) \,|\, F_\circ(m^* \ovee b) \,\rangle \\
 & & \Big\downarrow{\scriptstyle monoidality\ of\ F_\circ} \\
\Big\downarrow{\scriptstyle monoidality\ of\ F_\bullet} & & \langle\, F_\bullet(a) \,|\, F_\circ(m^*) \ovee F_\circ(b) \,\rangle \\
 & & \Big\downarrow{\scriptstyle \widetilde{F}} \\
\langle\, F_\bullet(m) \owedge F_\bullet(a) \,|\, F_\circ(b) \,\rangle & \xrightarrow{\quad \chi^L_{F_\bullet(m)} \quad} & \langle\, F_\bullet(a) \,|\, F_\bullet(m)^* \ovee F_\circ(b) \,\rangle
\end{array}
\tag{12}
$$

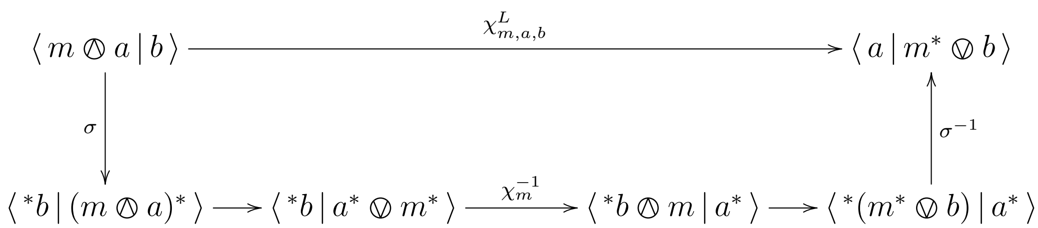
The 2-dimensional cells. The 2-dimensional cells are defined as previously for the 2-category ***HelChir***.

3.3 Proof of isomorphism

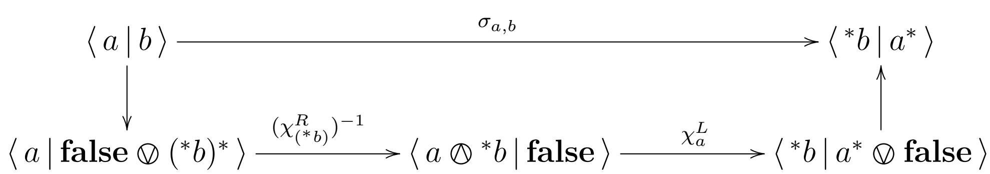
In order to clarify the comparison between the two definitions of helical dialogue chirality, we decide to call **Def-a** the original definition 1 and **Def-b** its alternative but equivalent formulation in Proposition 1.

Def-a $\Rightarrow$ Def-b. Every helical dialogue category in the sense of Definition 1 is equipped with a natural bijection σ, In order to obtain a helical dialogue category in the sense of Proposition 1, one defines the natural bijection χ^R as χ and the natural bijection χ^L as the family of bijections $\chi^L_{m,a,b}$

as the composite morphism:



Def-b $\Rightarrow$ **Def-a.** Conversely, given a helical dialogue chirality in the sense of Proposition 1, the natural bijection σ is defined as the unique family of bijections $\sigma_{a,b}$ making the diagram below commute:



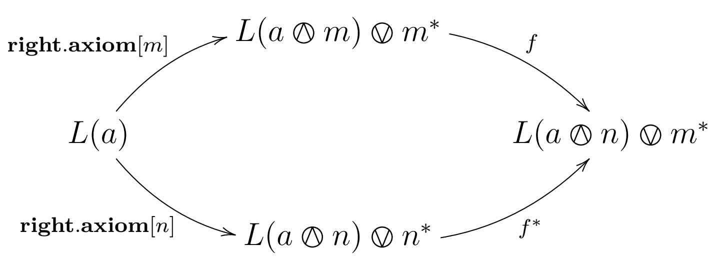
A series of chase diagrams establish that the relationship is one-to-one.

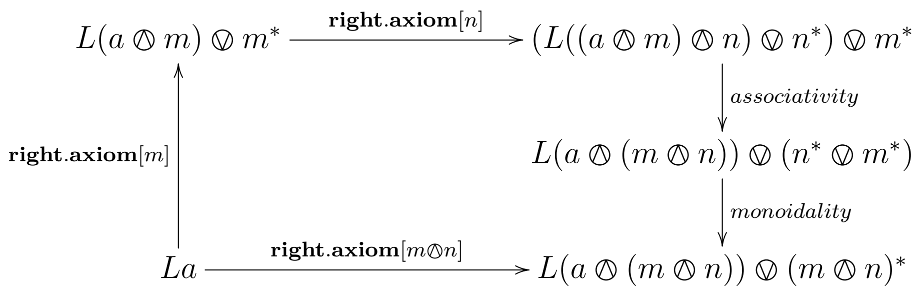
4 Helical chiralities by transjunctions

Construire une 2-categorie de nouveau, et montrer le lien avec la dualite.

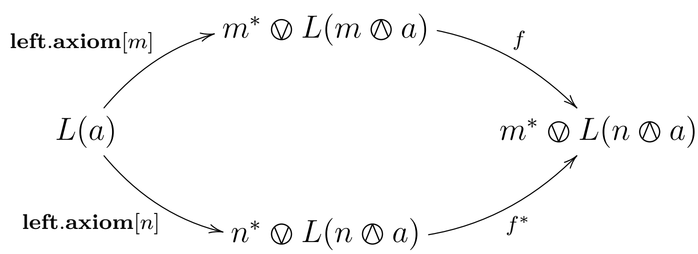
4.1 Definition

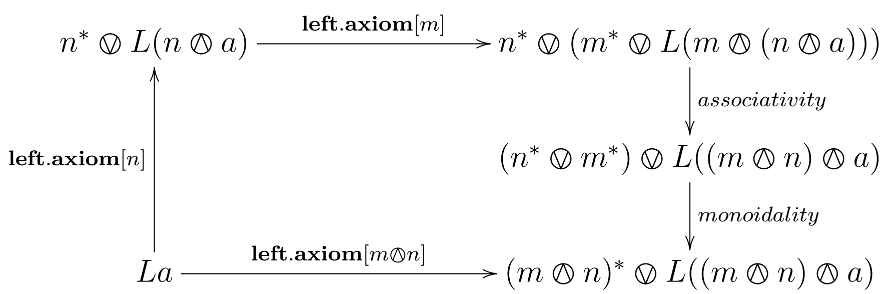
It is not difficult to deduce a formulation of helical categories based on transjunctions. This starts with the two coherence diagrams for the right axiom combinator



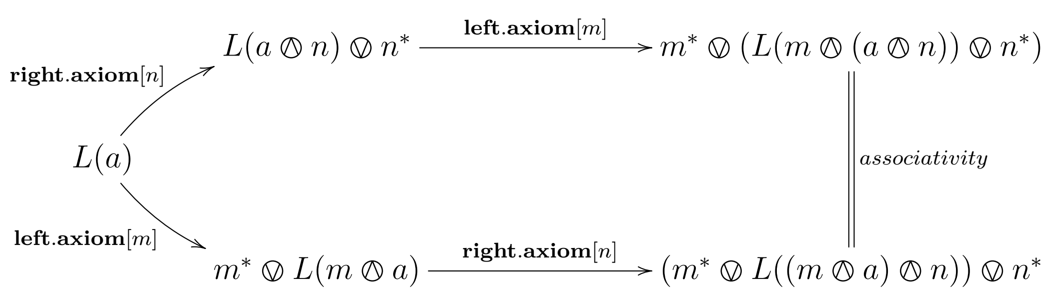


followed by the two coherence diagrams for the left axiom combinator

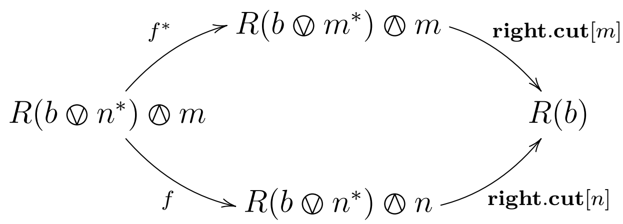


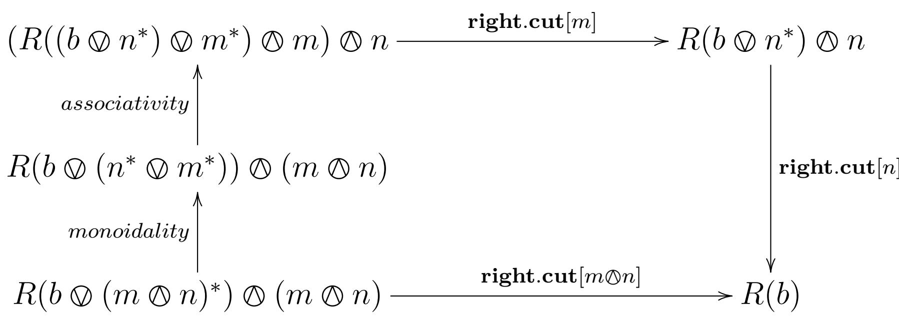


for every morphism $f : m \longrightarrow n$ of the category $\mathscr{A}$. Finally, the important additional coherence diagram tells that the tensor products permute:

