A micrological study of negation

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Abstract

The purpose of this paper is to develop a purely combinatorial presentation of dialogue categories, based on the symmetric notion of linearly distributive chirality. As such, the micrological analysis may be understood as performing a chemical decomposition into primary elements of the molecular notion of negation in tensorial logic.

1 Introduction

One primary purpose of our work in tensorial logic is to transfer the ideas of linear logic to this more primitive logic of tensor and negation, where negation $A \mapsto \neg A$ is not required to be involutive anymore. An amusing but also disturbing aspect of this research program is that it looks perfectly meaningless on the surface, at least when one considers it with the prevailing spectacles of linear logic. Indeed, seen from this angle, the project immediately bumps against the objection that tensorial logic lacks the nice "classical" symmetries of linear logic. The misunderstanding is deeply rooted in the tradition, and it cannot be overtaken without recasting the very foundations of linear logic in a radically different way. This task is accomplished in the present paper by explaining that the so-called "classical" symmetries of linear logic are not specific to "classical" systems like classical logic or linear logic. On the contrary, we advocate that these symmetries are an intrinsic principle of logic itself, and apply for instance to a system like intuitionistic logic. Once this new picture of logic spelled out, the rigid separation between classical logic and intuitionistic logic appears suddenly obsolete – and our research program becomes meaningful

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again. In particular, tensorial logic benefits of the symmetries of classical logic, just like any other reasonable logical system.

As already mentioned, this revised picture of logic emerges from the project of adapting to tensorial logic the “classical” symmetries of linear logic. Our starting point in this investigation is provided by the notion of $*$-autonomous category — the categorical alter-ego of linear logic — traditionally defined as a symmetric monoidal closed category $\mathscr{C}$ equipped with a dualizing object $\bot$. Recall that an object $\bot$ of a symmetric monoidal closed category $\mathscr{C}$ is called *dualizing* when the canonical morphism

$$x \quad \longrightarrow \quad (x \multimap \bot) \multimap \bot$$

transporting x into its double negation is an isomorphism, for every object x of the category. The terminology itself of “dualizing object” comes from the fact that the negation functor

$$(-)^{\bot} \quad : \quad A \quad \mapsto \quad A \multimap \bot \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}^{op}$$

defines in that case an equivalence

$$\mathscr{C} \quad \underset{(-)^{\bot}}{\overset{(-)^{\bot}}{\rightleftarrows}} \quad \mathscr{C}^{op} \qquad \textit{equivalence} \tag{1}$$

between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. This establishes that every $*$-autonomous category $\mathscr{C}$ is self-dual, in the technical sense that it is equivalent to its opposite category.

Although this definition of $*$-autonomous is prevailing, it is not “symmetric” in the sense that it starts from the conjunction ($\otimes$) and the implication ($\multimap$) provided by the underlying monoidal closed category, rather than from the conjunction ($\otimes$) and the disjunction ($\invamp$) provided by linear logic. In particular, in this non-symmetric formulation of linear logic, the disjunction is deduced from the implication by the equality

$$x \invamp y \quad := \quad ((y \multimap \bot) \otimes (x \multimap \bot)) \multimap \bot$$

which holds in any $*$-autonomous category. From a purely esthaetic point of view, this non-symmetric presentation of a perfectly self-dual logic like linear logic looks awkward, and one wonders whether it may be replaced by a more symmetric presentation. The matter was resolved in a very interesting and elegant way by Cockett and Seely with the notion of *linearly distributive category*. Recall from [2, 3] that a linearly distributive category is defined as a category $\mathscr{C}$ equipped with two monoidal structures $(\mathscr{C}, \otimes, 1)$ and $(\mathscr{C}, \parr, \bot)$ together with a pair of distributivity laws

$$\begin{array}{lcccc} \kappa^R_{x,y,z} & : & (x \parr y) \otimes z & \longrightarrow & x \parr (y \otimes z) \\ \kappa^L_{x,y,z} & : & x \otimes (y \parr z) & \longrightarrow & (x \otimes y) \parr z \end{array} \qquad (2)$$

between the two tensor products, which satisfy a series of coherence diagrams recalled in the paper. The four authors establish then a nice theorem, which states that a $*$-autonomous category is the same thing as a (symmetric) linearly distributive category where every object m comes equipped with a dual object m^* together with two maps

$$\begin{array}{lcccc} \mathbf{AX}[m] & : & 1 & \longrightarrow & m^* \parr m \\ \mathbf{CUT}[m] & : & m \otimes m^* & \longrightarrow & \bot \end{array} \qquad (3)$$

satisfying a series of well-chosen coherence diagrams. This alternative presentation of $*$-autonomous categories is purely combinatorial, and provides a perfectly symmetric formulation of the conjunction ($\otimes$) and the disjunction ($\parr$) of linear logic. A fascinating aspect of the approach is that the notion of linearly distributive category does not require the category $\mathscr{C}$ to be self-dual. As a matter of fact, in this particular presentation of $*$-autonomous categories, self-duality comes only at a later stage, when one requires that every object x of the linearly distributive category $\mathscr{C}$ comes equipped with a dual object x^*. As already explained, this forces the category $\mathscr{C}$ to be $*$-autonomous, and thus self-dual.

Now, the notion of *dialogue category* provides a categorical counterpart to tensorial logic, in the same way as the notion of $*$-autonomous category does for linear logic. So, if one really believes in this apparently extravagant idea that tensorial logic is a refinement of linear logic — rather than simply a fragment of it — then there should exist a way to present dialogue categories in a similarly symmetric fashion. As we already pointed out, the problem is that this idea does not make sense when one looks at it with the spectacles of linear logic. Recall that a dialogue category is a monoidal category equipped with an object $\bot$ and two natural isomorphisms

$$\begin{array}{lcccc} \varphi_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \cong & \mathscr{C}(y, x \multimap \bot) \\ \psi_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \cong & \mathscr{C}(x, \bot \multimapinv y) \end{array}$$

The dialogue category is called symmetric when the underlying monoidal category is symmetric. So, if one wants to adapt the notion of linearly distributive category to dialogue categories, one needs

a. to understand how and where the conjunction ($\otimes$) and the disjunction ($\parr$) should be interpreted in a dialogue category,

b. to separate very carefully the self-duality (for what it means) from the conjunction ($\otimes$) and the disjunction ($\parr$) of the dialogue category, in the same way as the self-duality is removed from $*$-autonomous categories in order to get linearly distributive categories.

Again, all this does not make a lot of sense at this stage, at least in the way it is formulated: one does not see how to interpret the disjunction ($\parr$) in a dialogue category, and probably even worse, one does not see how to interpret the self-duality. So, one needs to change spectacles at this point, and to analyze the problem in a deeper fashion, nurtured by 2-dimensional algebra. The first thing to observe is that every dialogue category comes with an adjunction

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}^{op} \qquad (L \dashv R) \tag{4}$$

between the two negation functors

$$L \;:\; x \;\mapsto\; \bot \multimapinv x \qquad\qquad R \;:\; x \;\mapsto\; x \multimap \bot$$

This adjunction induces in turn a monad

$$T \quad : \quad x \quad \mapsto \quad (\bot \multimapinv x) \multimap \bot \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

defined by double negation on the dialogue category $\mathscr{C}$. Here, it is worth mentioning that a $*$-autonomous category is the same thing as a symmetric dialogue category where the unit η of the double negation monad T instantiated at x

$$\eta_x \quad : \quad x \quad \longrightarrow \quad (\bot \multimapinv x) \multimap \bot$$

is invertible for every object x of the category. Equivalently, a $*$-autonomous category is a symmetric dialogue category where the adjunction (4) is an equivalence. This establishes that one does not need to start from a monoidal closed category in order to define a $*$-autonomous category: the weaker notion of dialogue category is sufficient for that task.

At this point, a tentative solution for problem a. emerges from the contemplation of the adjunction (4) between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. The idea is to interpret the conjunction ($\otimes$) as the tensor product taken in the dialogue category $\mathscr{C}$ and the disjunction ($\parr$) as the tensor product taken in the opposite dialogue category $\mathscr{C}^{op(0,1)}$. Here, we write $\mathscr{C}^{op(0,1)}$ for the monoidal category $\mathscr{C}$ where the orientation of tensors (dimension 0) and of morphisms (dimension 1) have been reversed. After this appealing but also very speculative resolution of problem a. we are facing two additional and quite serious difficulties:

c. on the one hand, the conjunction ($\otimes$) and the disjunction ($\parr$) do not live in the same category, since the conjunction ($\otimes$) lives in the category $\mathscr{C}$ and the disjunction ($\parr$) lives in its opposite category,

d. on the other hand, it is difficult to understand in what sense the conjunction ($\otimes$) is really different from the disjunction ($\parr$) since they are both defined as the tensor product of the category $\mathscr{C}$ and of the category $\mathscr{C}^{op}$, and consequently, they only differ modulo the apparentlyconventional difference between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$.

We will see how to resolve the three difficulties b. c. and d. in three steps. The first step is purely notational. It consists in writing

$$\mathscr{A} \quad = \quad (\mathscr{A}, \owedge, true)$$

for the monoidal category $\mathscr{C}$ and

$$\mathscr{B} \quad = \quad (\mathscr{B}, \ovee, false)$$

for its opposite category $\mathscr{C}^{op(0,1)}$. This resolves problem d. or at least overcomes it, since the two categories $\mathscr{A}$ and $\mathscr{B}$ are considered from now on as intrinsically different, although they are "secretly" related by the identity

$$\mathscr{B}^{op(0,1)} \quad = \quad \mathscr{A}. \tag{5}$$

Accordingly, and for clarity's sake, we choose to write conjunction as $\owedge$ in the category $\mathscr{A}$ and disjunction as $\ovee$ in the category $\mathscr{B}$. In order to resolve problem c. we recast the original adjunction (4) as an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathscr{B} \tag{6}$$

between the categories $\mathscr{A}$ and $\mathscr{B}$. In that way, the conjunction ($\owedge$) and the disjunction ($\ovee$) live in *different* categories $\mathscr{A}$ and $\mathscr{B}$ related by the functors L and R of the adjunction (6).

Our ongoing analysis is sufficiently advanced at this stage to provide the ground for a relaxed notion of "linearly distributive category" adapted to dialogue categories. The idea is that the linearly distributive category $\mathscr{C}$ together with its two tensor products $\otimes$ and $\parr$ should be replaced in the case of tensorial logic by a pair of monoidal categories

$$(\mathscr{A}, \oslash, true) \qquad\qquad (\mathscr{B}, \varobslash, false)$$

equipped with an adjunction (6) relating them. In addition, the original pair of distributivity laws (2) should be replaced by the pair of distributivity laws

$$\begin{array}{cccccc} \kappa^{\oslash} & : & R(b \varobslash L(a)) \oslash m & \longrightarrow & R(b \varobslash L(a \oslash m)) \\ \kappa^{\varobslash} & : & L(a \oslash R(b \varobslash n)) & \longrightarrow & L(a \oslash R(b)) \varobslash n \end{array} \tag{7}$$

which happen to exist (and this is the fundamental point) in *every* dialogue category. This leads to the notion of *linearly distributive chirality* introduced in the technical core of the paper (§5). The notion refines the notion of linearly distributive category, in the same way as the notion of dialogue category refines the notion of $*$-autonomous category. Note that the resulting notion of chirality is *mixed* rather than pure, in the sense that the side $\mathscr{B}$ of the chirality is not required to be equivalence to the opposite of the side $\mathscr{A}$, see [7] for a discussion about the distinction between mixed and. pure chiralities. The notion of linearly distributive chirality is justified by the two basic observations:

- every dialogue category $\mathscr{C}$ defines a linearly distributive chirality with sides $\mathscr{A}$ and $\mathscr{B}$ defined as $\mathscr{A} = \mathscr{C}$ and $\mathscr{B} = \mathscr{C}^{op(0,1)}$ and where the functors L and R are defined as the negation functors as indicated above,

- a linearly distributive category $\mathscr{C}$ is the same thing as a linearly distributive chirality where the two sides $\mathscr{A}$ and $\mathscr{B}$ are defined as the category $\mathscr{C}$ and where the two functors L and R are defined as the identity functor between the category $\mathscr{C}$ and itself.

It is worth stressing the perfect symmetry between the two distributivity laws $\kappa^{\oslash}$ and $\kappa^{\varobslash}$. This symmetry is witnessed by the fact that each of the two laws may be obtained from the other one by applying the involution

$$R \leftrightarrow L \qquad x \oslash y \leftrightarrow x \varobslash y \qquad a \leftrightarrow b \qquad m \leftrightarrow n$$

and by reversing the orientation of the map. In the particular case of a linearly distributive chirality $(\mathscr{C}, \mathscr{C})$ with $L = R = id_{\mathscr{C}}$ induced by a linearly distributive category, this perfect symmetry between $\kappa^{\owedge}$ and $\kappa^{\ovee}$ boils down to a "self-symmetry" of the original linear distributivity (2) or more precisely to the symmetry between the left and right variants κ^R and κ^L.

Now that we have a counterpart to the notion of linearly distributive category, we are very close to the resolution of problem b. and to understand how to adapt the notion of self-duality of $*$-autonomous categories. The idea is to take the identity (5) very seriously, and to relax it to a monoidal equivalence

$$\mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \quad \text{(monoidal equivalence)} \tag{8}$$

between the monoidal categories $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$ respectively equipped with conjunction ($\owedge$) and the disjunction ($\ovee^{op}$) oriented right-to-left. One motivation for relaxing the identity (5) into an equivalence (8) is to include the original notion of *duality* in a linearly distributive category [3]. In the case of linearly distributive categories, the negation functor is not required to be strictly involutive — but only up to equivalence — as it is the case in a $*$-autonomous category. In that way, the notion of duality on a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ includes the two cases we have in mind:

- in the case of a dialogue category $\mathscr{C}$, the equivalence is defined by taking the identity functors between $\mathscr{C}$ and itself,
- in the case of a $*$-autonomous category $\mathscr{C}$, the equivalence is defined by taking the two negation functors between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op(0,1)}$.

