On dialogue games and coherent strategies*

Paul-André Melliès

CNRS, Laboratoire PPS (Preuves, Programmes, Systèmes), UMR 7126
Université Paris Diderot, Sorbonne Paris Cité, F-75205 Paris, France.
mellies@pps.univ-paris-diderot.fr

Abstract

We explain how to see the set of positions of a dialogue game as a coherence space in the sense of Girard or as a bistructure in the sense of Curien, Plotkin and Winskel. The coherence structure on the set of positions results from a Kripke translation of tensorial logic into linear logic extended with a necessity modality. The translation is done in such a way that every innocent strategy defines a clique or a configuration in the resulting space of positions. This leads us to study the notion of configuration designed by Curien, Plotkin and Winskel for general bistructures in the particular case of a bistructure associated to a dialogue game. We show that every such configuration may be seen as an interactive strategy equipped with a backward as well as a forward dynamics based on the interplay between the stable order and the extensional order. In that way, the category of bistructures is shown to include a full subcategory of games and coherent strategies of an interesting nature.

1 Introduction

An important dichotomy in the denotational semantics of a programming language like PCF is provided by the distinction between the *qualitative* and the *quantitative* interpretations of the language. The distinction is important but recent since the first quantitative model emerged in the work by Girard on quantitative domains [9] only a few months before the discovery of linear logic. All the denotational models of PCF were qualitative before that. This includes the domain-theoretic models either based on continuous functions between Scott domains [24] or on stable functions between dI-domains [2] as well as the precursor of game semantics based on sequential algorithms between concrete data structures [3]. The difference between qualitative and quantitative models is best understood today by translating the intuitionistic types of PCF into formulas of linear logic. There, the distinction between the two classes of models boils down to the way the exponential modality ! of linear logic is interpreted. As shown by Ehrhard in his work on differential linear logic, the quantitative models of linear logic are usually better behaved and closer to a mathematical understanding of resource (formal series, differential calculus) because they incorporate the number of times a procedure is called by its environment. On the other hand, the qualitative models are precious tools for automatic verification of software because they interpret finite types (typically limited to booleans or to finite approximations of the natural numbers) as finite mathematical structures, and thus provide mechanical procedures to *decide* specific properties of programs.

Quite interestingly, most interactive models based on game semantics are quantitative, rather than qualitative. There is a good reason for that: once understood how to track the several copies of a game A in the game $!A$ by using indices or pointers, it is generally easier to describe the behaviour of a PCF program in exactly the same way as it proceeds in time, typically in a Krivine machine. As a consequence, the number of times a program of type $(!A) \multimap B$ calls its procedure of type A is generally reflected in the game model. As we already mentioned, a remarkable counter-example to this general principle is provided by the sequential algorithm model of PCF [3] which is indeed operational and qualitative at the same time. Lamarche and Curien [15, 5] have shown very early in the history of game semantics how to reformulate this

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model of PCF as a model of intuitionistic linear logic based on *sequential data structures* — which we prefer to call here *simple games* in order to distinguish them among the more general *dialogue games*. Because of the qualitative nature of the model, simple games are defined as alternating decision trees, without the need for extra indexing or pointer structure. The key idea of the model is to define the simple game $!A$ as the tree of partial explorations of a given strategy σ of the simple game A. The contraction $!A \to !A \circledast !A$ of linear logic is then interpreted by a clever *memoisation* procedure which keeps track of the portion of the strategy σ of the game A on the left explored by the two environments playing independently on the copies of $!A$ on the right. In this way, one obtains a qualitative model of intuitionistic linear logic whose co-Kleisli category embeds in the category of sequential algorithms originally introduced by Berry and Curien. Note that we write $A \circledast B$ for the tensor product of simple games defined by interleaving, in order to distinguish it from the tensor product of dialogue games $A \otimes B$.

The interest in this specific Curien-Lamarche modality ! has been recently revived by the observation that the category of dialogue games and innocent strategies defined by Hyland and Ong [12] may be reconstructed as a bi-Kleisli category from the category of simple games, using a quantitative (or repetitive) version of the modality [11]. For the sake of completeness, we find instructive to take the reverse point of view here, and to see the category **Simple** of simple games as a specific full subcategory of a category **Dialogue** of dialogue games and innocent strategies. This category **Dialogue** should be understood as a resource-aware and linear variant of the original category in [12]. At this point, it is worth recalling the definition of a dialogue category, see [23] for instance:

▸ **Definition 1** (Dialogue category). A dialogue category $\mathcal{C}$ is a symmetric monoidal category equipped with an object $\perp$ together with a functor

$$\neg \quad : \quad \mathcal{C}^{op} \quad \longrightarrow \quad \mathcal{C} \tag{1}$$

and a family of bijections

$$\varphi_{A,B} \quad : \quad \mathcal{C}(A \otimes B, \perp) \quad \cong \quad \mathcal{C}(A, \neg B)$$

natural in A and B. A dialogue category is called affine when it is equipped with a natural family of morphisms (called weakening)

$$e_A \quad : \quad \neg A \quad \longrightarrow \quad 1.$$

The category **Dialogue** of dialogue games and total innocent strategies may be concisely defined as the free affine dialogue category with finite sums (and tensors distributing over these finite sums). A more concrete definition will appear in §3 but the conceptual definition is convenient at this introductory stage. Similarly,

▸ **Definition 2** (Negation category). A negation category $\mathcal{C}$ is a category equipped with a functor (1) and with a family of bijections

$$\nu_{A,B} \quad : \quad \mathcal{C}(A, \neg B) \quad \cong \quad \mathcal{C}(B, \neg A)$$

natural in A and B.

The category **Simple** of simple games and total sequential strategies may be concisely defined as the free negation category with finite sums. Note that **Simple** coincides with the free category $Fam(\mathcal{G})$ with finite sums (or finite family construction) generated by the category $\mathcal{G}$ of finite Opponent starting games and total strategies considered in [5]. As a dialogue category, the category **Dialogue** is also a negation category. This implies the existence of a negation and finite sum preserving functor

$$embedding \quad : \quad \textbf{Simple} \quad \longrightarrow \quad \textbf{Dialogue}$$

The functor is full and faithful, and injective on objects. As such, it identifies the category **Simple** of simple games to a full subcategory of the category **Dialogue** of dialogue games. The category **Simple** is symmetric monoidal closed with tensor product and linear implication noted $\circledast$ and $\multimap$ respectively. As such, it defines a dialogue category with negation defined as $\neg A = A \multimap \bot$ where $\bot$ is the simple game with one initial Player move $*$ (which may be also seen as a unique initial position $*$ of the game) followed by a single Opponent move q. Again, by the universal characterization of the category **Dialogue**, this induces a finite sum, tensor and negation preserving functor

$$\textit{pathification} \quad : \quad \mathbf{Dialogue} \longrightarrow \mathbf{Simple}$$

which we call *pathification* because it transports every dialogue game A to a simple game entirely defined by its alternating paths. Despite its name, the *pathification* functor is a brutal transformation on the original dialogue game, since it destroys the asynchronous structure of the asynchronous game A and only retains its alternating paths. On the other hand, every simple game is already a tree, and thus the composite functor

$$\mathbf{Simple} \xrightarrow{\textit{embedding}} \mathbf{Dialogue} \xrightarrow{\textit{pathification}} \mathbf{Simple}$$

is equal to the identity. One preliminary observation of the paper is that the tensor product $A \circledast B$ between simple games factors as

$$A \circledast B \quad = \quad \textit{pathification}(\textit{embedding}(A) \otimes \textit{embedding}(B))$$

and similarly, that the Curien-Lamarche exponential modality ! factors as

$$\mathbf{Simple} \overset{\textit{shriek}}{\dashrightarrow} \mathbf{Dialogue} \xrightarrow{\textit{pathification}} \mathbf{Simple} \tag{2}$$

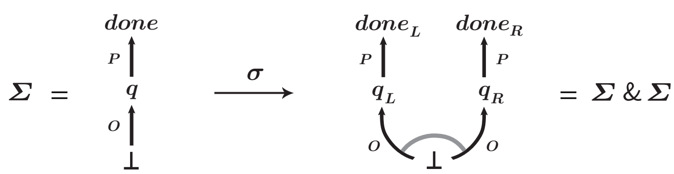
Note that the transformation *shriek* is entirely described by the recursive equation

$$\textit{shriek}\left(\bigoplus_{i \in I} \neg \bigoplus_{j \in J_i} \neg A_{ij}\right) \quad = \quad \bigoplus_{i \in I} \bigotimes_{j \in J_i} \neg\neg\ \textit{shriek}\,(A_{ij})$$

whose purpose is to replace every cartesian product (or negated sum indexed by $j \in J_i$) by the corresponding tensor product. We will illustrate the construction in (3) and (5). In particular, for every pair of simple games A, B, there exists a bijection

$$\mathbf{Simple}(!A, B) \quad \cong \quad \mathbf{Dialogue}(\textit{shriek}(A), \textit{embedding}(B)).$$

This basic observation seems to underlie a lot of work in the field of game semantics, in particular the graph-theoretic formulation of the sequential algorithm model by Hyland and Schalk [13]. A fundamental difficulty (or phenomenon) arises at this point of our analysis: the transformation *shriek* is *not* functorial — and this is precisely the reason why we prefered to indicate it with a dotted line in (2). In order to understand what is going wrong, let us define 1 as the simple game with a unique Player move $*$ (or initial position) and the Sierpinski game $\Sigma = \neg\neg 1$ as the simple game with a unique initial Player move $*$ (or initial position) followed by a unique Opponent move *done*, itself followed by a unique Player move *done*. The cartesian product $\Sigma \,\&\, \Sigma$ in the category **Simple** is equal to the simple game $\neg\big((\neg 1) \oplus (\neg 1)\big)$. Now, consider the morphism