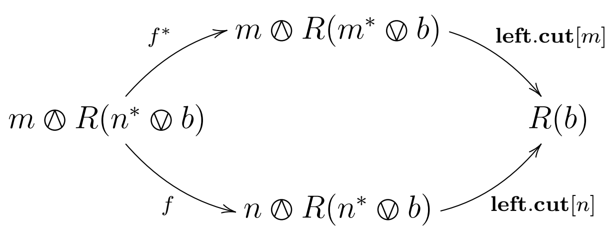


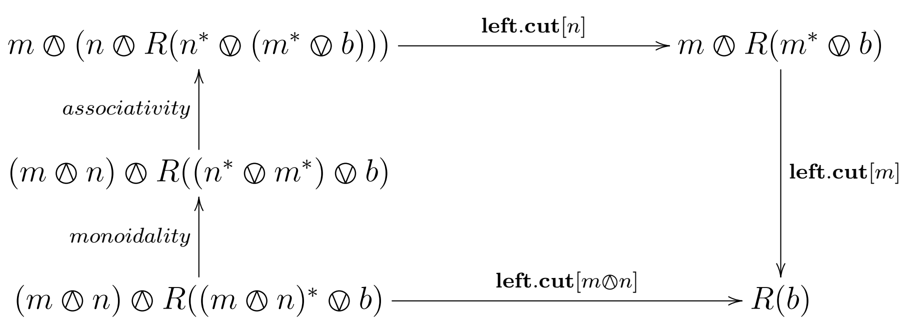
4.2 Cut





followed by the two coherence diagrams for the left cut combinator



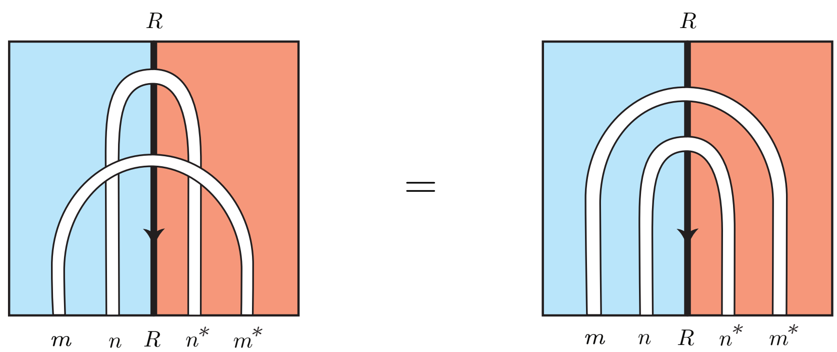


for every morphism $f : m \longrightarrow n$ of the category $\mathscr{A}$. Finally, the important additional coherence diagram tells that the tensor products permute:

$$
\begin{array}{ccc}
(m \owedge R(m^* \ovee (a \ovee n^*))) \owedge n & \xrightarrow{\mathbf{left}.\mathbf{cut}[m]} & n^* \ovee R(n \owedge a) \\
\Big| & & \Big\downarrow {\scriptstyle \mathbf{right}.\mathbf{cut}[n]} \\
{\scriptstyle associativity} & & R(b) \\
\Big| & & \Big\uparrow {\scriptstyle \mathbf{left}.\mathbf{cut}[m]} \\
m \owedge (R((m^* \ovee a) \ovee n^*) \owedge n) & \xrightarrow{\mathbf{right}.\mathbf{cut}[n]} & m^* \ovee R(m \owedge a)
\end{array}
\qquad (13)
$$

4.3 Graphically

The coherence diagram (13) is depicted as



4.4 The 2-category

The 1-dimensional cells. A homorphism of helical dialogue chiralities is defined

5 Eta law for negation

5.1 Flexible negation

The following equation holds when the negation is twist-free +

$$
\begin{array}{ccc}
{}^*(R(\mathbf{false})) \ovee L(R(\mathbf{false}) \owedge \mathbf{true}) & \longrightarrow & L(\mathbf{true}) \ovee L(R(\mathbf{false})) \\
\uparrow{\scriptstyle axiom-left} & & \downarrow{\scriptstyle \varepsilon} \\
L(\mathbf{true}) & \xrightarrow{\quad id \quad} & L(\mathbf{true}) \\
\downarrow{\scriptstyle axiom-right} & & \uparrow{\scriptstyle \varepsilon} \\
L(\mathbf{true} \owedge R(\mathbf{false})) \ovee {}^*(R(\mathbf{false})) & \longrightarrow & L(R(\mathbf{false})) \ovee L(\mathbf{true})
\end{array}
$$

6 Discursive pairs

6.1 Definition

A discursive pair is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

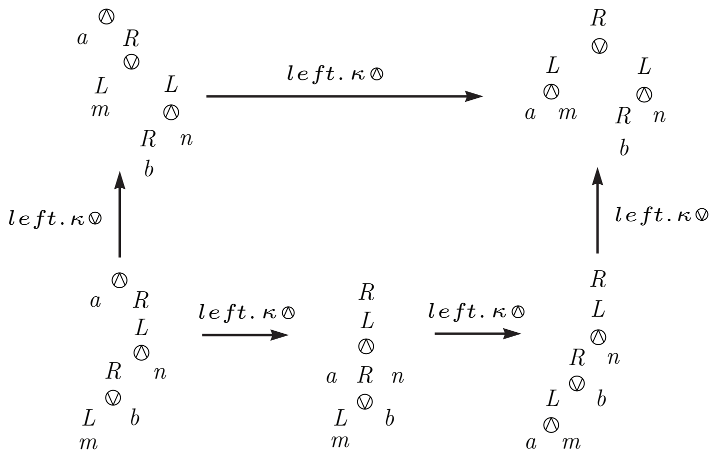
equipped with an adjunction

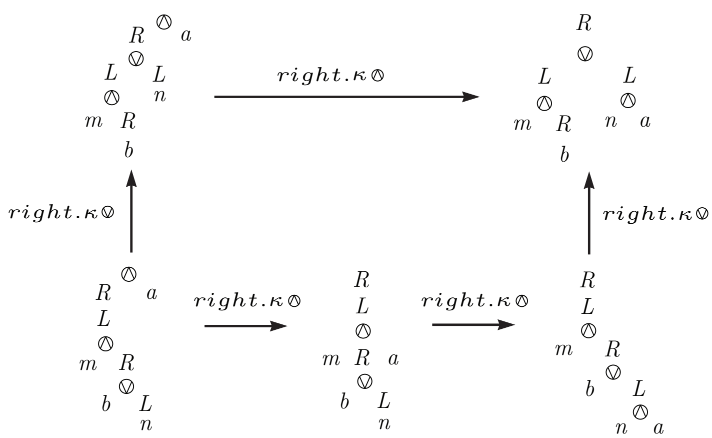
$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \qquad (L \dashv R)$$

together with the four bimonads

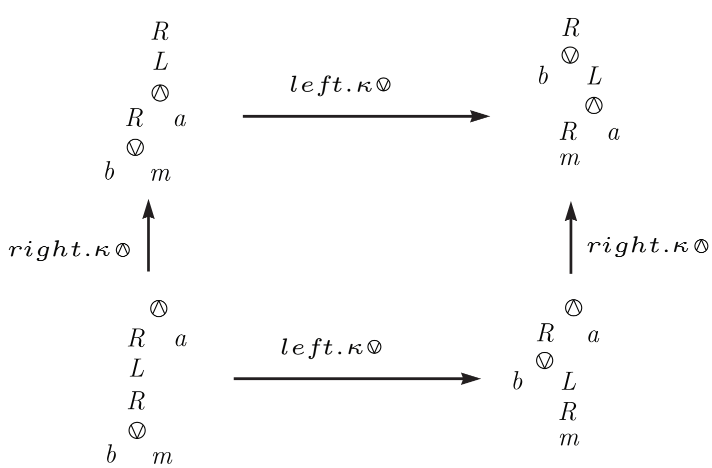
$$
\begin{array}{rcccc}
\mathbf{left}.\kappa^{\owedge} & : & m \owedge R(L(a) \ovee b) & \longrightarrow & R(L(m \owedge a) \ovee b) \\
\mathbf{left}.\kappa^{\ovee} & : & L(R(n \ovee b) \owedge a) & \longrightarrow & n \ovee L(R(b) \owedge a) \\
\mathbf{right}.\kappa^{\owedge} & : & R(b \ovee L(a)) \owedge m & \longrightarrow & R(b \ovee L(a \owedge m)) \\
\mathbf{right}.\kappa^{\ovee} & : & L(a \owedge R(b \ovee n)) & \longrightarrow & L(a \owedge R(b)) \ovee n
\end{array} \tag{14}
$$

between the $\owedge$-tensor product and the $\mathscr{B}$-monad of $\mathscr{A}$ on the one hand, and between the $\ovee$-tensor product and the $\mathscr{A}$-comonad of $\mathscr{B}$ on the other hand. Besides the resulting series of commutative diagrams, we ask that the two diagrams below commute





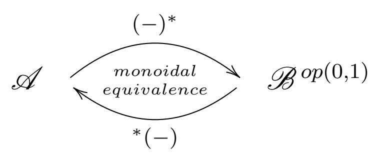
for all objects a, m, n of the category $\mathscr{A}$ and all object b of the category $\mathscr{B}$. Plus a series of other diagrams required in the proof of the following lemma.