At this stage, we have finally resolved the four obstructions a. b. c. and d. to the very idea of presenting the notion of dialogue category in a nice and symmetric fashion. In order to complete our program, there remains to formulate a variant of duality refining (3) between an object m in the category $\mathscr{A}$ and an object m^* in the category $\mathscr{B}$. We will see that this two-sided notion of duality is based on the existence of two combinators

$$\begin{array}{rcccc} \mathbf{AX}[m] & : & \mathit{true} & \longrightarrow & R(m^* \ovee L(m)) \\ \mathbf{CUT}[m] & : & L(m \owedge R(m^*)) & \longrightarrow & \mathit{false} \end{array}$$

which happen to coincide in the one-sided case with the combinators of (3) since L and R are defined in that case as the identity functors between the linearly distributive category $\mathscr{C}$ and itself. The existence of such a duality for every object m of the category $\mathscr{A}$ implies that $\mathscr{A}$ defines a dialogue category, at least when $\mathscr{A}$ and $\mathscr{B}$ are not just monoidal, but also symmetric monoidal.

What have we learned from this brief excursion accross the universe of logical symmetries? Well, it revealed to us that every logical category $\mathscr{A} = \mathscr{C}$ (typically provided by a category of proofs) comes with an opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$ (typically provided by a category of refutations) and that every formula may be either seen from the point of view of $\mathscr{A}$ or from the point of view of $\mathscr{B}$. This change of frame between $\mathscr{A}$ and $\mathscr{B}$ is precisely what the "classical" involutive negation of logic $x \mapsto x^*$ captures. It also appeared under inspection that this symmetry between $\mathscr{A}$ and $\mathscr{B}$ has nothing to do with classical logic or with linear logic. There lies a general "relativity" principle in logic which reflects the fact that every dispute involves a Player (or Prover) and an Opponent (or Refutator) and that each of them looks at the formula from its own point of view. As we have seen, classical logic or linear logic is simply the very particular case when the universe $\mathscr{A}$ of the Player happens to coincide with the universe $\mathscr{B}$ of the Opponent, up to equivalence.

It should be finally acknowledged how much this symmetric account of logic owes to the technical shift from 1-dimensional to 2-dimensional categorical semantics. This shift to 2-categories enables to articulate the fact that the primitive symmetry between Player and Opponent lives at a 2-dimensional level, rather than at a 1-dimensional level — and that the primary duality of logic is thus provided by the 2-dimensional operation of "reversing" a category $\mathscr{C}$ into its opposite category $\mathscr{C}^{op(0,1)}$. This purely algebraic observation reflects this linguistic and possibly ethological principle that *symmetry comes before logic*. Understood from that point of view, every logical system offers a specific symbolic regime in order to accomodate this primitive symmetry between Player and Opponent, and to make the two sides of the dispute interact.

This new scenography of logic based on 2-categories is arguably more uniform and harmonious than the existing one with its rigid separation between classical and intuitionistic logic. It relies on a microcosm principle formulated in [7] and inspired by a similar phenomenon in higher dimensional algebra stressed by Baez and Dolan [1]. When applied to logic, the microcosm principle tells us that any contravariant operation — like negation or implication — relies in the end on the primary duality between $\mathscr{C}$ and $\mathscr{C}^{op(0,1)}$. In retrospect, linear logic appears as the miraculous encounter between this primary 2-dimensional duality between $\mathscr{C}$ and $\mathscr{C}^{op(0,1)}$ and the purely 1-dimensional (that is, categorical) notion of $*$-autonomous category where the category $\mathscr{C}$ is equivalent to its opposite category $\mathscr{C}^{op(0,1)}$. One purpose of the present work is to dissect this specific situation in order to extend it to tensorial logic, and to bring to light its primary structure based on the 2-dimensional duality $\mathscr{C} \mapsto \mathscr{C}^{op(0,1)}$ and its refraction in a 1-dimensional category.

Plan of the paper. We start by recalling in §2 the notion of linearly distributive category introduced by Cockett and Seely in [2] together with the notion of right duality we will be specifically interested in. We recall in §3 the notion of dialogue chirality introduced by the author in [7] as a symmetric and deformed notion of dialogue category. The notion of commutator between parametric monads is also recalled in §4. We reach in §5 the technical core of the paper, where we introduce the notion of linearly distributive chirality, and formulate the corresponding notion of right duality. We establish at the end of the section that the notion of dialogue chirality coincides with the notion of linearly distributive chirality with a right duality when the underlying monoidal categories are symmetric. We conclude in §6 by observing that a mismatch remains between the two notions of dialogue chirality and linearly distributive chirality in the case of non-symmetric monoidal categories. We leave the task of resolving this matter to the companion paper [9].

2 Linearly distributive categories

2.1 Definition

Recall from [2, 3] that a (planar) linearly distributive category $\mathscr{C}$ is defined as a category equipped with two monoidal structures, the first one called "tensor product" given by the bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ with unit 1 and natural isomorphisms

$$\alpha^{\otimes}_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C),$$

$$\lambda^{\otimes}_{A} : 1 \otimes A \longrightarrow A, \qquad \rho^{\otimes}_{A} : A \otimes 1 \longrightarrow A,$$

and the second one called "cotensor product" given by the bifunctor $\parr : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ with unit $\bot$ and natural isomorphisms

$$\alpha^{\parr}_{A,B,C} : (A \parr B) \parr C \longrightarrow A \parr (B \parr C),$$

$$\lambda^{\parr}_{A} : \bot \parr A \longrightarrow A, \qquad \rho^{\parr}_{A} : A \parr \bot \longrightarrow A.$$

Linearly distributive categories are moreover equipped with two natural morphisms

$$\kappa^{L}_{A,B,C} : A \otimes (B \parr C) \longrightarrow (A \otimes B) \parr C$$

$$\kappa^R_{A,B,C} : (A \parr B) \otimes C \longrightarrow A \parr (B \otimes C)$$

required to satisfy a series of commutativity axioms, consisting of six pentagons and four triangles. The six pentagons may be separated into three series of two pentagons, where the first series describes how the distributive law κ^R interacts with the associativity laws:

$$\begin{array}{ccccc}
((A \parr B) \otimes C) \otimes D & \xrightarrow{\kappa^R} & (A \parr (B \otimes C)) \otimes D & \xrightarrow{\kappa^R} & A \parr ((B \otimes C) \otimes D) \\
\downarrow{\scriptstyle \alpha^{\otimes}} & & & & \downarrow{\scriptstyle \alpha^{\otimes}} \\
(A \parr B) \otimes (C \otimes D) & & \xrightarrow{\qquad\qquad \kappa^R \qquad\qquad} & & A \parr (B \otimes (C \otimes D))
\end{array}$$

$$\begin{array}{ccccc}
(A \parr (B \parr C)) \otimes D & \xrightarrow{\kappa^R} & A \parr ((B \parr C) \otimes D) & \xrightarrow{\kappa^R} & A \parr (B \parr (C \otimes D)) \\
\downarrow{\scriptstyle \alpha^{\parr}} & & & & \downarrow{\scriptstyle \alpha^{\parr}} \\
((A \parr B) \parr C) \otimes D & & \xrightarrow{\qquad\qquad \kappa^R \qquad\qquad} & & (A \parr B) \parr (C \otimes D)
\end{array}$$

the second series describes how the distributive law κ^L interacts with the associativity laws:

$$\begin{array}{ccccc}
(A \otimes B) \otimes (C \parr D) & & \xrightarrow{\qquad\qquad \kappa^L \qquad\qquad} & & ((A \otimes B) \otimes C) \parr D \\
\downarrow{\scriptstyle \alpha^{\otimes}} & & & & \downarrow{\scriptstyle \alpha^{\otimes}} \\
A \otimes (B \otimes (C \parr D)) & \xrightarrow{\kappa^L} & A \otimes ((B \otimes C) \parr D) & \xrightarrow{\kappa^L} & (A \otimes (B \otimes C)) \parr D
\end{array}$$

$$\begin{array}{ccccc}
A \otimes (B \parr (C \parr D)) & & \xrightarrow{\qquad\qquad \kappa^L \qquad\qquad} & & (A \otimes B) \parr (C \parr D) \\
\downarrow{\scriptstyle \alpha^{\parr}} & & & & \downarrow{\scriptstyle \alpha^{\parr}} \\
A \otimes ((B \parr C) \parr D) & \xrightarrow{\kappa^L} & (A \otimes (B \parr C)) \parr D & \xrightarrow{\kappa^L} & ((A \otimes B) \parr C) \parr D
\end{array}$$

and the last series describes how κ^L and κ^R interact together with the associativity laws:

$$\begin{array}{ccccc}
(A \otimes (B \parr C)) \otimes D & \xrightarrow{\alpha^{\otimes}} & A \otimes ((B \parr C) \otimes D) & \xrightarrow{\kappa^R} & A \otimes (B \parr (C \otimes D)) \\
\downarrow{\scriptstyle \kappa^L} & & & & \downarrow{\scriptstyle \kappa^L} \\
((A \otimes B) \parr C) \otimes D & & \xrightarrow{\qquad\qquad \kappa^R \qquad\qquad} & & (A \otimes B) \parr (C \otimes D)
\end{array}$$

$$
\begin{array}{ccccc}
(A \parr B) \otimes (C \parr D) & \xrightarrow{\kappa^L} & ((A \parr B) \otimes C) \parr D & \xrightarrow{\kappa^R} & (A \parr (B \otimes C)) \parr D \\
\big\downarrow{\scriptstyle \kappa^R} & & & & \big\downarrow{\scriptstyle \alpha^{\parr}} \\
A \parr (B \otimes (C \parr D)) & & \xrightarrow{\qquad\qquad \kappa^L \qquad\qquad} & & A \parr ((B \otimes C) \parr D)
\end{array}
$$

Similarly, the four coherence triangles may be separated in two series of two triangles, where the first series describes how the distributive law κ^R interacts with the units:

$$
\begin{array}{ccc}
& A \otimes B & \\
{\scriptstyle \lambda^{\parr}}\swarrow & & \searrow{\scriptstyle \lambda^{\parr}} \\
(\bot \parr A) \otimes B & \xrightarrow{\kappa^R} & \bot \parr (A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
& A \parr B & \\
{\scriptstyle \rho^{\otimes}}\swarrow & & \searrow{\scriptstyle \rho^{\otimes}} \\
(A \parr B) \otimes 1 & \xrightarrow{\kappa^R} & A \parr (B \otimes 1)
\end{array}
$$

and the second series describes how the distributive law κ^L interacts with the units:

$$
\begin{array}{ccc}
1 \otimes (A \parr B) & \xrightarrow{\kappa^L} & (1 \otimes A) \parr B \\
{\scriptstyle \lambda^{\otimes}}\searrow & & \swarrow{\scriptstyle \lambda^{\otimes}} \\
& A \parr B &
\end{array}
\qquad
\begin{array}{ccc}
A \otimes (B \parr \bot) & \xrightarrow{\kappa^L} & (A \otimes B) \parr \bot \\
{\scriptstyle \rho^{\parr}}\searrow & & \swarrow{\scriptstyle \rho^{\parr}} \\
& A \otimes B &
\end{array}
$$

One requires that these diagrams commute for all objects A, B, C, D of the linearly distributive category $\mathscr{C}$.

2.2 Right duality in linearly distributive categories

A general notion of negation in a linearly distributive category $\mathscr{C}$ is introduced by Cockett and Seely in [2]. Here, we find convenient to keep only half of it, this leading us to the notion of *right duality* in a linearly distributive category $\mathscr{C}$, defined below.

Definition 1 *A right duality in a linearly distributive category $\mathscr{C}$ consists of the following data:*

- *an object A^*,*
- *two morphisms $ax^R_A : 1 \longrightarrow A^* \parr A$ and $cut^R_A : A \otimes A^* \longrightarrow \bot$*

for every object A of the category $\mathscr{C}$. The morphisms are moreover required to make the diagrams

$$
\begin{array}{ccc}
A \otimes 1 & \xrightarrow{A \otimes ax^R} & A \otimes (A^* \parr A) \\
& & \downarrow{\scriptstyle \kappa^L} \\
\downarrow{\scriptstyle \rho^{\otimes}} & & (A \otimes A^*) \parr A \\
& & \downarrow{\scriptstyle cut^R \parr A} \\
A & \xleftarrow{\lambda^{\parr}} & \bot \parr A
\end{array}
\qquad\qquad
\begin{array}{ccc}
1 \otimes A^* & \xrightarrow{ax^R \otimes A^*} & (A^* \parr A) \otimes A^* \\
& & \downarrow{\scriptstyle \kappa^R} \\
\downarrow{\scriptstyle \lambda^{\otimes}} & & A^* \parr (A \otimes A^*) \\
& & \downarrow{\scriptstyle A^* \parr cut^R} \\
A^* & \xleftarrow{\rho^{\parr}} & A^* \parr \bot
\end{array}
$$

commute.