(3)

in the category **Simple**, where σ denotes the strategy which starts at the initial position $(\bot, \bot)$ and consists of the two sequences of moves below:

$$
\begin{array}{lcccccccccc}
s_L & : & (\bot,\bot) & \xrightarrow{O} & (\bot, q_L) & \xrightarrow{P} & (q, q_L) & \xrightarrow{O} & (done, q_L) & \xrightarrow{P} & (done, done_L) \\
s_R & : & (\bot,\bot) & \xrightarrow{O} & (\bot, q_R) & \xrightarrow{P} & (q, q_R) & \xrightarrow{O} & (done, q_R) & \xrightarrow{P} & (done, done_R)
\end{array}
\tag{4}
$$

together with their even-length prefixes. We indicate with a grey orb in (3) the fact that the moves q_L and q_R are incompatible and thus cannot appear in the same play of the game $\Sigma \mathbin{\&} \Sigma$. By definition, *shriek* transports the simple game Σ into itself, and the simple game $\Sigma \mathbin{\&} \Sigma = \neg\big((\neg 1) \oplus (\neg 1)\big)$ into the dialogue game $\Sigma \otimes \Sigma = (\neg\neg 1) \otimes (\neg\neg 1)$. We claim that *shriek* is not functorial because it cannot transport the strategy σ to any innocent strategy

$$
\begin{tikzcd}
& done & & done_L & & done_R & \\
\boldsymbol{\Sigma} = & q \arrow[u, "P"] & \arrow[r, "\tau"] & {} & q_L \arrow[ul, phantom, ""] & q_R \arrow[u, "P"'] & = \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma} \\
& \bot \arrow[u, "O"] & & & \bot & &
\end{tikzcd}
\tag{5}
$$

in the category **Dialogue**. Note that we remove the grey orb between the moves q_L and q_R in the case of the dialogue game $\Sigma \otimes \Sigma$ to indicate that the two moves are compatible in the game. Imagine that there exists such an innocent strategy $\tau = shriek(\sigma)$. In order to make our argument work, we will make the mild hypothesis that any reasonable functorial definition of *shriek* should transport the projection $\pi_i : \Sigma \mathbin{\&} \Sigma \to \Sigma$ to the expected strategy $shriek(\pi_i) : \Sigma \otimes \Sigma \to \Sigma$ which plays a copycat strategy between Σ and the first or second component of $\Sigma \otimes \Sigma$ depending on the value of $i = 1, 2$. With this additional hypothesis, it is easy to deduce from the equality $\pi_i \circ \sigma = id_\Sigma$ (for $i = 1, 2$) and from the totality of τ that the strategy $\tau = shriek(\sigma)$ coincides with the strategy consisting of the two plays

$$
\begin{array}{lcccccc}
s_{LR} & : & (\bot,\bot,\bot) & \xrightarrow{O^{(*)}} & (\bot, q_L, \bot) & \xrightarrow{P^{(*)}} & (q, q_L, \bot) \\
& & & \xrightarrow{O} & (done, q_L, \bot) & \xrightarrow{P} & (done, done_L, \bot) \\
& & & \xrightarrow{O} & (done, done_L, q_R) & \xrightarrow{P} & (done, done_L, done_R) \\[1ex]
s_{RL} & : & (\bot,\bot,\bot) & \xrightarrow{O} & (\bot, \bot, q_R) & \xrightarrow{P} & (q, \bot, q_R) \\
& & & \xrightarrow{O} & (done, \bot, q_R) & \xrightarrow{P} & (done, \bot, done_R) \\
& & & \xrightarrow{O^{(**)}} & (done, q_L, done_R) & \xrightarrow{P^{(**)}} & (done, done_L, done_R)
\end{array}
$$

together with their even-length prefixes. One recognizes here the contraction strategy $\Sigma \to \Sigma \circledast \Sigma$ of the Curien-Lamarche model for the simple game $\Sigma = {!}\,\Sigma$. Our whole point is that the strategy τ is *not* innocent as a strategy $\Sigma \to \Sigma \otimes \Sigma$ because it does not play in the same way in the move $P^{(*)}$ of the play s_{LR} and in the move $P^{(**)}$ of the play s_{RL} although the Player views are the same seen from the move $O^{(*)}$ and from the move $O^{(**)}$.

In order to repair the situation, we introduce the notion of *coherent strategy* which relaxes the familiar notion of innocent strategy between dialogue games in such a way that (1) there exists a functor

$$
\textbf{Dialogue} \xrightarrow{\ embedding\ } \textbf{Coherent}
$$

which enables one to transport every strategy $\sigma : A \to B$ between simple games into a coherent strategy using the composite functor

$$
\textbf{Simple} \xrightarrow{\ embedding\ } \textbf{Dialogue} \xrightarrow{\ embedding\ } \textbf{Coherent}
$$

and moreover (2) a functor

$$
\textbf{Simple} \xrightarrow{\ shriek\ } \textbf{Coherent}
\tag{6}
$$

making the diagram below commute

$$\begin{tikzcd}
& \textbf{Dialogue} \arrow[dd, "embedding"] \\
\textbf{Simple} \arrow[ur, dotted, "shriek"] \arrow[dr, "shriek"'] & \\
& \textbf{Coherent}
\end{tikzcd} \tag{7}$$

One main purpose of this paper is thus to introduce the notion of *coherent strategy* on a dialogue game. We proceed in the same (slightly unconventional) way as the notion emerged in our work. First, we recall in §2 the relationship between dialogue games and tensorial logic, and then define in §3 the notion of innocent strategy we have in mind. Then, we introduce in §4 a Kripke translation of tensorial logic into linear logic extended with a necessity modality (noted $\square$) which enables us to interpret the set of positions of a dialogue game as a coherence space in the sense of Girard or as a bistructure in the sense of Curien, Plotkin and Winskel [6]. After briefly recalling in the Appendix this model of bistructures, we show in §5 that the configurations σ of the bistructure $[A]$ of positions of a dialogue game A are positional strategies extending the familiar notion of innocent strategies. These strategies are precisely what we call the *coherent strategies* of a dialogue game. Accordingly, the category **Coherent** is defined as the category of coherent strategies between dialogue games. We thus obtain a series of functorial translations:

$$\textbf{Simple} \xrightarrow{embedding} \textbf{Dialogue} \longrightarrow \textbf{Coherent} \xrightarrow{forgetful} \textbf{Bistr}$$

where **Bistr** denotes the category of bistructures and configurations introduced by Curien, Plotkin and Winskel [6] and where *forgetful* adapts to **Coherent** the forgetful functor U from the category $\mathcal{M}$ of coalgebras of the comonad $\square$ to the category **Bistr**. One interesting observation is that the functor

$$\textbf{Simple} \xrightarrow{shriek} \textbf{Coherent} \xrightarrow{forgetful} \textbf{Bistr} \xrightarrow{\square} \textbf{Bistr} \tag{8}$$

transports every simple game A to the bistructure $!\,[A]$ where ! denotes the qualitative exponential modality of **Bistr**. From this follows that the functor lifts to a functor between the Kleisli categories associated to **Simple** and to **Bistr**. We deduce that every sequential algorithm $\sigma : !A \to B$ defines a stable and extensional function $\Gamma(A) \to \Gamma(B)$ between the associated bidomains of configurations.

2 Dialogue games and tensorial logic

Tensorial logic is a primitive logic of tensor and negation which refines linear logic by relaxing the hypothesis that negation is involutive. At the same time, tensorial logic may be seen as a resource-aware version of polarized linear logic developed by Laurent [17] which itself was based on the ideas by Girard on polarities in classical logic [10]. In particular, it extends the connection between polarized linear logic and dialogue games formulated in [16] to the positional and resource-aware notion of dialogue game defined below.

▸ **Definition 3.** A dialogue game is defined as a family of rooted trees (or forest) where every node m is equipped with an equivalence relation $\mathbf{conflict}[m]$ on its set of children. A node of the forest A is called a *move* of the dialogue game. One writes $m \vdash_A n$ when the node n is a child of the node m in the forest A, and one declares in that case that the move m justifies the move n. Accordingly, a root of the forest A is called an *initial move* of the dialogue game because it is not justified by any other move. A position of the dialogue game A is then defined as a (non-empty) subtree x of the forest A containing only pairwise non-conflicting moves. The set of positions of a dialogue game is denoted $\mathbf{Pos}\,(A)$. By convention, we declare that every move of odd depth is Player, and every move of even depth is Opponent. In other words, every initial move is Player, and every branch of the forest is then alternating between Opponent and Player moves.

We will make use of the fact that every finite dialogue game A may be alternatively seen as a formula of tensorial logic:

$$A, B \quad ::= \quad 0 \mid 1 \mid A \oplus B \mid A \otimes B \mid \neg A$$

modulo the equations:

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C) \qquad\qquad 0 \otimes A \cong 0$$

together with the associativity and commutativity of $\oplus$ and $\otimes$ and the fact that the formulas 0 and 1 are their respective units. Let us briefly explain how the correspondence works. The dialogue game 0 is the empty forest and the sum $A \oplus B$ of two dialogue games is obtained by putting the two forests A and B side by side. As already mentioned, the game 1 is the tree with a unique Player move $*$. The negation $\neg A$ of a dialogue game A is defined by "lifting" the game A with a move $*$ which justifies the initial moves of the game A. The equivalence relation $\mathbf{conflict}[*]$ is defined as the total relation, hence every two moves justified by the unique initial move $*$ are conflicting in the game $\neg A$. The tensor product of two dialogue games is required to satisfy the distributivity law

$$\bigoplus_{i \in I} A_i \quad \otimes \quad \bigoplus_{j \in J} B_j \quad = \quad \bigoplus_{(i,j) \in I \times J} A_i \otimes B_j$$