Note that these coherence diagrams are not justified by any of the previous discussions.

7 Dualities

A duality in a discursive pair $(\mathscr{A}, \mathscr{B})$ is defined as a monoidal equivalence



together with four families of morphisms

$$\mathbf{right}.\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R(L(m) \ovee m^*)$$

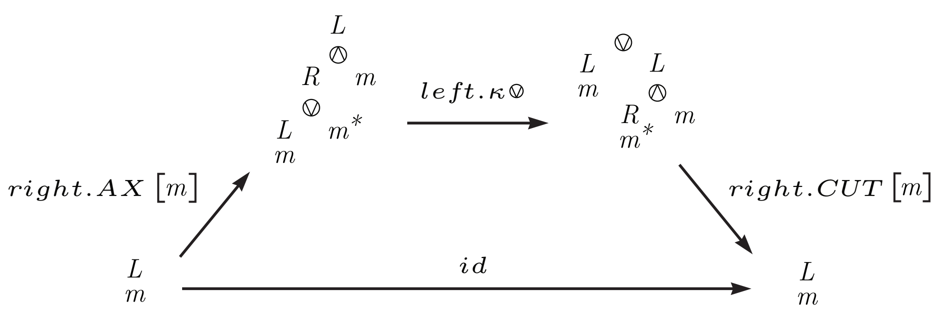
$$\mathbf{right}.\mathbf{CUT}[m] \quad : \quad L(R(m^*) \owedge m) \quad \longrightarrow \quad \mathbf{false}$$

$$\mathbf{left}.\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R(m^* \ovee L(m))$$

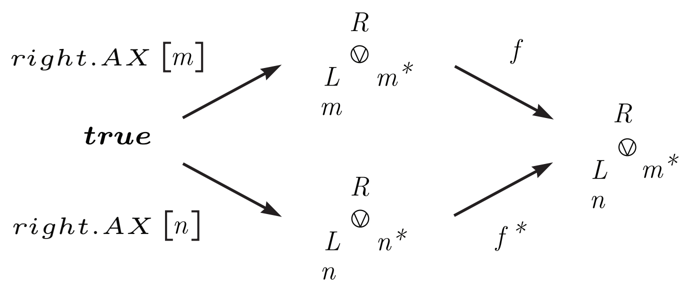
$$\mathbf{left}.\mathbf{CUT}[m] \quad : \quad L(m \owedge R(m^*)) \quad \longrightarrow \quad \mathbf{false}$$

each of them parametrized by the objects m of the category $\mathscr{A}$. These morphisms are required to make the three coherence diagrams below commute.

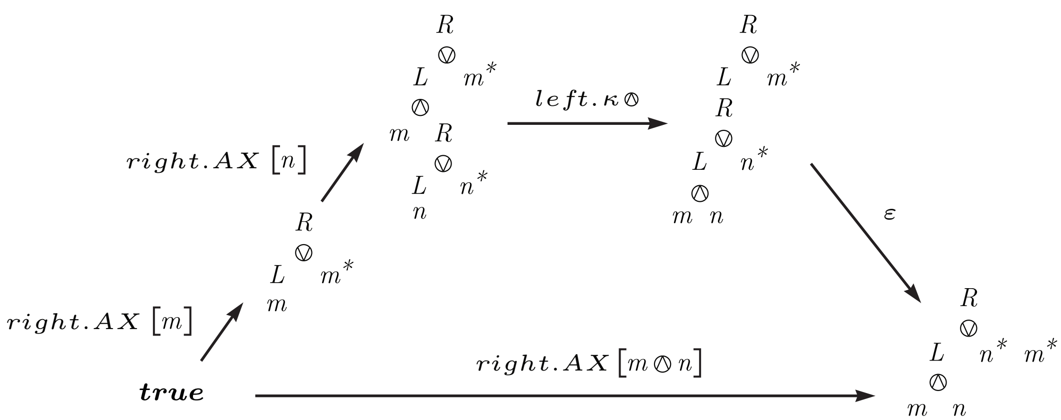
The right coherence diagrams. The first coherence diagram adapts the usual triangular axiom of adjunctions:

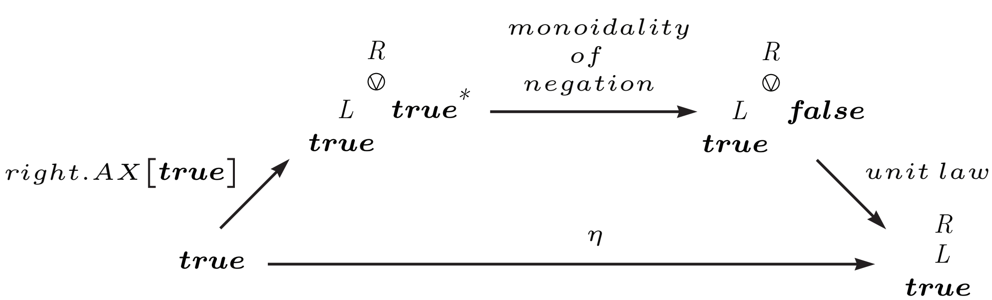


The second coherence diagram means that the family of combinators $\mathbf{AX}[-]$ is dinatural:



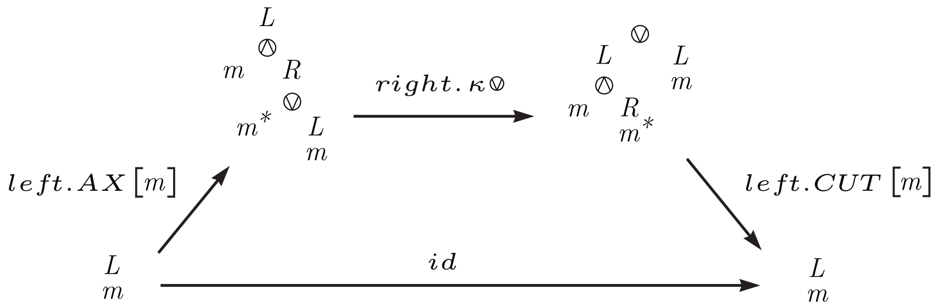
The third coherence diagram expresses a monoidality of the family $\mathbf{AX}[-]$:

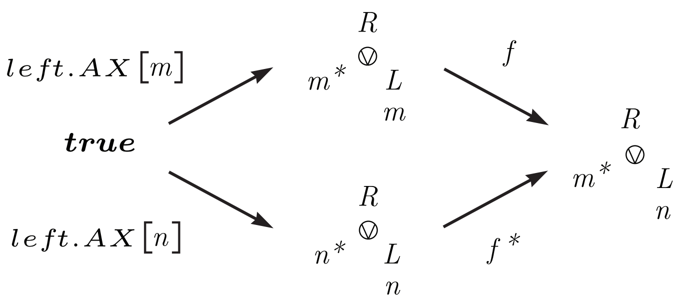


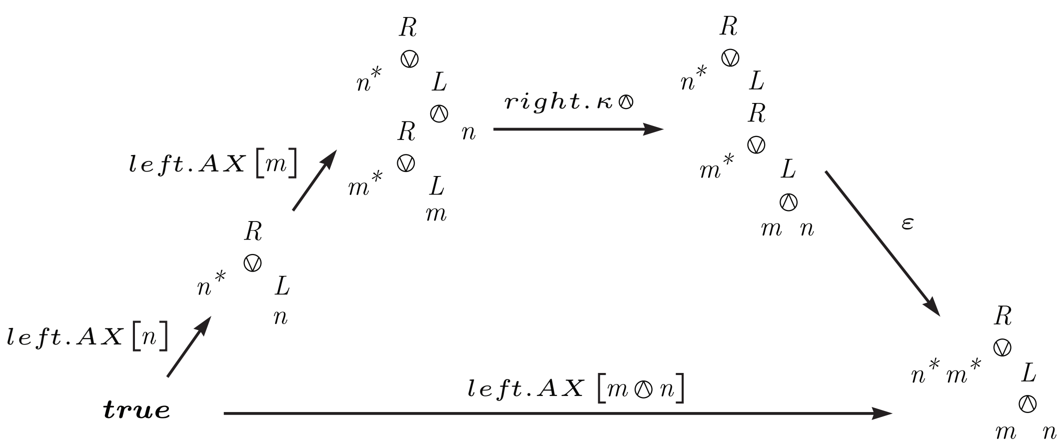


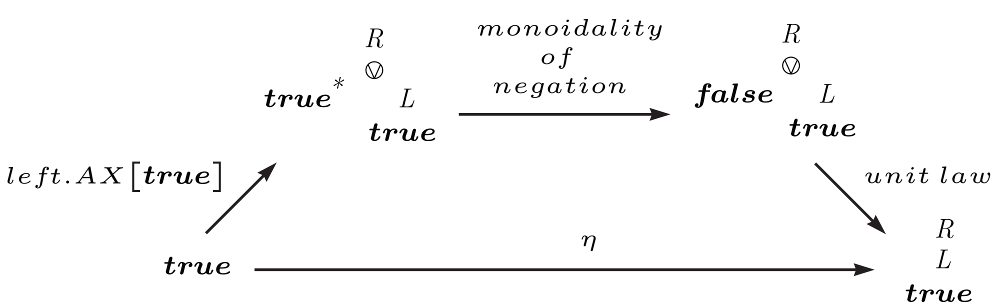
The four coherence diagrams hold for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

The left coherence diagrams. We need to give the same coherence diagrams on the left side.

