The parametric continuation monad

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Abstract

Every dialogue category comes equipped with a continuation monad defined by applying the negation functor twice. We advocate here that this double negation monad should be seen as part of a larger parametric monad, or lax action, with parameter in the opposite category. This change of point of view has one main benefit: it reveals that the strength of the continuation monad is the asymmetric fragment of a more fundamental symmetric structure — provided by a distributivity law between the parametric continuation monad and the canonical action of the dialogue category over itself. The purpose of this work is to describe the formal properties of this distributivity law, and to prepare the way for a purely combinatorial presentation of dialogue categories.

1 Introduction

Origins of tensorial logic. The starting point of tensorial logic was the somewhat unexpected discovery that the distributivity law of linear logic

$$(A \parr B) \otimes C \quad \longrightarrow \quad A \parr (B \otimes C) \tag{1}$$

could be unified with the tensorial strength of the continuation monad

$$(\neg\neg\, A) \otimes B \quad \longrightarrow \quad \neg\neg\,(A \otimes B) \tag{2}$$

by shifting to a more primitive logic of tensor and negation, which we called tensorial logic because its own distributivity law

$$\kappa_{X,B,C} \quad : \quad \neg\,(\neg\, B \otimes X) \otimes C \quad \longrightarrow \quad \neg\,(\neg\,(\,B \otimes C\,) \otimes X\,) \tag{3}$$

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may be understood *at the same time* as a tensorial strength parametrized by the object X. This principle of logic is sufficiently general to hold in every dialogue category. Recall that a dialogue category $\mathscr{C}$ is a monoidal category equipped with an object $\perp$ and a pair of natural isomorphisms

$$\begin{array}{lclcl} \varphi_{A,B} & : & \mathscr{C}(A \otimes B, \perp) & \cong & \mathscr{C}(B, A \multimap \perp) \\ \psi_{A,B} & : & \mathscr{C}(A \otimes B, \perp) & \cong & \mathscr{C}(A, \perp \multimapinv B) \end{array}$$

each of them providing a representation of the presheaves

$$A, B \mapsto \mathscr{C}(A \otimes B, \perp) \quad : \quad \mathscr{C}^{op} \longrightarrow Set.$$

A typical illustration is provided by the category of finite and infinite dimensional vector spaces on a given field k, with the object $\perp$ defined as the field k itself. When the dialogue category is symmetric, the two objects $A \multimap \perp$ and $\perp \multimapinv A$ are isomorphic, and are thus often identified for simplicity, both of them written $\neg A$ in that case. Note that every dialogue category comes equipped with a monad

$$A \mapsto \perp \multimapinv (A \multimap \perp) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

obtained by double negation. This monad is traditionally called the *continuation monad* of the category because it is related to the continuation-passing style translations used in programming language semantics. Note that the tensorial strength (2) of the continuation monad is recovered in any dialogue category $\mathscr{C}$ by instantiating (3) at the parameter X equal to the tensorial unit I.

Linear logic is to a large extent based on the decision of forgetting this continuation monad. Accordingly, the categorical counterpart of linear logic is provided by the notion of $*$-autonomous category, which plays the same role as the notion of dialogue category for tensorial logic. Seen from that point of view, a $*$-autonomous category is conveniently defined as a particular case of symmetric dialogue category, where the unit

$$A \quad \longrightarrow \quad \neg\neg A$$

of the continuation monad happens to be invertible — this reflecting the fact that negation is involutive in linear logic. At this point, the distributivity law (1) of linear logic is recovered by taking the parameter $X = \neg A$ in the distributivity law (3) of tensorial logic

$$\kappa_{\neg A,B,C} \quad : \quad \neg(\neg B \otimes \neg A) \otimes C \quad \longrightarrow \quad \neg(\neg(B \otimes C) \otimes \neg A)$$

and by recalling that the multiplicative disjunction of linear logic is defined as

$$A \parr B \quad = \quad \neg\,(\neg\, B \otimes \neg\, A) \tag{4}$$

in any $*$-autonomous category.

This convergence of (1) and (2) leads to the methodological question of understanding the algebraic nature of their unifying principle (3). This mathematical investigation should shed light on (1) and (2) in retrospect and benefit at the same time from what is already known about these two important and well-studied instances. Typically, it seems natural to think of the distributivity law (3) as a parametric refinement of the tensorial strength (2). This leads us to decompose (3) in two independent ingredients:

- a functor

  $$\circledast \quad : \quad (X, A) \quad \mapsto \quad X \circledast A = \neg\,(\,X \otimes \neg\, A\,) \quad : \quad \mathscr{C}^{\,op} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

  extending the continuation monad of the dialogue category $\mathscr{C}$,

- a natural transformation

  $$\kappa_{X,A,B} \quad : \quad (X \circledast A) \otimes B \quad \longrightarrow \quad X \circledast (A \otimes B)$$

  generalizing the tensorial strength of the continuation monad.

This decomposition reduces our original problem to understanding in turn the algebraic nature of this specific functor $\circledast$ and of this specific natural transformation κ. As we will see, the exercise is not difficult in itself — although it should be done with care — but it is foundational, because it reveals a series of 2-dimensional structures which secretly regulate the logical discourse, and more specifically its use of negation.

Parametric continuation monad. Let us start by the functor $\circledast$ which we would like to see here as a parametric version of the continuation monad. A preliminary step in that direction is to observe is that every dialogue category $\mathscr{C}$ comes equipped with an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C}^{\,op} \arrow[l, bend left, "R"] \end{tikzcd}$$

where L and R denote the expected negation functors:

$$L \quad : \quad a \mapsto a \multimap \bot \qquad\qquad\qquad R \quad : \quad b \mapsto \bot \multimapinv b.$$

In order to analyze the algebraic nature of this adjunction, it appears convenient to rename the monoidal categories $\mathscr{C}$ and $\mathscr{C}^{\,op}$ in the following way:

- the category $\mathscr{A}$ is the new name for $\mathscr{C}$ and its tensor product and unit are noted $\owedge$ and $\mathtt{true}$ in order to stress the logical interpretation of $\otimes$ and I as a linear conjunction and its neutral element,

- the category $\mathscr{B}$ is the new name for $\mathscr{C}^{\,op(0,1)}$ whose tensor product and unit are denoted $\ovee$ and $\mathtt{false}$ in order to stress the logical interpretation of $\otimes$ and I as a linear disjunction and its neutral element.

Here, the notation $\mathscr{C}^{\,op(0,1)}$ means that the orientation of the morphisms (of dimension 1) is reversed in $\mathscr{C}$ as well as the orientation of the tensor product (of dimension 0). This orientation of disjunction enables to rewrite the formula (4) as follows:

$$A \parr B \quad = \quad R\,(\,LA \ovee LB\,) \tag{5}$$

and thus to interpret $\ovee$ as a primitive variant of $\parr$, with the functors L and R playing the role of coercions interpreted as identity functors in the case of linear logic.

Now, suppose that we are given a lax 2-monad T on the 2-category CAT of categories. As the reader will see, we will be more specifically interested in the weak 2-monad

$$T \quad : \quad \mathscr{X} \quad \mapsto \quad \mathscr{B} \times \mathscr{X} \quad : \quad \mathbf{CAT} \quad \longrightarrow \quad \mathbf{CAT} \tag{6}$$

whose lax T-algebras are the lax actions

$$* \quad : \quad \mathscr{B} \times \mathscr{X} \quad \longrightarrow \quad \mathscr{X}$$

of the category $\mathscr{B} = \mathscr{C}^{\,op(0,1)}$ on the left, possibly seen as the lax actions of the category $\mathscr{C}^{\,op(1)}$ on the right. A particular example of such a lax action is provided by the weak action

$$\ovee \quad : \quad \mathscr{B} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

of the monoidal category $\mathscr{B}$ over itself. Now, given an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd} \tag{7}$$

and such a lax 2-monad T, a general *transfer theorem* establishes that every lax T-algebra structure

$$* \quad : \quad T\mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

on the category $\mathscr{B}$ induces a lax T-algebra structure on the category $\mathscr{A}$, defined as follows:

$$\circledast \quad : \quad T\mathscr{A} \quad \xrightarrow{TL} \quad T\mathscr{B} \quad \xrightarrow{*} \quad \mathscr{B} \quad \xrightarrow{R} \quad \mathscr{A}.$$

This transfer theorem may be established by purely equational means, and thus works in any 2-category $\mathscr{V}$ equipped with a lax 2-monad T. It appears that when T is the identity 2-monad in such a 2-category $\mathscr{V}$, the notion of lax T-algebra coincides with the notion of formal monad. So, the transfer theorem applied to a formal adjunction (21) implies that every formal monad S in $\mathscr{B}$ is transported to a formal monad S' in $\mathscr{A}$, defined as $S' = R \circ S \circ L$. Observe that in the case $S = id$, one recovers the fact that $R \circ L$ defines a formal monad in $\mathscr{A}$.

The transfer theorem may be also applied to the free monoidal category monad T in the 2-category **CAT**, and to the specific adjunction $L \dashv R$ between the negations of a dialogue category. In that case, it enables to transfer the monoidal structure $(\oslash, \mathrm{false})$ of the category $\mathscr{B}$ to a lax monoidal structure of the category $\mathscr{A}$. This lax monoidal structure is provided by the family of n-ary disjunctions

$$[\, A_1 \parr \cdots \parr A_n \,] \quad = \quad R\,(\, LA_1 \oslash \cdots \oslash LA_n \,)$$

which generalizes to any dialogue category the familiar definition (5) of $\parr$ in linear logic. An essential aspect of this definition of disjunction in tensorial logic is that it requires to replace the binary disjunction of linear logic by a family of n-ary disjunctions. This is done in order to recover the expected associativity property. The point is that the tensorial version of binary disjunction is not associative in the expected sense, since the two formulas

$$[\,[\, A \parr B \,] \parr C \,] \qquad [\, A \parr [\, B \parr C \,]\,]$$

are not required to be isomorphic. However, the family of n-ary disjunctions is associative in a more subtle and oriented way. Typically, there are canonical proofs of tensorial logic connecting the two clusters of binary disjunctions above to the ternary disjunction:

$$[\,[\, A \parr B \,] \parr C \,] \quad \longrightarrow \quad [\, A \parr B \parr C \,] \quad \longleftarrow \quad [\, A \parr [\, B \parr C \,]\,]$$

One benefit of this 2-categorical analysis is to explain in what sense the linear disjunction $\bindnasrepma$ living in the dialogue category $\mathscr{A} = \mathscr{C}$ is derived by *deformation* — one should probably say by *adjunction* in that case — from the disjunction $\varovee$ living in the opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$.

Finally, the transfer theorem may be applied to the weak 2-monad (20) and to the weak action of the monoidal category $(\mathscr{B}, \varovee, \mathsf{false})$ on itself on the left. From this, one derives a lax action on the left

$$\circledast \quad : \quad \mathscr{B} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

of the monoidal category $(\mathscr{B}, \varovee, \mathsf{false})$ on the category $\mathscr{A}$ which happens to coincide with the functor

$$b \circledast a \quad = \quad R\,(\,b \varovee La\,)$$

we started from. We will call *parametric monad* in $\mathscr{C}$ with parameters in $\mathscr{M}$ such a lax action of a monoidal category $\mathscr{M}$ on a category $\mathscr{C}$. The terminology is justified by the fact that every such parametric monad $\circledast$ includes a monad in $\mathscr{C}$ defined as $(I \circledast -)$ where I is the unit of the monoidal category $\mathscr{M}$. Note that in the case of the parametric continuation monad, one recovers in this way the familiar continuation monad as

$$\mathbf{false} \circledast a \quad = \quad R\,(\,\mathbf{false} \varovee La\,) \quad \cong \quad R \circ L\,(a).$$

We will see moreover that the coherence diagrams defining a parametric monad are a direct adaptation of the familiar definition of monad.

