Axiomatic Rewriting Theory VI

Residual Theory Revisited

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Abstract. Residual theory is the algebraic theory of confluence for the λ-calculus, and more generally *conflict-free* rewriting systems (=without critical pairs). The theory took its modern shape in Lévy's PhD thesis, after Church, Rosser and Curry's seminal steps. There, Lévy introduces a *permutation equivalence* between rewriting paths, and establishes that among all confluence diagrams $P \longrightarrow N \longleftarrow Q$ completing a span $P \longleftarrow M \longrightarrow Q$, there exists a *minimum* such one, modulo permutation equivalence. Categorically, the diagram is called a *pushout*.
In this article, we extend Lévy's residual theory, in order to enscope "border-line" rewriting systems, which admit critical pairs but enjoy a strong Church-Rosser property (=existence of pushouts.) Typical examples are the *associativity rule* and the *positive braid* rewriting systems. Finally, we show that the resulting theory reformulates and clarifies Lévy's optimality theory for the λ-calculus, and its so-called "extraction procedure".

1 Introduction

Knuth and Bendix [KB] define a *critical pair* as a pair of rewriting rules which induces overlapping redexes. The definition applies with little variation to word, term, or higher-order term rewriting systems [KB,K_2,Ni,MN].

The syntactical definition captures the idea that a critical pair is a "singularity" of the system, and a *risk of non-confluence.* Technically speaking, the definition is mainly motivated by Knuth and Bendix's theorem, which states that a (word, term, higher order) rewriting system is *locally* confluent, see [K_2], iff all its critical pairs are *convergent* (=confluent).

For instance, the two rules:

$$A \longrightarrow B \qquad A \longrightarrow C \tag{1}$$

form a critical pair, because they induce overlapping redexes $A \longrightarrow B$ and $A \longrightarrow C$. The critical pair is not convergent, and thus, the term rewriting system is not locally confluent.

On the other hand, the two rules L and R which express the logical connective "parallel or"

$$\mathrm{por}(\mathbf{true}, \mathrm{x}) \xrightarrow{\mathrm{L}} \mathbf{true} \qquad \mathrm{por}(\mathrm{x}, \mathbf{true}) \xrightarrow{\mathrm{R}} \mathbf{true} \tag{2}$$

form also a critical pair, because they induce overlapping redexes l and r from the term $\mathrm{por}(\mathbf{true}, \mathbf{true})$:

$$\mathrm{por}(\mathbf{true}, \mathbf{true}) \;\underset{r}{\overset{l}{\rightrightarrows}}\; \mathbf{true} \tag{3}$$

But this critical pair (2) is convergent, and thus, the rewriting system is locally confluent. It is even *confluent* — this is proved by applying Newman's lemma, or better, by observing that the system is an instance of van Oostrom and van Raamsdonk's *weakly orthogonal* rewriting systems [OR].

Does this mean that the pair (1) is more "critical" than the pair (2) ? Well, from the strict point of view of confluence, yes: the pair (1) is certainly "more" critical than the pair (2). But on the other hand, the standardization theory developed in [HL,Bo,M_2] provides excellent reasons to consider that the pair (2) is as "critical" as (1). We explain this briefly. Suppose that one extends the term rewriting system (2) with the rule

$$\Omega \longrightarrow \mathbf{true}$$

The extended rewriting system is still confluent. Let us count the number of *head* rewriting paths from

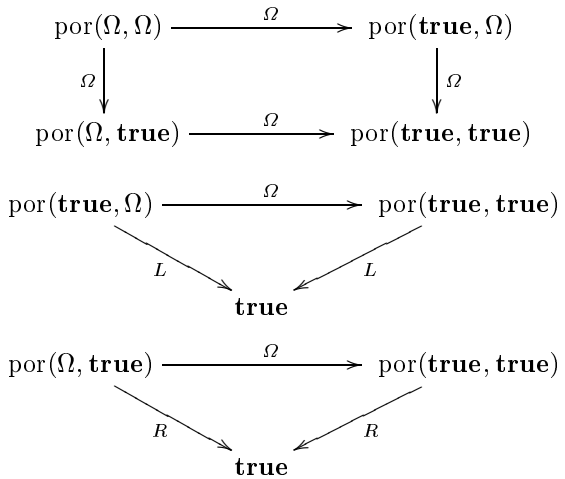
$\mathrm{por}(\Omega,\Omega)$ to the "value" term $V = \mathbf{true}$. These head rewriting paths are characterized in [M_3] for a class of rewriting systems enscoping all (possibly conflicting, possibly higher-order) term rewriting systems. The characterization shows that there exists at most *one* head rewriting path in a *conflict-free* (that is: critical-pair free, and left-linear) term rewriting system; think of the λ-calculus. In contrast, one counts *two* head rewriting paths in the system (2):

$$\begin{array}{l} \mathrm{por}(\Omega,\Omega) \xrightarrow{\Omega} \mathrm{por}(\mathbf{true},\Omega) \xrightarrow{\mathrm{L}} \mathbf{true} \\ \mathrm{por}(\Omega,\Omega) \xrightarrow{\Omega} \mathrm{por}(\Omega,\mathbf{true}) \xrightarrow{\mathrm{R}} \mathbf{true} \end{array} \qquad (4)$$

Each path mirrors the causal cascade leading to the rule L or R which computes the "parallel or" operation to **true**.

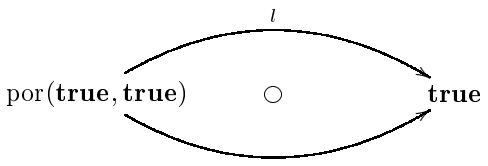
This should convince the reader that the pair (2) is not "critical" for syntactical reasons only (as in Knuth and Bendix's definition) but also for *structural* reasons related to causality of computations.

In [M_1,M_2,M_3], we analyze causality of computations by purely *diagrammatic* methods. Rewriting systems are replaced by their reduction graphs, which we equip with 2-dimensional *tiles*, in order to reflect redex permutations. Typically, the reduction graph of $\mathrm{por}(\mathbf{true},\mathbf{true})$ in (2) is equipped with three tiles:



That way, *generic* properties of rewriting systems (seen as 2-dimensional graphs) may be established. For instance, the *standardization* theorem [CF,Lé$_1$,HL,Bo] states that there exists a unique *standard* (= outside-in) path in each "homotopy" equivalence class. It is proved diagrammatically in [GLM,M_1,M_2] from fairly general axioms on redex permutations in rewriting systems.

It is worth noting here that tiling diagram (3) in the 2-dimensional graph of $\mathrm{por}(\Omega,\Omega)$ would break the standardization theorem, since the two standard paths (4) would appear then in the same "homotopy" class. This dictates to keep diagram (3) untiled in the 2-dimensional graph of $\mathrm{por}(\Omega,\Omega)$, and to understand it as a *2-dimensional hole*.



At this point, each path of (4) appears to be the canonical (that is: standard) representative of one homotopy class from $\mathrm{por}(\Omega,\Omega)$ to **true**. Reformulated in this way, rewriting theory looks simpler, and closer to algebraic topology. This is certainly promising [1].

But let us come back to our main point. On one hand, Knuth and Bendix's definition of a critical pair is purely syntactic; on the other hand, it captures subtle diagrammatic and causal singularities, like the critical pair (2).

However, and this is the starting point of the article, we claim that in one case at least, Knuth and Bendix's syntactic definition of critical pair is too crude.

Consider the term rewriting system with a unique binary symbol: the tensor product noted $\otimes$, and a unique rewriting rule: the *associativity rule* below.

$$x \otimes (y \otimes z) \xrightarrow{\alpha} (x \otimes y) \otimes z \qquad (5)$$

This rewriting system and various extensions have been studied thoroughly by category theorists, in order to

[1] A few readers will find example (2) funny, because tiling the critical pair (3) defines a 2-dimensional graph which happens to enjoy *universal confluence*. This is by pure chance, really. For instance, this universality property does not extend to weakly orthogonal rewriting systems.

establish so-called *coherence theorems* [Mc]. We recall that the rewriting system (5) is confluent and strongly normalizing; and that its normal forms are the variables, or the terms $x \otimes V$ obtained by "appending" a variable x to a normal form V.