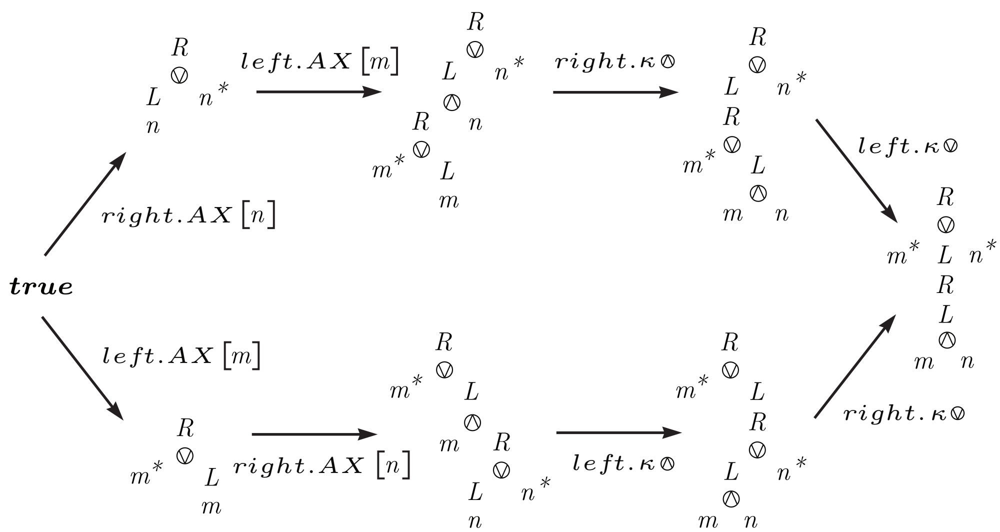

right.κ⊘
left.AX [m]
left.CUT [m]
id


left.AX [m]
left.AX [n]
true
f
f *


right.κ⊘
left.AX [m]
left.AX [n]
ε
true
left.AX [m⊘n]


monoidality
of
negation
left.AX [true]
unit law
η
true
false

The mixed coherence diagrams. We also ask this one, which ensures that the two left and right axioms commute:



(15)

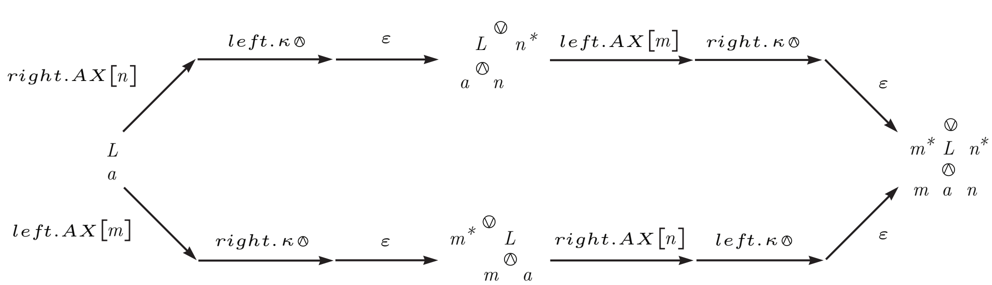
Remark. These coherence diagrams should be dualized and repeated for the combinator CUT. There is apparently no way to recover them from the coherence diagrams for the combinator AX.

8 Main theorem

8.1 Preliminary result on helicality

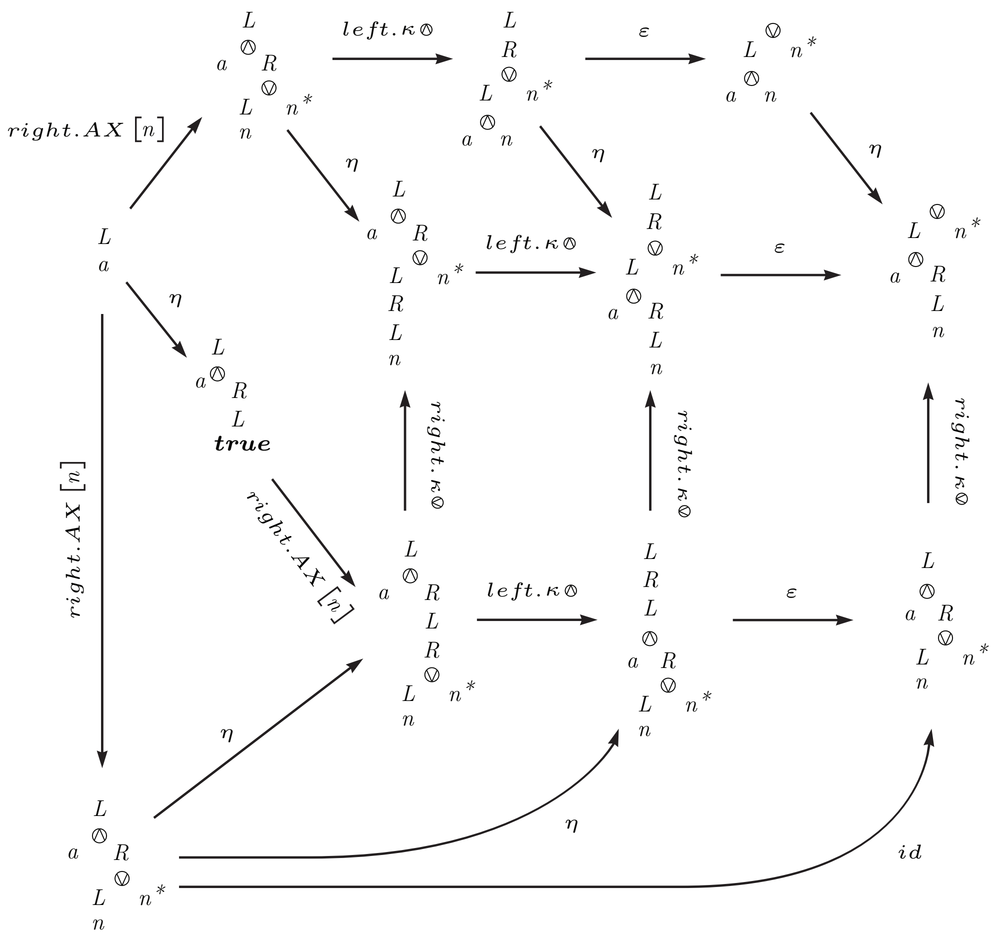
One main benefit of introducing the coherence diagram (**??**) is that we can establish the following property, which states that the dialogue category is helical.

Proposition 2 *Suppose given a discursive pair with a duality. In that case, the following diagram*

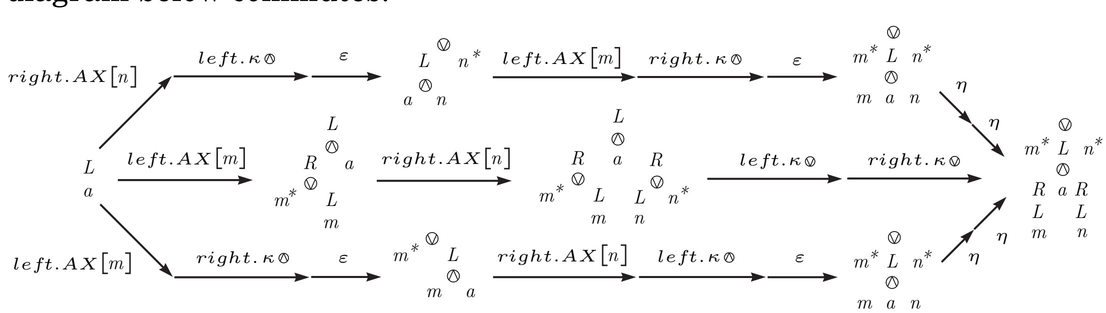


commutes.

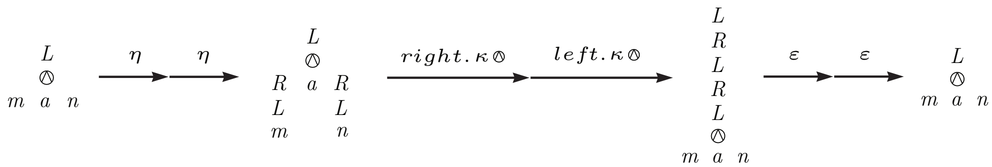
The proof that the left and the right axioms commute is based on the commutative diagram below:



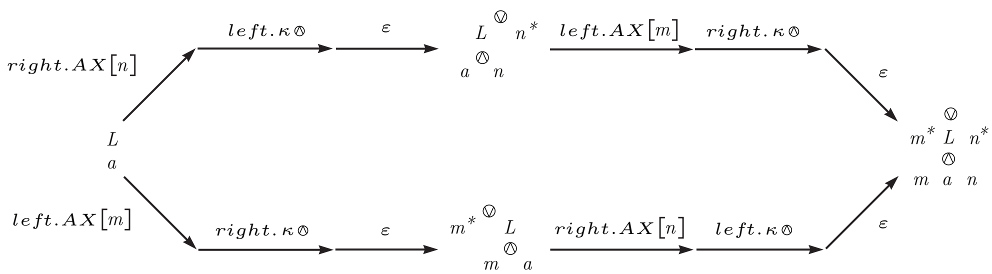
Note that we only need a special case of Diagram (**??**) with η in front of $\mathrm{AX}[n]$. This commutative diagram enables to establish in turn that the diagram below commutes:



In order to conclude, there simply remains to establish that the two morphisms η appearing in the previous diagram are monos. This immediately follows from the fact that



is equal to the identity., this esta that each η morphism involved in the previous diagram are monos. We conclude that the expected diagram



commutes.