Commutation between monads. The notion of parametric monad leads to a new and more symmetric way to think of a monadic strength — this leading to a natural unification with the notion of distributivity law between monads.

$$\sigma_{A,B} \quad : \quad T(A) \otimes B \longrightarrow T(A \otimes B)$$

The notion of strength is regulated by four coherence diagrams, which may be organized in two independent series, each of them consisting of two coherence diagrams. First, the monad vs. the tensor product

Second, the functor vs. the multiplication of the tensor product:

$$\begin{tikzcd}
T(A) \otimes B \otimes C \arrow[rr, "\sigma_{A,B\otimes C}"] \arrow[dr, "\sigma_{A,B} \otimes C"'] & & T(A \otimes B \otimes C) \\
& T(A \otimes B) \otimes C \arrow[ur, "\sigma_{A\otimes B,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes I \arrow[rr, "\rho_{TA}"] \arrow[dr, "\sigma_{A,I}"'] & & TA \\
& T(A \otimes I) \arrow[ur, "T\rho_A"'] &
\end{tikzcd}$$

Plan of the paper. We start by §4 §2 §5 §6

2 Parametric monads

2.1 Formal adjunctions

Recall that an adjunction in a 2-category $\mathscr{W}$ consists of a pair An, B of 0-dimensional cells, of a pair

$$L \,:\, \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad R \,:\, \mathscr{B} \longrightarrow \mathscr{A}$$

of 1-dimensional cells, and of a pair

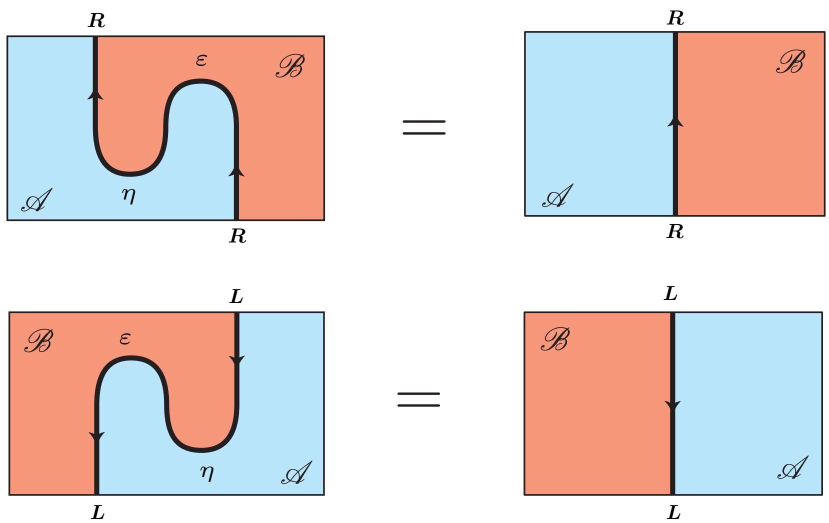
$$\eta \,:\, 1_{\mathscr{A}} \Rightarrow R \circ L \qquad\qquad \varepsilon \,:\, L \circ R \Rightarrow 1_{\mathscr{B}}$$

of 2-dimensional cells. One requires moreover that the 2-dimensional cells obtained by pasting:

$$\begin{tikzcd}
& \mathscr{A} \arrow[rr, "1"] \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & \mathscr{A} \\
\mathscr{B} \arrow[ur, "R"] \arrow[rr, "1"'] & & \mathscr{B} \arrow[ur, "R"'] \arrow[u, phantom, "\Downarrow\eta"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \arrow[rr, "1"] \arrow[dr, "L"'] & & \mathscr{A} \arrow[dr, "L"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
& \mathscr{B} \arrow[ur, "R"'] \arrow[rr, "1"'] \arrow[u, phantom, "\Downarrow\eta"] & & \mathscr{B}
\end{tikzcd}$$

coincide with the identity on the 1-dimensional cells L and R, respectively. In that case, one writes $L \dashv R$ and one says that the 1-cell L is left adjoint to the 1-cell R, and conversely, that the 1-cell R is right adjoint to the 1-cell L.

These equations are depicted in string diagrams, with the 0-cells $\mathscr{A}$ and $\mathscr{B}$ are colored blue and red respectively.



2.2 Formal adjunctions continued

Suppose given a formal adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

in a 2-category $\mathscr{W}$. It is well-known that every such adjunction induces a monad $R \circ L$ on the 0-cell $\mathscr{A}$ and a comonad $L \circ R$ on the 0-cell $\mathscr{B}$. Less known is the fact that this monad is part of a much broader structure, originally noticed by Jean Bénabou, and which we describe now.

Let $\mathrm{End}(\mathscr{A})$ denote the category with 1-cells from the 0-cell $\mathscr{A}$ to itself as objects, and 2-cells between them as morphisms. The category $\mathrm{End}(\mathscr{A})$ may be alternatively defined as the hom-category $\mathscr{W}(\mathscr{A}, \mathscr{A})$. The category $\mathrm{End}(\mathscr{A})$ is strictly monoidal, with composition $\circ$ as tensor product, and the identity 1-cell 1_A as tensor unit. Note that a monoid in this category $\mathrm{End}(\mathscr{A})$ is the same thing as a monad in $\mathscr{W}$ on the 0-cell $\mathscr{A}$. Similarly, a comonoid in the category $\mathrm{End}(\mathscr{B})$ is the same thing as a comonad in $\mathscr{W}$ on the 0-cell $\mathscr{B}$. Now, the main observation is that the two 1-cells L and R induce in turn two functors

$$\begin{array}{rcccc} [L,R] & : & \mathrm{End}(\mathscr{A}) & \to & \mathrm{End}(\mathscr{B}) \\ & & F & \mapsto & L \circ F \circ R \end{array}$$

$$\begin{array}{rcccc} [R,L] & : & \mathrm{End}(\mathscr{B}) & \to & \mathrm{End}(\mathscr{A}) \\ & & G & \mapsto & R \circ G \circ L \end{array}$$

defined by pre and postcomposition. The two functors are moreover involved in an adjunction

$$\begin{tikzcd} \mathrm{End}(\mathscr{A}) \arrow[rr, bend left, "{[L,R]}"] \arrow[rr, phantom, "\perp"] & & \mathrm{End}(\mathscr{B}) \arrow[ll, bend left, "{[R,L]}"] \end{tikzcd} \tag{8}$$

whose unit and counit are defined in the expected way:

$$\begin{array}{rcl} [\eta]_F = \eta \circ F \circ \eta & : & F \Rightarrow R \circ L \circ F \circ R \circ L, \\ [\epsilon]_G = \varepsilon \circ G \circ \varepsilon & : & L \circ R \circ G \circ L \circ R \Rightarrow G. \end{array}$$

The adjunction simply says that there exists a one-to-one correspondence between the 2-cells

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[dr, phantom, "\Downarrow"] & \mathscr{A} \arrow[d, "L"] \\ \mathscr{B} \arrow[u, "R"] \arrow[r, "G"'] & \mathscr{B} \end{tikzcd}$$

and the 2-cells

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow"] & \mathscr{A} \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{B} \arrow[u, "R"'] \end{tikzcd}$$

in the 2-category $\mathscr{W}$, and that this correspondence is natural wrt. action on F in $\mathrm{End}(A)$ and action on G in $\mathrm{End}(B)$. The adjunction is just an avatar of what Kelly and Street called "mate 2-cells" in their 2-categorical theory of adjunctions.

Now, it appears that the right adjoint functor $[R, L]$ is lax monoidal from $\mathrm{End}(B)$ to $\mathrm{End}(A)$. This means that there exists a morphism

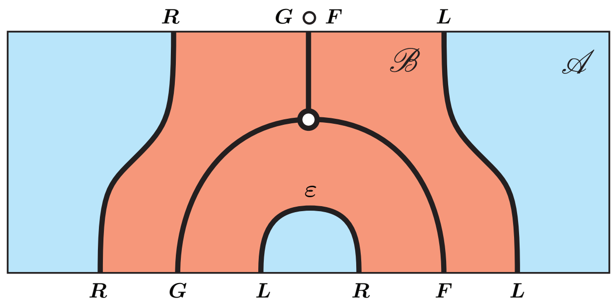
$$m_1 \quad : \quad 1_A \longrightarrow R \circ 1_B \circ L$$

and a family of morphisms

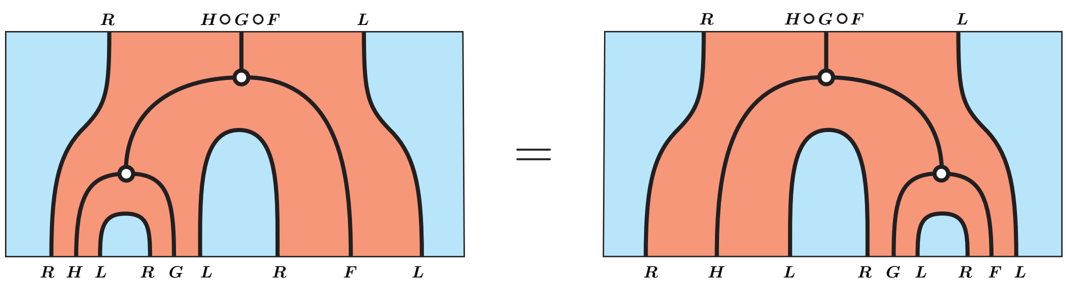
$$m_{G,G'} \quad : \quad (R \circ G \circ L) \circ (R \circ G' \circ L) \longrightarrow R \circ (G \circ G') \circ L$$

making a series of coherence diagrams commute in the category $\mathrm{End}(A)$. The morphisms m_1 and $m_{G,F}$ are provided in this case by the unit η and the counit ε of the formal adjunction $L \dashv R$, respectively. The construction is nicely depicted in the language of string diagrams. The morphism $m_{G,F}$ is typically depicted as follows.

$m_{G,F}$ =



The proof that the coherence diagrams required of a lax monoidal functor commute works exactly in the same way as the proof that the endofunctor $R \circ L$ defines a monoid in the category $\mathrm{End}(\mathscr{A})$. The main coherence diagram is typically reflected by the pictorial equality below:



2.3 Parametric monads

This discussion motivates to parametrize by a monoidal category $(J, \otimes, e)$ the usual notion of formal monad, in the following way.

Definition 1 (parametric monad) *A parametric J-monad on a 0-cell $\mathscr{A}$ of a 2-category $\mathscr{W}$ is defined as a lax monoidal functor*

$$(T, m) : J \longrightarrow \mathrm{End}(A).$$

The monoidal category J is called the parameter category of the J-monad; and an object j of J is called a parameter.

Hence, a parametric J-monad (T, m) consists of

- a 1-cell $T_j : \mathscr{A} \longrightarrow \mathscr{A}$ for every parameter j and a 2-cell $T_f : T_j \Rightarrow T_k$ for every morphism $f : j \longrightarrow k$ between such parameters,
- a 2-cell $m_e : 1_A \Rightarrow T_e$ called the unit of the parametric monad,
- a 2-cell $m_{j,k} : T_j \circ T_k \Rightarrow T_{j\otimes k}$ called the (j,k)-component of the multiplication of the parametric monad, for every pair of parameters j and k.