Every such right duality defines a contravariant functor

$$A \mapsto A^* \quad : \quad \mathscr{C}^{op} \longrightarrow \mathscr{C}$$

which transports every morphism $f : A \longrightarrow B$ of the linearly distributive category $\mathscr{C}$ to the morphism $f^* : B^* \longrightarrow A^*$ constructed in the following way:

$$
\begin{array}{ccccccc}
B^* & & A^* \parr (A \otimes B^*) & \xrightarrow{A^* \parr (f \otimes B^*)} & A^* \parr (B \otimes B^*) & \xrightarrow{A^* \parr cut^R} & A^* \parr \bot \\
\downarrow{\scriptstyle (\lambda^{\otimes})^{-1}} & & \uparrow{\scriptstyle \kappa^R} & & \uparrow{\scriptstyle \kappa^R} & & \downarrow{\scriptstyle \rho^{\parr}} \\
1 \otimes B^* & \xrightarrow{ax^R \otimes B^*} & (A^* \parr A) \otimes B^* & \xrightarrow{(A^* \parr f) \otimes B^*} & (A^* \parr B) \otimes B^* & & A^*
\end{array}
$$

The functoriality of $A \mapsto A^*$ follows easily from the coherence diagrams.

2.3 Alternative formulation of $*$-autonomous categories

As explained in the introduction, one main purpose of the notion of linearly distributive category is to isolate the properties of $\otimes$ and $\parr$ from the properties of the duality in a $*$-autonomous category. In particular, the notion of right duality is motivated by the following property, which states that every linearly distributive category $\mathscr{C}$ equipped with a right duality is monoidal closed on the left.

Proposition 1 *In any linearly distributive category $\mathscr{C}$ with a right duality,*

- *the functor $(A \otimes -)$ is left adjoint to the functor $(A^* \parr -)$*
- *the functor $(- \parr B)$ is right adjoint to the functor $(- \otimes B^*)$*

for all objects A, B of the category. In particular, every such category $\mathscr{C}$ is monoidal closed on the left, with implication defined as

$$A \multimap B \quad = \quad A^* \parr B.$$

The proof is essentially immediate. One associates to every morphism

$$f \quad : \quad A \otimes X \quad \longrightarrow \quad B$$

the morphism $\varphi_{A,X,B}(f)$ defined as the composite

$$X \xrightarrow{(\lambda^{\otimes})^{-1}} 1 \otimes X \xrightarrow{ax^R} (A^* \parr A) \otimes X \xrightarrow{\kappa^R} A^* \parr (A \otimes X) \xrightarrow{f} A^* \parr Y$$

Conversely, one associates to every morphism

$$g \quad : \quad X \quad \longrightarrow \quad A^* \parr Y$$

the morphism $\psi_{A,X,B}(f)$ defined as the composite

$$A \otimes X \xrightarrow{g} A \otimes (A^* \parr Y) \xrightarrow{\kappa^L} (A \otimes A^*) \parr Y \xrightarrow{cut^R} \bot \parr Y \xrightarrow{\rho^{\parr}} Y$$

The coherence properties of the linearly distributive category $\mathscr{C}$ and of the right duality $A \mapsto A^*$ imply together that the correspondence is one-to-one and natural in X and Y. This establishes the property.

A linearly distributive category $\mathscr{C}$ is called symmetric when its monoidal structure $(\mathscr{C}, \otimes, 1)$ is equipped with a symmetry. An immediate but important corollary of Proposition 1 is that

Corollary 2 *The notion of $*$-autonomous category coincides with the notion of symmetric linearly distributive category $\mathscr{C}$ with a right duality.*

3 Dialogue chiralities

We start the section by recalling in §3.1 the two-sided formulation as so-called *dialogue chiralities* of the elementary notion of dialogue category. We then investigate a series of equivalent formulations of dialogue chiralities, either based on a family of adjunctions in §3.2 or on a family of transjunctions in §3.3. Meanwhile, we recall in §3.4 the notion of transjunction introduced by the author in [8] together with their pictorial representation in string diagrams. We conclude the section in §3.5 by pointing out how the formulation of dialogue chiralities based on transjunctions recovers and refines the familiar axiom-cut and $\otimes$-$\parr$ cut-elimination rules of multiplicative proof-nets in linear logic, as well as their η-expansion rules for axiom links.

3.1 Original definition of dialogue chiralities

We start from the original definition of dialogue chiralities given in [7]. Recall that a dialogue chirality is a pair of monoidal categories

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with a monoidal equivalence

$$\mathscr{A} \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \mathscr{B}^{op(0,1)} \quad \text{(monoidal equivalence)} \tag{9}$$

together with an adjunction

$$\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B} \quad (\bot) \tag{10}$$

whose unit and counit are denoted as

$$\eta \;:\; Id \longrightarrow R \circ L \qquad\qquad \varepsilon \;:\; L \circ R \longrightarrow Id$$

and, finally, with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \owedge a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \ovee b \,\rangle$$

natural in m, a, b, which we call *currification* in honor of the logician Haskell Curry. Here, the bracket $\langle\, a \,|\, b \,\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R(b)\,).$$

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle\, (m \owedge n) \owedge a \,|\, b \,\rangle & & \xrightarrow{\quad \chi_{m \owedge n} \quad} & & \langle\, a \,|\, (m \owedge n)^* \ovee b \,\rangle \\
\Big\downarrow {\scriptstyle associativity} & & & & \Big\uparrow {\scriptstyle associativity,\ monoidality\ of\ negation} \\
\langle\, m \owedge (n \owedge a) \,|\, b \,\rangle & \xrightarrow{\chi_m} & \langle\, n \owedge a \,|\, m^* \ovee b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \,|\, n^* \ovee (m^* \ovee b) \,\rangle
\end{array} \tag{11}$$

commute for all objects a, m, n of the category $\mathscr{A}$, and all objects b of the category $\mathscr{B}$.

3.2 A formulation based on adjunctions

A preliminary step towards the algebraic presentation of dialogue chiralities performed in §5 is to replace the currification isomorphism χ by a family of adjunctions, in the following way.

Proposition 3 *A dialogue chirality is the same thing as a pair of monoidal categories* $(\mathscr{A}, \owedge, true)$ *and* $(\mathscr{B}, \ovee, false)$ *equipped with a monoidal equivalence (9) and an adjunction (4) together with an adjunction*

$$L(m \owedge -) \quad \dashv \quad R(m^* \ovee -) \tag{12}$$

for every object m of the category $\mathscr{A}$, whose unit and counit are denoted

$$\eta[m] \,:\, a \longrightarrow R(m^* \ovee L(m \owedge a)) \qquad \varepsilon[m] \,:\, L(m \owedge R(m^* \ovee b)) \longrightarrow b$$

The family $\eta[-]$ is moreover required to be natural and monoidal, this meaning that the diagrams below

$$\begin{array}{ccc} a & \xrightarrow{\eta[m]} R(m^* \ovee L(m \owedge a)) \xrightarrow{f} & R(m^* \ovee L(n \owedge a)) \\ a & \xrightarrow{\eta[n]} R(n^* \ovee L(n \owedge a)) \xrightarrow{f^*} & R(m^* \ovee L(n \owedge a)) \end{array} \tag{13}$$

$$\begin{array}{ccc} a & \xrightarrow{\eta[m \owedge n]} & R((m \owedge n)^* \ovee L((m \owedge n) \owedge a)) \\ \downarrow{\scriptstyle \eta[n]} & & \\ R(n^* \ovee L(n \owedge a)) & & \downarrow{\scriptstyle \text{associativity \& monoidality of negation}} \\ \downarrow{\scriptstyle \eta[m]} & & \\ R(n^* \ovee LR(m^* \ovee L(m \owedge (n \owedge a)))) & \xrightarrow{\varepsilon} & R(n^* \ovee (m^* \ovee L(m \owedge (n \owedge a)))) \end{array} \tag{14}$$

should commute for all objects a, m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

Remark. The family of adjunctions (12) instantiated at the unit *true* induces an adjunction $L \dashv R$ between the functors L and R, whose unit and counit are defined as the expected families of morphisms

$$\eta'_a \;:\; a \xrightarrow{\eta[true]} R(true^* \ovee L(true \owedge a)) \xrightarrow{\text{associativity \& monoidality}} RL(a)$$

$$\varepsilon'_b \;:\; LR(b) \xrightarrow{\text{associativity \& monoidality}} L(true \owedge R(true^* \ovee b)) \xrightarrow{\varepsilon[true]} b$$

An important question is to understand whether this adjunction necessarily coincides with the original adjunction (4) between L and R. The answer is positive. Indeed, it is not difficult to see that the coherence diagram (14) implies that the two adjunctions coincide in the sense that $\eta = \eta'$ and $\varepsilon = \varepsilon'$. The main idea is to instantiate the coherence diagram (14) with $m = n = e$, and to apply the coherence laws of the monoidal categories $\mathscr{A}$ and $\mathscr{B}$ in order to show that the diagram

$$\begin{array}{ccc} a & \xrightarrow{\quad \eta' \quad} & RLa \\ {\scriptstyle \eta'}\downarrow & & \| \\ RLa & & \| \\ {\scriptstyle \eta'}\downarrow & & \| \\ RLRLa & \xrightarrow{\quad \varepsilon \quad} & RLa \end{array}$$

commutes. It easily follows from this and from the properties of adjunctions that $\eta = \eta'$ and $\varepsilon' = \varepsilon$. This means in particular that the coherence diagram

$$\begin{array}{ccc} R(true^* \ovee L(true \owedge a)) & \xrightarrow{\quad monoidality \quad} & R(false \ovee L(true \owedge a)) \\ {\scriptstyle \eta[true]}\uparrow & & \downarrow{\scriptstyle associativity} \\ a & \xrightarrow{\quad \eta \quad} & RL(a) \end{array} \tag{15}$$

is a consequence of the two coherence diagrams (13) and (14) formulated in the statement of Proposition 3.

Remark. Once established that the adjunction $(L, R, \eta', \varepsilon')$ coincides with the adjunction $(L, R, \eta, \varepsilon)$, it is not difficult to deduce from their companion diagrams (13) and (14) that the two coherence diagrams

$$\begin{array}{ccccc} & & L(n \owedge R(n^* \ovee b)) & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle \varepsilon[n]} & \\ L(m \owedge R(n^* \ovee b)) & & & & b \\ & {\scriptstyle f^*}\searrow & & \nearrow{\scriptstyle \varepsilon[m]} & \\ & & L(m \owedge R(m^* \ovee b)) & & \end{array}$$

$$\begin{array}{ccc}
L(m \owedge (n \owedge R(n^* \ovee (m^* \ovee b)))) & \xrightarrow{\eta} & L(m \owedge RL(n \owedge R(n^* \ovee (m^* \ovee b)))) \\
 & & \downarrow{\scriptstyle \varepsilon[n]} \\
\Big\downarrow{\scriptstyle \textit{associativity \& monoidality of negation}} & & L(m \owedge R(m^* \ovee b)) \\
 & & \downarrow{\scriptstyle \varepsilon[m]} \\
L((m \owedge n) \owedge R((m \owedge n)^* \ovee b)) & \xrightarrow{\varepsilon[m \owedge n]} & b
\end{array}$$

commute for all object b of the category $\mathscr{B}$ and all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$. At this point, it is worth mentioning that there is an element of choice in the definition of dialogue chiralities formulated in Proposition 3 since the two coherence diagrams for $\varepsilon[-]$ may very well replace the two corresponding diagrams (13) and (14) for $\eta[-]$.

3.3 A formulation based on transjunctions

The formulation of dialogue chiralities described in §3.2 is fine, but may be marginally improved. The idea is to replace the original combinators $\eta[-]$ and $\varepsilon[-]$ presenting the adjunctions by somewhat simpler combinators inspired by proof-theory:

$$\begin{array}{rcccc}
\mathbf{axiom}[m] & : & L(a) & \longrightarrow & m^* \ovee L(m \owedge a) \\
\mathbf{cut}[m] & : & m \owedge R(m^* \ovee b) & \longrightarrow & R(b)
\end{array}$$

This pair of combinators is defined from $\eta[-]$ and $\varepsilon[-]$ in the following way:

$$\begin{array}{rcccccc}
\mathbf{axiom}[m] & : & L(a) & \xrightarrow{\eta[m]} & LR(m^* \ovee L(m \owedge a)) & \xrightarrow{\varepsilon} & m^* \ovee L(m \owedge a) \\
\mathbf{cut}[m] & : & m \owedge R(m^* \ovee b) & \xrightarrow{\eta} & RL(m \owedge R(m^* \ovee b)) & \xrightarrow{\varepsilon[m]} & R(b)
\end{array}$$

Conversely, the combinators $\eta[-]$ and $\varepsilon[-]$ may be recovered from the original combinators **axiom**$[-]$ and **cut**$[-]$ in the following way:

$$\begin{array}{rcccccc}
\eta[m] & : & a & \xrightarrow{\eta} & RL(a) & \xrightarrow{\mathbf{axiom}[m]} & R(m^* \ovee L(m \owedge a)) \\
\varepsilon[m] & : & L(m \owedge R(m^* \ovee b)) & \xrightarrow{\mathbf{cut}[m]} & LR(b) & \xrightarrow{\varepsilon} & b
\end{array}$$

It is immediate that this back-and-forth translation between the pair $\eta[-]$, $\varepsilon[-]$ and the pair **axiom**$[-]$, **cut**$[-]$ defines a one-to-one relationship between the two pairs of combinators. This idea leads to an alternative formulation of dialogue chiralities, based this time on the transjunction between the two functors

$$m \owedge - \; : \; \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad m^* \ovee - \; : \; \mathscr{B} \longrightarrow \mathscr{B}$$

along the adjunction $L \dashv R$ between $\mathscr{A}$ and $\mathscr{B}$. The reader unaware of the notion of transjunction will find the notion recalled in §3.4. One main reason for introducing this notion is that it enables to replace the original combinators $\eta[-]$ and $\varepsilon[-]$ by the logically flavoured combinators **axiom**$[-]$ and **cut**$[-]$.