Corollary 3 *Suppose given a helical discursive pair with a helical duality. In that case, the induced dialogue chirality is helical.*

There remains to show that...

8.2 From chiralities to discursive pairs and back

Given a helical chirality, one constructs a helical discursive pair equipped with a duality.

Proposition 4 *The helical chirality deduced from the helical discursive pair and its duality coincide with the original helical chirality.*

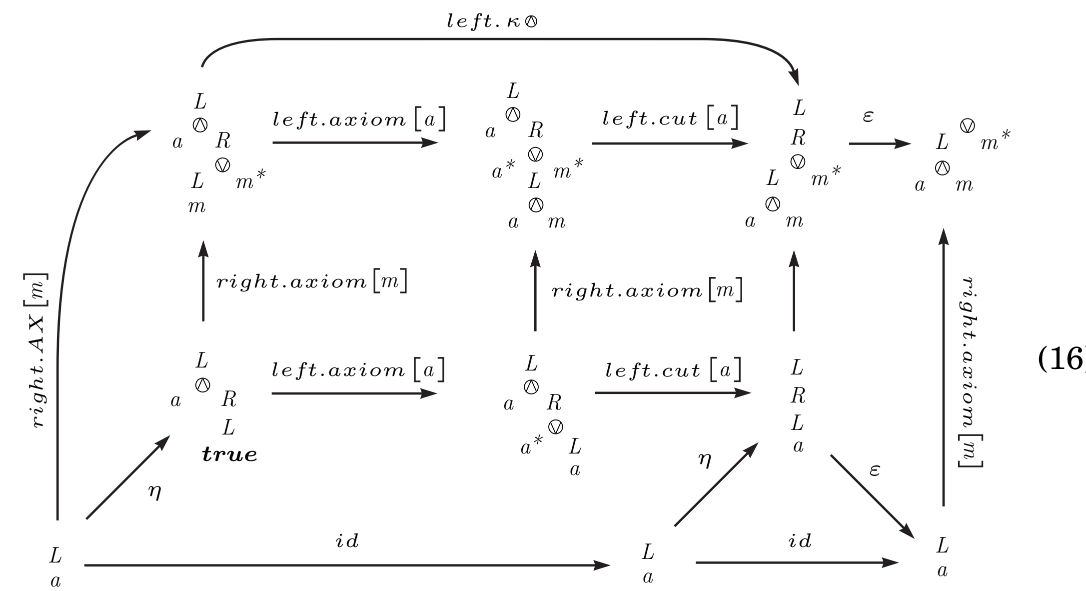
This is easy. This essentially reduces to establishing that the morphism

$$\mathbf{right}.\mathbf{axiom}[m] \quad : \quad L(a) \quad \longrightarrow L(a \oslash m) \obslash m^*$$

in the original helical chirality $(\mathscr{A}, \mathscr{B})$ coincides with the morphism

$$L(a) \xrightarrow{\mathbf{right}.\mathbf{AX}[m]} \xrightarrow{\mathbf{left}.\kappa^{\oslash}} \xrightarrow{\varepsilon} L(a \oslash m) \obslash m^*$$

recovered from the associated discursive pair. This is established by the simple diagram chase below



(16)

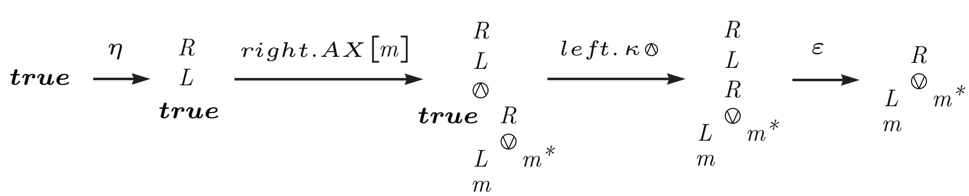
One then needs to check that the morphisms **left**.**axiom**$[m]$, **right**.**cut**$[m]$ and **left**.**cut**$[m]$ coincide with their reconstruction in the discursive pair. Each of the three facts is established by one of the three possible symmetric variants of the diagram (16).

8.3 From discursive pairs to chiralities and back

Suppose given a helical discursive pair equipped with a duality. We have seen how to construct a helical chirality from it. The question we would like to address here is whether the associated discursive pair coincides with the original one. The first step is to check that the morphism

$$\mathbf{right}.\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R(L(m) \oslash m^*)$$

of the original duality coincides with the morphism

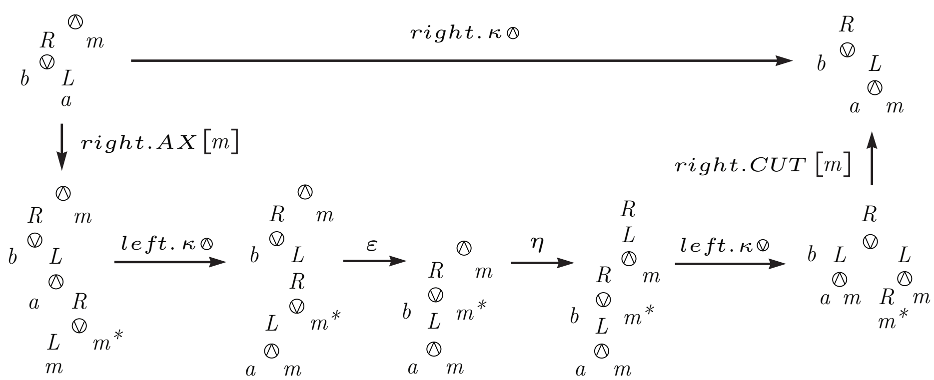


which corresponds to the morphism reconstructed from the chirality. This is essentially immediate. One needs to do the same for the three other components **left**.**AX**$[m]$, **right**.**CUT**$[m]$ and **left**.**CUT**$[m]$ of the original duality. This is done in just the same way, by applying the appropriate symmetry to the case just treated.

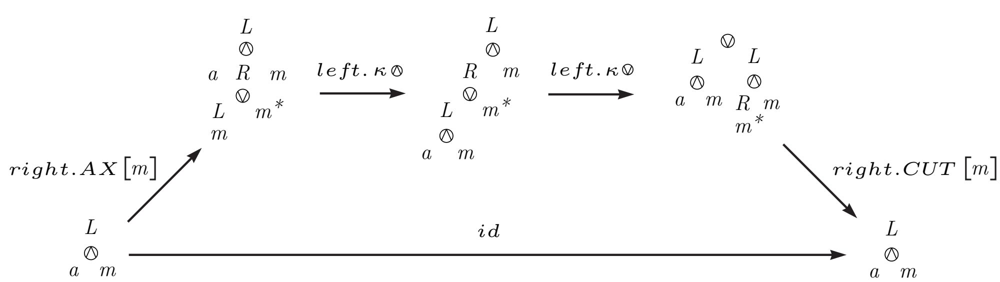
Now, the main difficulty lies in the second part of the proof, which consists in establishing that the distributivity law

$$\mathbf{right}.\kappa^{\oslash} \quad : \quad R(b \varobar L(a)) \oslash m \quad \longrightarrow \quad R(b \varobar L(a \oslash m))$$

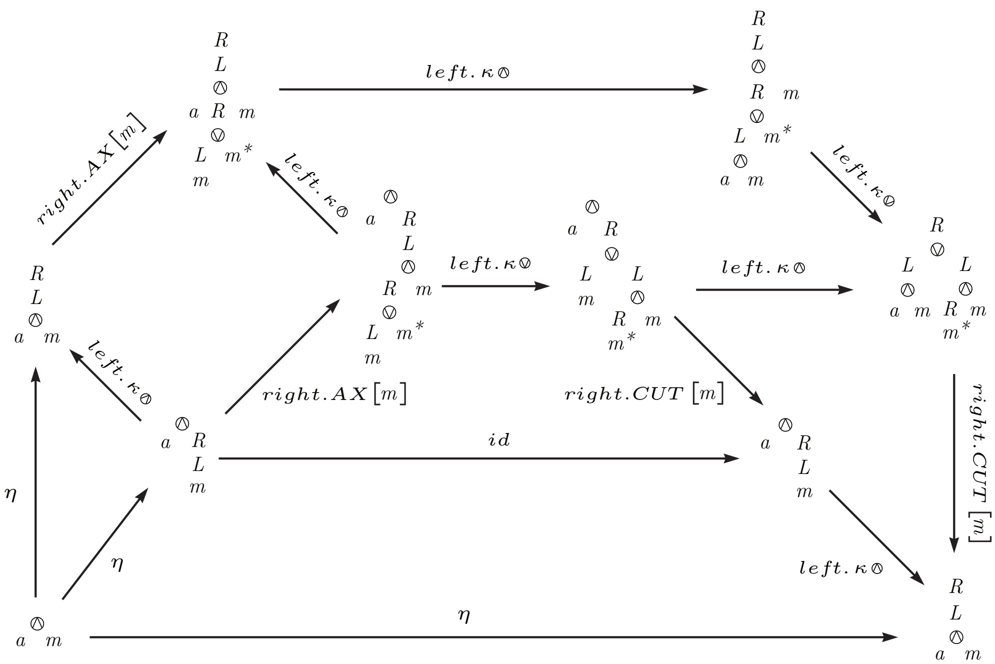
of the original discursive pair coincides with the morphism reconstructed from the associated chirality. This amounts to establishing that the diagram below