Here, we consider the local confluence diagrams induced by (5). They are of two kinds, depending whether the co-initial redexes u and v overlap, or do not overlap. First, the rewriting rule being right-linear (= it does not duplicate) two non-overlapping redexes u and v induce a square diagram, completed by the obvious *residual redexes* u' and v':

$$\begin{array}{ccc} M & \xrightarrow{u} & P \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ Q & \xrightarrow{u'} & N \end{array} \qquad (6)$$

Then, two redexes u and v overlap in one case only:

$$\begin{array}{l} w \otimes (x \otimes (y \otimes z)) \xrightarrow{w \otimes \alpha} w \otimes ((x \otimes y) \otimes z) \\ w \otimes (x \otimes (y \otimes z)) \xrightarrow{\alpha} (w \otimes x) \otimes (y \otimes z) \end{array} \qquad (7)$$

The local confluence diagram below is famous in the categorical litterature: it is called Mac Lane's pentagon.

$$\begin{array}{ccc} w \otimes (x \otimes (y \otimes z)) & \xrightarrow{w \otimes \alpha} & w \otimes ((x \otimes y) \otimes z) \\ & & \downarrow{\scriptstyle \alpha} \\ {\scriptstyle \alpha}\downarrow & & (w \otimes (x \otimes y)) \otimes z \\ & & \downarrow{\scriptstyle \alpha \otimes z} \\ (w \otimes x) \otimes (y \otimes z) & \xrightarrow{\alpha} & ((w \otimes x) \otimes y) \otimes z \end{array} \qquad (8)$$

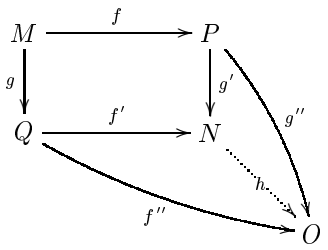
Mac Lane introduced it in order to axiomatize the concept of *monoidal category*. We write $\equiv$ for the "homotopy" equivalence relation on paths obtained by tiling every confluence diagrams (6) and (8). By strong normalization, and Newman's lemma [Ne,K_2], every two sequences f, g of associativity rules from a term M to its normal form V are equal modulo homotopy equivalence $\equiv$. This is roughly the content of Mac Lane's coherence theorem for monoidal category, see [M_5] for further details.

At this point, we stress that the rewriting system (5) and homotopy equivalence $\equiv$ verify a remarkable property, which goes far beyond confluence. We call this property *universal confluence*. It states that, for every two coinitial rewriting paths $f : M \longrightarrow P$ and $g : M \longrightarrow Q$, there exists two paths $f' : Q \longrightarrow N$ and $g' : P \longrightarrow N$ such that:

- the rewriting paths $f; g' : M \longrightarrow N$ and $g; f' : M \longrightarrow N$ are equal modulo $\equiv$,

$$\begin{array}{ccc} M & \xrightarrow{f} & P \\ {\scriptstyle g}\downarrow & \equiv & \downarrow{\scriptstyle g'} \\ Q & \xrightarrow[f']{} & N \end{array}$$

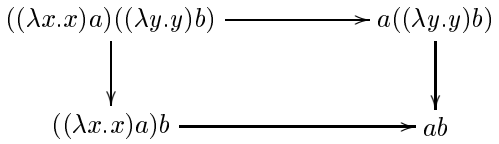
- for every diagram path $f'' : Q \longrightarrow O$ and $g'' : P \longrightarrow O$ such that $f; g'' \equiv g; f''$, there exists a unique rewriting path h (unique modulo homotopy) such that $f'' \equiv f'; h$ and $g'' \equiv g'; h$



The property is established in Dehornoy's theory of Garside groups, using "reversing word" techniques [D_1,DL]. Categorically speaking, the property states that the category of terms and rewriting paths modulo homotopy $\equiv$ enjoys existence of *pushout diagrams*.

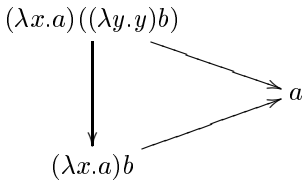
Quite interestingly, the same universal confluence property holds in the λ-calculus, and more generally in any conflict-free rewriting system. There, the homotopy equivalence relation $\equiv$ is obtained by tiling all confluence diagrams generated by Lévy's residual theory. In the λ-calculus, the equivalence relation $\equiv$ is best

illustrated by three paradigmatic confluence diagrams:

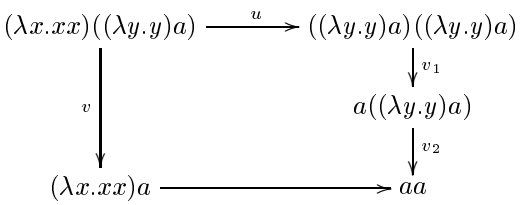


(9)

and



and



(10)

So, the associativity rule (5) on one hand, and all conflict-free rewriting systems on the other hand, enjoy the same remarkable universal confluence property.

However, Knuth and Bendix's definition considers that the associativity rule (5) forms a critical pair (with itself) because the two redexes (7) *overlap*. Obviously, this syntactic reason does not take into account Mac Lane's pentagon (8) and its striking similarity to, say, diagram (10).

So, one feels that Lévy's residual theory should be extended to incorporate example (5). The main task of the article is precisely to formulate this extended theory. One technical difficulty on the way is that, despite what we have just suggested, Mac Lane's pentagon is not *exactly* similar to diagram (10). Let us explain why.

In diagram (10), the path $v_1; v_2$ *develops* the set of *residuals* of the β-redex v after the β-redex u. In particular, the β-redex v_2 is residual after v_1 of a β-redex

$$w : ((\lambda y.y)a)((\lambda y.y)a) \longrightarrow ((\lambda y.y)a)a$$

itself residual of the β-redex v after u. This enables Lévy to deduce all local confluence diagrams from his theory of *redexes*, *residuals* and *developments*.

On the contrary, in diagram (8) the path

$$w\otimes((x\otimes y)\otimes z) \xrightarrow{\alpha} (w\otimes(x\otimes y))\otimes z \xrightarrow{\alpha\otimes z} ((w\otimes x)\otimes y)\otimes z \tag{11}$$

is not *generated* by the set of residuals of the redex

$$w \otimes (x \otimes (y \otimes z)) \xrightarrow{\alpha} (w \otimes x) \otimes (y \otimes z) \tag{12}$$

after the redex

$$w \otimes (x \otimes (y \otimes z)) \xrightarrow{w\otimes\alpha} w \otimes ((x \otimes y) \otimes z) \tag{13}$$

In particular, the redex

$$(w \otimes (x \otimes y)) \otimes z \xrightarrow{\alpha\otimes z} ((w \otimes x) \otimes y) \otimes z$$

is *not* the residual of any redex after the redex:

$$w \otimes ((x \otimes y) \otimes z) \xrightarrow{\alpha} (w \otimes (x \otimes y)) \otimes z$$

In fact, after a short thought, one must admit that the path (11) *as a whole* is residual of the redex (12) after the redex (13).

Admitting that *paths* may be residuals of *redexes* is not a small change in Lévy's residual theory. If one sees redexes as the computational *atoms* of rewriting theory, then these rewriting paths deduced as residuals of redexes, should be understood as its *molecules*. We shall call them *treks* to carry the intuition that generally, in case studies, these rewriting paths implement a unique elementary task — just as the starship Enterprise does in the famous TV show Star Trek — however the number of rewriting steps necessary to complete this task.

A few years ago, the author considered such "treks" in order to prove the (universal) confluence property of Artin positive braid monoid [A]. Unfortunately, the resulting proof [M_4,KOV_1] was too involved to generalize properly to other rewriting systems. Later discussions with Patrick Dehornoy, Jan Willem Klop and Vincent van Oostrom helped the author to develop the theory exposed here.

So, the theory will keep nice and tractable, in sharp contrast to [M$_4$]. The trick is to avoid defining the residuals of a trek *after a trek* $(*)$ because this is too difficult to formalize in most rewriting systems. Instead, we limit ourselves to defining the residuals of a trek *after a redex,* then deduce $(*)$ from a few coherence properties.

This makes all the difference!

Synopsis: we expose the classical theory of redexes and residuals in section 2, and revisit the theory in section 3. For comparison's sake, we wrote sections 2 and 3 in a very similar fashion. We illustrate the theory on associativity and Mac Lane's pentagon in section 4; on braids and Garside's theorem in section 5; on Lévy's optimality theory for the λ-calculus in section 6. Finally, we conclude and indicate further research directions.

2 Classical residual theory (Lévy)

In this section, we recall the residual theory developed by J.-J. Lévy in his PhD thesis [Lé$_1$], see also [HL,M$_1$].

Consider a graph $\mathcal{G}$, that is a set V of *terms* and a set E of *redexes*, together with two functions $\partial_0, \partial_1 : E \longrightarrow V$ indicating the *source* and *target* of each redex. We write

$$u : M \longrightarrow M'$$

when u is a redex and $\partial_0 u = M$ and $\partial_1 u = M'$. A path is sequence

$$f \;=\; (M_1, u_1, M_2, ..., M_m, u_m, M_{m+1}) \qquad (14)$$

where M and N are terms, and for every $1 \leq i \leq m$

$$u_i : M_i \longrightarrow M_{i+1}$$

Remark: every term M induce a path (M), also noted $\mathbf{id}_M$, called the *identity* path of M.

We overload the arrow notation, and write $f : M_1 \longrightarrow M_{m+1}$ in a situation like (14). Moreover, we write $f;g : M \longrightarrow N$ for the composite of two paths $f : M \longrightarrow P$ and $g : P \longrightarrow N$.

In any graph $\mathcal{G}$, we write $R(M)$ for the set of redexes outgoing from the term M.