Proposition 4 *A dialogue chirality may be alternatively defined as a pair of categories $(\mathscr{A}, \obslash, true)$ and $(\mathscr{B}, \oslash, false)$ equipped with a monoidal equivalence (9) and an adjunction (4) together with a family of transjunctions*

$$\mathbf{axiom}[m] \,:\, L(a) \longrightarrow m^* \oslash L(m \obslash a) \qquad\qquad \mathbf{cut}[m] \,:\, m \obslash R(m^* \oslash b) \longrightarrow R(b)$$

natural in b and m. The family **cut**$[-]$ *is moreover required to be natural and monoidal, in the sense that the two diagrams*

$$\begin{array}{ccc} & m \obslash R(m^* \oslash b) & \\ {}^{f^*}\nearrow & & \searrow^{\mathbf{cut}[m]} \\ m \obslash R(n^* \oslash b) & & R(b) \\ {}_{f}\searrow & & \nearrow_{\mathbf{cut}[n]} \\ & n \obslash R(n^* \oslash b) & \end{array} \tag{16}$$

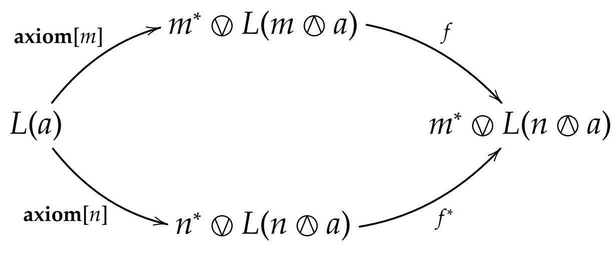
$$\begin{array}{ccc} m \obslash (n \obslash R(n^* \oslash (m^* \oslash b))) & \xrightarrow{\mathbf{cut}[n]} & m \obslash R(m^* \oslash b) \\ \uparrow{\scriptstyle associativity} & & \\ (m \obslash n) \obslash R((n^* \oslash m^*) \oslash b) & & \downarrow{\scriptstyle \mathbf{cut}[m]} \\ \uparrow{\scriptstyle monoidality} & & \\ (m \obslash n) \obslash R((m \obslash n)^* \oslash b) & \xrightarrow{\mathbf{cut}[m \obslash n]} & R(b) \end{array} \tag{17}$$

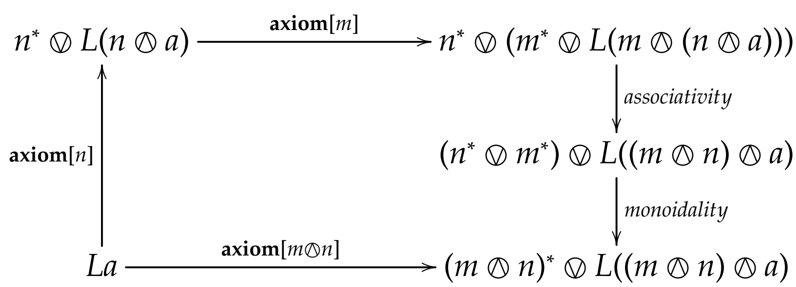
commute for all objects a, m, n and morphisms $f : m \to n$ of the category $\mathscr{A}$.

Remark. In the same way as in the definition of dialogue chiralities based on adjunctions in §3.2, the coherence diagram

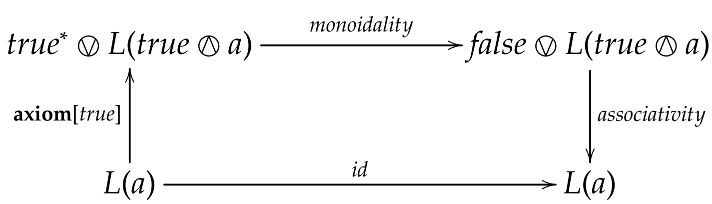
$$\begin{array}{ccc} R(b) & \xrightarrow{id} & R(b) \\ {\scriptstyle \mathbf{cut}[true]}\uparrow & & \uparrow{\scriptstyle associativity} \\ true \obslash R(true^* \oslash b) & \xrightarrow{monoidality} & true \obslash R(false \oslash b) \end{array} \tag{18}$$

follows from the coherence diagrams (16) and (17). Moreover, one can easily check that in the same way as in §3.2, the two coherence diagrams for the combinator **cut**[−] are equivalent to the corresponding coherence diagrams for **axiom**[−] below:





Note that the coherence diagram



also follows from the coherence diagrams (16) and (17).

3.4 Transjunctions

The notion of transjunction was introduced by the author in [8] in order to reflect the structure of negation in dialogue categories.

Definition 2 (transjunction) *Suppose given a pair of adjunctions*

whose units and counits are denoted η_1, η_2 and $\varepsilon_1, \varepsilon_2$ respectively. A transjunction $F \dashv G$ between a pair of functors

$$F : \mathscr{A}_1 \to \mathscr{A}_2 \qquad\qquad G : \mathscr{B}_2 \to \mathscr{B}_1$$

along the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

$$\textbf{axiom} : L_1 \Rightarrow G \circ L_2 \circ F \qquad\qquad \textbf{cut} : F \circ R_1 \circ G \Rightarrow R_2$$

making the two diagrams

$$\begin{array}{ccc} F \circ R_1 \circ L_1 & \xRightarrow{\ \textbf{axiom}\ } & F \circ R_1 \circ G \circ L_2 \circ F \\ {\scriptstyle \eta_1} \Uparrow & (a) & \Downarrow {\scriptstyle \textbf{cut}} \\ F & \xRightarrow{\ \eta_2\ } & R_2 \circ L_2 \circ F \end{array} \qquad \begin{array}{ccc} G \circ L_2 \circ F \circ R_1 \circ G & \xRightarrow{\ \textbf{cut}\ } & G \circ L_2 \circ R_2 \\ {\scriptstyle \textbf{axiom}} \Uparrow & (b) & \Downarrow {\scriptstyle \varepsilon_2} \\ L_1 \circ R_1 \circ G & \xRightarrow{\ \varepsilon_1\ } & G \end{array} \tag{19}$$

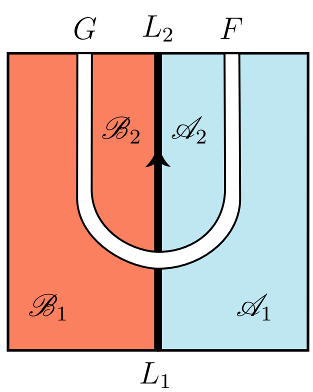
commute.

This notion of transjunction is ultimately justified by the following statement.

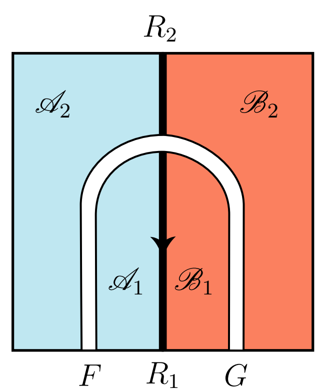
Proposition 5 *A transjunction $F \dashv G$ along the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is the same thing as an adjunction $F \circ L_1 \dashv R_2 \circ G$.*

The two equations (a) and (b) may be alternatively depicted as string diagrams living in the 2-category ***Cat*** of categories, functors and natural transformations. First of all, the generators **axiom** and **cut** of a transjunction $F \dashv G$ along the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ mentioned in Definition 2 are depicted as follows:

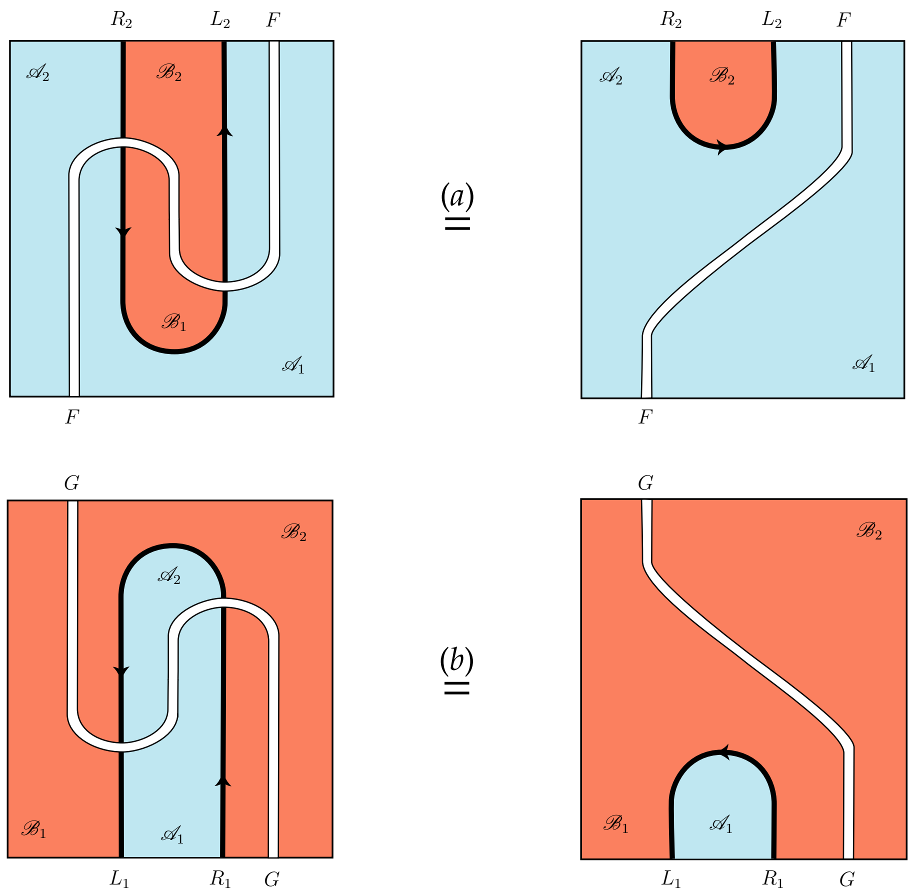
axiom =



cut =



The two commutative diagrams (a) and (b) are then depicted as follows:



An important methodological point to observe at this point is that the two equations (a) and (b) refine the familiar cut-axiom equation of multiplicative proof-nets encountered in linear logic. In the particular case of a $*$-autonomous category $\mathscr{C}$, the two sides $\mathscr{A}$ and $\mathscr{B}$ of the dialogue chirality coincide with $\mathscr{C}$, the two functors R and L are the identity functor, and the family of transjunctions mentioned in Proposition 4 thus boils down to a family of adjunctions $m \otimes - \dashv m^* \parr -$ with unit and counit defined as

$$\mathbf{axiom}[m] \;:\; a \longrightarrow m^* \parr (m \otimes a) \qquad\qquad \mathbf{cut}[m] \;:\; m \otimes (m^* \parr b) \longrightarrow b.$$

Each transjunction discussed in Proposition 4 should be thus seen as a refinement of this adjunction of $*$-autonomous categories in the presence of the non involutive negation functors L and R of dialogue categories.

A notion of homorphism between transjunctions may be then introduced, this giving rise to a category of transjunctions.

Definition 3 (homomorphism) *A homomorphism between two transjunctions $F \dashv G$ and $F' \dashv G'$ along the same pair of adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations*

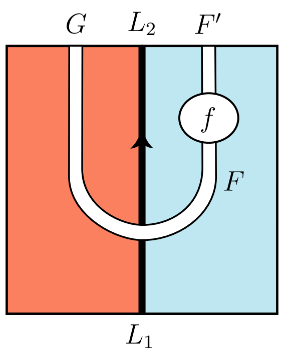
$$f \quad : \quad F \quad \Rightarrow \quad F' \qquad\qquad g \quad : \quad G' \quad \Rightarrow \quad G$$

making the two diagrams

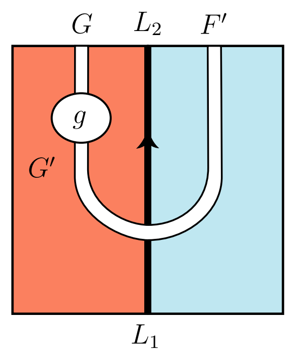
$$\begin{array}{ccc} G \circ L_2 \circ F & \xRightarrow{\;f\;} & G \circ L_2 \circ F' \\ {\scriptstyle \textbf{axiom}}\Uparrow & (a) & \Uparrow{\scriptstyle g} \\ L_1 & \xRightarrow{\;\textbf{axiom}'\;} & G' \circ L_2 \circ F' \end{array} \qquad\qquad \begin{array}{ccc} F' \circ R_1 \circ G' & \xRightarrow{\;\textbf{cut}'\;} & R_2 \\ {\scriptstyle f}\Uparrow & (b) & \Uparrow{\scriptstyle \textbf{cut}} \\ F \circ R_1 \circ G' & \xRightarrow{\;g\;} & F \circ R_1 \circ G \end{array}$$

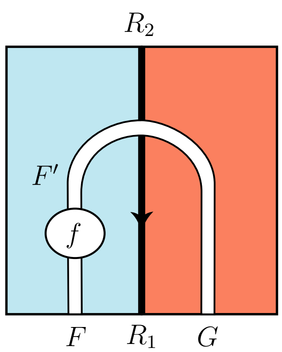
commute.