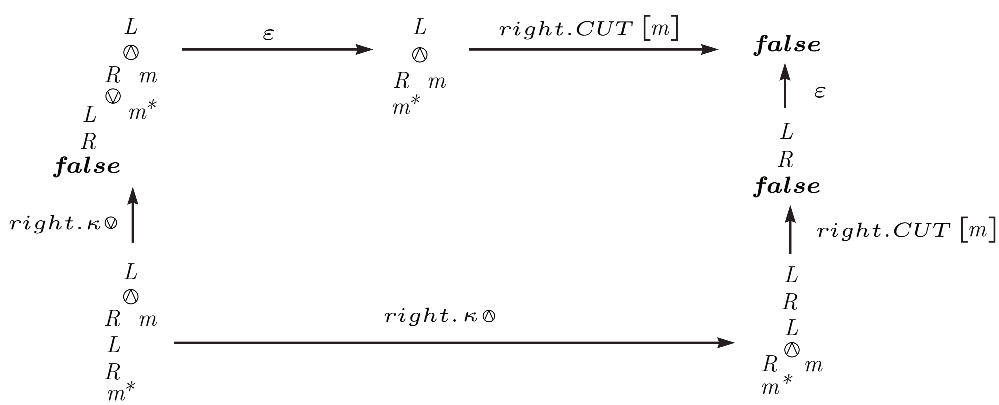
commutes. This is true, but not so easy to establish, although it boils down to producing the appropriate diagram chase. We start by establishing that



commutes as follows:



We then observe that the diagram below



commutes, as an instance of Then, we get the final diagram chase:

8.4 Main result (step 4)

Proposition 5 *There is a one-to-one relationship between the two following notions:*

- *a helical dialogue chirality,*
- *a helical discursive pair equipped with a duality.*

References

[1] Richard Blute, Robin Cockett, Robert Seely, and Todd Trimble. Natural deduction and coherence for weakly distributive categories. Journal of Pure and Applied Algebra, 113:229?296, 1996.

[2] Robin Cockett, Robert Seely. Polarized category theory, modules, and game semantics. Theory and Applications of Categories 18 (2) (2007) 4–101.

[3] Max Kelly, Ross Street. Review of the elements of 2-categories, in Kelly (ed.), Category Seminar, LNM 420.

[4] Paul-André Melliès. Categorical semantics of linear logic. Published in *Interactive models of computation and program behaviour*. Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, Paul-André Melliès. Panoramas et Synthèses 27, Société Mathématique de France, 2009.

[5] Paul-André Melliès. Dialogue categories and chiralities. Submitted. Manuscript available on the author's webpage.

[6] Paul-André Mellies. Braided notions of dialogue categories. Submitted. Manuscript available on the author's webpage.

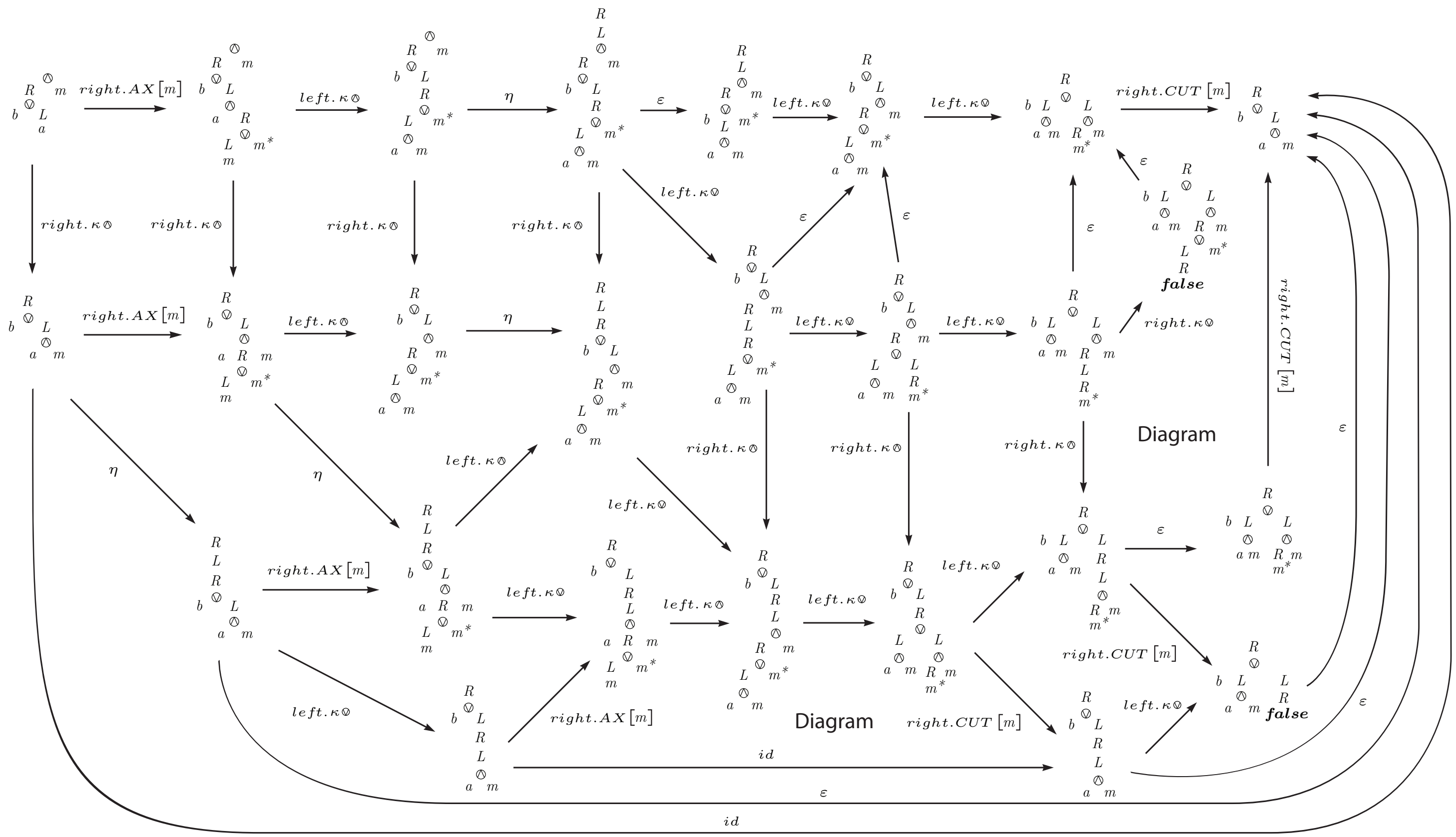


Figure 1: Commutative diagram

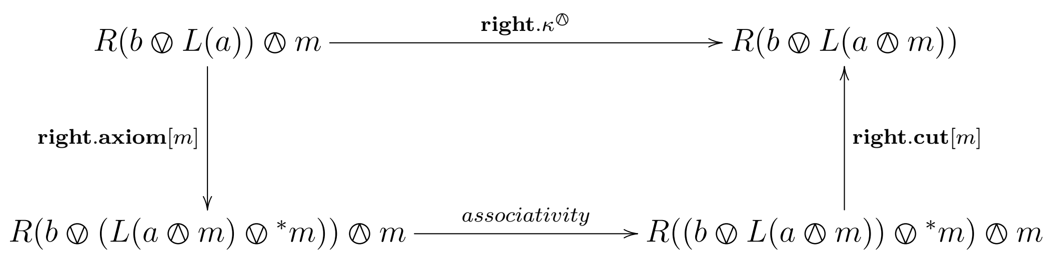
9 Appendix

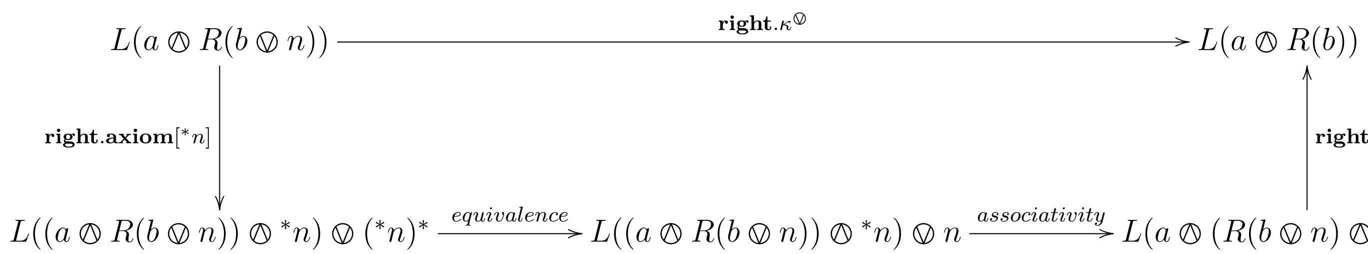
9.1 Main proposition

Proposition 6 *Every helical chirality* $(\mathscr{A}, \mathscr{B})$ *comes equipped with the natural transformations*