Definition 1 (residual relation). *A* residual relation *over a graph $\mathcal{G}$ is a relation $/_u$ for every redex $u : M \longrightarrow N$, relating redexes in $R(M)$ to redexes in $R(N)$.*

The residual relation $/_-$ is extended to paths, by declaring that the relation $u /_f v$ holds when

- either $f = \mathbf{id}_M$ and $u = v$ are redexes outgoing from M,
- or $f = u_1; \cdots; u_n$ and there exists redexes $w_2, ..., w_n$ such that $u /_{u_1} w_2 /_{u_2} ... /_{u_{n-1}} w_n /_{u_n} v$.

Notation: given a redex u and a path f, we write $u /_f$ for the set $\{ v \mid u /_f v \}$.

Every residual relation is supposed to verify four properties, formulated in this section. The two first properties are quite elementary:

1. **self-destruction:** for every redex u, the set $u /_u$ is empty,
2. **finiteness:** for every two coinitial redexes u and v, the set $u /_v$ is finite,

Definition 2 (multi-redex). *A* multi-redex *is a pair (M, U) consisting of:*

- *a term M,*
- *a finite set U of redexes outgoing from M.*

Remark: every redex $u : M \longrightarrow N$ may be seen alternatively as the multi-redex $(M, \{u\})$.

Notation: we say that a redex u is element of a multi-redex (M, U) when $u \in U$.

The residual relation $/_f$ is extended to multi-redexes by declaring that $(M, U) /_f (N, V)$ when:

- $f : M \longrightarrow N$,
- V is the set of residuals of elements of U after f, that is:

$$V \;=\; \{v \mid \exists u \in U,\ u /_f v\}$$

Definition 3 (development). *A rewriting path*

$$f : M = M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1}$$

develops *a multi-redex* (M, U) partially *when every redex* u_i *is element of the multi-redex*

$$(M, U) \,/_{u_1;\cdots;u_{i-1}} .$$

The rewriting path $f : M \longrightarrow M_{n+1}$ develops *the multi-redex* (M, U) *when* f *develops* (M, U) *partially and:*

$$(M, U) \,/_f \; = (M_{n+1}, \emptyset).$$

Notation: we write

$$U \Vdash_M f$$

when the rewriting path f develops the multi-redex (M, U). We like this notation because it captures the idea that, in some sense, the path f "realizes" the multi-redex (M, U).

The two last properties required on a residual relation $/_$ appear in [GLM,M_1]. They axiomatize the *finite development lemma* [Hyl,K_1,Ba,M_1,O_4] and the *cube lemma* [Lé$_1$,HL,Bo,Ba,P,OV].

3. **finite developments:** for every multi-redex (M, U), there does not exist any infinite sequence

   $$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} ...M_n \xrightarrow{u_n} M_{n+1}...$$

   of redexes such that, for every index i, the path

   $$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} ...M_{i-1} \xrightarrow{u_{i-1}} M_i$$

   develops the multi-redex (M, U) partially.
4. **permutation:** for every two coinitial redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$, there exists two paths h_u and h_v such that:
   - $u \,/_v \Vdash_Q h_u$ and $v \,/_u \Vdash_P h_v$,
   - h_u and h_v are cofinal and induce the same residual relation $/_{u;h_v} \; = \; /_{v;h_u}$.

   Diagrammatically

$$\begin{array}{ccc} M & \xrightarrow{u} & P \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle h_v} \\ Q & \xrightarrow{h_u} & N \end{array} \qquad (15)$$

In his PhD thesis, Lévy introduces a *permutation equivalence* between rewriting paths. The idea is to tile all diagrams (15) and consider the induced "homotopy equivalence".

Definition 4 (Lévy). *We write* $f \equiv^1 g$ *when the paths* f *and* g *factor as*

$$f = M' \xrightarrow{f_1} M \xrightarrow{u} P \xrightarrow{h_v} N \xrightarrow{f_2} N'$$
$$g = M' \xrightarrow{f_1} M \xrightarrow{v} Q \xrightarrow{h_u} N \xrightarrow{f_2} N'$$

where

- u *and* v *are coinitial redexes,*
- $u \,/_v \Vdash_Q h_u$ *and* $v \,/_u \Vdash_P h_v$,
- h_u *and* h_v *induce the same residual relation* $/_{u;h_v} \; = \; /_{v;h_u}$.

Definition 5 (Lévy). *The* permutation *or* homotopy *equivalence relation* $\equiv$ *is the least equivalence relation (reflexive, transitive, symmetric) on rewriting paths, containing the relation* $\equiv^1$.

Lévy proves his two main theorems about residual systems in [Lé$_1$,HL], see also [Ba,K_1,P,St,M_1]:

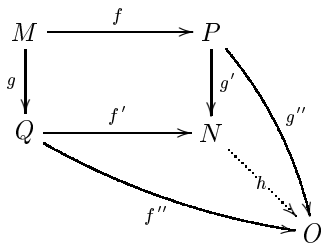
Theorem 1 (pushouts). *For every two coinitial rewriting paths* $f : M \longrightarrow P$ *and* $g : M \longrightarrow Q$, *there exists two paths* $f' : Q \longrightarrow N$ *and* $g' : P \longrightarrow N$ *such that:*

- *the rewriting paths* $f; g' : M \longrightarrow N$ *and* $g; f' : M \longrightarrow N$ *are equal modulo* $\equiv$,

$$\begin{array}{ccc} M & \xrightarrow{f} & P \\ {\scriptstyle g}\downarrow & \equiv & \vdots{\scriptstyle g'} \\ Q & \xrightarrow[f']{} & N \end{array}$$

- *for every diagram path* $f'' : Q \longrightarrow O$ *and* $g'' : P \longrightarrow O$ *such that* $f; g'' \equiv g; f''$, *there exists a unique rewriting path* h *(unique modulo homotopy)*

such that $f'' \equiv f'; h$ *and* $g'' \equiv g'; h$



Theorem 2 (epis). *Every path* $f : M \longrightarrow P$ *enjoys the left-simplification rule below:*

$$f; g \equiv f; h \;\Rightarrow\; g \equiv h$$

for every two paths $g, h : P \longrightarrow N$.

Theorem 1 states that the category $\mathcal{G}/\equiv$ of terms and rewriting paths modulo homotopy has *pushouts*, while Theorem 2 states that every morphism of the category is an epimorphism, see [Mc] for a nice exposition of these categorical concepts.

3 Classical theory revisited: treks and residuals

The residual theory we develop here repeats the pattern of section 2, except that *redexes* are replaced by *treks* in the definition of residuals. We also change the notation $/_u$ into $[\![u]\!]$ to help the reader put the two theories in contrast.

In any graph $\mathcal{G}$, we write $R(M)$ for the set of redexes outgoing from the term M.

Definition 6 (trek graph). *A trek graph* $\mathcal{R} = (\mathcal{G}, T, \leq)$ *is a graph equipped, for every term* M*, with a partially ordered set* $(T(M), \leq_M)$ *of* treks*, whose minimal elements are the redexes of* M*:*

$$R(M) \;= min_{\leq_M} T(M)$$

Notation: we say that the trek $\mathfrak{t} \in T(M)$ *enables* the redex $u \in R(M)$ when $u \leq \mathfrak{t}$. Remark: every graph defines a trek graph, by setting $T(M) = R(M)$ and $\leq_M$ the identity.

Definition 7 (residual relation). *A* residual relation *over a trek graph* $\mathcal{R}$ *is a relation*

$$[\![u]\!] \subset T(M) \times T(N)$$

for every redex $u : M \longrightarrow N$*, which is order preserving in the following sense: for every treks* $\mathfrak{t}_1, \mathfrak{t}_2 \in T(M)$ *and* $\mathfrak{t}'_1 \in T(N)$ *verifying*

$$\mathfrak{t}_1 \leq_M \mathfrak{t}_2 \quad \textit{and} \quad \mathfrak{t}_1 [\![u]\!] \mathfrak{t}'_1,$$

there exists a trek $\mathfrak{t}'_2 \in T(N)$*, such that:*

$$\mathfrak{t}_2 [\![u]\!] \mathfrak{t}'_2 \quad \textit{and} \quad \mathfrak{t}'_1 \leq_N \mathfrak{t}'_2. \tag{16}$$

Just like in section 2, the residual relation $[\![u]\!]$ is extended from redexes to paths, by declaring that the relation $\mathfrak{s} [\![f]\!] \mathfrak{t}$ holds when

- either $f = \mathbf{id}_M$ and $\mathfrak{s} = \mathfrak{t}$ are treks of $T(M)$,
- or $f = u_1; \cdots; u_n$ and there exists treks $\mathfrak{r}_2, ..., \mathfrak{r}_n$ such that $\mathfrak{s} [\![u_1]\!] \mathfrak{r}_2 [\![u_2]\!] ... [\![u_{n-1}]\!] \mathfrak{r}_n [\![u_n]\!] \mathfrak{t}$.

Notation: given a trek $\mathfrak{s} \in T(M)$ and a path $f : M \longrightarrow N$, we write

$$\mathfrak{s} [\![f]\!] \;=\; \{\; \mathfrak{t} \in T(N) \mid \mathfrak{s} [\![f]\!] \mathfrak{t} \;\}$$

Just like in section 2, four properties are required of every residual relation. The two first properties appear below.