Pictorially, such a homomorphism (f, g) is defined as a pair of natural transformations $f : F \Rightarrow F'$ and $g : G' \Rightarrow G$ satisfying the diagrammatic equalities below:

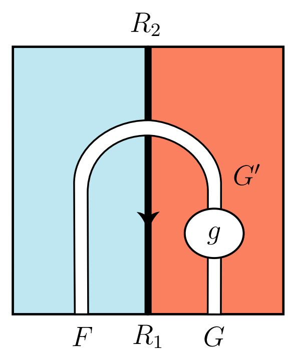


$\overset{(a)}{=}$





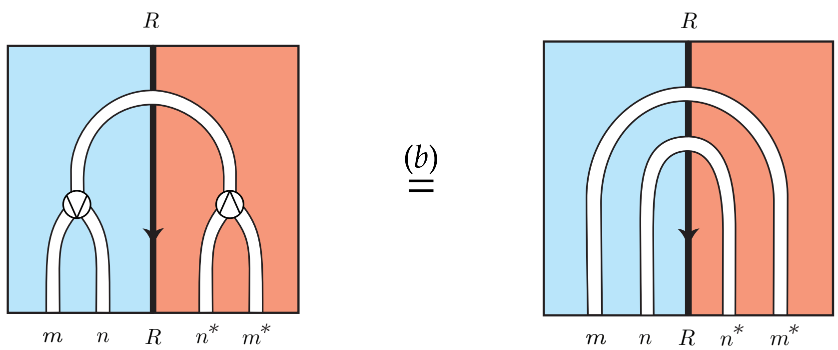
$\overset{(b)}{=}$



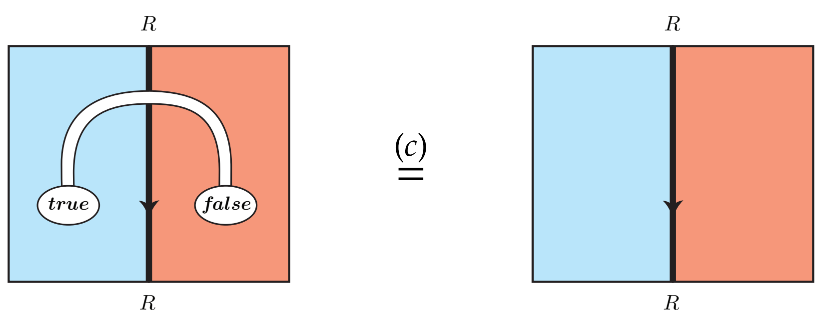
3.5 Tensorial conjunction vs. disjunction

The guiding philosophy of our present work is to refine the constructions of linear logic to the more primitive and more general situation of tensorial logic, where negation is not required to be involutive anymore. We have already seen in the previous section §3.4 how to extend the cut-axiom rule of linear logic in order to accomodate a non involutive pair of negations L and R. This refinement is natural, but also far from obvious. A nice outcome of our algebraic approach is that the solution is provided by the first part of Proposition 4. Let us turn now to the second part of Proposition 4 which is also very informative. The first diagram (16) ensures

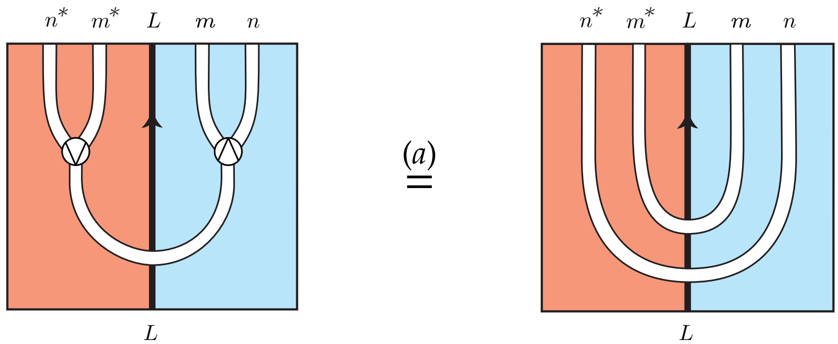
The second diagram 17

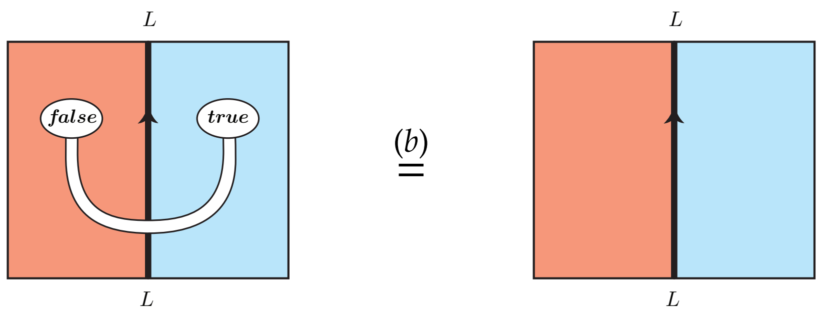


The third diagram 18 is depicted as follows:



These equations are reminiscent of the $\otimes$ vs. $\wp$ as well as the 1 vs. $\perp$ cut-elimination rewriting steps in proof-nets of linear logic. Similar diagrams for the axiom link The η-expansion laws of proof-nets in linear logic.





Note that one recovers the familiar equations of linear logic by “depolarizing” the notion of dialogue chirality $(\mathscr{A}, \mathscr{B})$ to a situation where the two sides $\mathscr{A}$ and $\mathscr{B}$ are equal, and where the functors L and R are equal to the identity.

4 Commutators

In this section, we recall the notion of commutator between parametric monads introduced in [8].

4.1 Parametric monads

This discussion motivates to parametrize by a monoidal category $(J, \otimes, e)$ the usual notion of formal monad, in the following way.

Definition 4 (parametric monad) *A parametric J-monad on a 0-cell $\mathscr{A}$ of a 2-category $\mathscr{W}$ is defined as a lax monoidal functor*

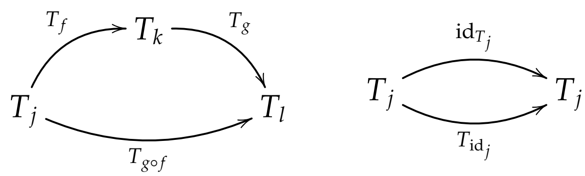
$$(T, m) : J \longrightarrow \mathrm{End}(A).$$

The monoidal category J is called the parameter category of the J-monad; and an object j of J is called a parameter.

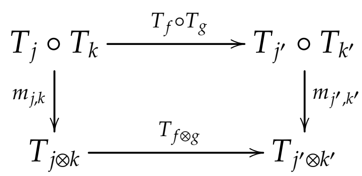
Hence, a parametric J-monad (T, m) consists of

- a 1-cell $T_j : \mathscr{A} \longrightarrow \mathscr{A}$ for every parameter j and a 2-cell $T_f : T_j \Rightarrow T_k$ for every morphism $f : j \longrightarrow k$ between such parameters,
- a 2-cell $m_e : 1_A \Rightarrow T_e$ called the unit of the parametric monad,
- a 2-cell $m_{j,k} : T_j \circ T_k \Rightarrow T_{j\otimes k}$ called the (j,k)-component of the multiplication of the parametric monad, for every pair of parameters j and k.

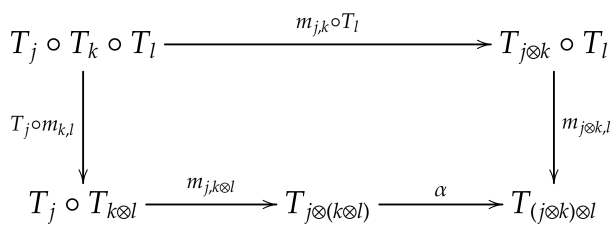
These data are moreover required to make a series of coherence diagrams commute in the category End($\mathscr{A}$). First, the diagrams



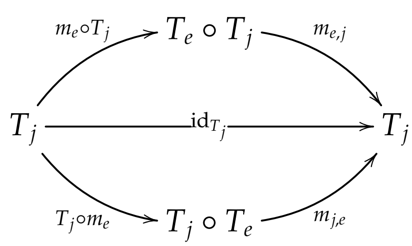
expressing the functoriality of T; then, the diagrams



expressing the naturality of m; and finally the diagrams



and



expressing the monoidality of m; this for all indices j, j', k, k', l and morphisms f, g, h in the parameter category J.

4.2 Commutators between parametric monads

At this point, we recall the notion of *commutator* between parametric monads introduced in [8]. As we have shown there, the notion of commutator unifies and generalizes the two fundamental notions of *monadic strength* on the one hand, and of *distributivity law* between two monads on the other hand. We recall at the same time that every dialogue chirality is equipped with such a structure. From now on, we suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathscr{W}$ equipped with a parametric $\mathscr{J}$-monad

$$T = \bullet \quad : \quad \mathscr{J} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

and a parametric $\mathscr{M}^{op(0)}$-monad

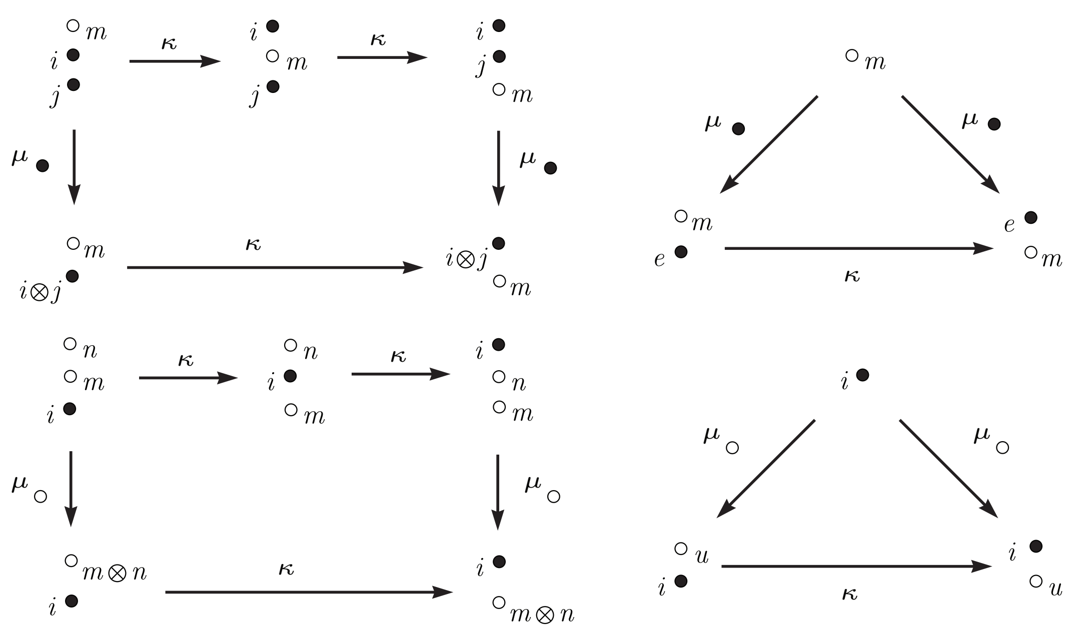
$$S = \circ \quad : \quad \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

with parameters taken in the monoidal categories $(\mathscr{J}, \otimes, e)$ and $(\mathscr{M}, \otimes, u)$.

Definition 5 (commutator) *A commutator between two parametric monads $T = \bullet$ and $S = \circ$ is defined as a natural transformation*

$$\kappa \quad : \quad S\,T \quad \Rightarrow \quad T\,S \quad : \quad \mathscr{J} \times \mathscr{M}^{op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

making the four diagrams below commute



for all objects i, j of the category $\mathscr{J}$ and all objects m, n of the category $\mathscr{M}$.