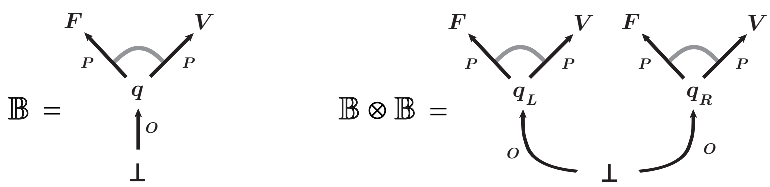
For that reason, the tensor product of two finite dialogue games is entirely described by the equality

$$A \quad = \quad \bigotimes_{i \in I} \neg A_i$$

where the dialogue game A is defined as the *coalesced sum* of the trees $\neg A_i$. This coalesced sum A is the dialogue game with unique initial move $*$ obtained (1) by taking the disjoint sum of the trees $\neg A_i$ and then (2) by identifying the unique initial move $*_i$ of each tree $\neg A_i$ to the unique initial move $*$ of the game A. By definition of a coalesced sum, this dialogue game A is a tree whose unique initial move justifies the moves justified by the root $*_i$ in the game A_i. Its compatibility relation is defined as follows:

$$\mathbf{conflict}[*_A] \quad = \quad \biguplus_{i \in I} \mathbf{conflict}[*_i]$$

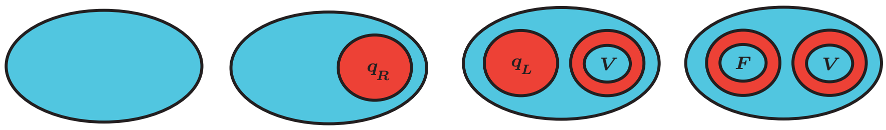
Typically, the boolean formula $1 \oplus 1$ is interpreted as the forest with only two nodes V and F (for *Vrai* and *Faux*, true and false in French) whereas its double negation $\mathbb{B} = \neg\neg\,(1 \oplus 1)$ and the tensor product $\mathbb{B} \otimes \mathbb{B}$ define the following dialogue games:



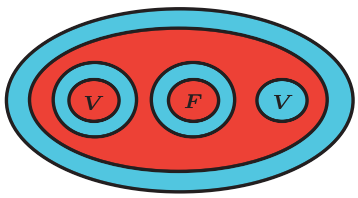
A dialogue game is called *simple* when the conflict relation is full over every move of the game. For instance, the dialogue game $\mathbb{B}$ is simple whereas the dialogue game $\mathbb{B} \otimes \mathbb{B}$ is not simple because the two moves q_L and q_R are not in the same equivalence class of **conflict**$[*]$. Note that the set of positions of a dialogue game may be defined inductively as follows:

$$\begin{array}{rcl}
\mathbf{Pos}\,(0) & = & \varnothing \\
\mathbf{Pos}\,(1) & = & \{\,*\,\} \\
\mathbf{Pos}\,(A \oplus B) & = & \mathbf{Pos}\,(A) \;+\; \mathbf{Pos}\,(B) \\
\mathbf{Pos}\,(A \otimes B) & = & \mathbf{Pos}\,(A) \;\times\; \mathbf{Pos}\,(B) \\
\mathbf{Pos}\,(\neg\, A) & = & \mathbf{Pos}\,(A) \;+\; \{\,*\,\}
\end{array}$$

Every such position x may be nicely depicted by drawing every move m in it as a circle (or as an ellipse) containing the circles corresponding to the moves n justified by m. The colour convention is to depict the Player moves as blue circles, and the Opponent moves as red circles. Typically, the four positions $\{\perp\}$, $\{\perp, q_R\}$, $\{\perp, q_R, V_R, q_L\}$ and $\{\perp, q_L, F_L, q_R, V_R\}$ of the game $\mathbb{B} \otimes \mathbb{B}$ are respectively depicted as



Similarly, the maximal position of the dialogue game $(\mathbb{B} \otimes \mathbb{B}) \multimap \mathbb{B} = \neg(\mathbb{B} \otimes \mathbb{B} \otimes \neg\,(1 \oplus 1))$ is depicted as



(9)

where, by convention, we write $A \multimap B$ for the dialogue game $\neg\,(A \otimes B')$ when $B = \neg\, B'$. The intuition behind these pictures is that every move m of a dialogue game is a *memory cell* of a more advanced technology than in the case of concrete data structures, since it may contain several *independent* cells, each of them filled by a value. Quite obviously, each of these cells corresponds to a specific equivalence class of **conflict**$[m]$. This is typically the case of the Opponent move q in the position (9) which is filled by the three independent "values" q_L, q_R and *done*. Note that one recovers the traditional notion of memory cell when the dialogue game is simple, since in that case every memory cell is filled by at most one value.

3 Innocent strategies

In order to define the notion of innocent strategy on dialogue games, we find convenient to recall the asynchronous formulation of innocence formulated in [20]. The starting point of the approach is the idea that every dialogue game A defines an asynchronous transition system

- whose nodes are the positions of the game,
- with a transition $m : x \to y$ between two positions whenever $y = x \uplus \{m\}$ where m is a move of the game and $\uplus$ means disjoint sum,
- with a permutation $(x \to y_1 \to z) \sim (x \to y_2 \to z)$ whenever $y_1 = x \uplus \{m\}$, $y_2 = x \uplus \{n\}$ and $z = x \uplus \{m, n\}$ for two different moves m and n of the game.

Every transition $m : x \to y$ is polarized either as Player or Opponent depending on the polarity of the move m added to the position x in order to obtain the position y. Every initial Player move $*$ of the dialogue game defines an initial position $\{*\}$ of the associated asynchronous transition system. By convention, we generally identify the initial position $\{*\}$ with the initial Player move $*$.

▸ **Definition 4.** A sequential play of a dialogue game A is defined as a path

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} \cdots \xrightarrow{m_k} x_k$$

starting from an initial position $*$ of the asynchronous transition system and then alternating between Opponent and Player moves. In particular, every move m_k is Opponent when k is odd and Player when k is even. The position x is called the *target position* of the play s. A play is called *empty* when $k = 0$. There is a one-to-one correspondence between the initial positions of a dialogue game and its empty plays.

▸ **Definition 5.** A sequential strategy σ of a dialogue game A is defined as set of even-length sequential plays which

- has a starting point: σ contains the empty play $*$ for exactly one initial position $*$,
- is closed under even-length prefix: $s \cdot m \cdot n \in \sigma$ implies that $s \in \sigma$,
- is deterministic: $s \cdot m \cdot n_1 \in \sigma$ and $s \cdot m \cdot n_2 \in \sigma$ implies that $n_1 = n_2$

for all plays s and all moves m, n, n_1, n_2 of the dialogue game.

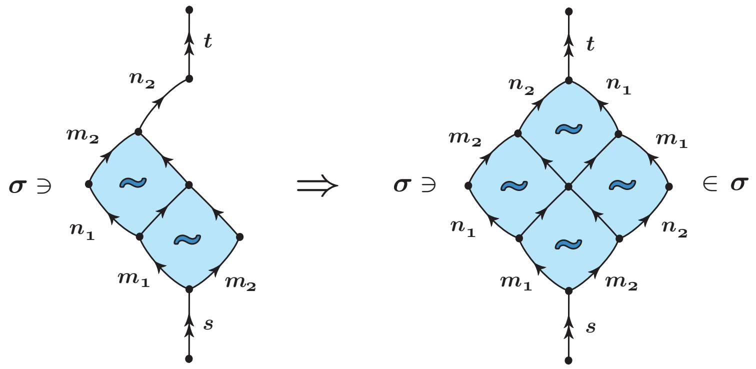
▸ **Definition 6.** A sequential strategy is called backward innocent when every play $s \in \sigma$, every path t, every pair of Opponent moves m_1, m_2, and every pair of Player moves n_1, n_2 which satisfy the properties:

$$s \cdot m_1 \cdot n_1 \cdot m_2 \cdot n_2 \cdot t \in \sigma \quad \text{and} \quad \neg(n_1 \vdash m_2) \quad \text{and} \quad \neg(m_1 \vdash m_2)$$

satisfy also the properties:

$$\neg(n_1 \vdash n_2) \quad \text{and} \quad \neg(m_1 \vdash n_2) \quad \text{and} \quad s \cdot m_2 \cdot n_2 \cdot m_1 \cdot n_1 \cdot t \in \sigma.$$

Backward innocence may be depicted as the following diagrammatic property:



(10)

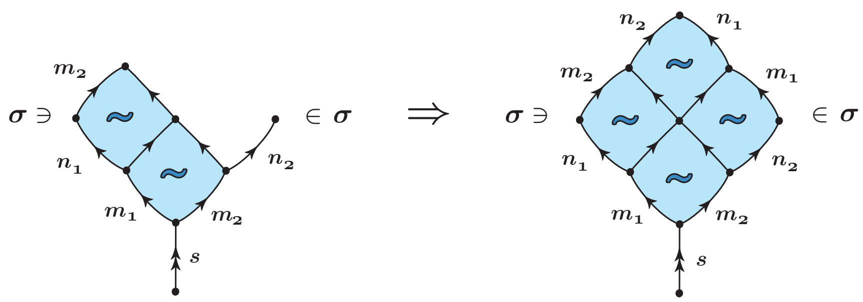
▸ **Definition 7.** A strategy σ is forward innocent when every play $s \in \sigma$, every pair of Opponent moves m_1, m_2, and every pair of Player moves n_1, n_2 satisfying the properties:

$$s \cdot m_1 \cdot n_1 \in \sigma \quad \text{and} \quad s_1 \cdot m_2 \cdot n_2 \in \sigma \quad \text{and} \quad m_1 \neq m_2$$

satisfy also the properties:

$$n_1 \neq n_2 \quad \text{and} \quad s \cdot m_1 \cdot n_1 \cdot m_2 \cdot n_2 \in \sigma.$$

Forward innocence may be depicted as the following diagrammatic property:



(11)

▸ **Definition 8.** A strategy is called innocent when it is backward and forward innocent.

One important property of innocence established in [20] is that every innocent strategy is positional, in the sense that it is entirely described by its set of halting positions. By halting position of the innocent strategy σ, we mean a position x of the dialogue game such that there exists a play $s \in \sigma$ with target position x.

▸ **Definition 9.** A sequential strategy σ of a dialogue game A is called total when for every play $s \in \sigma$, and for every Opponent move m such that $s \cdot m$ is a play of the dialogue game A, there exists a Player move n such that $s \cdot m \cdot n \in \sigma$.