1. **self-destruction:** for every redex $u : M \longrightarrow N$, the set $u[\![u]\!]$ is empty,
2. **finiteness:** for every trek $\mathfrak{t} \in T(M)$ and redex $u \in R(M)$, the set $\mathfrak{t}[\![u]\!]$ is finite,

Definition 8 (multi-trek). *A* multi-trek *is a pair* $(M, \mathfrak{S})$ *consisting of:*

- *a term* M*,*
- *a finite subset* $\mathfrak{S}$ *of* $T(M)$*.*

Remark: every redex $u : M \longrightarrow N$ may be seen alternatively as the multi-trek $(M, \{u\})$.

The residual relation $[\![f]\!]$ is extended from treks to multi-treks by declaring that $(M, \mathfrak{S})[\![f]\!](N, \mathfrak{T})$ when:

- $f : M \longrightarrow N$,
- $\mathfrak{T} = \{\mathfrak{t} \in T(N) \mid \exists \mathfrak{s} \in \mathfrak{S},\ \mathfrak{s}[\![f]\!]\mathfrak{t}\} = \bigcup_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{s}[\![f]\!]$

Remark: property 2 (finiteness) ensures that a multi-trek $(M, \mathfrak{S})$ has a (possibly empty) residual $(N, \mathfrak{T})$ after a path $f : M \longrightarrow N$.

Definition 9 (development). *A rewriting path*

$$f : M = M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-1}} M_n \xrightarrow{u_n} M_{n+1}$$

develops *a multi-trek* $(M, \mathfrak{S})$ partially *when, for every index* $1 \leq i \leq n$*, there exists a trek*

$$\mathfrak{t}_i \in (M, \mathfrak{S})[\![u_1; \cdots; u_{i-1}]\!]$$

such that

$$u_i \leq_{M_i} \mathfrak{t}_i.$$

The rewriting path $f : M \longrightarrow M_{n+1}$ develops *the multi-trek* $(M, \mathfrak{S})$ *when* f *develops* $(M, \mathfrak{S})$ *partially and:*

$$(M, \mathfrak{S})[\![f]\!](M_{n+1}, \emptyset)$$

Notation: as in section 2, we write

$$\mathfrak{S} \Vdash_M f$$

when the rewriting path f develops the multi-trek $(M, \mathfrak{S})$.

Remark: the order-preservation property (16) of definition 7 ensures that a multi-trek and its downward closure in $(T(M), \leq_M)$ have the same developments.

The next properties 3. and 4. repeat the axioms of finite developments and permutation formulated in section 2.

3. **finite developments:** for every multi-trek $(M, \mathfrak{S})$, there does not exist an infinite sequence

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \ldots M_n \xrightarrow{u_n} M_{n+1} \ldots$$

of redexes such that, for every index i, the path

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \ldots M_{i-1} \xrightarrow{u_{i-1}} M_i$$

develops the multi-trek $(M, \mathfrak{S})$ partially.

4. **permutation:** for every two coinitial redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$, there exists two paths h_u and h_v such that:
   - $u[\![v]\!] \Vdash_Q h_u$ and $v[\![u]\!] \Vdash_P h_v$,
   - h_u and h_v are cofinal and induce the same residual relation $[\![u; h_v]\!] = [\![v; h_u]\!]$.

   Diagrammatically

$$\begin{array}{ccc} M & \xrightarrow{u} & P \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle h_v} \\ Q & \xrightarrow{h_u} & N \end{array} \qquad (17)$$

Here, we adapt the definition (see section 2) of Lévy permutation equivalence between paths. Again, the idea is to tile all diagrams (17) and consider the "homotopy equivalence" on path.

Definition 10 (tile). *We write* $f \equiv^1 g$ *when the paths* f *and* g *factor as*

$$f = M' \xrightarrow{f_1} M \xrightarrow{u} P \xrightarrow{h_v} N \xrightarrow{f_2} N'$$
$$g = M' \xrightarrow{f_1} M \xrightarrow{v} Q \xrightarrow{h_u} N \xrightarrow{f_2} N'$$

where

- u *and* v *are coinitial redexes,*
- $u[\![v]\!] \Vdash_Q h_u$ *and* $v[\![u]\!] \Vdash_P h_v$*,*
- $u; h_v$ *and* $v; h_u$ *induce the same residual relation* $[\![u; h_v]\!] = [\![v; h_u]\!]$.

Definition 11 (homotopy). *The* permutation *or* homotopy *equivalence relation* $\equiv$ *is the least equivalence relation (reflexive, transitive, symmetric) on rewriting paths, containing the relation* $\equiv^1$.

The axiomatization above enables to extend Lévy's theorems 1 and 2 to "border-line" systems like the associativity rule, or the positive braids rewriting system.

Theorem 3 (pushouts). *The category $\mathcal{G}/\equiv$ of terms and rewriting paths modulo homotopy equivalence, has pushouts.*

Theorem 4 (epis). *Every morphism of $\mathcal{G}/\equiv$ is an epimorphism.*

Proof. The proof follows the pattern of [Lé$_1$,HL,M$_1$].

The main step is to prove property $(\star)$ which states that every two developments of a multi-trek $(M, \mathfrak{S})$ are equal modulo homotopy. The property is proved by induction using the finite developments property 4. Define the *depth* of a multi-trek as the least ordinal verifying that the depth of $(M, \mathfrak{S})$ is strictly larger than the depth of $(N, \mathfrak{T})$ when:

- $(M, \mathfrak{S})[\![u]\!](N, \mathfrak{T})$ for some redex u,
- there exists a trek $\mathfrak{t} \in \mathfrak{S}$ such that $u \leq_M \mathfrak{t}$.

Now, consider a multi-trek $(M, \mathfrak{S})$. Suppose that property $(*)$ holds on every multi-trek of depth strictly smaller than the depth of $(M, \mathfrak{S})$. This will be our induction hypothesis. We claim that property $(*)$ holds also on $(M, \mathfrak{S})$. To prove this, we consider two developments f and g of $(M, \mathfrak{S})$, what we noted: $\mathfrak{S} \Vdash_M f$ and $\mathfrak{S} \Vdash_M g$. The claim $f \equiv g$ is obvious when one of the two paths f, g is empty, because then, $\mathfrak{S}$ is empty, and consequently, $f = g = \mathbf{id}_M$. When the two paths f and g are non-empty, f, g factor as $f = u; f'$ and $g = u; g'$ where $u : M \longrightarrow P, v : M \longrightarrow Q$ are redexes, and f', g' are paths. As first redexes appearing in the paths f and g, the two redexes u and v are partial developments of $(M, \mathfrak{S})$. This means that there exists treks $\mathfrak{t}_1 \in \mathfrak{S}$ and $\mathfrak{t}_2 \in \mathfrak{S}$ such that

$$u \leq_M \mathfrak{t}_1 \qquad u \leq_M \mathfrak{t}_2 \tag{18}$$

Let $(P, \mathfrak{S}_1)$ and $(Q, \mathfrak{S}_1)$ denote the multi-treks residual of $(M, \mathfrak{S})$ after u and v respectively:

$$(M, \mathfrak{S})[\![u]\!](P, \mathfrak{S}_1) \qquad (M, \mathfrak{S})[\![v]\!](Q, \mathfrak{S}_2)$$

It follows from (18) that the depth of $(M, \mathfrak{S})$ is strictly larger than the depth of $(P, \mathfrak{S}_1)$ and $(Q, \mathfrak{S}_2)$. Besides,

$$\mathfrak{S}_1 \Vdash_P f' \qquad \mathfrak{S}_2 \Vdash_Q g' \tag{19}$$

Now, one of two cases may occur: the redexes u and v are equal, or they are different. We treat the two cases separately. (1) When $u = v$, the two multi-treks $(P, \mathfrak{S}_1)$ and $(Q, \mathfrak{S}_2)$ are equal, and thus, the two paths f' and g' develop the multi-trek $(P, \mathfrak{S}_u)$. It follows from induction hypothesis on $(P, \mathfrak{S}_u)$ that $f' \equiv g'$ We conclude with the series of equivalence

$$f = u; f' \equiv u; g' = g$$

(2) When $u \neq v$, we deduce from our permutation property 4, that there exists two paths h_u and h_v such that:

- $u[\![v]\!] \Vdash_Q h_u$ and $v[\![u]\!] \Vdash_P h_v$,
- h_u and h_v are cofinal and induce the same residual relation $[\![u; h_v]\!] = [\![v; h_u]\!]$.

It follows from $[\![u; h_v]\!] = [\![v; h_u]\!]$ that there exists a multi-trek $(N, \mathfrak{S}_{12})$ such that:

$$(Q, \mathfrak{S}_2)[\![h_u]\!](N, \mathfrak{S}_{12}) \qquad (P, \mathfrak{S}_1)[\![h_v]\!](N, \mathfrak{S}_{12}) \tag{20}$$

At this point, we need to prove the intermediate result below, which follows easily from the order preservation property required of a residual relation in definition 16.