$$\begin{array}{cccccc} \textbf{right}.\kappa^{\owedge} & : & R(b \ovee L(a)) \owedge m & \longrightarrow & R(b \ovee L(a \owedge m)) \\ \textbf{right}.\kappa^{\ovee} & : & L(a \owedge R(b \ovee n)) & \longrightarrow & L(a \owedge R(b)) \ovee n \\ \textbf{left}.\kappa^{\owedge} & : & m \owedge R(L(a) \ovee b) & \longrightarrow & R(L(m \owedge a) \ovee b) \\ \textbf{left}.\kappa^{\ovee} & : & L(R(n \ovee b) \owedge a) & \longrightarrow & n \ovee L(R(b) \owedge a) \end{array}$$

natural in a, m *and* b*, defined as*



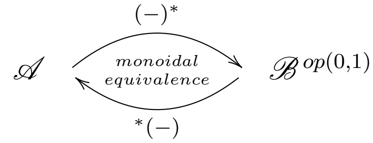


and similarly for left.$\kappa^{\owedge}$ *and* left.$\kappa^{\ovee}$. *They define together a helical dialogue chirality.*

10 Dualities

10.1 Definition

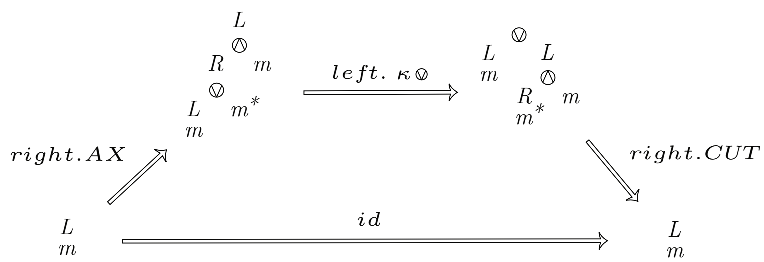
A duality on a linearly distributive pair $(\mathscr{A}, \mathscr{B})$ is defined as a monoidal equivalence

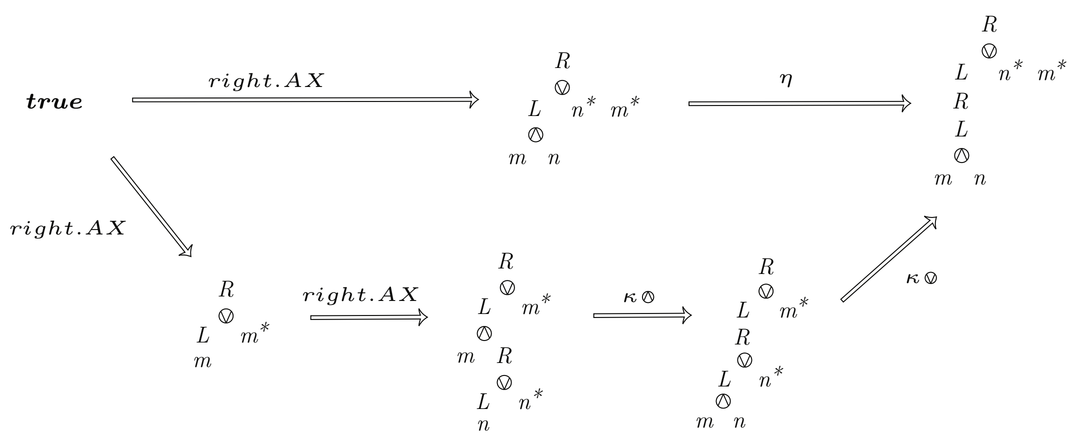


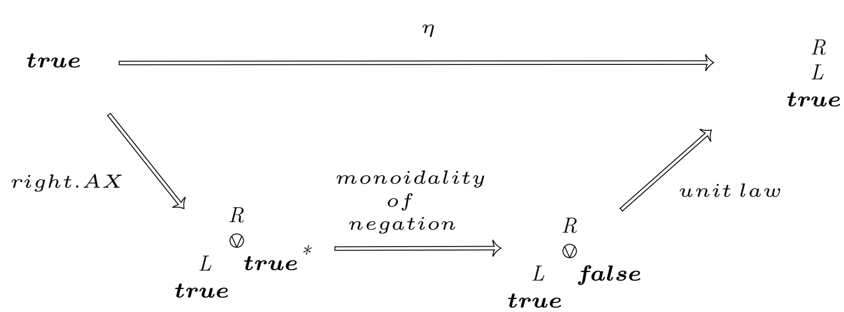
together with four families of morphisms

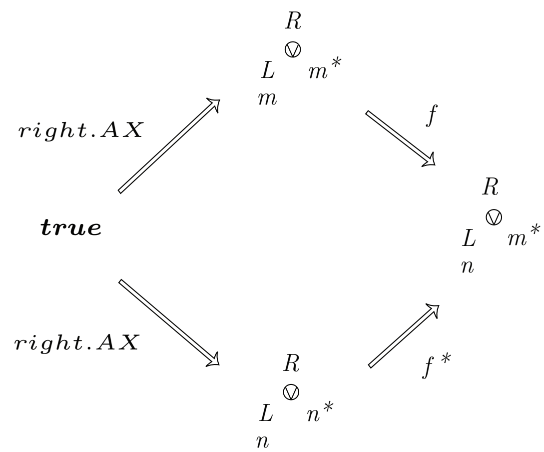
$$
\begin{array}{rcccl}
\mathbf{right}.\mathbf{AX}[m] & : & \mathbf{true} & \longrightarrow & R(L(m) \ovee m^*) \\
\mathbf{right}.\mathbf{CUT}[m] & : & L(R(m^*) \owedge m) & \longrightarrow & \mathbf{false} \\
\mathbf{left}.\mathbf{AX}[m] & : & \mathbf{true} & \longrightarrow & R(m^* \ovee L(m)) \\
\mathbf{left}.\mathbf{CUT}[m] & : & L(m \owedge R(m^*)) & \longrightarrow & \mathbf{false}
\end{array}
$$

each of them parametrized by the objects m of the category $\mathscr{A}$. These morphisms are moreover required to make the coherence diagrams below commute.









for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

10.2 Main proposition

Proposition 7 *Every helical dialogue chirality comes equipped with a duality defined as follows:*

$$
\begin{array}{lcl}
\mathbf{right}.\mathbf{AX}[m] & : & \mathbf{true} \xrightarrow{\eta} RL(\mathbf{true}) \xrightarrow{\mathbf{right}.\mathbf{axiom}[m]} R(L(m) \ovee m^*) \\
\mathbf{right}.\mathbf{CUT}[m] & : & L(R(m^*) \owedge m) \xrightarrow{\mathbf{right}.\mathbf{cut}[m]} L(R(\mathbf{false}) \xrightarrow{\varepsilon} \mathbf{false} \\
\mathbf{left}.\mathbf{AX}[m] & : & \mathbf{true} \xrightarrow{\eta} RL(\mathbf{true}) \xrightarrow{\mathbf{left}.\mathbf{axiom}[m]} R(m^* \ovee Lm) \\
\mathbf{left}.\mathbf{CUT}[m] & : & L(m \owedge R(m^*)) \xrightarrow{\mathbf{left}.\mathbf{cut}[m]} L(R(\mathbf{false}) \xrightarrow{\varepsilon} \mathbf{false}
\end{array}
$$

Remark. For simplicity, we do not mention the monoidal coercions when they are obvious. Typically, in full rigor, the right axiom map is defined as follows:

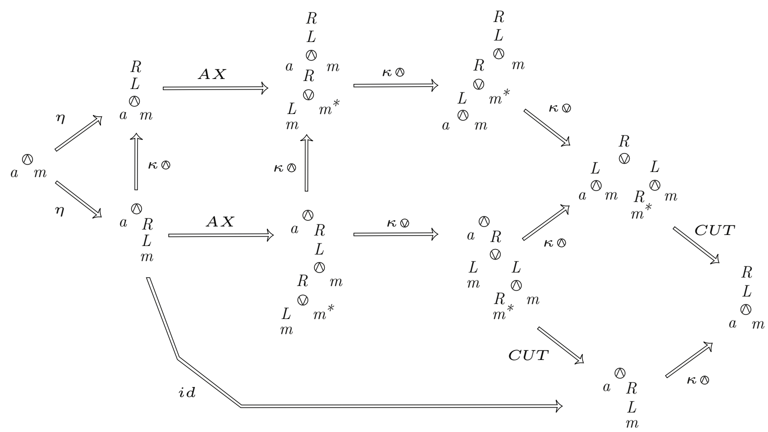
$$
\mathbf{true} \xrightarrow{\eta} RL(\mathbf{true}) \xrightarrow{\mathbf{right}.\mathbf{axiom}[m]} R(L(\mathbf{true} \owedge m) \ovee m^*) \xrightarrow{unit} R(L(m) \ovee m^*)
$$