Note that the notion of total strategy considered here is weaker than in [23] since we do not require that every maximal position (x, y) of the strategy in $\mathbf{Dialogue}(A, B)$ reaches two maximal positions x and y of the dialogue games A and B. A typical illustration is provided by the strategy $e_A : \neg A \to 1$ which contains exactly the empty play on the unique initial position $*$. Its unique maximal position is the pair $(*, *)$ of initial positions in $\neg A$ and 1 although the position $*$ is not maximal in the dialogue game $\neg A$.

One main application of tensorial logic is the following characterization of the category **Dialogue** of dialogue games and total innocent strategies. The proof of the proposition may be done directly in a proof-theoretic style or by extending to finite sums the combinatorial presentation of innocence in [23].

▸ **Proposition 10.** *The category* **Dialogue** *is the free affine dialogue category with finite sums (and tensor product distributing over these finite sums) generated by the empty category.*

Although Proposition 10 looks like a purely conceptual statement, it provides a very useful tool in order to relate game semantics to various models of tensorial or linear logic. In particular, it states that there exists a canonical (and functorial) interpretation of dialogue games and total innocent strategies

$$[-] \quad : \quad \mathbf{Dialogue} \longrightarrow \mathcal{D} \tag{12}$$

in any affine dialogue category $\mathcal{D}$ with finite sums, where the tensor distributes over finite sums. Moreover, by its mere construction, the functor $A \mapsto [A]$ preserves the monoidal structure, the finite sums, the negation and the weakening map $e_A : \neg A \to 1$ up to coherent isomorphism.

4 A Kripke translation of tensorial logic into linear logic + necessity

One preliminary insight of the paper is that the construction $A \mapsto \mathbf{Pos}\,(A)$ which transports a dialogue game to its set of positions may be understood as an instance of the semantic functor (12). After all, a simple example of such an affine dialogue category $\mathcal{D}$ is provided by the category **Rel** of sets and relations with weakening $e_A : A^{\perp} \to 1$ defined as the empty relation. As in the case of any such affine $*$-autonomous category, the tensorial negation $\neg A$ is interpreted as the involutive negation:

$$[\neg A] \quad = \quad [A]^{\perp}. \tag{13}$$

In the specific case of **Rel**, this implies that $[A]$ coincides with the set of *maximal positions* of the dialogue game A. This preliminary observation leads to the idea of replacing the inappropriate interpretation (13) of tensorial negation by the following one

$$[\neg A] \quad = \quad \Box\, [A]^{\perp} \tag{14}$$

where the modality $\Box$ would be typically defined as

$$\Box A \quad = \quad A \ \&\ \bot \tag{15}$$

in order to add a point to the relational interpretation of A. The idea is tempting, but there remains to justify it from a logical and algebraic point of view. In order to understand where we stand, it is worth recalling that tensorial logic enjoys the same position with respect to linear logic as intuitionistic logic does with respect to classical logic. From that point of view, it makes sense to translate tensorial logic into linear logic in just the same way as one translates intuitionistic logic into classical logic. A typical solution is to adapt the well-known Kripke translation of intuitionistic logic in the modal logic S4 consisting of classical logic extended with a necessity modality $\Box$. Recall that the Kripke translation is based on the following interpretation of the intuitionistic implication:

$$[\, A \Rightarrow_{int} B \,] \qquad = \quad \Box\,(\, [A]^{\perp} \vee [B]\,) \tag{16}$$

Note that one recovers an intuitionistic variant of (14) by taking the formula B equal to false in (16). Consequently, our next purpose will be to design a linear logic extended with a necessity modality $\Box$ in such a way as to make our tensorial version of the Kripke translation (14) work. We could proceed syntactically and define a sequent calculus for the logic, which we will call linear S4 for simplicity. Since this is essentially equivalent, we prefer to remain at an algebraic level, and to define a categorical semantics of linear S4. To that purpose, we introduce the following notion:

▸ **Definition 11.** A necessity modality on a symmetric monoidal category $\mathcal{L}$ is defined as a symmetric monoidal comonad $\Box$. By this, one means a comonad $\Box$ thus equipped with two natural families of morphisms

$$\varepsilon_A : A \longrightarrow \Box A \qquad\qquad \delta_A : \Box A \longrightarrow \Box\Box A$$

making the expected associativity and unit diagrams commute, together with a natural family of coercions

$$m_{A,B} : \Box A \otimes \Box B \to \Box(A \otimes B) \qquad\qquad m_1 : 1 \longrightarrow \Box 1$$

making $\Box$ a lax symmetric monoidal functor, and compatible with the structure of the comonad.

It is well-known and not difficult to check that in that case, the comonad factors as

$$\Box \quad = \quad \mathit{Forget} \circ \mathit{Necessary}$$

where *Forget* and *Necessary* define a symmetric monoidal adjunction

$$\begin{tikzcd} (\mathcal{M}, \otimes, 1) \arrow[r, bend left, "Forget"] \arrow[r, phantom, "\perp"] & (\mathcal{L}, \otimes, 1) \arrow[l, bend left, "Necessary"] \end{tikzcd} \tag{17}$$

and the category $\mathcal{M}$ is typically defined as the category of Eilenberg-Moore coalgebras of the comonad. The adjunction is called a *symmetric monoidal adjunction* because it is the same thing as a formal adjunction in the 2-category of symmetric monoidal categories and symmetric monoidal functors in the lax sense, see [21] for details. The notion of symmetric monoidal adjunction is important in tensorial logic because it enables one to transport the tensorial negations of the category $\mathcal{L}$ into the category $\mathcal{M}$. Suppose for instance that the category $\mathcal{L}$ is $*$-autonomous. In this case, the category $\mathcal{M}$ inherits a tensorial negation

$$\neg\, A \qquad = \quad \mathit{Necessary}\,(\,(\mathit{Forget}\, A)^{\perp}) \tag{18}$$

from the linear negation in the category $\mathcal{L}$. Hence, $\mathcal{M}$ defines a dialogue category. This establishes that every $*$-autonomous category $\mathcal{L}$ equipped with a necessity modality $\Box$ induces a model of tensorial logic, simply defined as its dialogue category $\mathcal{M}$ of Eilenberg-Moore coalgebras. Note that the category $\mathcal{M}$ has finite sums as soon as the underlying category $\mathcal{L}$ has finite sums. One shows moreover that the dialogue category $\mathcal{M}$ is affine when the necessity modality $\Box$ is affine in the following sense.

▸ **Definition 12.** An affine necessity modality $\Box$ on a symmetric monoidal category $\mathcal{L}$ is a necessity modality equipped with a family of coalgebra maps $e_A : \Box A \longrightarrow 1$ natural in A.

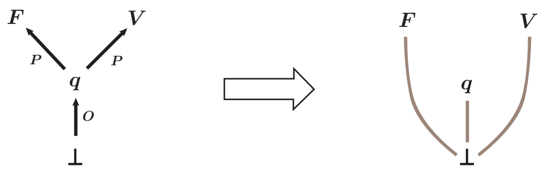
The notion of affine necessity modality is quite familiar in models of linear logic. In particular, the exponential modality ! of a linear category $\mathcal{L}$ defines an affine necessity modality, see [21] for details. The ongoing discussion establishes that

▸ **Proposition 13.** *Every $*$-autonomous category with finite sums equipped with an affine necessity modality $\Box$ induces a functor*

$$(\mathbf{Dialogue}, \otimes, 1) \quad \longrightarrow \quad (\mathcal{M}, \otimes, 1)$$

where $\mathcal{M}$ denotes the category of Eilenberg-Moore coalgebras of the comonad $\Box$.

It is not very difficult to check that equation (15) defines an affine necessity modality $\Box$ in the category **Rel**, with weakening $e_A : A \,\&\, 1 \to 1$ defined as the projection on the second component. Much more interesting is the fact that the same equation (15) defines an affine necessity modality in the category **Coh** of coherence spaces. The resulting semantic functor $A \mapsto [A]$ enables us to identity the set of positions $\mathbf{Pos}\,(A)$ as the web of the coherence space $[A]$. By way of illustration, the dialogue game $\mathbb{B} = \neg\neg(1 \oplus 1)$ is transported to the following coherence space:



(19)

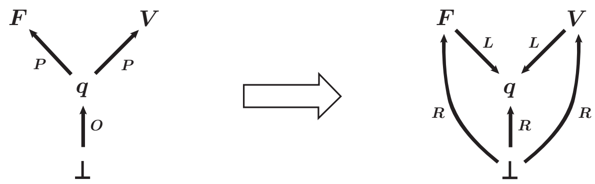
where the initial position $\perp$ is coherent with the three other positions $q = \{\perp, q\}$, $F = \{\perp, q, F\}$ and $V = \{\perp, q, V\}$ which are themselves pairwise incoherent. One main benefit of our logical approach to game semantics is that every innocent strategy σ playing on the dialogue game A is shown to be interpreted as a *clique* of halting positions $[\sigma]$ in the coherence space of positions $[A]$. This fact that the set of halting positions of an innocent strategy σ defines a clique in $[A]$ is reasonable, but it does not seem so easy to establish by a direct and purely combinatorial proof.