Lemma. *Suppose that $(M, \mathfrak{S})$ and $(M, \mathfrak{T})$ are two multi-treks on the same term M verifying:*

$$\forall \mathfrak{s} \in \mathfrak{S}, \exists \mathfrak{t} \in \mathfrak{T}, \mathfrak{s} \leq_M \mathfrak{t}$$

Then, every partial development of $(M, \mathfrak{S})$ is a partial development of $(M, \mathfrak{T})$.

Associated to (18), the lemma ensures that the developments $u; h_v$ and $v; h_u$ of $(M, \{u, v\})$ are partial developments of the multi-trek $(M, \mathfrak{S})$. From this, it follows that h_v and h_u are partial developments of $(P, \mathfrak{S}_1)$ and $(Q, \mathfrak{S}_2)$ respectively. Let h denote an arbitrary development of $(N, \mathfrak{S}_{12})$:

$$\mathfrak{S}_{12} \Vdash_N h$$

Then, it follows from (20) that:

$$\mathfrak{S}_1 \Vdash_P h_v; h \quad \mathfrak{S}_2 \Vdash_Q h_u; h$$

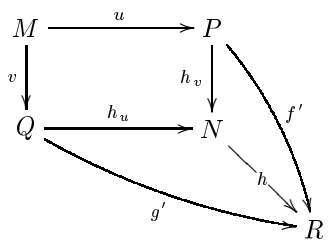
From this, as well as (19) and our induction hypothesis on $(P, \mathfrak{S}_1)$ and $(Q, \mathfrak{S}_2)$, we obtain that:

$$f' \equiv h_v; h \quad \text{and} \quad g' \equiv h_u; h$$

We conclude by the series of equivalence:

$$f = u; f' \equiv u; h_u; h \equiv v; h_u; h \equiv v; g' = g$$

based on the diagram below: The diagram below will certainly help to explain the proof above.



This concludes our proof by induction, that every multi-trek $(M, \mathfrak{S})$ verifies property $(*)$.

The proof of property $(*)$ is the difficult bit. We indicate briefly the the remaining steps of the proof, which mimick Lévy's proof of his original Theorems 1 and 2. A graph $\mathcal{M}$ is constructed,

- whose vertices are the vertices of the graph $\mathcal{G}$,
- whose edges $M \longrightarrow N$ are the multi-treks $(M, \mathfrak{S})$ whose developments $f : M \longrightarrow N$ have target the term N.

Property $(*)$ ensures that the definition of edges is correct, and enables to define a residual relation $[\![-]\!]$ between the edges of $\mathcal{M}$, as follows:

$$(M, \mathfrak{S})[\![(M, \mathfrak{T})]\!](M, \mathfrak{S}')$$

iff

$$(M, \mathfrak{S})[\![f]\!](M, \mathfrak{S}')$$

for every development f of $(M, \mathfrak{T})$. We write alternatively $U[\![V]\!]W$ or $U[\![V]\!] = W$ for this residual relation $[\![-]\!]$ between the edges U, V, W of $\mathcal{M}$. The relation is easily generalized to a residual relation $[\![-]\!]$ between paths D, E of $\mathcal{M}$, by the expected definitions:

$$(U; D)[\![V]\!] = U[\![V]\!]; D[\![V[\![U]\!]]\!]$$

when U and V are coinitial, and

$$D[\![U; E]\!] = D[\![U]\!][\![E]\!]$$

when U and D are coinitial.

The category $\mathcal{M}/\equiv$ is defined as the free category over $\mathcal{M}$ quotiented by the following equations:

$$(M, \emptyset) \equiv \mathbf{id}_M$$

for every term M, and

$$D; (E[\![D]\!]) \equiv E; (D[\![E]\!])$$

for every pair of coinitial paths D and E.

The point of the construction is that the resulting category $\mathcal{M}/\equiv$ enjoys existence of pushouts, and is a category of epimorphisms. This is easily derived from the algebraic properties of Lévy residual relation $[\![-]\!]$ between paths, see for instance [Lé$_1$,HL,St,M$_1$]. The very last and elementary step of the proof consists in showing that the two categories $\mathcal{M}/\equiv$ and $\mathcal{G}/\equiv$ coincide. Theorems 3 and 4 follow immediately from this.

4 Illustration 1: the associativity rule and Mac Lane's pentagon

Here, we apply the theory of section 3 to the rewriting system (5) introduced in section 1, with unique rewriting rule:

$$x \otimes (y \otimes z) \xrightarrow{\alpha} (x \otimes y) \otimes z \tag{21}$$

First, let us define what we mean by *trek* in that case. Given $n \geq 1$ variables $y_1, ..., y_n$, we define the normal form $[y_1, ..., y_n]$ by induction on n:

$$[y_1] = y_1 \qquad [y_1, ..., y_n] = [y_1, ..., y_{n-1}] \otimes y_n$$

A trek $\mathfrak{t}$ of $T(M)$ is defined as a pattern of

$$x \otimes [y_1, ..., y_n] \tag{22}$$

for $n \geq 2$, and its occurrence o in the term M. So, a trek $\mathfrak{t} \in T(M)$ is a pair (o, n) where the occurrence o is a sequence $\epsilon_1 \cdots \epsilon_k$ of L's and R's (for left and right) identifying the position of the "root" tensor product of the trek in the term.

Notation: We say that the trek $\mathfrak{t}$ has a *head*: the tensor product $x \otimes [y_1, ..., y_n]$; and a *spine*: the $n - 1$ tensor products inside $[y_1, ..., y_n]$.

The order $(T(M), \leq_M)$ on treks is defined as follows:

$$(o, m) \leq_M (o', n) \iff o = o' \text{ and } m \leq n$$

So, $\mathfrak{t} \leq_M \mathfrak{t}'$ when $\mathfrak{t}$ has same head as $\mathfrak{t}'$, and shorter spine. The minimal treks (o, n) of $(T(M), \leq_M)$ are the redex patterns $n = 2$. Thus, $R(M) = \min_{\leq_M} T(M)$. This ensures that $\mathcal{R} = (\mathcal{G}, \mathcal{T}, \leq)$ defines a trek graph.

Now, we define the residual structure $[\![-]\!]$ on $\mathcal{R}$, in such a way that the trek (22) is "realized" or "developed" by the path f:

$$x \otimes [y_1, ..., y_n] \longrightarrow [x, y_1, ..., y_n]$$

Note already that the trek (22) enables a unique redex $u \in R(M)$ which rewrites

$$x \otimes [y_1, ..., y_n] \longrightarrow (x \otimes [y_1, ..., y_{n-1}]) \otimes y_n$$

Consider a redex $u : M \longrightarrow N$ and two treks $\mathfrak{t} \in T(M)$ and $\mathfrak{t}' \in T(N)$. The redex u induces a one-to-one function φ between the occurrences of M not overlapping the left-hand side of u, and the occurrence of N not overlapping the right-hand side of u.

The two treks $\mathfrak{t}$ and $\mathfrak{t}'$ are related by $\mathfrak{t}[\![u]\!]\mathfrak{t}'$ precisely when:

- $\mathfrak{t} = (o, n)$ and $\mathfrak{t}' = (\varphi(o), n)$ when the redex u and the trek $\mathfrak{t}$ do not overlap,
- $\mathfrak{t} = (o, n)$ and $\mathfrak{t}' = (o \cdot L, n-1)$ when $n \geq 3$ and the redex u overlaps with both the head and the spine of $\mathfrak{t}$. Observe that $\mathfrak{t} \vdash u$ in that case.
- $\mathfrak{t} = (o \cdot R, n)$ and $\mathfrak{t}' = (o, n)$ when the redex u overlap with the head (and not the spine) of $\mathfrak{t}$,
- $\mathfrak{t} = (o, n)$ and $\mathfrak{t}' = (o, n+1)$ when the redex u overlap with the spine (and not the head) of $\mathfrak{t}$.

Condition (16) of definition 7 is easily verified. This ensures that $[\![-]\!]$ defines a residual relation.

There remains to check that $[\![-]\!]$ verifies the four properties required of residual systems. Property 1. (self-destruction) is obvious. Property 2. (finiteness) may be strengthened to the property that a trek has one residual at most. Accordingly, we only need to establish property 3. (finite developments) on singleton multi-treks only. Anyway, property 3. follows from strong normalization of the rewriting system.

There remains to prove property 4. (permutation.) for every pair of coinitial redexes $u : M \longrightarrow P$ and $v : M \longrightarrow Q$. We proceed by case analysis.

Suppose first that u and v do not overlap, and let the redex u' be the residual of u after v, and the redex v' be the residual of v after u, in the usual residual theory (for redexes.) Then, obviously, the trek u' is residual of the trek u after v, and the trek v' is residual of the trek v after u in our new residual theory (for treks.) Moreover, $\{u'\} \Vdash_Q u'$ and $\{v'\} \Vdash_P v'$. Thus, diagram (6) is an instance of (17). For lack of space, we will only sketch how one proves that $[\![u; v']\!] = [\![v; u']\!]$. The proof is not very difficult, quite pleasant to proceed in fact, but requires to consider a few cases, based on whether u or v overlaps with the head and spine of a given trek $\mathfrak{t} \in T(M)$.