11 A reconstruction of helical chiralities

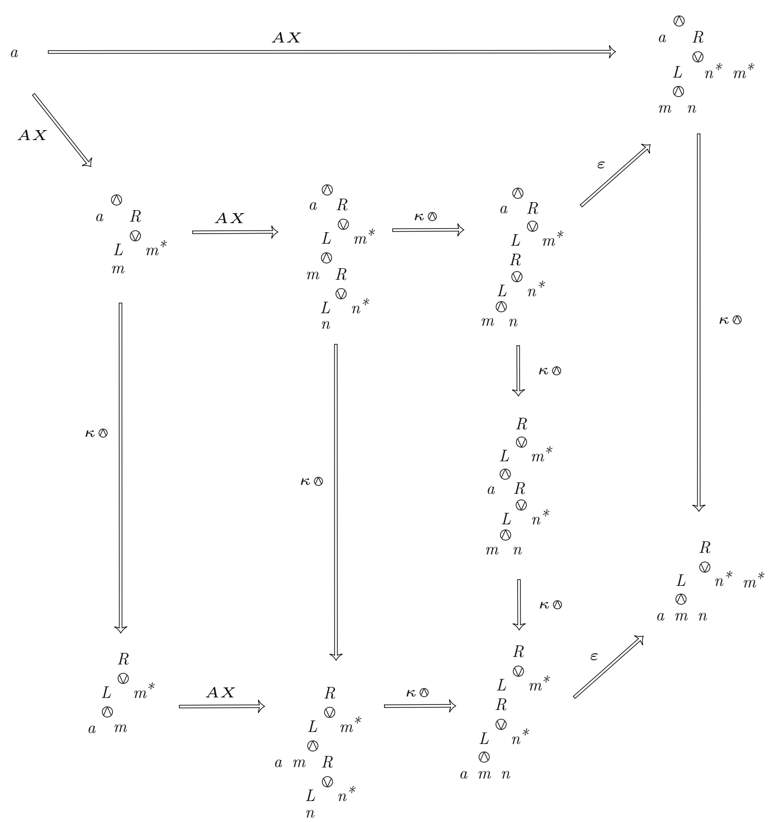
We show that a distributive pair equipped with a duality is the same thing as a dialogue chirality.

Theorem 1 *A helical chirality is the same thing as a linearly distributive pair equipped with a duality.*

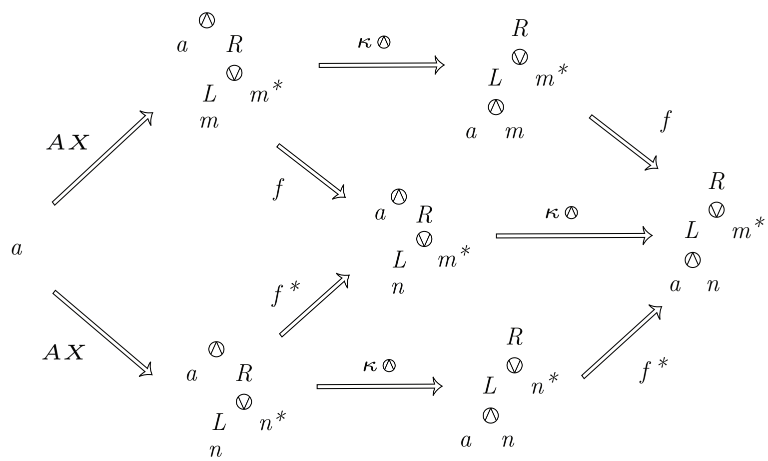
Proof of one of the triangular law



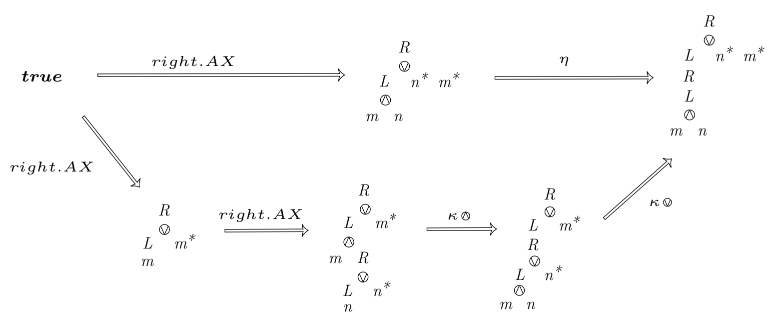
Proof of monoidality



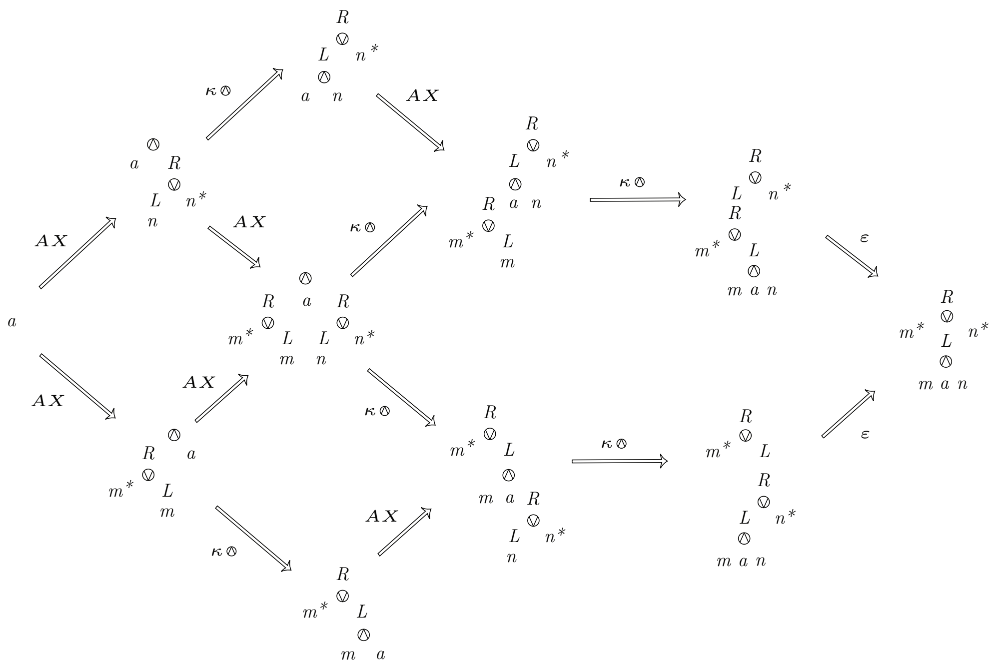
Proof of naturality



The proof of commutation is more involved. First of all, we observe that the diagram below commutes.

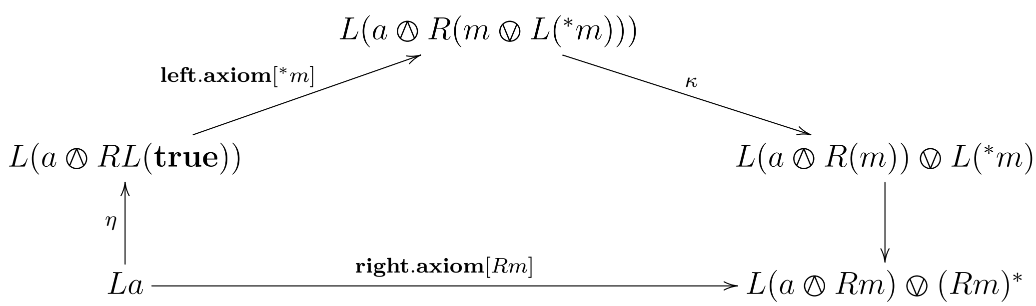


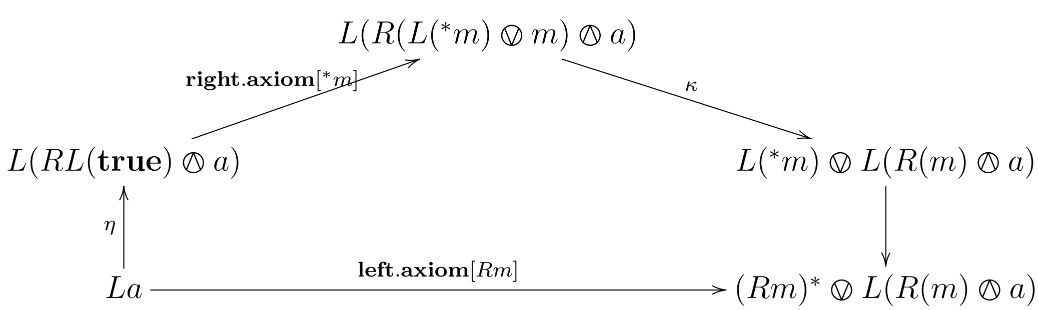
Then.



11.1 Eta-expansion of negation

Check that the diagram below commutes.





12 The balanced and symmetric cases

In that case, all the left structures are deduced using the braiding. This has to be done very carefully.