5 Dialogue categories and coherent strategies

Our next task is to apply our general method in order to interpret the positions of a dialogue game A as the web of a bistructure. The bistructure model of linear logic was introduced by Curien, Plotkin and Winskel about ten years ago [6] and it remains today one of the most clever and enigmatic models ever designed for linear logic. Its main achievement is to integrate the causality principles underlying Berry's notion of stable function — later revisited by Girard in his coherence space model of linear logic — to the information structure underlying the notion of continuous function between Scott domains [24]. The definition of bistructure is recalled in the appendix. In order to achieve our task on dialogue games, we introduce an affine necessity modality on bistructures:

$$\Box \quad : \quad \mathbf{Bistr} \quad \longrightarrow \quad \mathbf{Bistr}$$

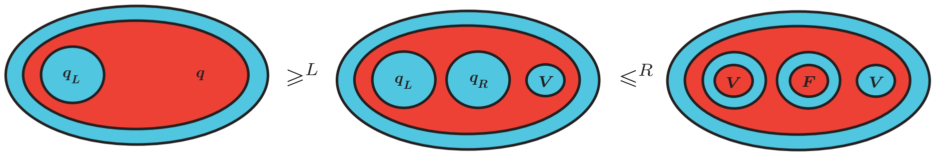
simply defined by extending a given bistructure E with one element $*$ in such a way that $* \leqslant^R e$ for all $e \in E$. Note that by definition of a bistructure, this implies that $* \frown e$ for all $e \in E$. We then apply Proposition 13 in order to interpret the set of positions of a dialogue game A as the web of a bistructure $[A]$, and an innocent strategy $\sigma : A \to B$ as a configuration $[\sigma]$ defining a morphism $[A] \to [B]$ in the category $\mathcal{M}$ of coalgebras of the comonad $\Box$. Typically, the bistructure associated to the dialogue game $\mathbb{B} = \neg\neg(1 \oplus 1)$ refines the coherence space (19) with the extra $\leqslant^L$ and $\leqslant^R$ ordering information:



The diagram should be read as follows: it states that $F, V \leqslant^L q$ and that $\perp \leqslant^R q, F, V$. An easy induction on the formulas of tensorial logic enables one to characterize the two orders $\leqslant^L$ and $\leqslant^R$ on the set of positions of a dialogue game A.

▸ **Proposition 14.** *For every dialogue game A, two positions $x, y \in \mathbf{Pos}(A)$ satisfy*
- *$x \leqslant^L y$ precisely when $y \subseteq x$ and the position y may be obtained from x by removing subtrees with Player moves as roots,*
- *$x \leqslant^R y$ precisely when $x \subseteq y$ and the position x may be obtained from y by removing subtrees with Opponent moves as roots.*

Proposition 14 is important because it provides an elementary and purely combinatorial account of the two orders $\leqslant^L$ and $\leqslant^R$. A typical illustration of these orderings is provided by the three positions of the dialogue game $\mathbb{B} \otimes \mathbb{B} \multimap \mathbb{B}$ considered earlier:



Unfortunately, the coherence relation $\frown_{[A]}$ between positions of a dialogue game A appears more difficult to formulate in a similarly simple combinatorial way. We will not try to do that here. Rather, we establish the following useful property.

▸ **Proposition 15.** *The set-theoretic intersection $x \cap y$ of two positions $x, y \in \mathbf{Pos}(A)$ included in a position $z \in \mathbf{Pos}(A)$ is itself a position of the dialogue game A. Moreover, the two positions x and y are coherent in the bistructure $[A]$ of positions in the sense that $x \frown_{[A]} y$ whenever they satisfy the inequalities:*

$$x \cap y \leqslant^R x \qquad\qquad x \cap y \leqslant^R y.$$

Dually, the two positions x and y are incoherent in the bistructure of positions in the sense that $x \asymp_{[A]} y$ whenever they satisfy the inequalities:

$$x \leqslant^L x \cap y \qquad\qquad y \leqslant^L x \cap y.$$

Proof. See the appendix. ◂

An interesting and non trivial consequence of Proposition 13 is the following statement:

▸ **Proposition 16.** *The set of halting positions $[\sigma]$ of a total innocent strategy σ playing on a dialogue game A defines a configuration of the bistructure of positions $[A]$.*

Once this result established, a natural question is to understand more generally the behaviour of *any* configuration σ of the bistructure $[A]$ of positions associated to a dialogue game A. We know already that every such configuration σ is secured, and thus has a *backward dynamics* which recovers from every position $x \in \sigma$ the causal cascade which produced it from the initial position of the dialogue game $*$. Indeed, in the case of a bistructure of positions $[A]$, securedness means that for every position x in the configuration σ and for every position y obtained by removing some Opponent information from x, there exists a position $z \in \sigma$ obtained by removing some Player information from y. This interpretation of securedness follows from Proposition 14. The somewhat surprising observation is that *every* configuration σ of a bistructure of positions $[A]$ is also equipped with a *forward dynamics* and thus behaves like a (usually not sequential) strategy. This last claim is formulated as the following result:

▸ **Proposition 17.** *For every configuration σ of the bistructure $[A]$ of positions of a dialogue game A, and for every pair of positions $x \in \sigma$ and $z \in \sigma$, such that $x \subseteq z$, and for every Opponent transition $m : x \to y$ to a position $y \in \mathbf{Pos}\,(A)$ such that $y \subseteq z$, there exists a (possibly empty) path of Player transitions $t : y \to y_1 \to \cdots \to y_n \to y'$ such that $y' \in \sigma$ and $y' \subseteq z$. The position y' is moreover unique.*

Proof. See the appendix. ◂

The result of Proposition 17 justifies to introduce the following definition.

▸ **Definition 18.** A coherent strategy on a dialogue game A is defined as a configuration on the bistructure of positions $[A]$. Accordingly, the category **Coherent** is defined as the category with dialogue games as objects and with configurations σ of the bistructure $[A] \multimap [B]$ making the diagram below commute

$$\begin{tikzcd} {[A]} \arrow[r, "\sigma"] \arrow[d, "d_A"'] & {[B]} \arrow[d, "d_B"] \\ \square\,[A] \arrow[r, "\square\sigma"] & \square\,[B] \end{tikzcd}$$

where d_A and d_B are the coalgebra structures wrt. the comonad $\square$ of the bistructures of positions $[A]$ and $[B]$.

as morphisms. By construction, the category **Coherent** is an affine dialogue category with finite sums, and its tensor product distributes over these finite sums. Moreover, there is a functor of dialogue category

$$\mathbf{Dialogue} \xrightarrow{\ \ embedding\ \ } \mathbf{Coherent}$$

and the category **Coherent** embeds fully and faithfully as a dialogue category in the dialogue category $\mathcal{M}$ of coalgebras of the comonad $\square$.

6 Sequential algorithms as stable extensional functions

The connection between dialogue games and bistructures provided by the functor $[-]$ only works at this stage for the linear fragment of tensorial logic. In particular, it does not include the quantitative exponential modality of dialogue games and innocent strategies. However, we explain that this connection is sufficient in order to interpret the qualitative exponential modality ! of simple games. The connection is provided by the following observation:

▸ **Proposition 19.** *For every simple game A, there exists an isomorphism*

$$\lambda_A \quad : \quad !\,[A] \quad \to \quad \square\ [shriek(A)] \tag{20}$$

in the category of bistructures, where ! denotes the qualitative exponential modality on bistructures introduced by Curien, Plotkin and Winskel.

Proof. See the appendix. ◂

Just as announced in the introduction, using this result, one constructs a functor

$$shriek \quad : \quad \mathbf{Simple} \quad \longrightarrow \quad \mathbf{Coherent}$$

making the diagram (7) commute. The functor *shriek* is constructed in such a way that the composite functor (8) coincides with the functor

$$\mathbf{Simple} \xrightarrow{\ [-]\ } \mathbf{Coherent} \xrightarrow{\ forgetful\ } \mathbf{Bistr} \xrightarrow{\ !\ } \mathbf{Bistr}$$

One observes moreover that the bistructure $[A]$ of positions of a simple game A is a B-bistructure in the sense of Curien, Plotkin and Winskel, see [6]. From this follows that its set of configurations $\Gamma(A)$ equipped with the stable order $\sqsubseteq^R$ and the extensional order $\sqsubseteq$ defines a bidomain in the sense of Berry [2]. From all this, one deduces that

▸ **Proposition 20.** *There exists a functor*

$$\Gamma \quad : \quad \mathit{Kleisli}(\mathbf{Simple}, !) \quad \longrightarrow \quad \mathit{Kleisli}(\mathbf{Bistr}, !)$$

between the co-Kleisli categories induced by the Curien-Lamarche modality $!$ *on simple games and the Curien-Plotkin-Winskel modality* $!$ *on bistructures. The definition of the functor* Γ *is based on the fact that every sequential algorithm*

$$\sigma \quad : \quad A \quad \Rightarrow \quad B \tag{21}$$

may be alternatively seen as a sequential strategy

$$\sigma \quad : \quad !\,A \quad \longrightarrow \quad B$$

in the category **Simple** *of simple games, which may be itself seen as an innocent strategy*

$$\varphi(\sigma) \quad : \quad \mathit{shriek}(A) \quad \longrightarrow \quad B$$

in the category **Dialogue** *of dialogue games. By definition, the functor* Γ *transports the sequential strategy (21) to the composite morphism*

$$!\,[A] \xrightarrow{\lambda_A} \Box\,[\,\mathit{shriek}(A)\,] \xrightarrow{\mathit{counit}} [\,\mathit{shriek}(A)\,] \xrightarrow{[\varphi(\sigma)]} [B]$$

in the category of bistructures, which itself corresponds to the stable and extensional function

$$\Gamma(\sigma) \quad : \quad \Gamma(A) \quad \Rightarrow \quad \Gamma(B)$$

between the bidomains of configurations $\Gamma(A)$ *and* $\Gamma(B)$ *induced by the bistructures of positions* $[A]$ *and* $[B]$ *of the simple games* A *and* B*.*

7 Conclusion

This work on coherent strategies between dialogue games is still at a pretty preliminary stage but we find useful to share the general methodology of our approach based on tensorial logic as well as the somewhat unexpected discovery that the category of bistructures contains a subcategory of dialogue games and coherent strategies. Our final result that every sequential algorithm between two simple games A and B induces a stable and extensional function $\Gamma(A) \to \Gamma(B)$ between the associated bidomains of configurations is related to the extensional description of sequential algorithms investigated by Curien, Laird and Streicher [14, 7, 18]. In particular, Streicher made the important observation that the set of *sequential strategies with errors* on a simple game defines a bidomain in the sense of Berry. In that line of research, it should be possible to refine our Proposition 20 in order to characterize the sequential algorithms between A and B as a specific class of stable and extensional functions, but we prefer to leave that aspect for future work. Note that such a characterization has already been given by Calderon and McCusker [4] for sequential strategies between simple games. Another question of interest would be to understand the relationship between the present work on dialogue games and bistructures with the tight connection between sequential games and Ehrhard's hypercoherence spaces [8, 22].