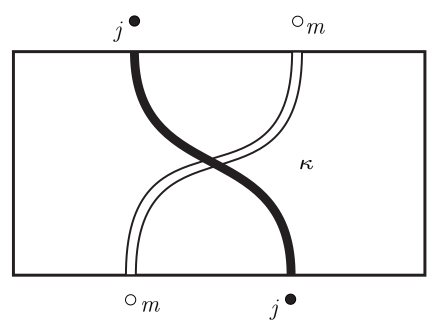
Remark. In the particular case when $\mathscr{W} = \mathbf{Cat}$, a commutator may be alternatively defined as a natural transformation

$$\kappa \quad : \quad (- \bullet -) \circ - \quad \Rightarrow \quad - \bullet (- \circ -) \quad : \quad \mathscr{J} \times \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

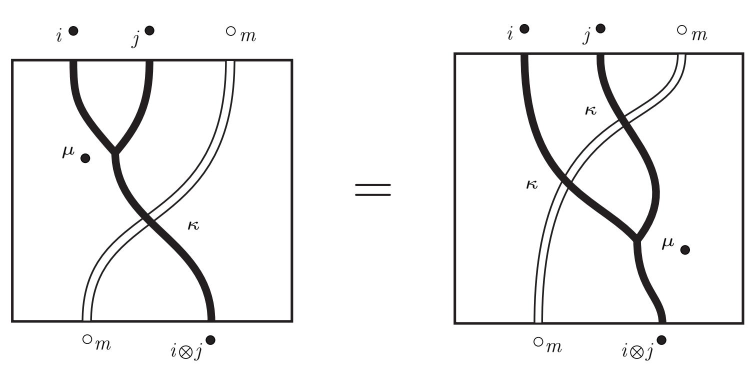
with components

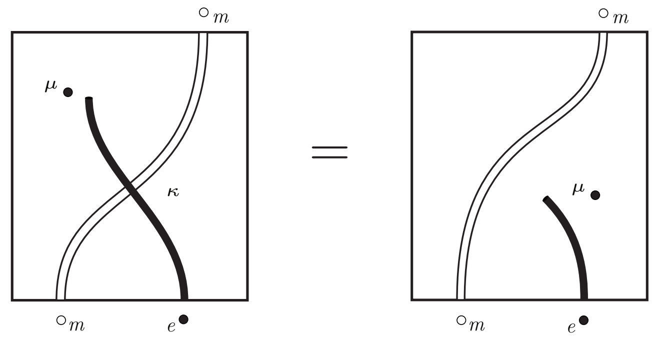
$$\kappa_{i,m,A} \quad : \quad (i \bullet A) \circ m \quad \longrightarrow \quad i \bullet (A \circ m)$$

parametrized by the objects i of the category $\mathscr{J}$, m of the category $\mathscr{M}$ and A of the category $\mathscr{C}$. As such, the parametric monads $\circ$ and $\bullet$ together with the commutator κ may be seen as a lax version of a $(\mathscr{J}, \mathscr{M})$-biaction (or bimodule). This may be depicted in string diagrams as follows. The idea is to depict the commutator κ as a braiding

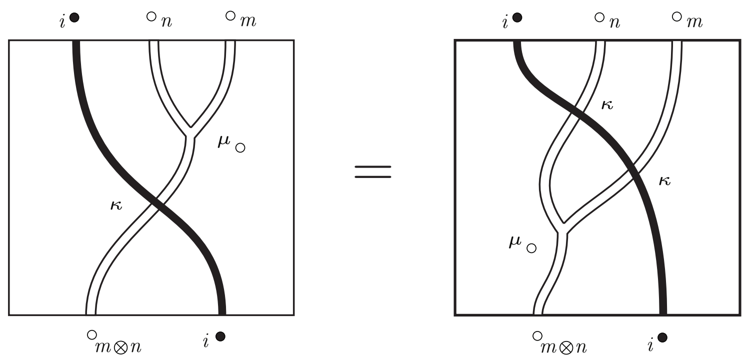


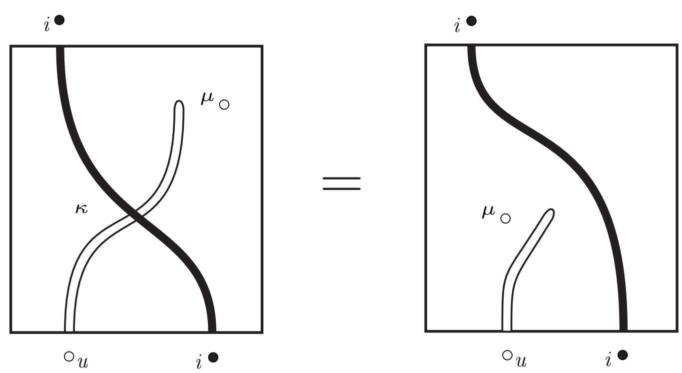
commuting the string representing the action $\bullet$ under the string representing $\circ$. This notation enables to depict the coherence diagrams of the commutator κ as a series of topologically intuitive equations, permuting the multiplication and unit of each parametric monad under or over the string representing the other parametric monad. Typically, the first series of equations in the definition of a commutator "permutes" the operations $\mu_\bullet$ over the string representing the action $\circ$





while the second series of equations "permutes" the operations $\mu_\circ$ under the string representing the action $\bullet$





Remark. The notion of commutator may be easily adapted to the case of a parametric comonad commuting with a parametric comonad, or a parametric monad commuting with a parametric comonad.

4.3 Dialogue chiralities on the right

We have recalled in §3 the original definition of dialogue chirality as it appears in [7]. However, it is mentioned in the very same paper [7] that there is an element of choice in the definition of dialogue chirality. In particular, if we call dialogue chirality *on the left* the notion of dialogue chirality recalled in §3, every dialogue category $\mathscr{C}$ induces such a dialogue chirality, obtained by defining L and R as the negation functors

$$L : x \mapsto \bot \multimapinv x \qquad\qquad R : x \mapsto x \multimap \bot.$$

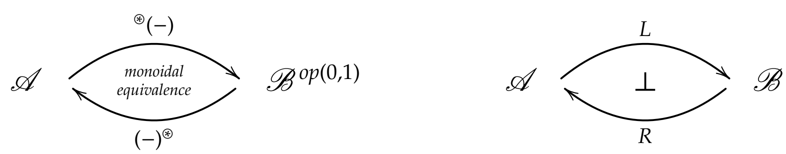
On the other hand, the other choice of negation functors

$$L : x \mapsto x \multimap \bot \qquad\qquad R : x \mapsto \bot \multimapinv x$$

induces a dialogue chirality *on the right*. The notion is symmetric to the notion of dialogue chirality *on the left* and is defined in the following way. A dialogue chirality (on the right) is a pair of monoidal categories

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with a monoidal equivalence and with an adjunction



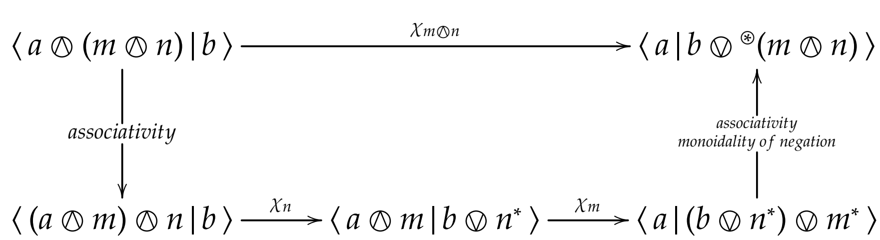
and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \owedge m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \ovee {}^{\circledast}m \,\rangle$$

natural in m, a, b. In the same way as in the case of a dialogue chirality on the left, the bracket $\langle\, a \,|\, b \,\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R(b)\,).$$

The family χ is moreover required to make the diagram



commute for all objects a, m, n of the category $\mathscr{A}$, and all objects b of the category $\mathscr{B}$. Recall from [7] that every dialogue chirality on the left induces a dialogue chirality on the right (and conversely) by applying the involution

$$\mathscr{A} \mapsto \mathscr{B}^{op(0,1)} \qquad \mathscr{B} \mapsto \mathscr{A}^{op(0,1)} \qquad L \mapsto R^{op(0,1)} \qquad R \mapsto L^{op(0,1)}.$$

We have seen in [8] that every dialogue chirality (on the right) comes equipped with two parametric monads

$$\begin{array}{ccccccc} T & : & (b,a) & \mapsto & R(b \ovee L(a)) & : & \mathscr{B} \times \mathscr{A} \longrightarrow \mathscr{A} \\ S & : & (a,m) & \mapsto & a \owedge m & : & \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A} \end{array}$$

together with a commutator

$$\kappa^{\owedge} \quad : \quad S\,T \quad \longrightarrow \quad T\,S$$

Symmetrically, every dialogue chirality (on the right) comes equipped with two parametric comonads

$$\begin{array}{ccccccc} J & : & (b,n) & \mapsto & b \ovee n & : & \mathscr{B} \times \mathscr{B} \longrightarrow \mathscr{B} \\ K & : & (a,b) & \mapsto & L(a \owedge R(b)) & : & \mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{B} \end{array}$$

together with a commutator

$$\kappa^{\ovee} \quad : \quad K\,J \quad \longrightarrow \quad J\,K.$$

5 Linearly distributive chiralities

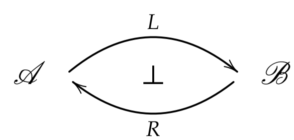
At this point, we are ready to articulate the formal definition of linearly distributive chirality discussed in the introduction. The reader will find it fully exposed in §5.1. We then proceed by analogy with linearly distributive categories, and introduce in §5.2 a notion of right duality adapted to linearly distributive chiralities. The main technical result of the paper appears in §5.3. We establish there that every linearly distributive chirality equipped with a right duality defines a dialogue chirality in the sense of §3.1. Finally, we conclude in §5.5 by showing the notion of linearly distributive chirality (and of right duality) coincides with the traditional notion of linearly distributive category (and of right duality) in the "depolarized" case when the two sides $\mathscr{A}$ and $\mathscr{B}$ of the chirality coincide, and the two functors L and R are equal to the identity functor.

5.1 Definition

A *linearly distributive chirality* is defined as a pair of monoidal categories

$$(\mathscr{A}, \owedge, true) \qquad\qquad (\mathscr{B}, \ovee, false)$$

equipped with an adjunction



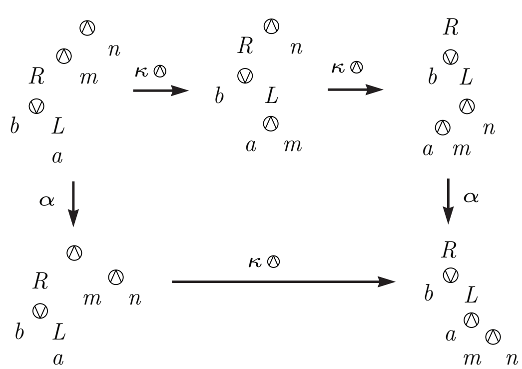
together with two commutators

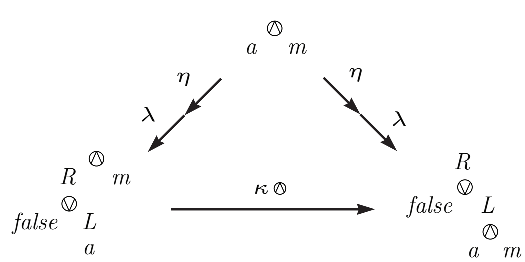
$$\begin{array}{cccccc} \kappa^{\owedge} & : & R(b \ovee L(a)) \owedge m & \longrightarrow & R(b \ovee L(a \owedge m)) \\ \kappa^{\ovee} & : & L(a \owedge R(b \ovee n)) & \longrightarrow & L(a \owedge R(b)) \ovee n \end{array} \qquad (20)$$

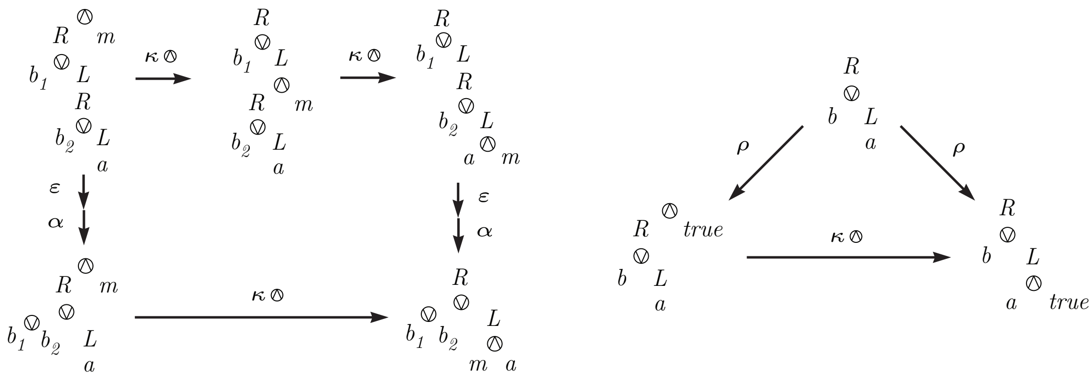
between

- the parametric $\mathscr{B}$-monad $T_b : a \mapsto R(L(b) \owedge a)$ and the parametric $\mathscr{A}$-monad $S_m : a \mapsto a \owedge m$ on the category $\mathscr{A}$ in the case of $\kappa^{\owedge}$,
- the parametric $\mathscr{B}$-comonad $K_n : b \mapsto b \ovee n$ and the parametric $\mathscr{A}$-comonad $L_a : b \mapsto L(a \ovee R(b))$ on the category $\mathscr{B}$ in the case of $\kappa^{\ovee}$.