These data are moreover required to make a series of coherence diagrams commute in the category $\mathrm{End}(\mathscr{A})$. First, the diagrams

$$\begin{tikzcd} & T_k \arrow[dr, bend left, "T_g"] & \\ T_j \arrow[ur, bend left, "T_f"] \arrow[rr, bend right, "T_{g\circ f}"'] & & T_l \end{tikzcd} \qquad \begin{tikzcd} T_j \arrow[r, bend left, "\mathrm{id}_{T_j}"] \arrow[r, bend right, "T_{\mathrm{id}_j}"'] & T_j \end{tikzcd}$$

expressing the functoriality of T; then, the diagrams

$$\begin{tikzcd} T_j \circ T_k \arrow[r, "T_f \circ T_g"] \arrow[d, "m_{j,k}"'] & T_{j'} \circ T_{k'} \arrow[d, "m_{j',k'}"] \\ T_{j\otimes k} \arrow[r, "T_{f\otimes g}"] & T_{j'\otimes k'} \end{tikzcd}$$

expressing the naturality of m; and finally the diagrams

$$\begin{tikzcd} T_j \circ T_k \circ T_l \arrow[rr, "m_{j,k}\circ T_l"] \arrow[d, "T_j\circ m_{k,l}"'] & & T_{j\otimes k} \circ T_l \arrow[d, "m_{j\otimes k,l}"] \\ T_j \circ T_{k\otimes l} \arrow[r, "m_{j,k\otimes l}"] & T_{j\otimes(k\otimes l)} \arrow[r, "\alpha"] & T_{(j\otimes k)\otimes l} \end{tikzcd}$$

and

$$\begin{tikzcd} & T_e \circ T_j \arrow[dr, bend left, "m_{e,j}"] & \\ T_j \arrow[ur, bend left, "m_e\circ T_j"] \arrow[rr, "\mathrm{id}_{T_j}"] \arrow[dr, bend right, "T_j\circ m_e"'] & & T_j \\ & T_j \circ T_e \arrow[ur, bend right, "m_{j,e}"'] & \end{tikzcd}$$

expressing the monoidality of m; this for all indices j, j', k, k', l and morphisms f, g, h in the parameter category J.

It was first observed by Jean Bénabou that a monoid in a monoidal category $(J, \otimes, e)$ is the same thing as a lax monoidal functor

$$\mathbb{1} \longrightarrow J$$

from the monoidal category $\mathbb{1}$ with a single object and a single morphism. Thus, a monad in the usual sense is just the same thing as an parametric monad whose parameter category is the monoidal category $\mathbb{1}$.

Lax monoidal functors compose: they define a category, and in fact a 2-category, whose 2-cells are provided by monoidal natural transformations. From this follows that lax monoidal functors preserve monoids. Hence, every monoid (j, p, u) in the parameter category J induces a monoid in the category $\mathrm{End}(A)$, and thus a monad T_j in the category $\mathscr{A}$ in the usual sense. The multiplication and unit of the monad T_j are defined as expected:

$$T_j \circ T_j \xrightarrow{m_{j,j}} T_{j \otimes j} \xrightarrow{Tp} T_j$$

$$1_A \xrightarrow{m_e} T_e \xrightarrow{Tu} T_j$$

Since the unit e is a monoid of the category J, every J-monad T induces a monad T_e. This is the particular case, for $K = \mathbb{1}$, of the fact that every lax monoidal functor $K \longrightarrow J$ and every J-monad T induce together a parametric K-monad, obtained by composition.

The discussion onwards may be summarized by the following

Proposition 1 *Every formal adjunction*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

in a 2-category $\mathscr{W}$ induces a formal J-monad T on the 0-cell $\mathscr{A}$, where the parameter category J is the hom-category $\mathrm{End}(\mathscr{B}) = \mathscr{W}(\mathscr{B}, \mathscr{B})$ with tensor product defined as composition of 1-cells $\mathscr{B} \longrightarrow \mathscr{B}$ in the 2-category.

The well-known fact that $R \circ L$ defines a monad follows then from the equality

$$R \circ L = T_{1_B}$$

where 1_B is the identity functor of the category $\mathscr{B}$, and the unit of the parameter category $\mathrm{End}(B)$.

There is a similar account of the comonadic aspects of adjunctions, in which a parametric J-comonad S in a category $\mathscr{B}$ is defined as a colax monoidal functor

$$(S, n) : J \longrightarrow \mathrm{End}(\mathscr{B}).$$

A parametric J-comonad in the category $\mathscr{B}$ is the same thing as a parametric J^{op}-monad in the opposite category $\mathscr{B}^{op}$.

It is not difficult to establish then that every adjunction $L \dashv R : \mathscr{A} \to \mathscr{B}$ induces a parametric comonad S in the category $\mathscr{B}$, parametrized this time by the monoidal category $J = \mathrm{End}(\mathscr{A})$. The parametric comonad S is obtained by equipping the functor $[L, R]$ with a colax monoidal structure n in the same way as the parametric monad T was obtained by equipping the functor $[R, L]$ with a lax monoidal structure m. The construction of the colax monoidal structure n works exactly in the same way. There exists also an alternative way to construct n. The adjunction (8) induces a one-to-one correspondence between the colax monoidal structures on the left adjoint functor $[L, R]$ and the lax monoidal structures on the right adjoint functor $[R, L]$. The colax monoidal structure n on the functor $[L, R]$ is then inherited from the lax structure m on the functor $[R, L]$. Note moreover that the adjunction (8) is monoidal in the lax sense iff the colax monoidal structure n is invertible; and monoidal in the colax sense iff the lax monoidal structure m is invertible.

2.4 Illustration: the parametric continuation monad

The notion of parametric monad is appropriate to describe the following situation. Suppose given an adjunction:

$$L : \eta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \varepsilon : R\mathscr{A}\mathscr{B}$$

in which the category B is monoidal — equipped with a tensor product (noted $\ovee$) and a unit (noted false). We have seen in Section 2.3 that the adjunction generates a J-monad in the category $\mathscr{A}$, parametrized by the category $J = \mathrm{End}(\mathscr{B})$ of endofunctors of $\mathscr{B}$. Now, the monoidal structure on $\mathscr{B}$ induces a strong monoidal functor

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{B}^{co} & \longrightarrow & \mathrm{End}(\mathscr{B}) \\ (b_1, b_2) & \mapsto & b_1 \ovee - \ovee b_2 \end{array}$$

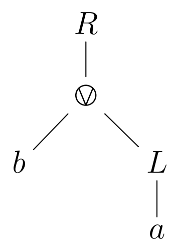
where $b_1 \ovee b \ovee b_2$ means either $(b_1 \ovee b) \ovee b_2$ or $b_1 \ovee (b \ovee b_2)$ depending on the taste of the reader. Precomposing the $\mathrm{End}(\mathscr{B})$-monad with the strong monoidal functor induces a parametric $\mathscr{B} \times \mathscr{B}^{co}$-monad in the category $\mathscr{A}$.

The component T_b of the resulting monad T is provided by the endofunctor

$$\begin{array}{rcccc} T_b & : & \mathscr{A} & \longrightarrow & \mathscr{A} \\ & & a & \mapsto & R(b \ovee La). \end{array}$$

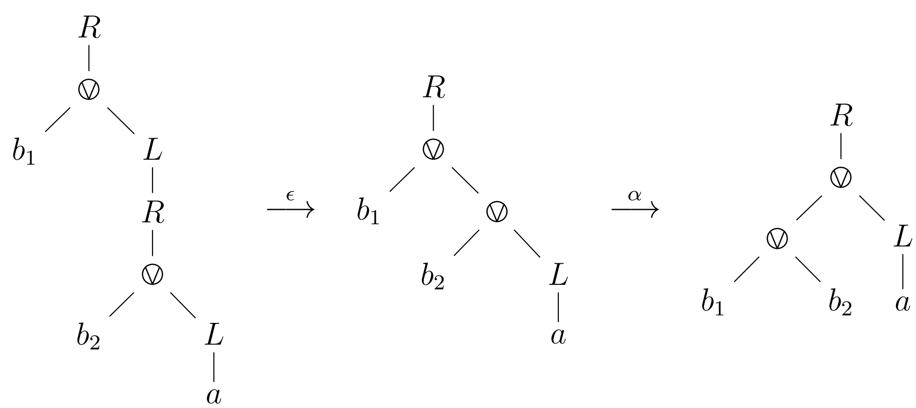
This functor will be often noted using a tree notation:

$a \quad \mapsto$

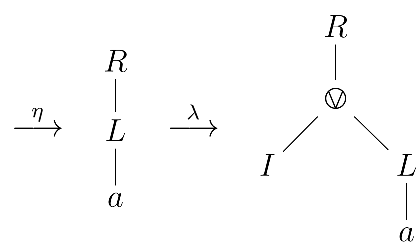


Let us clarify what parametricity means in this case. There are natural transformations

$m_2 \quad :$



$m_0 \quad : \quad a$



satisfying a series of expected coherence properties, expressing associativity, etc.

3 Transjunctions

3.1 Definition

Applying the translation to the triangular equations of adjunctions leads to the notion of transjunction.

Definition 2 (transjunction) *Suppose given a pair of adjunctions*

$$\begin{tikzcd} \mathscr{A}_1 \arrow[r, bend left, "L_1"] \arrow[r, phantom, "\perp"] & \mathscr{B}_1 \arrow[l, bend left, "R_1"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A}_2 \arrow[r, bend left, "L_2"] \arrow[r, phantom, "\perp"] & \mathscr{B}_2 \arrow[l, bend left, "R_2"] \end{tikzcd}$$

whose units and counits are denoted η_1, η_2 *and* $\varepsilon_1, \varepsilon_2$ *respectively. A transjunction* $F \dashv G$ *between a pair of functors*

$$F : \mathscr{A}_1 \to \mathscr{A}_2 \qquad G : \mathscr{B}_2 \to \mathscr{B}_1$$

along the adjunctions $L_1 \dashv R_1$ *and* $L_2 \dashv R_2$ *is defined as a pair of natural transformations*

$$\mathbf{axiom} : L_1 \Rightarrow G \circ L_2 \circ F \qquad \mathbf{cut} : F \circ R_1 \circ G \Rightarrow R_2$$

making the two diagrams

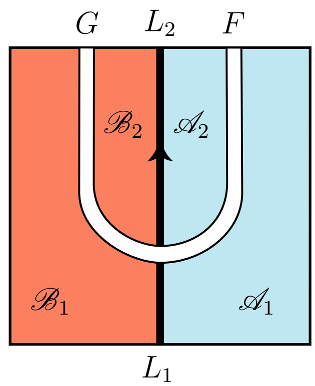
$$\begin{tikzcd} F \circ R_1 \circ L_1 \arrow[r, Rightarrow, "\mathbf{axiom}"] & F \circ R_1 \circ G \circ L_2 \circ F \arrow[d, Rightarrow, "\mathbf{cut}"] \\ F \arrow[u, Rightarrow, "\eta_1"] \arrow[r, Rightarrow, "\eta_2"] \arrow[ur, phantom, "(a)"] & R_2 \circ L_2 \circ F \end{tikzcd} \qquad \begin{tikzcd} G \circ L_2 \circ F \circ R_1 \circ G \arrow[r, Rightarrow, "\mathbf{cut}"] & G \circ L_2 \circ R_2 \arrow[d, Rightarrow, "\varepsilon_2"] \\ L_1 \circ R_1 \circ G \arrow[u, Rightarrow, "\mathbf{axiom}"] \arrow[r, Rightarrow, "\varepsilon_1"] \arrow[ur, phantom, "(b)"] & G \end{tikzcd}$$

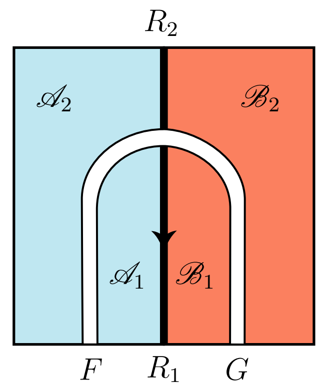
commute.

This notion of transjunction is ultimately justified by the following statement.