Now, suppose that u and v do overlap. We may suppose wlog that v has instance o, that u has instance $o \cdot R$, and that the pattern of the critical pair is (7):

$$u : \quad w \otimes (x \otimes (y \otimes z)) \xrightarrow{w \otimes \alpha} w \otimes ((x \otimes y) \otimes z)$$

$$v : \quad w \otimes (x \otimes (y \otimes z)) \xrightarrow{\alpha} (w \otimes x) \otimes (y \otimes z)$$

On one hand, the redex v overlaps with the head of the trek u. Thus $u = (o \cdot R, 2)$ has residual the trek $\mathfrak{t}_u = (o, 2)$:

$$\mathfrak{t}_u : \quad (w \otimes x) \otimes [y, z]$$

developed by the redex u':

$$(w \otimes x) \otimes [y, z] \xrightarrow{\alpha} [w, x, y, z]$$

On the other hand, the redex u overlaps with the spine of the trek v. Thus, $v = (o, 2)$ has residual the trek $\mathfrak{t}_v = (o, 3)$.

$$\mathfrak{t}_v : \quad w \otimes [x, y, z]$$

developed by the path h_v:

$$w \otimes [x, y, z] \xrightarrow{\alpha} (w \otimes [x, y]) \otimes z \xrightarrow{\alpha \otimes z} [w, x, y, z]$$

So, Mac Lane's pentagon (8) is an instance of diagram (17). There remains to prove $[\![u; h_v]\!] = [\![v; u']\!]$. Just as

in case of non-overlapping redexes u and v, the proof proceeds by case analysis, which we omit here for lack of space.

Since the square and pentagon diagrams (6) and (8) induce the same homotopy equivalence $\equiv$ as our residual theory, we conclude from theorems 3 and 4 that:

Theorem 5 (universal confluence). *The rewriting paths generated by (21) and quotiented by the homotopy diagrams (6) and (8) define a category of epimorphisms with pushouts.*

Remark: the ancestor of all λ-calculi with explicit substitutions like [ACCL] is Pierre-Louis Curien's Categorical Combinatory Logic [C] — whose main rule (and rewriting engine) is the associativity rule. We are currently experimenting a theory of treks for a λ-calculus with explicit substitutions, extending what we have done here, and establishing the first universal confluence theorem for a calculus of that kind.

5 Illustration 2: braids and Garside's theorem

We will be *very* concise here, for lack of space. For information, the reader is advised to read the crisp course notes [KOV$_1$] where a nice exposition of [M$_4$] appears, along with other materials.

The braids group $\mathbb{B}$ was introduced by Artin [A]. The group is presented by the alphabet

$$\Sigma \;=\; \{\sigma_i \mid i \in \mathbb{Z}\}$$

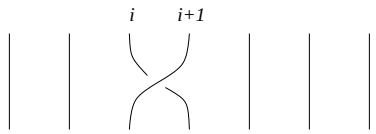
and the relations

$$\sigma_i\sigma_{i+1}\sigma_i \;=\; \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{for every } i \in \mathbb{Z} \qquad (23)$$

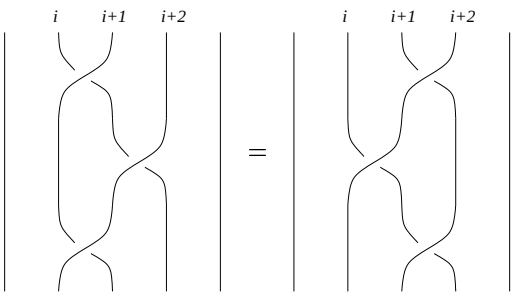
$$\sigma_i\sigma_j \;=\; \sigma_j\sigma_i \qquad \text{when } |i-j| \geq 2 \qquad (24)$$

In order to solve the word problem in $\mathbb{B}$, Garside [Ga] considered the monoid $\mathbb{B}^+$ of *positive* braids, with same presentation. He established that every two elements of $\mathbb{B}^+$ have a lower common multiple. The theorem may be restated as the universal confluence property of the rewriting system with elements b of $\mathbb{B}^+$ as terms, arrows $M \xrightarrow{\sigma_i} M\sigma_i$ as redexes, and commutation diagrams (23) and (24) as tiles.

Each σ_i is interpreted diagrammatically as the operation of permuting the strand i over the strand $i+1$ in a braid diagram:



In particular, the equation (23) mirrors the topological equality:



We define our trek graph as follows. A trek $\mathfrak{t}$ of M is a non-identity permutation of the set $\mathbb{Z}$ of integers, with finite domain[2]. Recall that the domain $\mathrm{dom}(\mathfrak{t})$ of a permutation $\mathfrak{t}$ is the set of integers $i \in \mathbb{Z}$ such that $\mathfrak{t}(i) \neq i$.

Intuitively, a trek $\mathfrak{t}$ is intended to "permute" all strands $j \in \mathbb{Z}$ over all strands $i \in \mathbb{Z}$ such that $i \leq j$ and $\mathfrak{t}(j) \leq \mathfrak{t}(i)$.

The order $(T(M), \leq_M)$ is defined as follows: two treks $\mathfrak{t}, \mathfrak{t}'$ are related by $\mathfrak{t} \leq_M \mathfrak{t}'$ precisely when for every pair of integers $i \leq j$:

$$\mathfrak{t}(j) \leq \mathfrak{t}(i) \Rightarrow \mathfrak{t}'(j) \leq \mathfrak{t}'(i)$$

The minimal treks of $(T(M), \leq_M)$ are the permutations $\dot{\sigma}_k$ with domain $\{k, k+1\}$, which may be identified to the redex σ_k. This ensures that $R(M) =$

[2] The treks define a well-known class of generators for braids, with remarkable permutation properties. Another interesting class of generators appears in [BKL].

$\min_{\leq_M} T(M)$, and thus that $\mathcal{R} = (\mathcal{G}, T, \leq)$ defines a trek graph.

Notation: let $\{I, J\}$ be a partition of a finite interval $[i, j]$ of $\mathbb{Z}$, with $i \in I$ and $j \in J$. We write $[I, J]$ for the unique permutation of $\mathbb{Z}$ such that:

- $\mathrm{dom}[I, J] = I \cup J$,
- $\forall (i, j) \in I \times J, [I, J](j) \leq [I, J](i)$,
- $\forall i, i' \in I,\ i \leq i' \ \Rightarrow\ [I, J](i) \leq [I, J](i')$,
- $\forall j, j' \in J,\ j \leq j' \ \Rightarrow\ [I, J](j) \leq [I, J](j')$.

Intuitively, the permutation $[I, J]$ "permutes" all strands of J over all strands of I, without altering the internal order of strands in I and J. For instance, $\sigma_k = [\{k\}, \{k+1\}]$.

The residual relation $[\![-]\!]$ on $\mathcal{R}$ is defined as follows.

1. $\mathrm{t}[\![\sigma_k]\!](\mathrm{t} \circ \sigma_k)$ when $\mathrm{t}(k+1) \leq \mathrm{t}(k)$,
2. $\mathrm{t}[\![\sigma_k]\!]([I, J] \circ \mathrm{t} \circ \sigma_k)$ when $\mathrm{t}(k) \leq \mathrm{t}(k+1)$, for $I = \{\mathrm{t}(i), i \leq k, \mathrm{t}(k) \leq \mathrm{t}(i)\}$ and $J = \{\mathrm{t}(j), k+1 \leq j, \mathrm{t}(j) \leq \mathrm{t}(k+1)\}$.

Condition (16) of definition 7 is easily verified. This ensures that $[\![-]\!]$ defines a residual relation.

There remains to check that the definition verifies the four properties of residual systems: *self-destruction* is immediate while *finiteness* may be strengthened to the property that a trek has a residual at most, just as for the associativity rule in section 4. *Finite development* is easily established too, by weighting treks.

The property of *permutation* is more interesting. Suppose that σ_i and σ_j are redexes outgoing from M. Two cases occur. Either $|j - i| \geq 2$, and then

$$\sigma_i[\![\sigma_j]\!]\sigma_i \quad \text{and} \quad \sigma_j[\![\sigma_i]\!]\sigma_j$$

induce the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\sigma_i} & M\sigma_i \\ {\scriptstyle\sigma_j}\downarrow & & \downarrow{\scriptstyle\sigma_j} \\ M\sigma_j & \xrightarrow{\sigma_i} & M\sigma_i\sigma_j \end{array} \qquad (25)$$

Or $j = i + 1$, and then

$$\sigma_i[\![\sigma_{i+1}]\!][\{i\}, \{i+1, i+2\}]$$
$$\sigma_{i+1}[\![\sigma_i]\!][\{i, i+1\}, \{i+2\}]$$

induce

$$\begin{array}{ccccc} M & & \xrightarrow{\sigma_i} & & M\sigma_i \\ & & & & \downarrow{\scriptstyle\sigma_{i+1}} \\ {\scriptstyle\sigma_{i+1}}\downarrow & & & & M\sigma_i\sigma_{i+1} \\ & & & & \downarrow{\scriptstyle\sigma_i} \\ M\sigma_{i+1} & \xrightarrow{\sigma_i} & M\sigma_i\sigma_{i+1} & \xrightarrow{\sigma_{i+1}} & M\sigma_i\sigma_{i+1}\sigma_i \end{array} \qquad (26)$$

which mirror exactly equations (23) and (24).