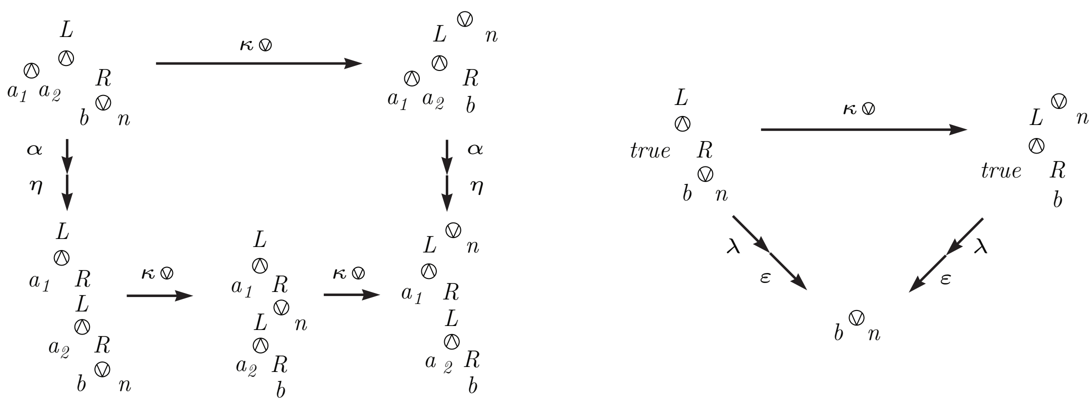
According to the definition of a commutator, this means that 2×4 diagrams are required to commute. We review them in turn. First of all, the fact that $\kappa^{\owedge}$ defines a commutator means that the four diagrams

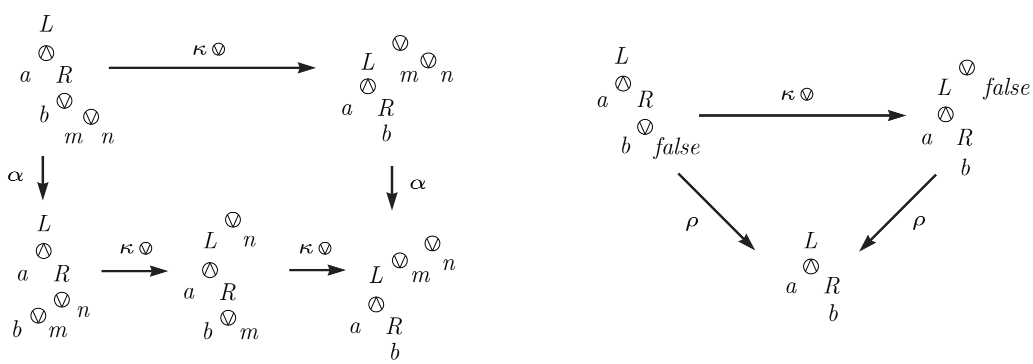




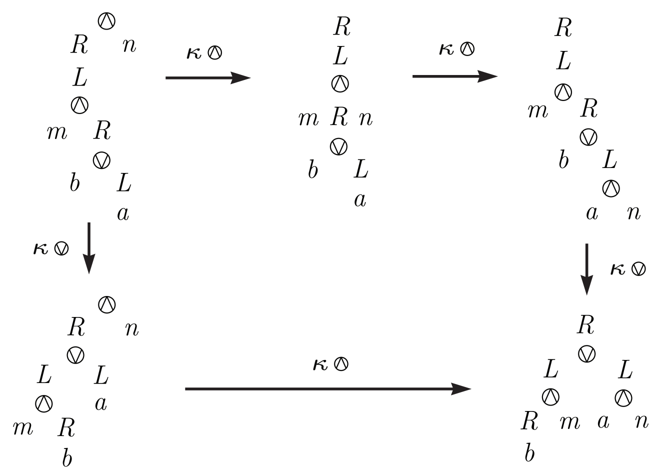


commute for all objects a, m, n in $\mathscr{A}$ and b, b_1, b_2 in $\mathscr{B}$. Symmetrically, the fact that $\kappa^{\oslash}$ defines a commutator means that the four diagrams

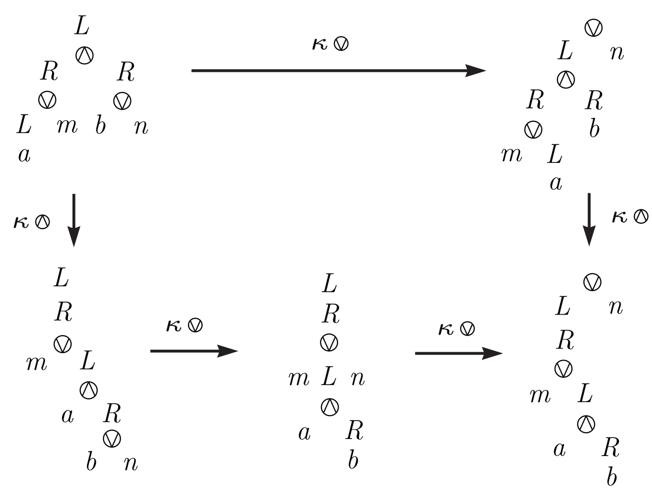




commute for all objects a, a_1, a_2 in $\mathscr{A}$ and b, m, n in $\mathscr{B}$. Besides this series of eight commutative diagrams, we ask that the diagram



commutes for all objects a, m, n of the category $\mathscr{A}$ and all object b of the category $\mathscr{B}$. Symmetrically, we ask that its mirror image

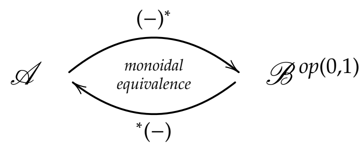


commutes for any object a of the category $\mathscr{A}$ and all objects b, m, n of the category $\mathscr{B}$.

Remark. It should be mentioned that in contrast to the eight other coherence diagrams, the algebraic status of these two coherence diagrams remains somewhat mysterious at the current stage. In particular, they are not justified by the previous discussions on commutators since they relate the two commutators $\kappa^{\oslash}$ and $\kappa^{\obslash}$.

5.2 Right duality in linearly distributive chiralities

A right duality in a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ is defined as a monoidal equivalence

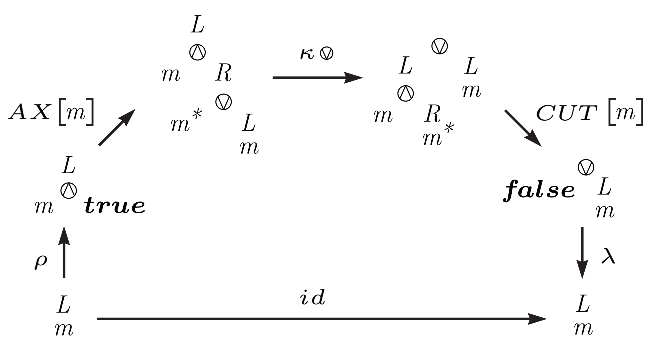


together with two families of morphisms

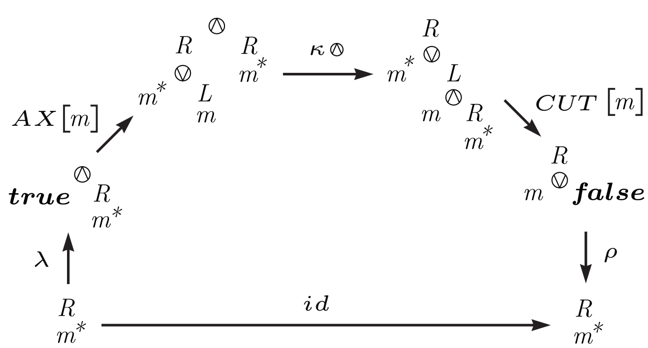
$$\mathbf{AX}[m] \quad : \quad true \quad \longrightarrow \quad R\,(\,m^{*} \oslash L\,(\,m\,)\,)$$

$$\mathbf{CUT}[m] \quad : \quad L\,(\,m \oslash R\,(\,m^{*}\,)\,) \quad \longrightarrow \quad false$$

each of them parametrized by the objects m of the category $\mathscr{A}$. These morphisms are required to make a series of 4×2 coherence diagrams commute, each of the four pairs consisting of a diagram and of its mirror image. The first pair of diagrams adapts the usual triangular axiom of adjunctions to the combinators **AX**[−] and **CUT**[−]:

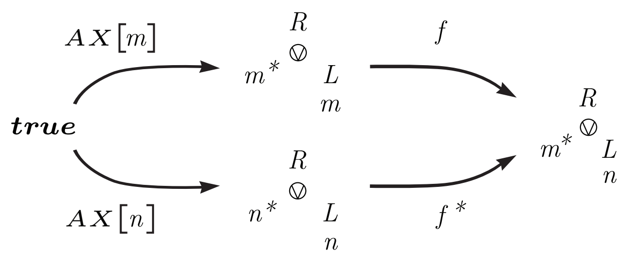


(21)

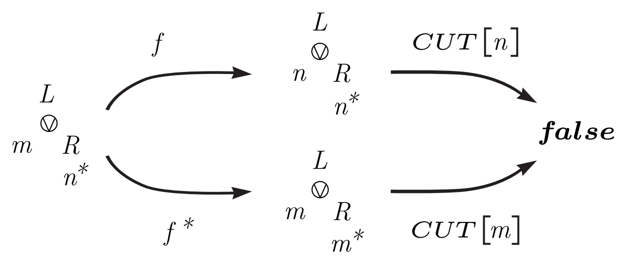


(22)

The second pair of coherence diagrams expresses that the two combinators **AX**[−] and **CUT**[−] are dinatural:

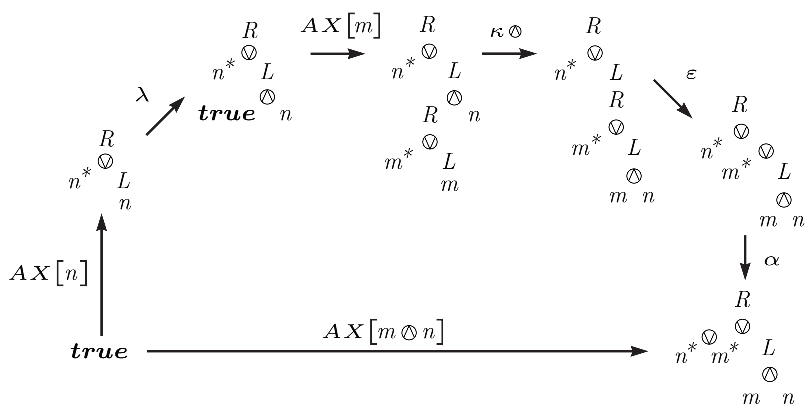


(23)

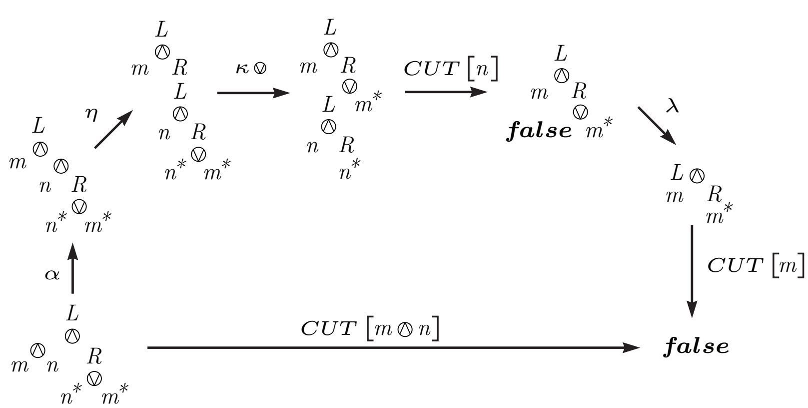


(24)

The third pair of coherence diagrams expresses that the combinators **AX**[−] and **CUT**[−] are monoidal:

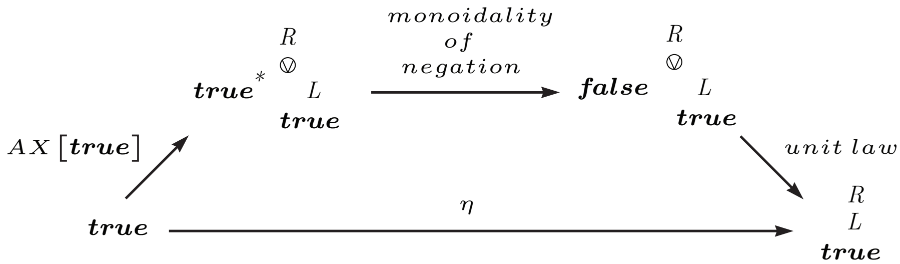


(25)

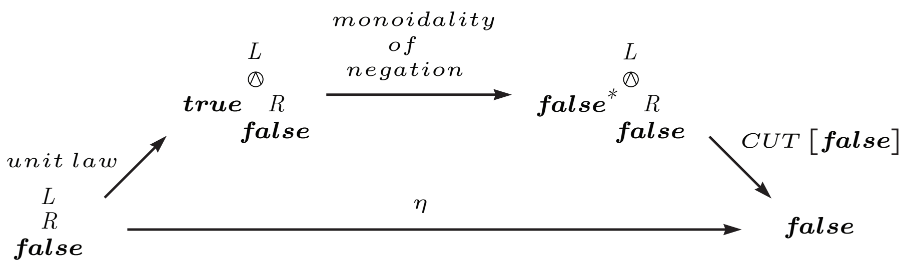


(26)

The fourth and last pair of coherence diagrams expresses that the combinators **AX**[−] and **CUT**[−] are monoidal, this time with respect to the units:



(27)



(28)

These diagrams are required to commute for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

5.3 Main result

At this stage, we are ready to establish the main technical result of the paper, which we identify as the cornerstone of the theory. As a matter of fact, the property is so important that the two notions of linearly distributive chirality and of right duality have been carefully carved in order to establish it.