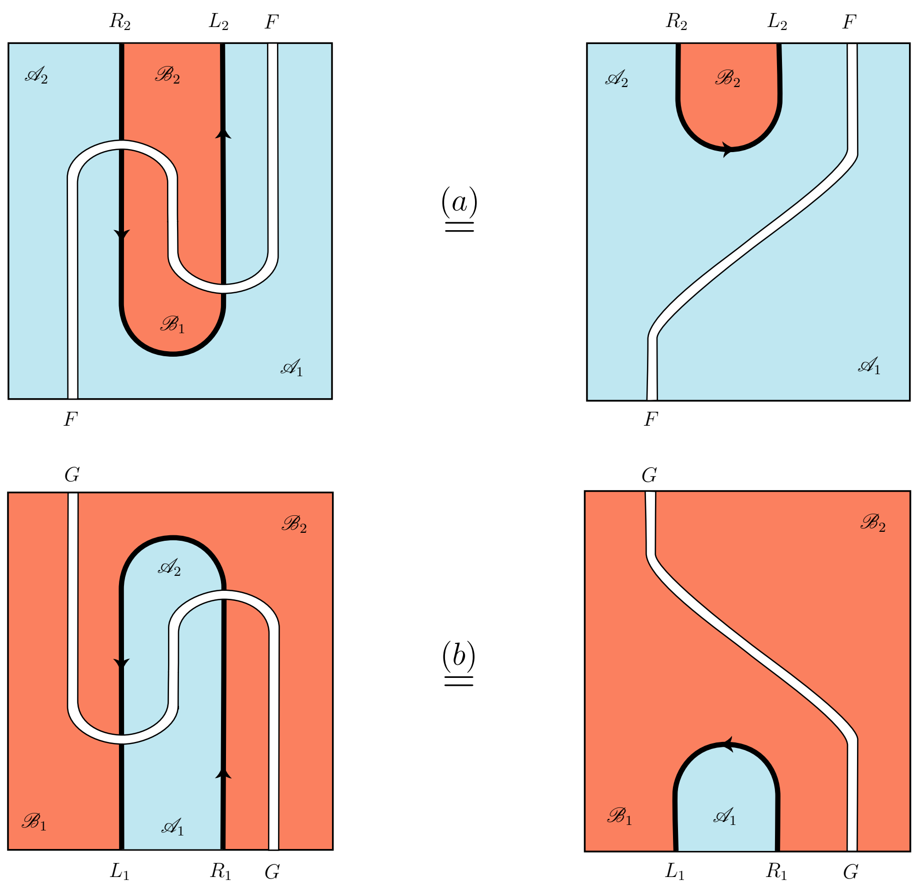
Proposition 2 *A transjunction* $F \dashv G$ *along the adjunctions* $L_1 \dashv R_1$ *and* $L_2 \dashv R_2$ *is the same thing as an adjunction* $F \circ L_1 \dashv R_2 \circ G$.

These various equations between natural transformations may be alternatively depicted as string diagrams living in the 2-category ***Cat*** of categories, functors and natural transformations. First of all, the generators axiom and cut of a transjunction $F \dashv G$ along the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ mentioned in Definition 2 are depicted as

axiom = 

cut = 

The two equalities regulating transjunctions are then depicted as follows:



3.2 Transjunction homomorphism

A notion of homorphism between transjunctions may be also introduced, this giving rise to a category of transjunctions.

Definition 3 (homomorphism) *A homomorphism between two transjunctions $F \dashv G$ and $F' \dashv G'$ along the same adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations*

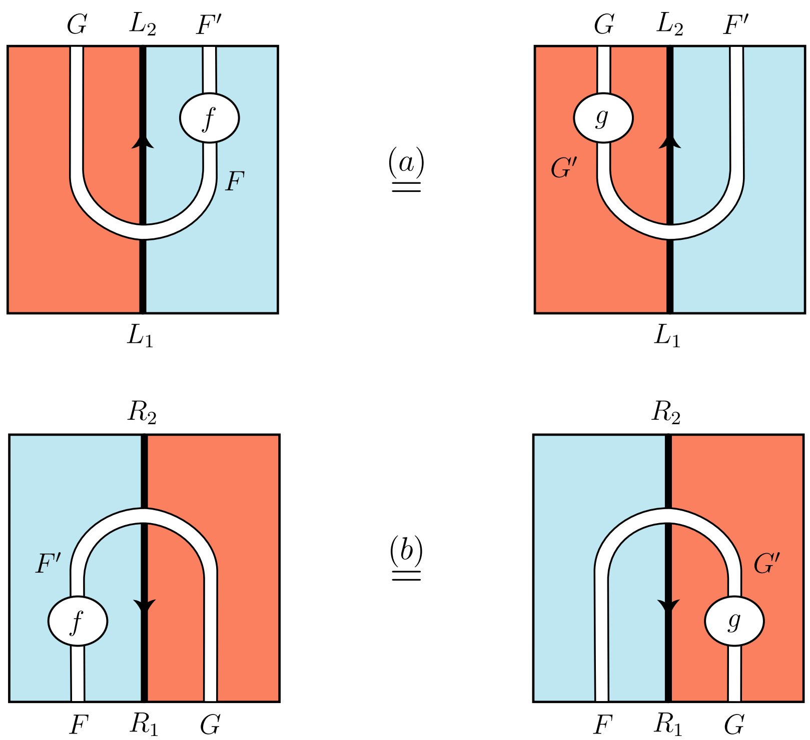
$$f \quad : \quad F \quad \Rightarrow \quad F' \qquad\qquad g \quad : \quad G' \quad \Rightarrow \quad G$$

making the two diagrams

$$\begin{tikzcd}
G \circ L_2 \circ F \arrow[r, Rightarrow, "f"] & G \circ L_2 \circ F' \\
L_1 \arrow[u, Rightarrow, "\textbf{axiom}"] \arrow[r, Rightarrow, "\textbf{axiom}'"] \arrow[ur, phantom, "(a)"] & G' \circ L_2 \circ F' \arrow[u, Rightarrow, "g"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F' \circ R_1 \circ G' \arrow[r, Rightarrow, "\textbf{cut}'"] & R_2 \\
F \circ R_1 \circ G' \arrow[u, Rightarrow, "f"] \arrow[r, Rightarrow, "g"] \arrow[ur, phantom, "(b)"] & F \circ R_1 \circ G \arrow[u, Rightarrow, "\textbf{cut}"']
\end{tikzcd}$$

commute.

A homomorphism (f, g) between from a transjunction $F \dashv G$ to a transjunction $F' \dashv G'$ along the same adjunctions is pair of natural transformations $f : F \Rightarrow F'$ and $g : G' \Rightarrow G$ satisfying the pictorial equalities below:



4 Dialogue chiralities

We start by recalling the two-sided formulation of the elementary notion of dialogue category. We will see that our axiomatization includes an η-expansion equality for the tensor product, but not for the linear negation. This will be repaired in the next section devoted to cyclic dialogue categories.

4.1 Original definition

Recall from [5] that a dialogue chirality is a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

equipped with a monoidal equivalence

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "(-)^{*}"] \arrow[r, phantom, "\substack{monoidal\\ equivalence}"] & \mathscr{B}^{op(0,1)} \arrow[l, bend left, "{}^{*}(-)"] \end{tikzcd} \tag{9}$$

with an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd} \tag{10}$$

whose unit and counit are denoted as

$$\eta \;:\; Id \longrightarrow R \circ L \qquad\qquad \varepsilon \;:\; L \circ R \longrightarrow Id$$

and, finally, with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee m^{*} \,\rangle$$

natural in m, a, b, called *currification* in honor of the logician Haskell Curry. Here, the bracket $\langle\, a \,|\, b \,\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R(b)\,).$$

The family χ is moreover required to make the diagram

$$\begin{tikzcd} \langle\, a \owedge (m \owedge n) \,|\, b \,\rangle \arrow[rr, "\chi_{m \owedge n}"] \arrow[d, "associativity"'] & & \langle\, a \,|\, b \ovee (m \owedge n)^{*} \,\rangle \\ \langle\, (a \owedge m) \owedge n \,|\, b \,\rangle \arrow[r, "\chi_{n}"] & \langle\, a \owedge m \,|\, b \ovee n^{*} \,\rangle \arrow[r, "\chi_{m}"] & \langle\, a \,|\, (b \ovee n^{*}) \ovee m^{*} \,\rangle \arrow[u, "\substack{associativity\\ monoidality\ of\ negation}"'] \end{tikzcd} \tag{11}$$

commute for all objects a, m, n of the category $\mathscr{A}$, and all objects b of the category $\mathscr{B}$.

4.2 A formulation based on adjunctions

A preliminary step towards the algebraic presentation of dialogue chiralities is provided by the following reformulation, where the currification isomorphism χ is replaced by a family of adjunctions.

Proposition 3 *A dialogue chirality is the same thing as a pair of monoidal categories* $(\mathscr{A}, \owedge, \mathbf{true})$ *and* $(\mathscr{B}, \ovee, \mathbf{false})$ *equipped with a monoidal equivalence (9) and an adjunction (10) together with an adjunction*

$$L(- \owedge m) \quad \dashv \quad R(- \ovee m^*) \tag{12}$$

for every object m *of the category* $\mathscr{A}$*, whose unit and counit are denoted*

$$\eta[m] \; : \; a \longrightarrow R(L(a \owedge m) \ovee m^*) \qquad\qquad \varepsilon[m] \; : \; L(R(b \ovee m^*) \owedge m) \longrightarrow b$$

The family $\eta[-]$ *is moreover required to be natural and monoidal, this meaning that the diagrams below*

$$\begin{tikzcd}
 & R(L(a \owedge m) \ovee m^*) \arrow[dr, bend left, "f"] & \\
a \arrow[ur, bend left, "\eta[m]"] \arrow[dr, bend right, "\eta[n]"'] & & R(L(a \owedge n) \ovee m^*) \\
 & R(L(a \owedge n) \ovee n^*) \arrow[ur, bend right, "f^*"'] &
\end{tikzcd} \tag{13}$$

$$\begin{tikzcd}
a \arrow[r, "\eta[m \owedge n]"] \arrow[d, "\eta[m]"'] & R(L(a \owedge (m \owedge n)) \ovee (m \owedge n)^*) \arrow[dd, "\substack{associativity\,\& \\ monoidality\ of\ negation}"] \\
R(L(a \owedge m) \ovee m^*) \arrow[d, "\eta[n]"'] & \\
R(L(R(L((a \owedge m) \owedge n) \ovee n^*)) \ovee m^*) \arrow[r, "\varepsilon"] & R((L((a \owedge m) \owedge n) \ovee n^*) \ovee m^*)
\end{tikzcd} \tag{14}$$

should commute for all objects a, m, n *and all morphisms* $f : m \to n$ *of the category* $\mathscr{A}$.

Remark. The family of adjunctions (12) instantiated at the unit true induces an adjunction $L \dashv R$ between the functors L and R, whose unit and counit are defined as the expected families of morphisms

$$\eta'_a \;:\; a \xrightarrow{\eta[\mathbf{true}]} R(L(a \oslash \mathbf{true}) \varobslash \mathbf{true}^*) \xrightarrow[\&\ monoidality]{associativity} RL(a)$$

$$\varepsilon'_b \;:\; LR(b) \xrightarrow[\&\ monoidality]{associativity} L(R(b \varobslash \mathbf{true}^*) \oslash \mathbf{true}) \xrightarrow{\varepsilon[\mathbf{true}]} b$$

An important question is thus to understand whether this adjunction coincides with the original adjunction (10) between L and R. It is not difficult to see that the coherence diagram (14) implies that the two adjunctions coincide in the sense that $\eta = \eta'$ and $\varepsilon = \varepsilon'$. The main idea is to instantiate the coherence diagram (14) with $m = n = e$, and to apply the coherence laws of the monoidal categories $\mathscr{A}$ and $\mathscr{B}$ in order to show that the diagram

$$\begin{tikzcd}
a \arrow[rr, "\eta'"] \arrow[d, "\eta'"'] & & RLa \arrow[dd, equal] \\
RLa \arrow[d, "\eta'"'] & & \\
RLRLa \arrow[rr, "\varepsilon"] & & RLa
\end{tikzcd}$$

commutes. It easily follows from this and from the properties of adjunctions that $\eta = \eta'$ and $\varepsilon' = \varepsilon$. This means in particular that the coherence diagram

$$\begin{tikzcd}
R(L(a \oslash \mathbf{true}) \varobslash \mathbf{true}^*) \arrow[r, "monoidality"] & R(L(a \oslash \mathbf{true}) \varobslash \mathbf{false}) \arrow[d, "associativity"] \\
a \arrow[u, "{\eta[\mathbf{true}]}"] \arrow[r, "\eta"] & RL(a)
\end{tikzcd} \tag{15}$$

is a consequence of the two coherence diagrams (13) and (14) formulated in the statement of Proposition 3.