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Appendix: a short account of bistructures

We recall below the notion of bistructure as well as the main definitions of the theory.

▸ **Definition 21.** A (countable) bistructure is a quadruple $(E, \leqslant^L, \leqslant^R, \frown)$ where E is a countable set called the *web* of the bistructure, $\leqslant^L$, $\leqslant^R$ are partial orders on E and $\frown$ is a binary reflexive, symmetric relation on E such that:

1. defining $\leqslant$ as the transitive closure of $(\leqslant^L \cup \leqslant^R)$, we have the following factorisation property:

$$e \leqslant e' \quad \Rightarrow \quad \exists e'' \in E, \quad e \leqslant^L e'' \leqslant^R e'$$

2. defining $\preceq$ as the transitive closure of $(\geqslant^L \cup \leqslant^R)$, we have the following properties:
   a. $\preceq$ is finitary, i.e., $\{e' | e' \preceq e\}$ is finite, for all $e \in E$,
   b. $\preceq$ is a partial order,
   3. (a) $\downarrow^L \subseteq \asymp$ and (b) $\uparrow^R \subseteq \frown$.

Here, the two compatibility relations are defined by:

$$\begin{array}{lcl} e \downarrow^L e' & \iff & \exists\, e'' \in E, \quad e'' \leqslant^L e \text{ and } e'' \leqslant^L e' \\ e \uparrow^R e' & \iff & \exists\, e'' \in E, \quad e \leqslant^R e'' \text{ and } e' \leqslant^R e''. \end{array}$$

and we write $\asymp$ for the reflexive closure of the complementary of $\frown$. We then recall below the definition of configuration.

▸ **Definition 22.** A configuration of a bistructure $(E, \leqslant^L, \leqslant^R, \frown)$ is a subset $\sigma \subseteq E$ which is:

- consistent: $\forall e, e' \in \sigma$, $e \frown e'$, and
- secured: $\forall e \in \sigma$, $\forall e' \leqslant^R e$, $\exists e'' \in \sigma$, $e' \leqslant^L e''$.

We write $\Gamma(E)$ for the set of configurations of a bistructure E, and $\Gamma_{fin}(E)$ for the subset of *finite* configurations. At this point, we recall how Curien, Plokin and Winskel [6] define a stable order $\sqsubseteq^R$ and an extensional order $\sqsubseteq$ on the configurations $\sigma, \tau \in \Gamma(E)$ of a given bistructure E.

▸ **Definition 23.** Let E be a bistructure. The stable order $\sqsubseteq^R$ and the extensional order $\sqsubseteq$ on configurations are defined as:

- $\sqsubseteq^R$ is set-theoretic inclusion,
- $\sigma \sqsubseteq \tau \iff \forall e \in \sigma, \exists e' \in \tau, e \leqslant^L e'$.

Note that it follows from the reflexivity of $\leqslant^L$ that $\sqsubseteq^R$ is included in $\sqsubseteq$. A third relation $\sqsubseteq^L$ is then defined as follows:

$$\sigma \sqsubseteq^L \tau \iff \sigma \sqsubseteq \tau \text{ and } (\forall \upsilon \in \Gamma(E), (\sigma \sqsubseteq \upsilon \text{ and } \upsilon \sqsubseteq^R \tau) \Rightarrow \tau = \upsilon)$$

Thus, $\sigma \sqsubseteq^L \tau$ means that τ is a $\sqsubseteq^R$-minimal configuration such that $\sigma \sqsubseteq \tau$. We also write $\sigma \uparrow^R \tau$ when there exists a configuration $\upsilon \in \Gamma(E)$ such that $\sigma \sqsubseteq^R \upsilon$ and $\tau \sqsubseteq^R \upsilon$.

We briefly recall from [6] that the category **Bistr** has bistructures as objects and configurations of $A \multimap B$ as morphisms $\sigma : A \to B$. The category is $*$-autonomous and has finite sums provided by the following definitions.

- the negation $E^\perp$ of a bistructure $(E, \leqslant^L, \leqslant^R, \frown)$ is defined as $(E, \geqslant^R, \geqslant^L, \asymp)$,
- the sum $E_1 \oplus E_2$ of two bistructures $(E_1, \leqslant_1^L, \leqslant_1^R, \frown_1)$ and $(E_2, \leqslant_2^L, \leqslant_2^R, \frown_2)$ is defined as $(E_1 + E_2, \leqslant_1^L + \leqslant_2^L, \leqslant_1^R + \leqslant_2^R, \frown_1 + \frown_2)$,
- the tensor product $E_1 \otimes E_2$ of two bistructures $(E_1, \leqslant_1^L, \leqslant_1^R, \frown_1)$ and $(E, \leqslant_2^L, \leqslant_2^R, \frown_2)$ is defined as $(E_1 \times E_2, \leqslant_1^L \times \leqslant_2^L, \leqslant_1^R \times \leqslant_2^R, \frown_1 \times \frown_2)$,
- the bistructure 0 has an empty web, and the bistructure 1 has a singleton web,
- the exponential $!E$ of a bistructure $(E, \leqslant^L, \leqslant^R, \frown)$ is defined as $(\Gamma_{fin}(E), \sqsubseteq^L, \sqsubseteq^R, \uparrow^R)$ where these structures are introduced in Definition 23.

Appendix: Proof of Proposition 15

Proof. The proof is established by an easy induction on the formula defining the dialogue game A. The property is obvious in the case of the two unit games 0 and 1. We treat in turn the inductive case of the game $A \otimes B$, of the game $A \oplus B$ and of the game $\neg A$.

First inductive case: the dialogue game $A \otimes B$.

By definition of the dialogue game $A \otimes B$, the two positions x and y are of the form $x = x_A \otimes x_B$ and $y = y_A \otimes y_B$. Suppose that the two positions x and y are included in a position $z = z_A \otimes z_B$. In that case, the positions x_A and y_A obtained by projecting x and y on the component A are included in the position z_A. By induction hypothesis, it follows that $x_A \cap y_A$ is a position of the dialogue game A. One establishes symmetrically that the intersection $x_B \cap y_B$ is a position of the dialogue game B. The set-theoretic intersection $x \cap y$ is equal to $(x_A \cap y_A) \otimes (x_B \cap y_B)$ which is a position of the dialogue game $A \otimes B$. We conclude that $x \cap y$ is a position of the game $A \otimes B$.

Now, suppose that two positions $x = x_A \otimes x_B$ and $y = y_A \otimes y_B$ are included in a position $z = z_A \otimes z_B$ and moreover that $x \cap y \leqslant^R x$ and $x \cap y \leqslant^R y$. In that case, the two positions x_A and y_A are included in the position z_A. Moreover, $x_A \cap y_A \leqslant^R x_A$ and $x_A \cap y_A \leqslant^R y_A$ since $x \cap y = (x_A \cap y_A) \otimes (x_B \cap y_B)$ and the order $\leqslant^R$ is defined in the bistructure $[A \otimes B] = [A] \otimes [B]$ as the componentwise product of $\leqslant^R$ in the bistructures $[A]$ and $[B]$. By induction hypothesis applied to the game A, it follows that $x_A \frown_{[A]} y_A$. One establishes symmetrically that $x_B \frown_{[B]} y_B$. From this, we conclude by definition of coherence in the bistructure $[A \otimes B] = [A] \otimes [B]$ that $x_A \otimes x_B \frown_{[A \otimes B]} y_A \otimes y_B$ and thus, that $x \frown_{[A \otimes B]} y$.

There remains to establish the last statement of the proposition. Suppose that two positions $x = x_A \otimes x_B$ and $y = y_A \otimes y_B$ are included in a position $z = z_A \otimes z_B$ and moreover that $x \leqslant^L x \cap y$ and $y \leqslant^L x \cap y$. The proof that $x \asymp_{[A \otimes B]} y$ is done in the same way as in the previous paragraph. In that case, the two positions x_A and y_A are included in the position z_A. Moreover $x_A \leqslant^L x_A \cap y_A$ and $y_A \leqslant^L x_A \cap y_A$ because $x \cap y = (x_A \cap y_A) \otimes (x_B \cap y_B)$ and the order $\leqslant^L$ is defined in the bistructure $[A \otimes B] = [A] \otimes [B]$ as a componentwise product of $\leqslant^L$ in the bistructures $[A]$ and $[B]$. By induction hypothesis applied to the game A, it follows that $x_A \asymp_{[A]} y_A$. One establishes symmetrically that $x_B \asymp_{[B]} y_B$. From this, we conclude by definition of coherence in the bistructure $[A \otimes B] = [A] \otimes [B]$ that $x_A \otimes x_B \asymp_{[A \otimes B]} y_A \otimes y_B$ and thus, that $x \asymp_{[A \otimes B]} y$.

Second inductive case: the dialogue game $A \oplus B$.

By definition of the dialogue game $A \oplus B$, the fact that the two positions x and y are included in a position z implies that the three positions x, y, z lie in the same component A or B of the game $A \oplus B$. We may suppose without loss of generality that the three positions x, y, z are positions of the component A. From this, it follows easily by induction hypothesis applied to the dialogue game A that the intersection $x \cap y$ is a position in the game A and thus in the game $A \oplus B$. The two remaining statements of the proposition are just as easy to establish by induction.

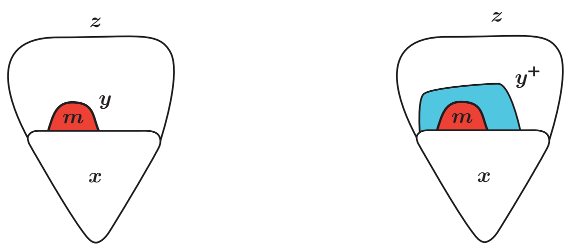
Third inductive case: the dialogue game $\neg A$.