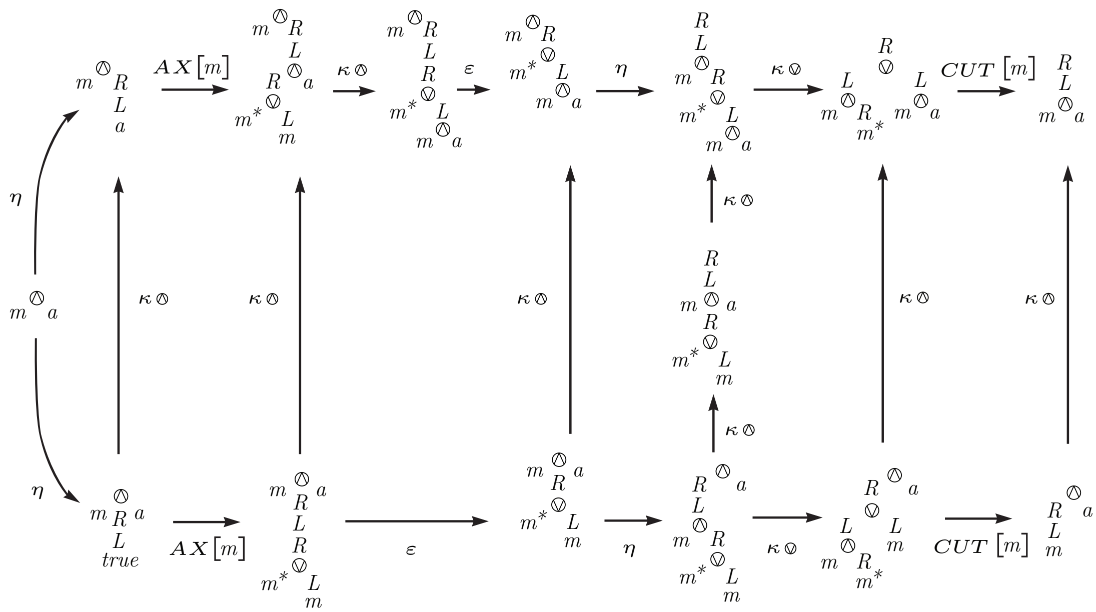
Proposition 6 *Every linearly distributive chirality* $(\mathscr{A}, \mathscr{B})$ *equipped with a right duality defines a dialogue chirality.*

We have described in §3 several equivalent formulations of the notion of dialogue chirality. In order to establish Proposition 6, we find convenient to follow the presentation based on transjunctions as it appears in §3.3. First of all, given a linearly distributive chirality with a right duality, the two categorical combinators **axiom**[−] and **cut**[−] are defined in such a way as to make the diagrams below commute:

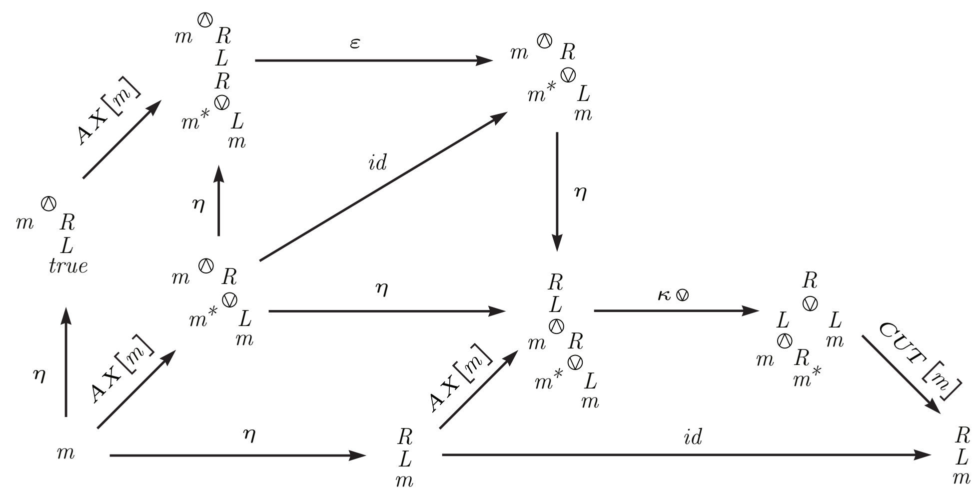
$$\begin{array}{ccc} L(R(m^* \oslash Lm) \obslash a) & \xrightarrow{\kappa^{\obslash}} & LR(m^* \oslash L(m \obslash a)) \\ \uparrow{\scriptstyle \mathbf{AX}[m]} & & \downarrow{\scriptstyle \varepsilon} \\ L(a) & \xrightarrow{\mathbf{axiom}[m]} & m^* \oslash L(m \obslash a) \end{array}$$

$$\begin{array}{ccc} RL(m \oslash R(m^* \obslash b)) & \xrightarrow{\kappa^{\obslash}} & R(L(m \oslash R(m^*)) \obslash b) \\ \uparrow \eta & & \downarrow \mathbf{CUT}[m] \\ m \oslash R(m^* \obslash b) & \xrightarrow{\mathbf{cut}[m]} & R(b) \end{array}$$

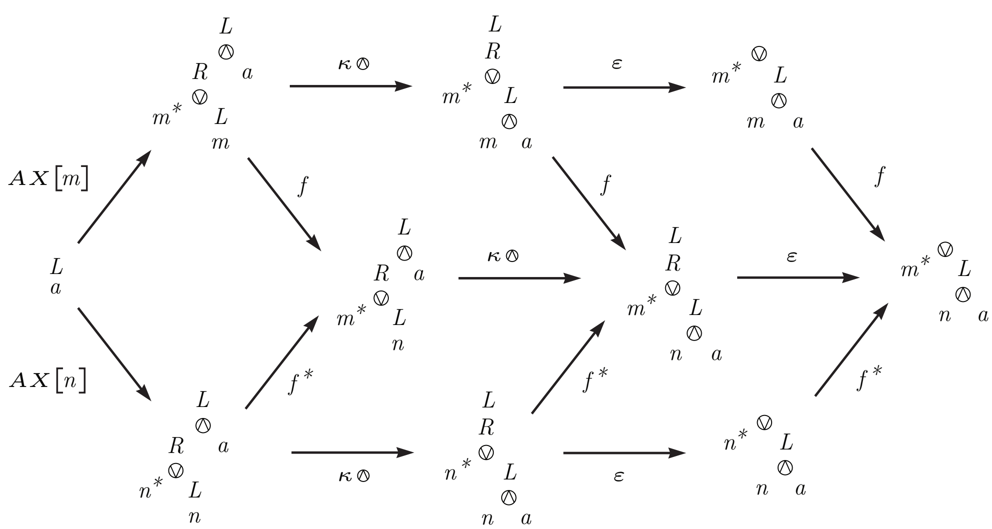
Note that the very definition of the families **axiom**[−] and **cut**[−] ensures that they are natural in a and b respectively. In order to establish the fact that each pair defines a transjunction, one needs to check that the two coherence diagrams (a) and (b) commute in (19). To that purpose, one starts by constructing the diagram chase below



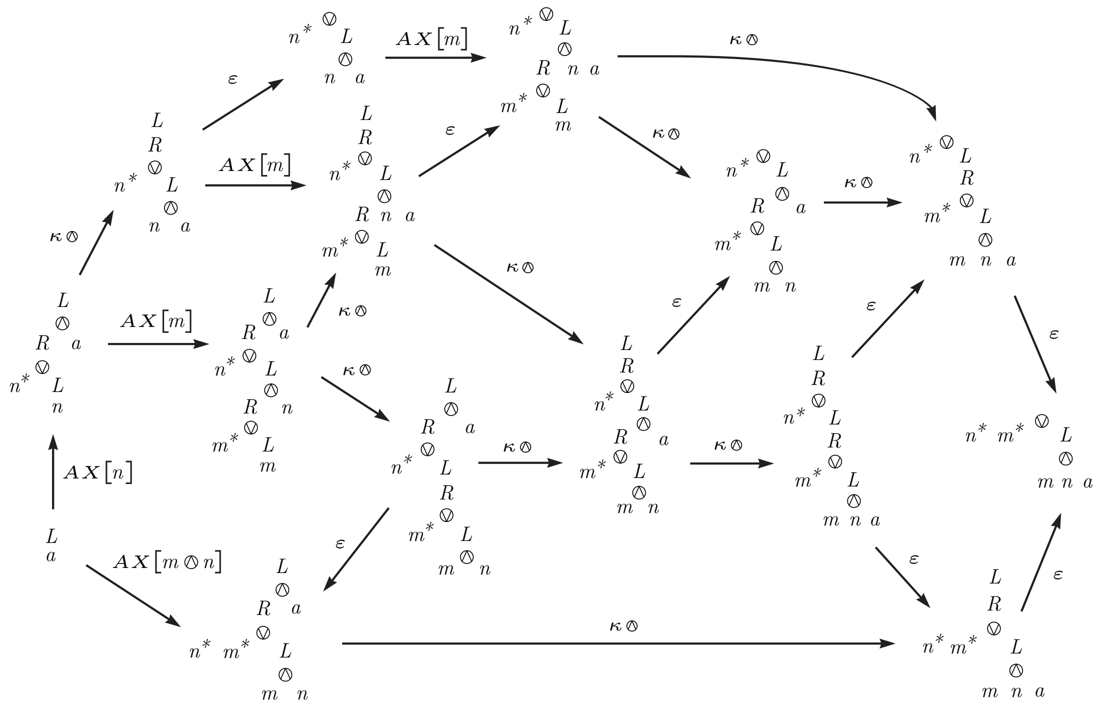
completed by the following one:



This establishes that the coherence diagram (*a*) commutes in (19). The fact that the other coherence diagram (*b*) commutes in (19) is proved in exactly the same way, by replacing each diagram chase by its mirror image. This establishes that the pair of morphisms **axiom**[*m*] and **cut**[*m*] defines a transjunction as required by the definition of a dialogue chirality in Proposition 4. At this stage, there remains to establish the two additional facts required in Proposition 4. This is achieved by the two diagram chases below, which establish that the family **axiom**[−] is dinatural in *m*:



and at the same time monoidal:



This concludes the proof that the family of transjunctions **axiom**[−] and **cut**[−] provides a dialogue chirality structure to the linearly distributive chirality $(\mathscr{A}, \mathscr{B})$.

An important consequence of Proposition 6 is that

Corollary 7 *In a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ equipped with a right duality, the functor*

$$L(m \oslash -) \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is left adjoint to the functor

$$R(m^* \varovee -) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$$

for every object m of the category $\mathscr{A}$.

Now, recall that a dialogue chirality $(\mathscr{A}, \mathscr{B})$ is called *symmetric* when the monoidal category $(\mathscr{A}, \oslash, true)$ is equipped with a symmetry. Similarly, a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ is called *symmetric* when the monoidal category $(\mathscr{A}, \oslash, true)$ is equipped with a symmetry. Although we are mainly interested here in the general case of non-symmetric monoidal categories, we find useful to observe that

Corollary 8 *The notion of symmetric dialogue chirality coincides with the notion of symmetric linearly distributive chirality with a right duality.*

Remark. It may come as a surprise to the careful reader that only four of the 4×2 coherence diagrams required of a right duality — namely equations (21), (22), (23) and (25) — are used in the proof of Proposition 6. We keep the four other coherence diagrams in the definition of a right duality because each of them makes perfect sense, either as the mirror image of another coherence diagram — in the case of (24) and (26) — or as the degenerate case for units of another coherence diagram — in the case of (27) and (28). On the other hand, it follows from Proposition 6 that the four coherence diagrams (24), (26), (27) and (28) are automatically valid when the coherence diagrams (21), (22), (23) and (25) hold in a linearly distributive chirality.

5.4 Depolarization

We start by introducing the notion of depolarized linearly distributive chirality.

Definition 6 *A linearly distributive chirality is called* depolarized *when its two sides $\mathscr{A}$ and $\mathscr{B}$ are equal and when the two functors L and R are identity functors.*

A careful inspection establishes that

Proposition 9 *A linearly distributive category is the same thing as a depolarized linearly distributive chirality.*

The main point to observe is that there is a one-to-one relationship between the coherence diagrams in §5.1 and the coherence diagrams in §2.1.

Proposition 10 *The notion of right duality in a linearly distributive category coincides with the notion of right duality in a depolarized linearly distributive chirality.*

An important example of linearly distributive chirality is provided by the notion of *linearly distributive category* introduced by Cockett and Seely in [2]. One recovers a theorem by Cockett and Seely

5.5 Illustration

6 Conclusion

An important observation at this point is that the linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ cannot be recovered from the dialogue chirality structure, because the dialogue chirality structure deals with right actions of $\mathscr{A}$ and $\mathscr{B}$, whereas the linearly distributive chirality structure (20) is concerned with their left actions. A natural way to resolve this problem is to start from a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ generated by left and right actions, and then to reconstruct an equivalent notion of dialogue chirality also based on left and right actions. However, the problem of properly correlating the left and right actions of $\mathscr{A}$ and $\mathscr{B}$ is far more subtle than it seems. In particular, it requires to introduce a natural isomorphism between the two negations

$$\mathbf{turn}_A \quad : \quad A \multimap \bot \quad \longrightarrow \quad \bot \multimapinv A$$

satisfying a series of appropriate coherence diagrams. This leads to the axiomatic investigation of helical, cyclic, braided or symmetric notions of dialogue category. This is done in a companion paper.

References

[1] John Baez and James Dolan. Higher-Dimensional Algebra III: n-Categories and the Algebra of Opetopes. Adv. Math. 135 (1998), 145-206.

[2] Robin Cockett, Robert Seely. Weakly Distributive Categories. Journal of Pure and Applied Algebra 114 (1997) 2, pp 133-173.

[3] Richard Blute, Robin Cockett, Robert Seely, and Todd Trimble. Natural deduction and coherence for weakly distributive categories. Journal of Pure and Applied Algebra, 113:229?296, 1996.

[4] Robin Cockett, Robert Seely. Polarized category theory, modules, and game semantics. Theory and Applications of Categories 18 (2) (2007) 4–101.

[5] Max Kelly, Ross Street. Review of the elements of 2-categories, in Kelly (ed.), Category Seminar, LNM 420.

[6] Paul-André Melliès. Categorical semantics of linear logic. Published in *Interactive models of computation and program behaviour*. Pierre-Louis Curien, Hugo Herbelin, Jean-Louis Krivine, Paul-André Melliès. Panoramas et Synthèses 27, Société Mathématique de France, 2009.

[7] Paul-André Melliès. Dialogue categories and chiralities. Submitted. Manuscript available on the author's webpage.

[8] Paul-André Melliès. The parametric continuation monad. Submitted. Manuscript available on the author's webpage.

[9] Paul-André Melliès. A micrological study of negation continued. In preparation.