Remark. Once established that the adjunction $(L, R, \eta', \varepsilon')$ coincides with the adjunction $(L, R, \eta, \varepsilon)$, it is not difficult to deduce from their companion diagrams (13) and (14) that the two coherence diagrams

$$
\begin{tikzcd}
 & L(R(b \ovee n^*) \owedge n) \arrow[dr, bend left, "\varepsilon[n]"] & \\
L(R(b \ovee n^*) \owedge m) \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f^*"'] & & b \\
 & L(R(b \ovee m^*) \owedge m) \arrow[ur, bend right, "\varepsilon[m]"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
L((R((b \ovee n^*) \ovee m^*) \owedge m) \owedge n) \arrow[r, "\eta"] \arrow[dd, "\substack{associativity\,\&\\ monoidality\ of\ negation}"] & L(RL(R((b \ovee n^*) \ovee m^*) \owedge m) \owedge n) \arrow[d, "\varepsilon[m]"] \\
 & L(R(b \owedge n) \ovee n^*) \arrow[d, "\varepsilon[n]"] \\
L(R(b \ovee (m \owedge n)^*) \owedge (m \owedge n)) \arrow[r, "\varepsilon[m \owedge n]"] & b
\end{tikzcd}
$$

commute for all object b of the category $\mathscr{B}$ and all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$. Note that there is an element of choice in the formulation of dialogue chiralities in Proposition 3 since the two coherence diagrams for $\varepsilon[-]$ may have very well replaced the two corresponding diagrams (13) and (14) for $\eta[-]$.

4.3 A formulation based on transjunctions

The formulation of dialogue chiralities described in §4.2 is fine, but may be marginally improved. The idea is to replace the original combinators $\eta[-]$ and $\varepsilon[-]$ presenting the adjunctions by somewhat simpler combinators inspired by proof-theory:

$$
\begin{array}{rcccc}
\mathbf{axiom}[m] & : & L(a) & \longrightarrow & L(a \owedge m) \ovee m^* \\
\mathbf{cut}[m] & : & R(b \ovee m^*) \owedge m & \longrightarrow & R(b)
\end{array}
$$

This pair of mirror-symmetric combinators is defined from $\eta[-]$ and $\varepsilon[-]$ in the following way:

$$
\begin{array}{rcccccc}
\mathbf{axiom}[m] & : & L(a) & \xrightarrow{\eta[m]} & LR(L(a \owedge m) \ovee m^*) & \xrightarrow{\varepsilon} & L(a \owedge m) \ovee m^* \\
\mathbf{cut}[m] & : & R(b \ovee m^*) \owedge m & \xrightarrow{\eta} & RL(R(b \ovee m^*) \owedge m) & \xrightarrow{\varepsilon[m]} & R(b)
\end{array}
$$

Conversely, the combinators $\eta[-]$ and $\varepsilon[-]$ may be recovered from the original combinators $\mathbf{axiom}[-]$ and $\mathbf{cut}[-]$ in the following way:

$$
\begin{array}{rclcccccc}
\eta[m] & : & a & \xrightarrow{\ \eta\ } & RL(a) & \xrightarrow{\mathbf{axiom}[m]} & R(L(a \owedge m) \ovee m^*) \\
\varepsilon[m] & : & L(R(b \ovee m^*) \owedge m) & \xrightarrow{\mathbf{cut}[m]} & LR(b) & \xrightarrow{\ \varepsilon\ } & b
\end{array}
$$

It is immediate that this back-and-forth translation between the pair $\eta[-]$, $\varepsilon[-]$ and the pair $\mathbf{axiom}[-]$, $\mathbf{cut}[-]$ defines a one-to-one relationship between the two pairs of combinators. This idea leads to an alternative formulation of dialogue chiralities, based this time on the transjunction between the two functors

$$
- \owedge m \ : \ \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad - \ovee m^* \ : \ \mathscr{B} \longrightarrow \mathscr{B}
$$

along the adjunction $L \dashv R$ between $\mathscr{A}$ and $\mathscr{B}$. Note that the transjunction is presented by the logical combinators $\mathbf{axiom}[-]$ and $\mathbf{cut}[-]$.

Proposition 4 *A dialogue chirality may be alternatively defined as a pair of categories* $(\mathscr{A}, \owedge, \mathbf{true})$ *and* $(\mathscr{B}, \ovee, \mathbf{false})$ *equipped with a monoidal equivalence (9) and an adjunction (10) together with a family of transjunctions*

$$
\mathbf{axiom}[m] : L(a) \longrightarrow L(a \owedge m) \ovee m^* \qquad\qquad \mathbf{cut}[m] : R(b \ovee m^*) \owedge m \longrightarrow R(b)
$$

natural in b *and* m. *The family* $\mathbf{cut}[-]$ *is moreover required to be natural and monoidal, in the sense that the two diagrams*

$$
\begin{tikzcd}
 & R(b \ovee m^*) \owedge m \arrow[dr, "\mathbf{cut}[m]"] & \\
R(b \ovee n^*) \owedge m \arrow[ur, "f^*"] \arrow[dr, "f"'] & & R(b) \\
 & R(b \ovee n^*) \owedge n \arrow[ur, "\mathbf{cut}[n]"'] &
\end{tikzcd}
\tag{16}
$$

$$
\begin{tikzcd}
(R((b \ovee n^*) \ovee m^*) \owedge m) \owedge n \arrow[r, "\mathbf{cut}[m]"] & R(b \ovee n^*) \owedge n \arrow[dd, "\mathbf{cut}[n]"] \\
R(b \ovee (n^* \ovee m^*)) \owedge (m \owedge n) \arrow[u, "associativity"] & \\
R(b \ovee (m \owedge n)^*) \owedge (m \owedge n) \arrow[u, "monoidality"] \arrow[r, "\mathbf{cut}[m \owedge n]"] & R(b)
\end{tikzcd}
\tag{17}
$$

commute for all objects a, m, n *and morphisms* $f : m \to n$ *of the category* $\mathscr{A}$.

Remark. Just as in the case of the alternative definition of dialogue chiralities based on adjunctions in §4.2, the expected coherence diagram

$$
\begin{tikzcd}
R(b) \arrow[rr, "id"] & & R(b) \\
R(b \ovee \mathbf{true}^*) \owedge \mathbf{true} \arrow[u, "\mathbf{cut}[\mathbf{true}]"] \arrow[rr, "monoidality"] & & R(b \ovee \mathbf{false}) \owedge \mathbf{true} \arrow[u, "associativity"']
\end{tikzcd}
\tag{18}
$$

follows from the coherence diagrams (16) and (17). Moreover, the two coherence diagrams for the combinator $\mathbf{cut}[-]$ are equivalent to the corresponding coherence diagrams for $\mathbf{axiom}[-]$ below:

$$
\begin{tikzcd}
& L(a \owedge m) \ovee m^* \arrow[dr, bend left, "f"] & \\
L(a) \arrow[ur, bend left, "\mathbf{axiom}[m]"] \arrow[dr, bend right, "\mathbf{axiom}[n]"'] & & L(a \owedge n) \ovee m^* \\
& L(a \owedge n) \ovee n^* \arrow[ur, bend right, "f^*"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
L(a \owedge m) \ovee m^* \arrow[r, "\mathbf{axiom}[n]"] & (L((a \owedge m) \owedge n) \ovee n^*) \ovee m^* \arrow[d, "associativity"] \\
& L(a \owedge (m \owedge n)) \ovee (n^* \ovee m^*) \arrow[d, "monoidality"] \\
La \arrow[uu, "\mathbf{axiom}[m]"] \arrow[r, "\mathbf{axiom}[m \owedge n]"] & L(a \owedge (m \owedge n)) \ovee (m \owedge n)^*
\end{tikzcd}
$$

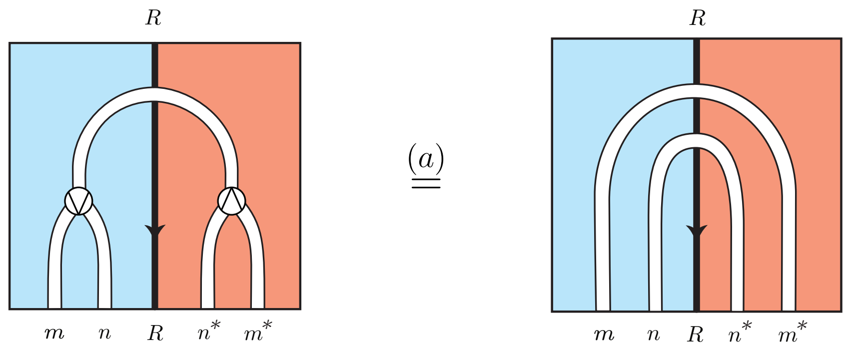
From this follows that any of the two pairs of coherence diagrams imply that the diagram

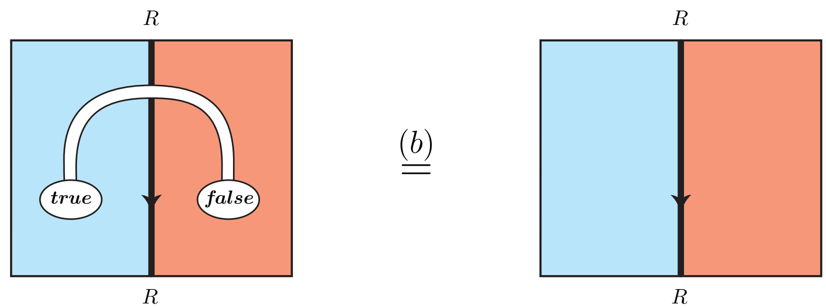
$$
\begin{tikzcd}
L(a \owedge \mathbf{true}) \ovee \mathbf{true}^* \arrow[r, "monoidality"] & L(a \owedge \mathbf{true}) \ovee \mathbf{false} \arrow[d, "associativity"] \\
La \arrow[u, "\mathbf{axiom}[\mathbf{true}]"] \arrow[r, "id"] & La
\end{tikzcd}
$$

commutes.

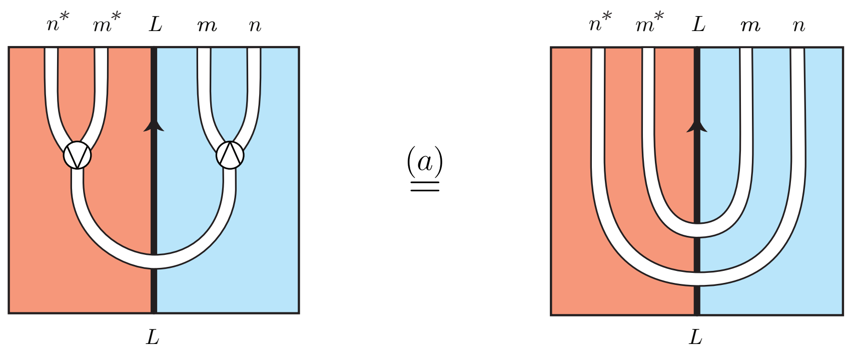
4.4 Dialogue chiralities in string diagrams

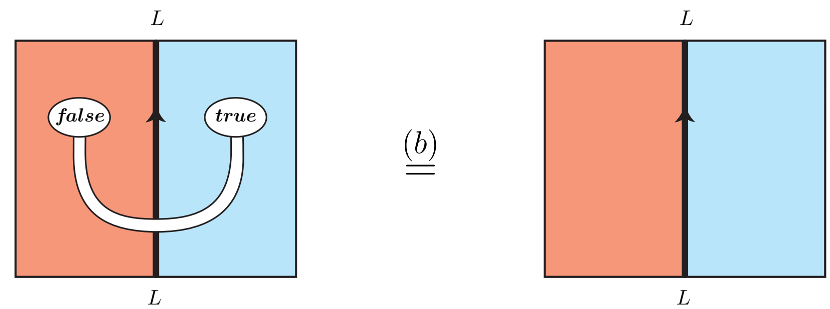
The equations for the cut link are depicted in the following way:





These equations are reminiscent of the $\otimes$ vs. $\wp$ as well as the 1 vs. $\perp$ cut-elimination rewriting steps in proof-nets of linear logic. Similar diagrams for the axiom link





The η-expansion laws of proof-nets in linear logic.