By definition of the dialogue game $\neg A$, we are in one of the two possible situations: either the three positions x, y, z are in the component A or one of the two positions x, y is the initial position $*$ itself. The first case is easily treated by induction hypothesis on A. In the second case, one of the two positions x and y is equal to the initial position $*$. For the sake of discussion, we may suppose without loss of generality that the position x is equal to the initial position $*$.

The intersection $x \cap y = *$ is a position of the dialogue game $\neg A$. Moreover, it follows from the definition of the bistructure $[\neg A]$ of positions of the game $\neg A$ as the bistructure $\square([A]^\perp)$ that the two positions x and y are coherent in the bistructure $[\neg A]$ since $x = *$ is the position added to the bistructure $[A]^\perp$ by the necessity modality. Note that, by definition of $\square([A]^\perp)$, the position $x = *$ also satisfies $* \leqslant^R y$. Moreover, if $y \leqslant^L *$ then $y = *$, and thus $x \asymp_{[\neg A]} y$. This concludes the proof by induction of Proposition 15. ◂

Appendix: Proof of Proposition 17

Proof. The proof is based on the very specific properties of the bistructure $[A]$ associated to a dialogue game A, and in particular on the two Propositions 14 and 15. Given the position $x \in \sigma$ and the Opponent transition $m : x \to y$ such that $y \subseteq z$ for $z \in \sigma$, let y^+ denote the smallest position (with respect to inclusion) containing the position y as a subset: $y \subseteq y^+$, and satisfying $y^+ \leqslant^R z$. This position y^+ exists and is defined according to Proposition 14 by removing from the position z all the subtrees with an Opponent root n not element of the position y. It is important to observe that the position y^+ only contains Player moves besides the moves already in the position y. The situation may be depicted as follows. Note that the Opponent move m in the position y is depicted in red, and the layer of Player moves between y and y^+ is depicted in blue.



By the securedness property of the configuration σ, there exists a position $y' \in \sigma$ such that $y^+ \leqslant^L y'$. By Proposition 14, the position y' is obtained from the position y^+ by removing a series of subtrees with Player roots. From this follows in particular that the position $x \cap y'$ is obtained from the position $x \cap y^+ = x$ by removing a series of subtrees with Player roots. Hence, $x \leqslant^L x \cap y'$.

First claim: the move m appears in the position y'.

We claim that the move m appears in the position y'. Suppose that this is no the case, and that the move m does not appear in the position y'. In that case, a simple argument shows that the position $x \cap y'$ is obtained from the position y' by removing only subtrees with roots in the position y^+ but not in the position y. An important point is that the roots of these subtrees removed from y' in order to obtain $x \cap y'$ are all Player moves. By Proposition 14, it thus follows that $y' \leqslant^L x \cap y'$. Recall moreover that the two positions x and y' are included in the position z and that $x \leqslant^L x \cap y'$. All this put together establishes thanks to Proposition 15 that the positions x and y' are incoherent in the bistructure $[A]$. Since the two positions x and y' are also coherent as elements of the clique σ, they are necessarily equal. This contradicts the definition of y and of y^+ and more specifically the fact that $y^+ \leqslant^L y'$. The point is that the position $x = y'$ can be obtained from the position y (and thus from the position y^+) only at the condition of removing the subtree with Opponent root m. From this, we conclude that the move m necessarily appears in the position y'.

Second claim: the position x is a subset of the position y'.

Now, we want to prove that $x \subseteq y'$. Suppose that this is not the case, and let the position x^+ be obtained by removing the subtree with Opponent root m from the position $y' \in \sigma$. By construction, one has $x^+ \leqslant^R y'$. One also has $x \cap x^+ = x \cap y'$ since m is not an element of x. From this follows that $x \leqslant^L x \cap x^+$ since we already know that $x \leqslant^L x \cap y'$. By securedness of σ, there exists a position $x' \in \sigma$ such that $x^+ \leqslant^L x'$. By Proposition 14, the position x' is obtained from the position x^+ by removing subtrees with Player roots. From this follows that the position $x \cap x'$ is obtained from the position $x \cap x^+$ by removing subtrees with Player roots. Hence, $x \cap x^+ \leqslant^L x \cap x'$ by Proposition 14 again. From this and $x \leqslant^L x \cap x^+$, we conclude by transitivity that $x \leqslant^L x \cap x'$. At this point, a simple argument shows that the position $x \cap x'$ is obtained from the position x' by removing subtrees whose roots stand among the Player moves in y^+ but not in y. The fact that these moves are all Player moves implies that $x' \leqslant^L x \cap x'$. The two inequalities $x \leqslant^L x \cap x'$ and $x' \leqslant^L x \cap x'$ together with the fact that the positions x and x' are included in the position z implies by Proposition 15 that x and x' are incoherent in the bistructure $[A]$. Since the two positions x and x' are also coherent as elements of the clique σ, they are equal. The equality $x = x'$ establishes that $x \subseteq y'$ since $x' \subseteq x^+ \subseteq y'$ by definition of the position $x' \in \sigma$.

From the two claims just established, we conclude that $y = x \uplus \{m\}$ is a subset of the position y'. By construction, the position y' is at the same time a subset of y^+ which only contains Player moves besides the moves already in the position y. From this, we deduce that the position y' only contains Player moves besides the moves already in position y, and thus that there exists a path of Player transitions from the position y to the position $y' \in \sigma$. ◀

Appendix: Proof of Proposition 19

Proof. We construct the isomorphism

$$\lambda_A \quad : \quad ! \, [A] \quad \rightarrow \quad \square \, [\mathit{shriek}(A)]$$

in the category of bistructures, for every simple game A. The first step of the construction is to characterize the configurations σ of the associated bistructure $[A]$ of positions. A preliminary observation is that there is a one-to-one relationship between (1) the positions of the simple game A seen as a dialogue game (2) the sequential plays

$$* \xrightarrow{m_1} x_1 \xrightarrow{m_2} \cdots \xrightarrow{m_k} x_k$$

of the simple game A and (3) the elements of the web of the bistructure $[A]$. Since every position x of the bistructure $[A]$ corresponds to a specific sequential play of the simple game A, every configuration σ is alternatively described by a set of sequential plays (or positions) $x \in \sigma$ of the simple game A. We claim that every configuration σ of the bistructure $[A]$ is closed under even-length prefix in the sense that every sequential play (or position) y which is even-length prefix of a sequential play (or position) $x \in \sigma$ is also an element of the configuration σ. In order to establish our claim, we first observe that by Proposition 14, a position y is an even-length prefix of the position x precisely when $y \leqslant^R x$. Suppose that we are in that case, and that $y \leqslant^R x$. By securedness of σ, we know that there exists a position $z \in \sigma$ such that $y \leqslant^L z$. We would like to prove that $z = y$. Suppose that it is not the case and that z is a strict prefix of y. In that case, $z \in \sigma$ is also a strict prefix of $x \in \sigma$. By Proposition 14, the position z is obtained from the position y by removing a subtree (in that case, a branch) with a Player root. Hence, the position z is also obtained from the position x by removing a subtree with a Player root since y is a prefix of x. From this, we conclude by Proposition 14 that $x \leqslant^L z$. By definition of a bistructure, the two positions x and z are thus incoherent in the bistructure $[A]$. The two positions x and z are also coherent as elements of the configuration σ. From this, we conclude that $x = z$. This contradicts the fact that the position z is a strict prefix of the position y and thus of the position x. From this, we conclude that $z = y$, and thus, that the configuration σ is closed under even-length prefix.

Similarly, we may establish a complementary property of the configuration σ, which states that every strict prefix $y \in \sigma$ of a position $x \in \sigma$ is of even length. The reason is that the two positions x and y of the configuration σ are coherent, whereas the relation $x \leqslant^L y$ (and thus $x \asymp_{[A]} y$) would hold if the position y was of odd length. This second observation leads us to introduce a useful variant of our Definition 5 of sequential strategy on a dialogue game A, see for instance [19].

▸ **Definition 24** (sequential strategy with errors). A sequential strategy σ with errors on a dialogue game A is defined as set of sequential plays which

- has a starting point: σ contains the empty play $*$ for exactly one initial position $*$,
- is closed under even-length prefix, in the sense that for every even-length prefix s of a sequential play t, one has $t \in \sigma \Rightarrow s \in \sigma$,
- has no intermediate errors, in the sense that for every odd-length prefix s of a sequential play t, one has $(s \in \sigma \text{ and } t \in \sigma) \Rightarrow s = t$,
- is deterministic, in the sense that for every even-length sequential play s, $s \cdot m \cdot n_1 \in \sigma$ and $s \cdot m \cdot n_2 \in \sigma$ implies that $n_1 = n_2$,

for all plays s and all moves m, n, n_1, n_2 of the dialogue game.

Note that by definition of a strategy σ with errors, every odd-length position s of the strategy σ is maximal among the positions in σ. Such an odd-length position of the strategy σ is called an error of the strategy. We have just established that every non-empty configuration σ of the bistructure $[A]$ of positions of a simple game A satisfies the three first properties of Definition 5. We prove the fourth property (determinism) below. Suppose given an even-length position x of the configuration σ, alternatively seen as a sequential play:

$$s \quad = \quad * \xrightarrow{m_1} x_1 \xrightarrow{m_2} \cdots \xrightarrow{m_{2k}} x_{2k} = x$$

and suppose that the two sequential plays $y_1 = s \cdot m \cdot n_1$ and $y_2 = s \cdot m \cdot n_2$ are positions in the configuration σ. We claim that $n_1 = n_2$. The proof is very easy, since it simply consists in observing that the two positions y_1 and y_2 are strictly incoherent in the bistructure $[A]$ when the moves n_1 and n_2 are different. Since the positions y_1 and y_2 are elements of the configuration σ, and thus coherent, we conclude that $n_1 = n_2$. This establishes that every non-empty configuration σ of the bistructure $[A]$ defines a sequential strategy with errors of the underlying simple game A. Conversely, it is easy to check that every sequential strategy σ with errors of the simple game A defines a non-empty configuration of the bistructure $[A]$ of configurations. From this we conclude that

Fact: there is a one-to-one relationship between the non-empty configurations of the bistructure $[A]$ and the sequential strategies with errors of the simple game A.