There remains to prove that the two paths f and g along the confluence diagrams (25) and (26) define the same residual relation $[\![f]\!] = [\![g]\!]$. This is easy to establish for diagram (25) and an inch more tedious for diagram (26). We skip the proof here for lack of space.

6 Illustration 3: Lévy's optimality and extraction theory

In his PhD thesis, Lévy [Lé$_1$,Lé$_2$] develops a theory of *optimality* for the λ-calculus. After defining *families* of β-redexes "of a common nature" which may be shared by an implementation, Lévy calls *optimal* any implementation which computes one such family at each step.

The moving part of the story is that no optimal implementation existed at the time. Lévy formulated its principle, and the implementation was designed twelve years later [Lam].

In [Lé$_1$,Lé$_2$], families are defined in three different ways, which turn out to be equivalent: zigzag, extraction, labelling. Here, we focus on zigzag and extraction, and forget labelling for lack of space.

Definition 12 (Lévy). *A β-redex with history is a pair (f, u) where $f : M \longrightarrow P$ is a rewriting path and $u : P \longrightarrow P'$ is a β-redex.*

For notation's sake, we feel free to identify the β-redex with history (f,u) and the path $M \xrightarrow{f} P \xrightarrow{u} P'$.

Definition 13 (residual). *Let $h : P \longrightarrow Q$ be a rewriting path, and let*

$$M \xrightarrow{f} P \xrightarrow{u} P' \text{ and } M \xrightarrow{g} Q \xrightarrow{v} Q'$$

be two β-redexes with history. We write

$$(f,u) \;/_h\; (g,v)$$

when

$$u \;/_h\; v \quad \text{and} \quad g \equiv f;h$$

In that case, we say that (f,u) is an ancestor of (g,v).

The zig-zag relation $\leftrightarrows$ is the least equivalence relation between redexes with history, containing this ancestor relation.

Definition 14 (family). *A family is an equivalence class of the equivalence relation $\leftrightarrows$ between β-redexes with history.*

One unexpected point about families is the lack of *common ancestor* property. We illustrate this with an example, adapted from [Lé$_2$]. Consider the three λ-terms

$$P = (\lambda x.xa)(\lambda y.y) \qquad Q = (\lambda y.y)a \qquad R = a$$

and the redex with history $\underline{v}$:

$$\underline{v}: \quad P \xrightarrow{u} Q \xrightarrow{v} R \tag{27}$$

The two redexes with history $\underline{v_1}$ and $\underline{v_2}$:

$$\underline{v_1}: \quad \Delta P \xrightarrow{\Delta} PP \xrightarrow{u_1} QP \xrightarrow{v_1} RP$$

$$\underline{v_2}: \quad \Delta P \xrightarrow{\Delta} PP \xrightarrow{u_2} PQ \xrightarrow{v_2} PR$$

are in the same family, but do not have an ancestor $\underline{v_0}$ in common. Intuitively, this common ancestor should be $\underline{v}$. But $\underline{v}$ computes the β-redex u, which is not a subcomputation of $\Delta;u_1$ and not a subcomputation of $\Delta;u_2$.

This means that the theory has to be slightly refined. Lévy [Lé$_2$] introduces a *non-deterministic* extraction procedure $\rhd$ acting on redexes with history.

At this point, it is useful to recall that the set $R(M)$ of β-redexes from a λ-term M may be ordered by prefix ordering $\preceq_M$ on occurrences. We also introduce a refinement of Lévy's homotopy relation on paths [GLM,M$_1$,M$_2$].

Definition 15 (reversible homotopy). *We write $f \simeq^1 g$ when f and g factor as*

$$f = M' \xrightarrow{f_1} M \xrightarrow{u} P \xrightarrow{v'} N \xrightarrow{f_2} N'$$
$$g = M' \xrightarrow{f_1} M \xrightarrow{v} Q \xrightarrow{u'} N \xrightarrow{f_2} N'$$

where

- *u and v are* disjoint *β-redexes, that is incomparable wrt. $\preceq_M$.*
- *u' is the unique residual of u after v,*
- *v' is the unique residual of v after u.*

The reversible homotopy *equivalence relation $\simeq$ is the reflexive transitive closure of $\simeq^1$.*

So, considering $\simeq$ amounts to tiling all diagrams of the form (9) in the reduction graph of the λ-calculus.

Definition 16 (Lévy). *The* extraction relation *is the union of the four relations below.*

1. $M \xrightarrow{f} P \xrightarrow{u} P' \xrightarrow{v'} N \;\rhd\; M \xrightarrow{f} P \xrightarrow{v} Q$ when $v \preceq_N u$ and $v \;/_u\; v'$,

2. $M \xrightarrow{f} P \xrightarrow{u} P' \xrightarrow{h} Q' \xrightarrow{v'} R' \;\rhd\; M \xrightarrow{f} P \xrightarrow{g} Q \xrightarrow{v} R$ when u and $g;v$ compute in "disjoint" occurrences, and $h;v'$ is the path residual of $g;v$ after u. In that case, $u \;/_g\; u' \;/_v\; u''$ and $u;h;v' \simeq g;v;u''$ induce the diagram below.

$$\begin{array}{ccccc} M & \xrightarrow{f} & C[A,B] & \xrightarrow{g} & C[A,B'] & \xrightarrow{v} & C[A,B''] \\ & & \downarrow{\scriptstyle u} & \simeq & \downarrow{\scriptstyle u'} & \simeq & \downarrow{\scriptstyle u''} \\ & & C[A',B] & \xrightarrow{h} & C[A',B'] & \xrightarrow{v'} & C[A',B''] \end{array}$$

3. $M \xrightarrow{f} P \xrightarrow{u} P' \xrightarrow{h} Q' \xrightarrow{v'} R' \rhd M \xrightarrow{f} P \xrightarrow{g} Q \xrightarrow{v} R$ *when the path* $g;v$ *computes inside the body* A *of the* β*-redex* $u : C[(\lambda x.A)B] \longrightarrow C[A[B/x]]$ *and the path* $h;v'$ *is the path residual of* $g;v$ *after* u*. In that case,* $u /_g u' /_v u''$ *induces the diagram below:*

$$\begin{array}{ccccc} C[(\lambda x.A)B] & \xrightarrow{g} & C[(\lambda x.A')B] & \xrightarrow{v} & C[(\lambda x.A'')B] \\ \downarrow{\scriptstyle u} & \equiv & \downarrow{\scriptstyle u'} & \equiv & \downarrow{\scriptstyle u''} \\ C[A[B/x]] & \xrightarrow{h} & C[A'[B/x]] & \xrightarrow{v'} & C[A''[B/x]] \end{array}$$

4. $M \xrightarrow{f} P \xrightarrow{u} P' \xrightarrow{h} Q' \xrightarrow{v'} R' \rhd M \xrightarrow{f} P \xrightarrow{g} Q \xrightarrow{v} R$ *when* $u : C[(\lambda x.A)B] \longrightarrow C[A[B/x]]$ *and the path* $g;v$ *computes inside the argument* B *of* u*, and the path* $h;v'$ *computes just as* g *inside* one *copy of the argument* B *of* $A[B/x]$*. In the diagram, we write* A_l *(the subscript* l *stands for* linear*) for the* λ*-term* A *where all instances of* x *are substituted by the* λ*-term* B*, except one instance which is substituted by a fresh variable* l*.*

$$\begin{array}{ccccc} C[(\lambda x.A)B] & \xrightarrow{g} & C[(\lambda x.A)B'] & \xrightarrow{v} & C[(\lambda x.A)B''] \\ \downarrow{\scriptstyle u} & & & & \\ C[A_l[B/l]] & \xrightarrow{h} & C[A_l[B'/l]] & \xrightarrow{v'} & C[A_l[B''/l]] \end{array}$$

Lévy shows that the procedure $\rhd$ is strongly normalizing, and confluent. Suppose moreover that (f,u) and (g,v) are two redexes with history whose paths f and g are *left-standard*, that is standard wrt. the *leftmost-outermost* order on β-redexes, see [CF,Lé$_1$,K$_1$,Ba,GLM,M$_2$]. Then, Lévy shows that (f,u) and (g,v) are in the same family iff they have the same $\rhd$-normal-form.

This result provides a pertinent *canonical form* theorem for Lévy families. Typically, stepping back to example (27) the redex with history $\underline{v}$ is not the common ancestor, but the common $\rhd$-normal-form of the two redexes with history $\underline{v_1}$ and $\underline{v_2}$.

We reformulate this part of Lévy optimality theory using the language of treks.

The set $P(M)$ of paths outgoing from a λ-term M, may be ordered as follows:

$$M \xrightarrow{f} P \;\lesssim\; M \xrightarrow{g} Q \iff \exists h : P \to Q, \quad f;h \simeq g$$

The order $(P(M),\lesssim)$ is a *local lattice*[3]: every two bounded elements f and g have a least upper bound $f \vee g$ and a greatest least bound $f \wedge g$. We recall that two paths $f,g \in P(M)$ are *bounded* in $(P(M),\lesssim)$ when there exists a path $h \in P(M)$ such that $f \lesssim h$ and $g \lesssim h$.