5 Commutations between parametric monads

At this point, we are ready to introduce the notion of *commutation* between parametric monads, and to establish at the same time that every dialogue category is equipped with such a structure. As we will see, the notion of commutation unifies and generalizes the celebrated notions of *monadic strength* on the one hand, and of *distributivity law* between two monads on the other hand.

5.1 Definition

We suppose given a category $\mathscr{C}$ equipped with a parametric $\mathscr{J}$-monad

$$\bullet \quad : \quad \mathscr{J} \times \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and a parametric $\mathscr{M}^{op(0)}$-monad

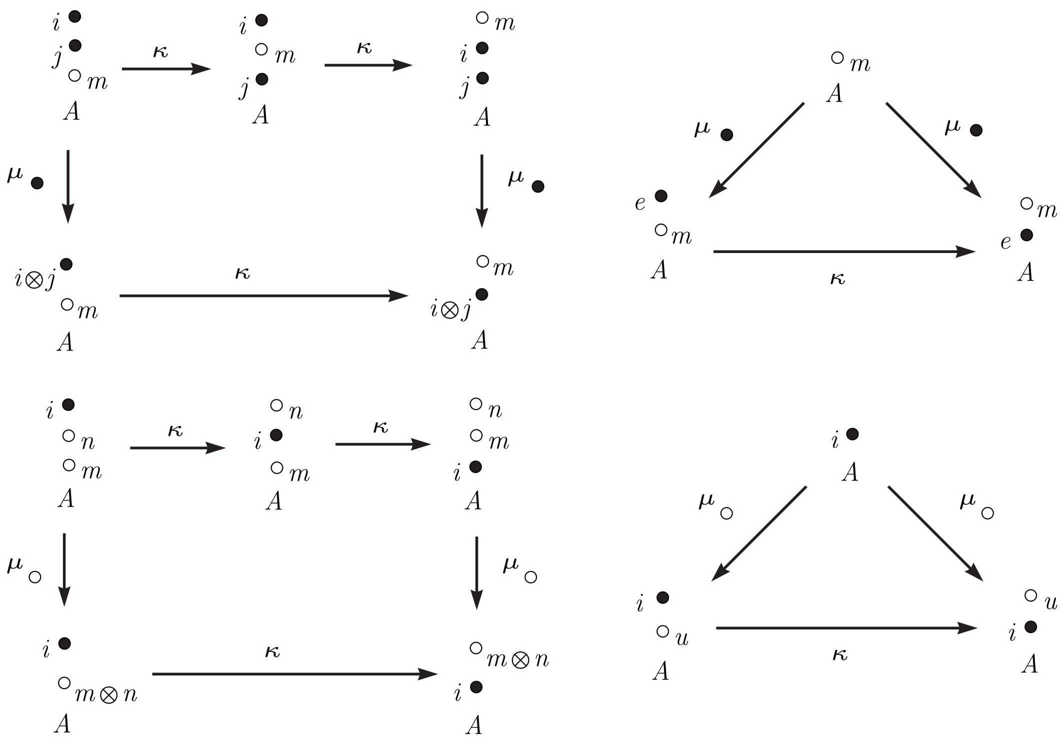
$$\circ \quad : \quad \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

with parameters taken in the monoidal categories $(\mathscr{J}, \otimes, e)$ and $(\mathscr{M}, \otimes, u)$.

Definition 4 (Commutation) *A commutation between the parametric monads $\bullet$ and $\circ$ is defined as a natural transformation*

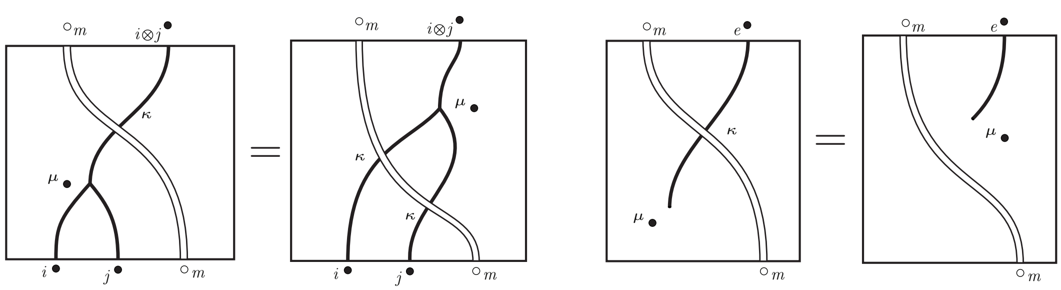
$$\kappa \quad : \quad - \bullet (- \circ -) \quad \Rightarrow \quad (- \bullet -) \circ - \quad : \quad \mathscr{J} \times \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

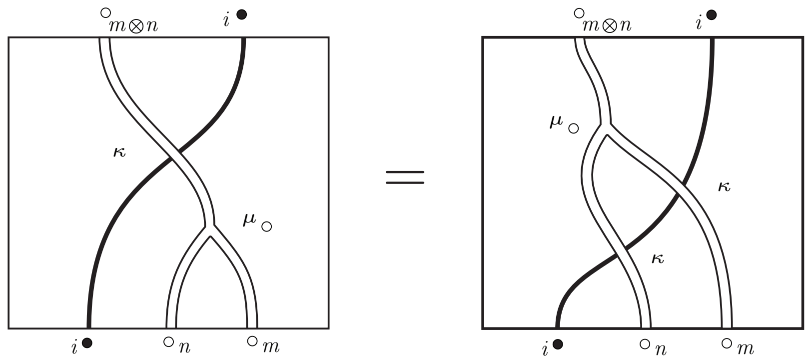
making the four diagrams below commute



for all objects A of the category $\mathscr{C}$, all objects i, j of the category $\mathscr{J}$ and all objects m, n of the category $\mathscr{M}$.

This may be depicted in string diagrams as follows: kappa as a braided permutation, compatible with multiplication and unit on both sides.





This may be also adapted to comonads vs. comonads, monads vs. comonads and comonads vs. monads.

5.2 Illustration

Every dialogue chirality is equipped with a commutation

$$\kappa^{\owedge} \quad : \quad R(b \ovee L(a)) \owedge m \quad \longrightarrow \quad R(b \ovee L(a \owedge m))$$

between

- the monoidal action of $\mathscr{A}$ over itself,
- the parametric $\mathscr{B}$-monad of $\mathscr{A}$.

defined as

$$\begin{tikzcd}
R(b \ovee L(a)) \owedge m \arrow[r, "\kappa^{\owedge}"] \arrow[d, "\mathbf{axiom}[m]"'] & R(b \ovee L(a \owedge m)) \\
R(b \ovee (L(a \owedge m) \ovee {}^*m)) \owedge m \arrow[r, "associativity"] & R((b \ovee L(a \owedge m)) \ovee {}^*m) \owedge m \arrow[u, "\mathbf{cut}[m]"']
\end{tikzcd}$$

Symmetrically, every dialogue chirality is equipped with a commutation

$$\kappa^{\ovee} \quad : \quad L(a \owedge R(b \ovee n)) \quad \longrightarrow \quad L(a \owedge R(b)) \ovee n$$

between

- the monoidal action of $\mathscr{B}$ over itself,
- the parametric $\mathscr{A}$-comonad of $\mathscr{B}$.

6 Discursive pairs

6.1 Definition

A discursive pair is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, \mathbf{true}) \qquad\qquad (\mathscr{B}, \ovee, \mathbf{false})$$

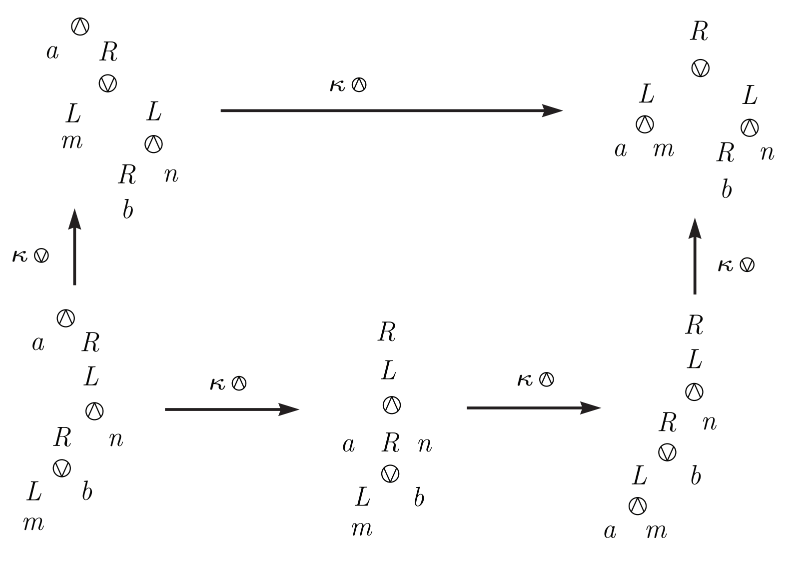
equipped with an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

together with the two bimonads

$$\begin{array}{cccccc} \kappa^{\owedge} & : & m \owedge R(L(a) \ovee b) & \longrightarrow & R(L(m \owedge a) \ovee b) \\ \kappa^{\ovee} & : & L(R(n \ovee b) \owedge a) & \longrightarrow & n \ovee L(R(b) \owedge a) \end{array} \tag{19}$$

between the $\owedge$-tensor product and the $\mathscr{B}$-monad of $\mathscr{A}$ on the one hand, and between the $\ovee$-tensor product and the $\mathscr{A}$-comonad of $\mathscr{B}$ on the other hand. Besides the resulting series of commutative diagrams, we ask that the diagram below commutes.



for all objects a, m, n of the category $\mathscr{A}$ and all object b of the category $\mathscr{B}$. Note that this coherence diagram is not justified by any of the previous discussions.

6.2 Right dualities

A duality in a discursive pair $(\mathscr{A}, \mathscr{B})$ is defined as a monoidal equivalence

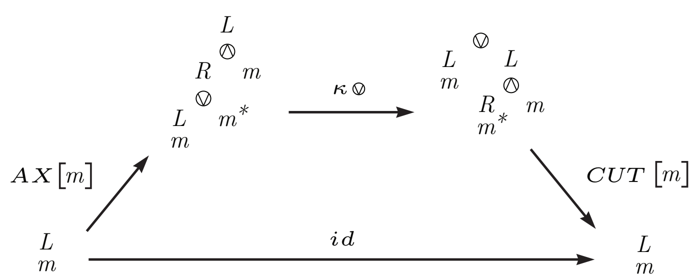
$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\substack{monoidal\\ equivalence}"] & \mathscr{B}^{op(0,1)} \arrow[l, bend left, "^*(-)"] \end{tikzcd}$$

together with two families of morphisms

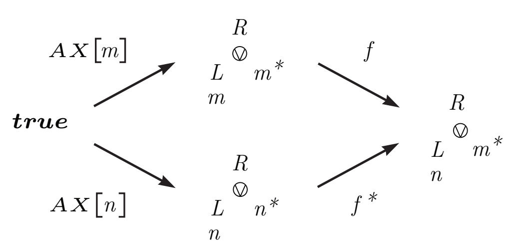
$$\mathbf{AX}[m] \quad : \quad \mathbf{true} \quad \longrightarrow \quad R(L(m) \obslash m^*)$$

$$\mathbf{CUT}[m] \quad : \quad L(R(m^*) \owedge m) \quad \longrightarrow \quad \mathbf{false}$$