At this point, an obvious but important observation is that every sequential strategy σ with errors of the simple game A may be alternatively seen as a non-empty subtree of A which only branches on Opponent moves. This subtree is entirely described by its set $\mathbf{maxpos}(\sigma)$ of maximal positions. Note that the positions in $\mathbf{maxpos}(\sigma)$ may be either of even-length or of odd-length. The sequential strategy σ with errors is then recovered from $\mathbf{maxpos}(\sigma)$ as

$$\sigma \quad = \quad \mathbf{maxpos}(\sigma) \quad \cup \quad \textbf{even-length-prefix}(\mathbf{maxpos}(\sigma))$$

where **even-length-prefix**(X) denotes the set of even-length prefixes of a position in X. This establishes that there is a one-to-one relationship between the non-empty configurations of $[A]$ and the non-empty subtrees of the simple game A which only branch on Opponent moves. Now, such a non-empty subtree which only branches on Opponent moves in the simple game A is the same thing as a position in the dialogue game *shriek*(A). From this, we conclude that:

Fact: there is a one-to-one relationship between the non-empty configurations of the bistructure $[A]$ and the positions of the dialogue game *shriek*(A).

We use the notation **config**(x) for the non-empty configuration σ of the bistructure $[A]$ associated to the position x in the dialogue game *shriek*(A). At this point, starting from Proposition 14, it is not difficult to establish that

$$\mathbf{config}(x) \sqsubseteq^R \mathbf{config}(y) \iff x \leqslant^R y$$

because **config**$(x) \subseteq$ **config**(y) precisely when $x \subseteq y$ and the position x may be obtained from the position y by removing subtrees with Opponent moves as roots ; that

$$\mathbf{config}(x) \sqsubseteq^L \mathbf{config}(y) \iff x \leqslant^L y$$

precisely when $y \subseteq x$ and the position y may be obtained from the position x by removing subtrees with Player moves as roots ; and finally that

$$\mathbf{config}(x) \uparrow^R \mathbf{config}(y) \iff x \frown_{[shriek(A)]} y$$

for every two positions x, y of the dialogue game *shriek*(A). This establishes that the bistructure $[shriek(A)]$ of positions of the dialogue game *shriek*(A) is *isomorphic* to the bistructure $!\,[A]$ restricted to its non-empty configurations. As for the empty configuration, one has that

$$\varnothing \;\sqsubseteq^R\; \sigma \qquad\qquad \varnothing \;\frown_{!\,A}\; \sigma$$

for every configuration σ of the bistructure $!\,[A]$. This concludes our proof that the bistructure $!\,[A]$ is isomorphic to the bistructure $\square\ [shriek(A)]$ for every simple game A. ◀

Appendix: Proof of Proposition 20

Proof. We show that the transformation Γ just defined is indeed functorial. The first step is to show that Γ transports the identity of a simple game A in the co-Kleisli category into the identity of $[A]$ in the co-Kleisli category. This identity is defined as

$$\varepsilon_A \quad : \quad !A \quad \longrightarrow \quad A$$

which corresponds to the innocent strategy

$$\varphi(\varepsilon_A) \quad : \quad shriek(A) \quad \longrightarrow \quad A$$

which plays in the expected way. This innocent strategy is transported to the configuration

$$[\varphi(\varepsilon_A)] \quad : \quad [shriek(A)] \quad \longrightarrow \quad [A]$$

whose positions are precisely the pairs (x', x) where x is a position of A and x' is the corresponding position in *shriek*(A). The morphism

$$!\,[A] \xrightarrow{\ \lambda_A\ } \square\,[\,shriek(A)\,] \xrightarrow{\ counit\ } [\,shriek(A)\,] \xrightarrow{\ [\varphi(\varepsilon)]\ } [B]$$

coincides with the dereliction morphism

$$\varepsilon_{[A]} \quad : \quad !\,[A] \quad \longrightarrow \quad [A]$$

in the category of bistructures. This establishes that Γ transports the identities into identities. Now, given two strategies

$$!\,A \xrightarrow{\quad\sigma\quad} B \qquad\qquad !\,B \xrightarrow{\quad\tau\quad} C$$

between simple games, we show that Γ transports their composite

$$!\,A \xrightarrow{\quad\sigma^\dagger\quad} !\,B \xrightarrow{\quad\tau\quad} C$$

computed in the category **Simple** into the composite of their image

$$!\,[A] \xrightarrow{\quad\Gamma(\sigma)\quad} [B] \qquad\qquad !\,[B] \xrightarrow{\quad\Gamma(\tau)\quad} [C]$$

computed as

$$!\,[A] \xrightarrow{\quad\Gamma(\sigma)^\dagger\quad} !\,[B] \xrightarrow{\quad\Gamma(\tau)\quad} [C]$$

in the category **Bistr**. Here, we write $\sigma^\dagger$ and $\Gamma(\sigma)^\dagger$ for the lifting of σ and of $\Gamma(\sigma)$ in the categories of simple games and of bistructures. Recall from [6] that the strengthening

$$\tau^\dagger \quad : \quad !\,E \quad \longrightarrow \quad !\,F$$

of a morphism

$$\tau \quad : \quad !\,E \quad \longrightarrow \quad F$$

in the category **Bistr** of bistructures is defined as the configuration

$$v^\dagger \quad = \quad \Big\{ \;\; (\bigcup_{i=1,n} \alpha_i \,,\, \{x_1, ..., x_n\}\;) \in\, !\,E \times !\,F \;\;\Big|\;\; (\alpha_i, x_i) \in \tau \quad \text{for } i = 1, \ldots, n \;\;\Big\}.$$

In particular, the strengthening of the configuration $\Gamma(\sigma)$ is defined as:

$$\Gamma(\sigma)^\dagger \;\; = \;\; \Big\{ \; (\bigcup_{i=1,n} \alpha_i \,,\, \{x_1, ..., x_n\}\;) \in\, !\,[A] \times !\,[B] \;\;\Big|\;\; (\alpha_i, x_i) \in \Gamma(\sigma) \quad \text{for } i = 1, \ldots, n \;\Big\}$$

Here, every element α of the web of the bistructure $[\,!\,A\,]$ is either the empty set $\varnothing$ or a strategy with errors of the simple game A, and every element x of the web of the bistructure $[B]$ is a position (and thus a path) in the simple game B. By construction, the configuration $\Gamma(\sigma)$ contains exactly the halting positions (α, x) of the sequential strategy

$$\varphi(\sigma) \quad : \quad \mathit{shriek}(A) \quad \longrightarrow \quad B$$

where for notational simplicity we identify every position $\alpha \in \mathbf{Pos}\,(\mathit{shriek}(A))$ with the non-empty configuration α in the bistructure $!\,[A]$ associated to it by the one-to-one relationship described in the proof of Proposition 19. In particular, every position (α, x) in the configuration $\Gamma(\sigma)$ satisfies the following property:

- the strategy α does not contain any error when the play x is of even-length,
- the strategy α contains exactly one error when the play x is of odd-length.

At this point, a careful argument establishes that

▸ **Proposition 25.** *For every sequential strategy*

$$\sigma \quad : \quad !A \quad \longrightarrow \quad B$$

between simple games A and B, the configuration

$$\Gamma(\sigma)^{\dagger} \quad : \quad ![A] \quad \longrightarrow \quad ![B]$$

is characterized by the fact that it contains exactly
- *the pair $(\varnothing, \varnothing)$,*
- *the pairs (α, β) consisting of two strategies with errors α and β such that the diagram*

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha^{\dagger}"'] \arrow[dr, "\beta"] & \\ !A \arrow[rr, "\sigma"] & & B \end{tikzcd} \tag{22}$$

commutes in the category of simple games and sequential strategies with errors.

Here, the category of simple games and sequential strategies with errors extends the category **Simple** and is defined in just the expected way. Note that an equivalent way to state Equation (22) is to ask that the diagram below commutes:

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha^{\dagger}"'] \arrow[dr, "\beta^{\dagger}"] & \\ !A \arrow[rr, "\sigma^{\dagger}"] & & !B \end{tikzcd}$$

The same description of $\Gamma(\tau)$ also holds. From this follows by relational composition in the category of bistructures that the configuration

$$![A] \xrightarrow{\Gamma(\sigma)^{\dagger}} ![B] \xrightarrow{\Gamma(\tau)^{\dagger}} ![C]$$

contains exactly
- the pair $(\varnothing, \varnothing)$,
- the pairs (α, γ) consisting of two strategies with errors α and γ such that the diagram

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha^{\dagger}"'] \arrow[dr, "\gamma^{\dagger}"] & \\ !A \arrow[rr, "\tau^{\dagger}\circ\sigma^{\dagger}"] & & !C \end{tikzcd} \tag{23}$$

commutes.

Note that the diagram (23) commutes precisely when the diagram

$$\begin{tikzcd} & 1 \arrow[dl, "\alpha^{\dagger}"'] \arrow[dr, "\gamma"] & \\ !A \arrow[rr, "\tau\circ(\sigma^{\dagger})"] & & C \end{tikzcd}$$

commutes. This last point establishes by Proposition 25 that the strengthening of the morphism

$$\Gamma(\tau \circ (\sigma^{\dagger})) \quad : \quad ![A] \quad \longrightarrow \quad [C]$$

coincides with the composite

$$![A] \xrightarrow{\Gamma(\sigma)^{\dagger}} ![B] \xrightarrow{\Gamma(\tau)^{\dagger}} ![C]$$

computed in the category of bistructures. From this we conclude that the transformation Γ defines a functor

$$\Gamma \quad : \quad \mathit{Kleisli}(\mathbf{Simple}, !) \quad \longrightarrow \quad \mathit{Kleisli}(\mathbf{Bistr}, !)$$

from the category of simple games and sequential algorithms to the category of bidomains and stable and extensional functions. ◂