Definition 17 (coprime). *A coprime* $f \in P(M)$ *is a path such that, for every two bounded paths* $g,h \in P(M)$*,*

$$f \lesssim g \vee h \quad \Rightarrow \quad f \lesssim g \text{ or } f \lesssim h$$

So, a path f is coprime when it cannot be decomposed into more elementary computations $f \simeq f_1 \vee \ldots \vee f_k$.

Standard coprimes enjoy an important "enclave" or "localization" property. Here, by *standard*, we mean standard wrt. the *prefix-ordering* $\preceq$ on β-redexes, see [GLM,M$_2$]. Let $v \in R(M)$ be the first β-redex computed by a coprime $f = v;g$. The localization property tells that if the β-redex $v \in R(M)$ occurs inside the body (resp. the argument) of a β-redex $u \in R(M)$, then the whole standard coprime f computes inside the body (resp. the argument) of the β-redex u.

This property leads to a new theory of residuals in the λ-calculus.

Definition 18 (β-trek). *A* β*-trek* $\mathfrak{t} \in T(M)$ *is a non-empty standard coprime of* $(P(M),\lesssim)$*, considered modulo* $\simeq$*.*

Observe that the minimal elements of $(T(M),\lesssim)$ are exactly the redexes of M. Thus, $R(M) = \min_{\lesssim} T(M)$. This ensures that $\mathcal{R} = (\mathcal{G},T,\lesssim)$ defines a trek graph.

The residual relation $\mathfrak{t}[\![u]\!]\mathfrak{t}'$ induced by a β-redex $u : M \longrightarrow N$ on treks $\mathfrak{t} \in T(M)$ and $\mathfrak{t}' \in T(N)$, is defined by induction on the length of $\mathfrak{t}$.

[3] The order $(P(M),\lesssim)$ is a bit more than a local lattice, because the lub and glb of f and g do not depend on the choice of slice h.

Definition 19 (residual).

1. *when the trek* $\mathfrak{t}$ *enables the* β*-redex* u*, then the* β*-trek* $\mathfrak{t}'$ *is defined as the unique* β*-trek such that* $\mathfrak{t} \simeq u; \mathfrak{t}'$*,*
2. *when* $\mathfrak{t}$ *enables a* β*-redex* v *occurring inside the body of* u*; then,* $\mathfrak{t}'$ *is the residual of* $\mathfrak{t}$ *after* u*, modulo* $\simeq$*,*
3. *when* $\mathfrak{t}$ *enables a* β*-redex* v *occurring inside the argument of* $u : C[(\lambda x.A)B] \longrightarrow C[A[B/x]]$*; then,* $\mathfrak{t}'$ *computes just as* $\mathfrak{t}$ *inside one copy of* B *in* $A[B/x]$*,*
4. *when* $\mathfrak{t}$ *enables a* β*-redex* v *not occurring inside the* β*-redex* u*, this* β*-redex* v *has a unique residual* v' *after* u*, and* $\mathfrak{t}$ *has the residual* $\mathfrak{s}$ *after* v*; the* β*-trek* $\mathfrak{t}'$ *is any path* $v'; \mathfrak{s}'$ *where* $\mathfrak{s}'$ *is a residual of* $\mathfrak{s}$ *after the development of* $u \,/_v$ *.*

The definition is fine because a β-trek has always residuals of smaller length. Moreover, condition (16) of definition 7 is easily verified. This ensures that $[\![-]\!]$ defines a residual relation.

The four properties of a residual system are established as follows. The properties of *self-destruction* and *finiteness* are immediate, while *finite development* follows from Lévy's finite family developments theorem [Lé$_1$,O$_3$]. The last property of *permutation* follows from elementary combinatorial arguments, that we do not expose here for lack of space.

At this point, two β-treks $\mathfrak{t}_1, \mathfrak{t}_2 \in T(M)$ may have the same β-trek $\mathfrak{t}_3 \in T(N)$ as residual after a β-redex $u : M \longrightarrow N$. Typically, in our example (27) the β-treks $\underline{v}$ and $\underline{v_1}$ have the β-trek

$$PP \xrightarrow{u_1} QP \xrightarrow{v_1} RP$$

as residual after the β-redex $\Delta P \longrightarrow PP$.

This motivates to limit oneself to *normal* β-treks.

Definition 20. *A trek is* normal *when it is a* $\triangleright$*-normal-form.*

It appears that normal β-treks define a sub-residual system of the β-trek system defined above, in the sense that (1) normal β-treks have only normal β-treks as residuals, and (2) that every β-redex is normal.

The main property of normal β-treks is that, for every normal β-trek $\mathfrak{t} \in T(N)$, and morphism $u : M \longrightarrow N$, there exists a *unique* normal β-trek $\mathfrak{s} \in T(M)$ such that $\mathfrak{t}$ is residual of $\mathfrak{s}$ after u. This normal β-trek is simply the $\triangleright$-normal-form of $u; \mathfrak{t}$.

So in a sense, there is no "creation" in the residual theory of normal treks. From this, a *canonical ancestor* theorem for normal treks follows, which clarifies very much the situation, since there lacks such a theorem for β-redexes.

Theorem 6 (canonical ancestor). *Every family is the set of residuals of a normal trek* $\mathfrak{t}$ *— modulo homotopy equivalence of the redexes with history.*

Remark: in contrast to section 4 and 5, where a redex may have a non-trivial trek as residual, β-redexes have only β-redexes as residuals. This hides the concept of β-trek very much, and explains why the theory of β-treks was not formulated earlier for the λ-calculus.

Since Lévy's early steps, the optimality theory has been extended to term rewriting systems [Ma] to interaction systems [AL] and to higher-order systems [O$_3$]. For instance, Asperti and Laneve gave several examples of higher-order rewriting systems, where the "zig-zag" notion of Lévy family does not match with the (more fundamental) "labelling" and "extraction" definitions. We conjecture that our language of treks and residuals repairs this, and provides the relevant "residual theory" to discuss optimality in higher-order rewriting systems.

7 Conclusion

In this article, we revisit Lévy's residual theory, in order to enscope "border-line" rewriting systems admitting "innocuous" critical pairs, like the associativity rule, or the positive braids rewriting systems. We show in addition that the resulting residual theory clarifies Lévy's theory of optimality.

In future work, we would like to express an effective syntactic criterion $(*)$ indicating when critical pairs are

"innocuous" in a given rewriting system, or equivalently, when the language of treks applies. The criterion would refine Knuth and Bendix's definition of critical pair, and formulate it in a more "combinatorial" style.

A nice line of research initiated by Gérard Huet [Hu] will certainly guide us. Its main task is to devise conditions on critical pairs, ensuring confluence. As far as we know, three such *syntactical* conditions were already expressed on term rewriting systems:

1. Gérard Huet [Hu] devises a liberal criterion on critical pairs, inducing confluence of left and right-linear term rewriting systems. His class of *strongly closed* rewriting systems is limited by the right-linearity (= non-duplication) condition on rewriting rules.
2. in the same article, Gérard Huet introduces a *critical pair condition* which avoids the right-linearity condition. Since then, the condition was generalized by Yoshihito Toyama [To] and Vincent van Oostrom [O_4].
3. Bernhard Gramlich [Gr] introduces a third criterion on critical pairs, which he calls a *parallel critical pair condition*. As in Huet-Toyama-van Oostrom's work, the condition implies confluence of left-linear term rewriting systems.

In his article, Bernhard Gramlich shows that the three conditions 1. 2. and 3. are not comparable with one another. For instance, the associativity rule (5) is an instance of 1. but not of 2. and 3.

This line of research has not only produced interesting syntactic conditions. Two *abstract* conditions were also exhibited:

1. Vincent van Oostrom [O_1,O_4] and Femke van Raamsdonk [R,OR] introduce *weakly orthogonal* rewriting systems and establish a confluence theorem for them.
2. Vincent van Oostrom and collaborators [O_2,BOK,KOV_2] establish a diagrammatic confluence theorem for a wide class of rewriting systems with *decreasing diagrams*.

Validity of these conditions, either syntactical or abstract, may be decided on any finite rewriting system. We are looking for a criterion $(*)$ just as effective.

There also remains to unify the approaches developed here and in Patrick Dehornoy's work on Garside groups [D_1]. At this stage, one main difference is the choice of "local" strong normalization property. Instead of our finite development hypothesis, Dehornoy requires strong normalization of each homotopy equivalence class of paths. This choice enables a simpler axiomatics, closer to Newman's spirit, which enjoys an effective criterion [D_2]. But it limits the scope of the theory. Typically, the pure λ-calculus is rejected because every path

$$(\lambda x.a)(\Delta\Delta) \longrightarrow \cdots \longrightarrow (\lambda x.a)(\Delta\Delta) \longrightarrow a$$

lies in the homotopy equivalence class of

$$(\lambda x.a)(\Delta\Delta) \longrightarrow a$$

where Δ is the duplicating combinator $\Delta = \lambda y.yy$.

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