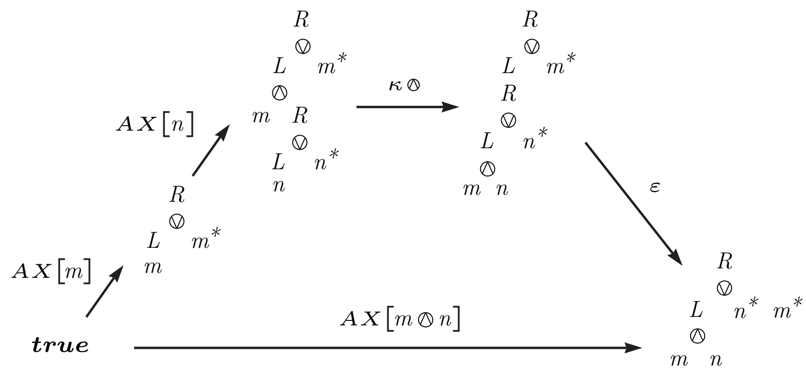
each of them parametrized by the objects m of the category $\mathscr{A}$. These morphisms are required to make the three coherence diagrams below commute. The first coherence diagram adapts the usual triangular axiom of adjunctions:

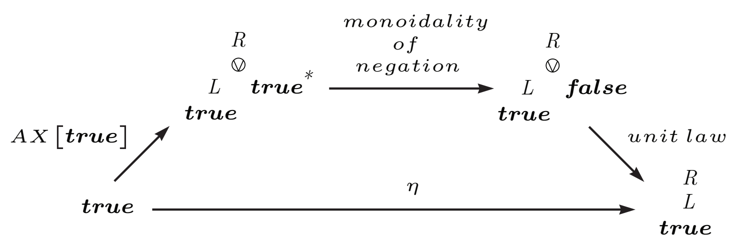


The second coherence diagram means that the family of combinators $\mathbf{AX}[-]$ is dinatural:



The third and fourth coherence diagrams express monoidality of the family $\mathbf{AX}[-]$:





The four coherence diagrams hold for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

6.3 Main result

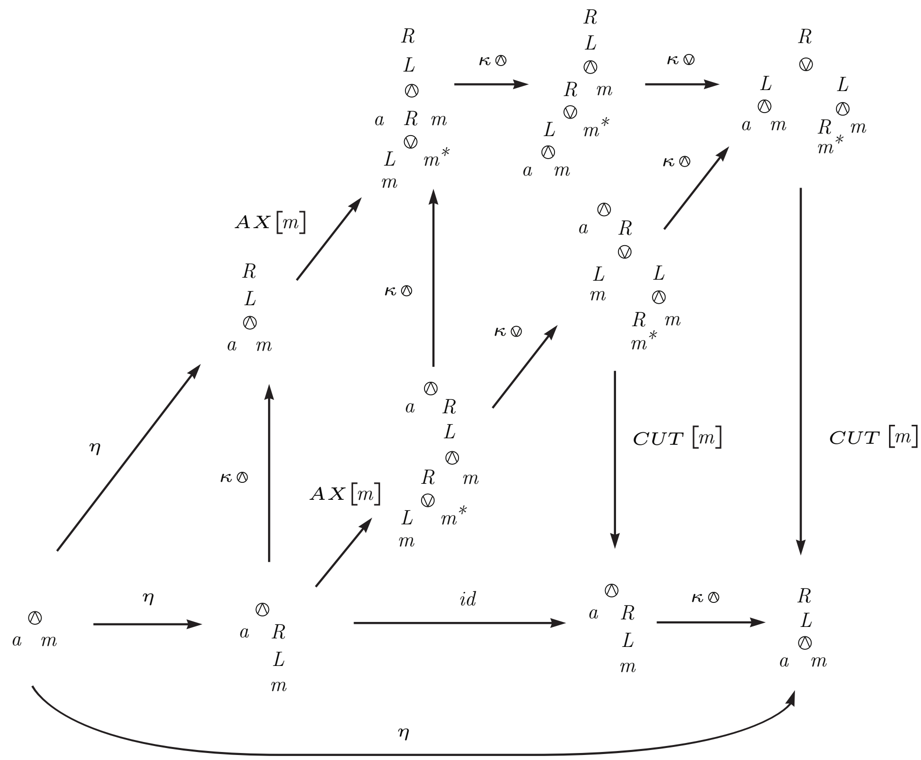
Proposition 5 *Every discursive pair $(\mathscr{A}, \mathscr{B})$ equipped with a duality defines a dialogue chirality.*

The two categorical combinators $\mathbf{axiom}[-]$ and $\mathbf{cut}[-]$ are defined in such as way as to make the diagrams below commute:

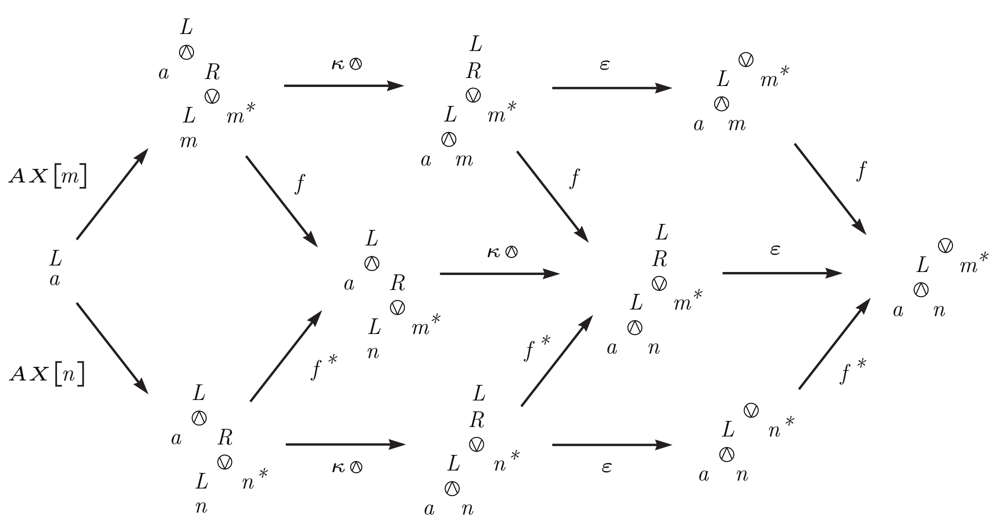
$$\begin{tikzcd}
L(a \owedge R(L(m) \ovee m^*)) \arrow[r, "\kappa^{\owedge}"] & LR(L(a \owedge m) \ovee m^*) \arrow[d, "\varepsilon"] \\
L(a) \arrow[u, "\mathbf{AX}[m]"] \arrow[r, "\mathbf{axiom}[m]"] & L(a \owedge m) \ovee m^*
\end{tikzcd}$$

$$\begin{tikzcd}
R(L(R(b \ovee m^*) \owedge m)) \arrow[r, "\kappa^{\ovee}"] & R(b \ovee L(R(m^*) \owedge m)) \arrow[d, "\mathbf{CUT}[m]"] \\
R(b \ovee m^*) \owedge m \arrow[u, "\eta"] \arrow[r, "\mathbf{cut}[m]"] & R(b)
\end{tikzcd}$$

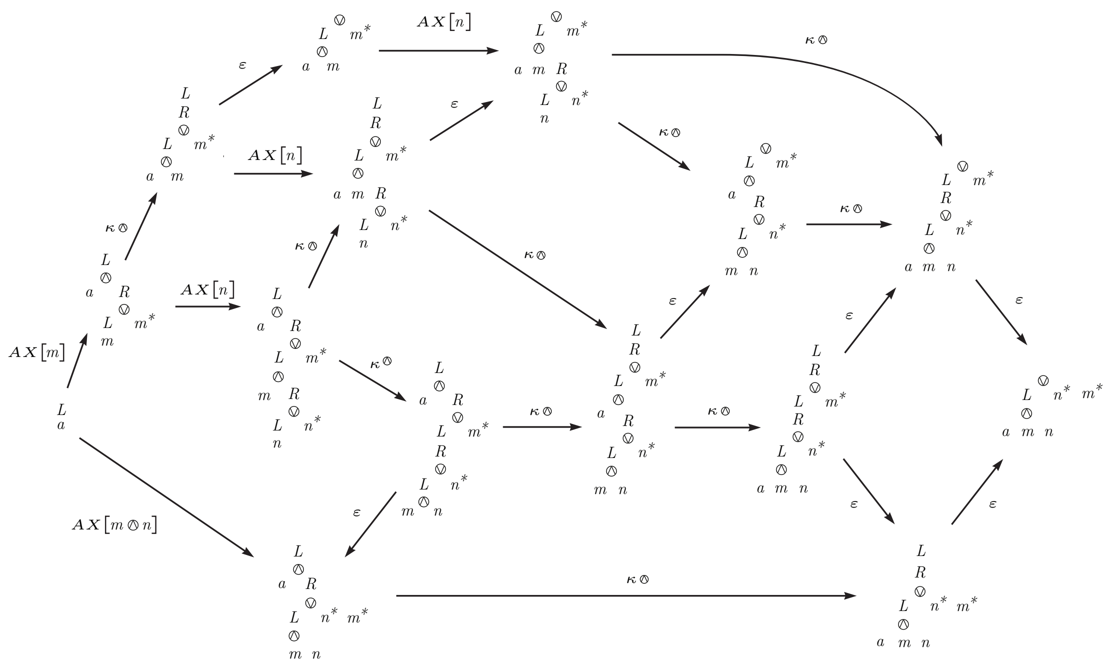
The very definition of the families axiom[−] and cut[−] ensures that they are natural in a and b respectively. The fact that each pair defines a transjunction is established by the diagram chase below



One establishes by a similar diagram chase the two additional facts that the family axiom[−] is dinatural in m:



and at the same time monoidal:



This concludes the proof that the pair axiom$[-]$ and cut$[-]$ provides a dialogue chirality structure to the original discursive pair $(\mathscr{A}, \mathscr{B})$. In particular,

Corollary 6 *In any discursive pair $(\mathscr{A}, \mathscr{B})$ equipped with a duality, the functor*

$$L(-\owedge m) \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is left adjoint to the functor

$$R(-\ovee m^*) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$$

for every object m of the category $\mathscr{A}$.

Remark. An important observation at this point is that the discursive pair $(\mathscr{A}, \mathscr{B})$ cannot be recovered from the chirality structure, since the chirality structure deals with right actions of $\mathscr{A}$ and $\mathscr{B}$, whereas the discursive structure (19) is concerned with their left actions. A natural way to resolve this problem is to start from a discursive pair $(\mathscr{A}, \mathscr{B})$ generated by left and right actions, and then to reconstruct an equivalent notion of dialogue chirality also based on left and right actions. However, the problem of properly correlating the left and right actions of $\mathscr{A}$ and $\mathscr{B}$ is far more subtle than it seems. In particular, it requires to introduce a natural isomorphism between the two negations

$$\mathbf{turn}_A \quad : \quad A \multimap \bot \quad \longrightarrow \quad \bot \circ\!\!- A$$

satisfying a series of appropriate coherence diagrams. This leads to the axiomatic investigation of helical, cyclic, braided or symmetric notions of dialogue category. This is done in a companion paper.

Remark. Note that the last coherence axiom required of the right duality is not used in the proof of Proposition 3. We keep it because it is a very natural requirement to ask, and because it is not clear that it follows from the three other coherence diagrams — in contrast to its counterpart diagram (15) in the definition of dialogue categories based on transjunctions.

6.4 Illustration

An important example of discursive pair is provided by the notion of *linearly distributive category* studied by Blute, Cockett, Seely and Trimble. One recovers a theorem by Cockett and Seely

Proposition 7 *A linearly distributive category is the same thing as a discursive pair where $\mathscr{A} = \mathscr{B}$ and where the two functors L and R are identity functors.*

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7 Director's cut – removed from the published paper

7.1 Part of the introduction

Now, suppose that we are given a lax 2-monad T on the 2-category CAT of categories. As the reader will see, we will be more specifically interested in the weak 2-monad

$$T \quad : \quad \mathscr{X} \quad \mapsto \quad \mathscr{B} \times \mathscr{X} \quad : \quad \mathbf{CAT} \quad \longrightarrow \quad \mathbf{CAT} \tag{20}$$

whose lax T-algebras are the lax actions

$$* \quad : \quad \mathscr{B} \times \mathscr{X} \quad \longrightarrow \quad \mathscr{X}$$

of the category $\mathscr{B} = \mathscr{C}^{\,op(0,1)}$ on the left, possibly seen as the lax actions of the category $\mathscr{C}^{\,op(1)}$ on the right. A particular example of such a lax action is provided by the weak action

$$\otimes \quad : \quad \mathscr{B} \times \mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

of the monoidal category $\mathscr{B}$ over itself. Now, given an adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathscr{B} \arrow[ll, bend left, "R"] \end{tikzcd} \tag{21}$$

and such a lax 2-monad T, a general *transfer theorem* establishes that every lax T-algebra structure

$$* \quad : \quad T\mathscr{B} \quad \longrightarrow \quad \mathscr{B}$$

on the category $\mathscr{B}$ induces a lax T-algebra structure on the category $\mathscr{A}$, defined as follows:

$$\circledast \quad : \quad T\mathscr{A} \quad \xrightarrow{TL} \quad T\mathscr{B} \quad \xrightarrow{*} \quad \mathscr{B} \quad \xrightarrow{R} \quad \mathscr{A}\,.$$

This transfer theorem may be established by purely equational means, and thus works in any 2-category $\mathscr{V}$ equipped with a lax 2-monad T. It appears that when T is the identity 2-monad in such a 2-category $\mathscr{V}$, the notion of lax T-algebra coincides with the notion of formal monad. So, the transfer theorem applied to a formal adjunction (21) implies that every formal monad S in $\mathscr{B}$ is transported to a formal monad S' in $\mathscr{A}$, defined as $S' = R \circ S \circ L$. Observe that in the case $S = id$, one recovers the fact that $R \circ L$ defines a formal monad in $\mathscr{A}$.

The transfer theorem may be also applied to the free monoidal category monad T in the 2-category **CAT**, and to the specific adjunction $L \dashv R$ between the negations of a dialogue category. In that case, it enables to transfer the monoidal structure $(\oslash, \mathrm{false})$ of the category $\mathscr{B}$ to a lax monoidal structure of the category $\mathscr{A}$. This lax monoidal structure is provided by the family of n-ary disjunctions

$$[\, A_1 \parr \cdots \parr A_n \,] \quad = \quad R\,(\, LA_1 \oslash \cdots \oslash LA_n \,)$$

which generalizes to any dialogue category the familiar definition (5) of $\parr$ in linear logic. An essential aspect of this definition of disjunction in tensorial logic is that it requires to replace the binary disjunction of linear logic by a family of n-ary disjunctions. This is done in order to recover the expected associativity property. The point is that the tensorial version of binary disjunction is not associative in the expected sense, since the two formulas

$$[\,[\, A \parr B \,] \parr C \,] \qquad\qquad [\, A \parr [\, B \parr C \,]\,]$$

are not required to be isomorphic. However, the family of n-ary disjunctions is associative in a more subtle and oriented way. Typically, there are canonical proofs of tensorial logic connecting the two clusters of binary disjunctions above to the ternary disjunction:

$$[\,[\, A \parr B \,] \parr C \,] \quad \longrightarrow \quad [\, A \parr B \parr C \,] \quad \longleftarrow \quad [\, A \parr [\, B \parr C \,]\,]$$

One benefit of this 2-categorical analysis is to explain in what sense the linear disjunction $\parr$ living in the dialogue category $\mathscr{A} = \mathscr{C}$ is derived by *deformation* — one should probably say by *adjunction* in that case — from the disjunction $\oslash$ living in the opposite category $\mathscr{B} = \mathscr{C}^{\,op(0,1)}$.

Finally, the transfer theorem may be applied to the weak 2-monad (20) and to the weak action of the monoidal category $(\mathscr{B}, \oslash, \mathrm{false})$ on itself on the left. From this, one derives a lax action on the left

$$\circledast \quad : \quad \mathscr{B} \times \mathscr{A} \quad \longrightarrow \quad \mathscr{A}$$

of the monoidal category $(\mathscr{B}, \oslash, \mathrm{false})$ on the category $\mathscr{A}$ which happens to coincide with the functor

$$b \circledast a \quad = \quad R\,(\, b \oslash La \,)$$

we started from.

7.2 Adjunction homomorphisms

A homomorphism between two adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of 1-dimensional cells

$$F : \mathscr{A}_1 \longrightarrow \mathscr{A}_2 \qquad\qquad G : \mathscr{B}_1 \longrightarrow \mathscr{B}_2$$

together with a pair of 2-dimensional cells

$$F \circ R_1 \Rightarrow R_2 \circ G \quad : \quad \mathscr{B}_1 \longrightarrow \mathscr{A}_2$$

$$L_2 \circ F \Rightarrow G \circ L_1 \quad : \quad \mathscr{A}_1 \longrightarrow \mathscr{B}_2$$

One requires moreover that the two 2-dimensional cells are mates, this meaning that the diagram below commutes:

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta_1"] \arrow[d, Rightarrow, "\eta_2"'] & F \circ R_1 \circ L_1 \arrow[d, Rightarrow] \\ R_2 \circ L_2 \circ F \arrow[r, Rightarrow] & R_2 \circ G \circ L_1 \end{tikzcd} \tag{22}$$

or equivalently, that the diagram below commutes:

$$\begin{tikzcd} L_2 \circ F \circ R_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & L_2 \circ R_2 \circ G \arrow[d, Rightarrow, "\varepsilon_2"] \\ G \circ L_1 \circ R_1 \arrow[r, Rightarrow, "\varepsilon_1"] & G \end{tikzcd} \tag{23}$$

It is not difficult to check that this additional coherence axiom is equivalent when $\mathscr{V} = \boldsymbol{Cat}$ to the requirement that the diagram below commutes

$$\begin{tikzcd} \mathscr{B}_1(L_1(a), b) \arrow[r, "\varphi_{a,b}"] \arrow[d, "G"'] & \mathscr{A}_1(a, R_1(b)) \arrow[d, "F"] \\ \mathscr{B}_2(G \circ L_1(a), G(b)) \arrow[d] & \mathscr{A}_2(F(a), F(R_1(b))) \arrow[d] \\ \mathscr{B}_2(L_2 \circ F(a), G(b)) \arrow[r, "\varphi_{F(a),G(b)}"] & \mathscr{A}_2(F(a), R_2 \circ G(b)) \end{tikzcd} \tag{24}$$

for all objects a of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$. Here, the greek letter φ is used generically in order to denote the natural bijection between the presheaves

$$\varphi_{a,b} \quad : \quad \mathscr{B}(La, b) \cong \mathscr{A}(a, Rb) \quad : \quad \mathscr{A}^{op} \times \mathscr{B} \quad \longrightarrow \quad Set$$

induced from an adjunction $L \dashv R$ with left adjoint $L : \mathscr{A} \longrightarrow \mathscr{B}$. Note that the reformulation of (22) as (24) also works in any 2-category $\mathscr{V}$, at the price however of carefully replacing the objects a, b of the categories $\mathscr{A}, \mathscr{B}$ by arbitrary 1-dimensional cells $a : \mathscr{X} \longrightarrow \mathscr{A}$ and $b : \mathscr{X} \longrightarrow \mathscr{B}$, playing there the role of generalized elements of the 0-dimensional cells $\mathscr{A}$ and $\mathscr{B}$. See [**?**] for details.

7.3 A 2-category of dialogue chiralities

The 1-dimensional cells. A 1-dimensional cell in

$$F \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a quadriple $(F_\bullet, F_\circ, \widetilde{F}, \overline{F})$ consisting of a lax monoidal functor

$$F_\bullet \quad : \quad \mathscr{A}_1 \quad \longrightarrow \quad \mathscr{A}_2$$

an oplax monoidal functor

$$F_\circ \quad : \quad \mathscr{B}_1 \quad \longrightarrow \quad \mathscr{B}_2$$

a monoidal natural isomorphism

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] \arrow[d, "(-)^*"'] \arrow[dr, phantom, "\widetilde{F}\ \Swarrow"] & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op(0,1)} \arrow[r, "F_\circ^{op(0,1)}"'] & \mathscr{B}_2^{op(0,1)}
\end{tikzcd}$$

and a natural transformation

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] & \mathscr{A}_2 \\
\mathscr{B}_1 \arrow[u, "R"] \arrow[r, "F_\circ"'] \arrow[ur, phantom, "\Searrow \overline{F}"] & \mathscr{B}_2 \arrow[u, "R"']
\end{tikzcd}$$

making the pair of functors $F_\bullet$ and $F_\circ$ together with the natural transformations

$$F_\bullet \circ R_1 \circ (- \oslash m^*) \quad \Rightarrow \quad R_2 \circ (- \oslash (F_\bullet m)^*) \circ F_\circ$$

$$L_2 \circ (- \owedge F_\bullet m) \circ F_\bullet \quad \Rightarrow \quad F_\circ \circ L_1 \circ (- \owedge m)$$

a homomorphism between the adjunctions

$$L_1(- \owedge m) \dashv R_1(- \ovee m^*) \qquad\qquad L_2(- \owedge F_\bullet m) \dashv R_2(- \ovee (F_\bullet m)^*)$$

for every object m of the category $\mathscr{A}_1$.

The 2-dimensional cells. A 2-dimensional cell in

$$\theta \quad : \quad F \Rightarrow G \quad : \quad (\mathscr{A}_1, \mathscr{B}_1) \quad \longrightarrow \quad (\mathscr{A}_2, \mathscr{B}_2)$$

is defined as a pair $(\theta_\bullet, \theta_\circ)$ of monoidal natural transformations $\theta_\bullet : F_\bullet \Rightarrow G_\bullet$ and $\theta_\circ : G_\circ \Rightarrow F_\circ$ satisfying the two equalities below:

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[rr, bend left, "F_\bullet"] \arrow[rr, bend right, "G_\bullet"'] \arrow[rr, phantom, "\Downarrow\theta_\bullet"] \arrow[d, "(-)^*"'] & & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op(0,1)} \arrow[rr, bend right, "G_\circ^{op(0,1)}"'] \arrow[rr, phantom, "\Swarrow\widetilde{G}"] & & \mathscr{B}_2^{op(0,1)}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A}_1 \arrow[rr, bend left, "F_\bullet"] \arrow[d, "(-)^*"'] \arrow[drr, phantom, "\Swarrow\widetilde{F}"] & & \mathscr{A}_2 \arrow[d, "(-)^*"] \\
\mathscr{B}_1^{op(0,1)} \arrow[rr, bend left, "F_\circ^{op(0,1)}"] \arrow[rr, bend right, "G_\circ^{op(0,1)}"'] \arrow[rr, phantom, "\Downarrow\theta_\circ^{op(0,1)}"] & & \mathscr{B}_2^{op(0,1)}
\end{tikzcd}
\qquad (25)$$

as well as the equation (26) below.

$$\begin{tikzcd}
\mathscr{A}_1 \arrow[r, "F_\bullet"] & \mathscr{A}_2 \\
\mathscr{B}_1 \arrow[u, "R"] \arrow[r, "F_\circ"'] \arrow[ur, phantom, "\overline{F}"] & \mathscr{B}_2 \arrow[u, "R"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A}_1 \arrow[r, bend left=60, "F_\bullet"] \arrow[r, "G_\bullet"'] \arrow[r, phantom, bend left=30, "\Downarrow\theta_\bullet"] & \mathscr{A}_2 \\
\mathscr{B}_1 \arrow[u, "R"] \arrow[r, "G_\circ"] \arrow[r, bend right=60, "F_\circ"'] \arrow[r, phantom, bend right=30, "\Downarrow\theta_\circ"] \arrow[ur, phantom, "\overline{G}"] & \mathscr{B}_2 \arrow[u, "R"']
\end{tikzcd}
\qquad (26)$$

The 1-dimensional cells. A homomorphism of chirality is a pair of functors

....

and a natural transformation...

...

such that

$$\cdots$$

defines a homomorphism of adjunctions

$$L_1(- \owedge m) \vdash R_1(- \ovee m^*) \quad \longrightarrow \quad L_2(- \owedge F(m)) \vdash R_1(- \ovee F'(m^*))$$

This including diagrams.