Axiomatic Rewriting Theory IV

A stability theorem in Rewriting Theory

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Abstract

One key property of the λ-calculus is that there exists a minimal *computation (the head-reduction) $M \xrightarrow{e} V$ from a λ-term M to the set of its head-normal forms. Minimality here means categorical "reflectivity" i.e. that every reduction path $M \xrightarrow{f} W$ to a head-normal form W factors (up to redex permutation) to a path $M \xrightarrow{e} V \xrightarrow{h} W$. This paper establishes a* stability *à la Berry or* poly-reflectivity *theorem [D, La, T] which extends the minimality property to Rewriting Systems with critical pairs. The theorem is proved in the setting of Axiomatic Rewriting Systems where sets of head-normal forms are characterised by their frontier property in the spirit of [GK].*

1 Axiomatic Rewriting Theory

Rewriting is a versatile model of computation which stretches from Turing Machines and Petri nets to λ-calculus and π-calculus. This versatility has generated in the past a variety of theories which are still poorly interconnected. Axiomatic Rewriting Theory [GLM, M, 1, 3] purports a unified treatment of Rewriting where every particular system is considered as part of a larger picture. General structures are exhibited as axioms and generic properties are established from them. The stability theorem presented here (also called fundamental theorem) is the most significant contribution of this approach. It inherits the conceptual clarity of its domain-theoretic ancestor, Berry stability theorem [Be], and the genericity of its own rewriting-theoretic principles. Informally, the theorem states that there exists from every term M a cone of minimal computations to the set of head-normal forms.

ARS. The framework of *Axiomatic Rewriting Systems* (ARSs) encompasses many existing Rewriting Systems like (left-linear) first-order rewriting systems with critical pairs, λ-calculus, call-by-value λ-calculus, $\lambda\sigma$-calculus of explicit substitutions, Petri nets, CCS and π-calculus.

Formally, an ARS is a couple $(\mathcal{G}, \rhd)$ which verifies a series of *standardisation* axioms formulated in [M, 1]. The couple $(\mathcal{G}, \rhd)$ is composed by

1. a graph $\mathcal{G}$, that is a quadruple $(\mathcal{T}, \mathcal{R}, \partial_0, \partial_1)$ consisting of a set $\mathcal{T}$ of vertices (= terms), a set $\mathcal{R}$ of arrows (= reduction steps, or redexes), and two functions $\partial_0, \partial_1 : \mathcal{R} \longrightarrow \mathcal{T}$ (= the source and target functions). We write $u : M \longrightarrow N$ when $\partial_0(u) = M$ and $\partial_1(u) = N$.

2. a binary relation $\rhd$ between paths[1] in $\mathcal{G}$.

The standardisation axioms try to capture the generic properties of *redex permutations* in a rewriting system. In particular, one axiom asks that two paths f and g are coinitial and cofinal when $f \rhd g$ and another axiom that f is of length 2 and g of length at least 1. So,

[1] We recall that a *path* in a graph $\mathcal{G}$ is a sequence $f = (M_1, r_1, M_2, ..., M_m, r_m, M_{m+1})$ where $r_i : M_i \longrightarrow M_{i+1}$ for every $i \in [1...m]$. We write $f : M_1 \longrightarrow M_{m+1}$. The length of f is m ; f is said to be empty when $m = 0$. Two paths $f : M \longrightarrow N$ and $g : P \longrightarrow Q$ are coinitial (resp. cofinal) when $M = P$ (resp. $N = Q$). $f;g : M \longrightarrow Q$ denotes the concatenation of two paths $f : M \longrightarrow P$ and $g : P \longrightarrow Q$.

every diagram $f \rhd g$ in an ARS has the form:

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Leftarrow"] & Q \arrow[d, "u'"] \\ P \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{array}{l} f = v; u' \\ g = u; h \end{array}$$

where u, v and u' are redexes and h is a path. For lack of space we do not present the axioms here and suggest [GLM, M, 1] for a complete description.

A CONCRETE EXAMPLE. As an illustration we explicate the construction of the ARS $(\mathcal{G}_\lambda, \rhd_\lambda)$ corresponding to the λ-calculus. $\mathcal{G}_\lambda$ is the transition graph of the calculus: its vertices are the λ-terms up to α-conversion, its arrows $u : M \longrightarrow P$ are the β-redexes from M to P. The relation $\rhd_\lambda$ relates two paths $f \rhd_\lambda g$ precisely when[2]:

1. $f = v; u'$ and $g = u; h$ where u, v and u' are β-redexes and h is a path,
2. the β-redex u is not nested under v wrt. the tree ordering,
3. u' is the (unique) residual of u after v and h develops the residuals of v after u.

Two paradigmatic examples of such *redex permutations* $f \rhd_\lambda g$ are:

$$\begin{tikzcd} PQ \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Leftarrow_1"] & P'Q \arrow[d, "u'"] \\ PQ' \arrow[r, "v'"'] & P'Q' \end{tikzcd} \qquad \begin{tikzcd} \Delta P \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Leftarrow_2"] & \Delta P' \arrow[d, "u'"] \\ PP \arrow[r, "v_1; v_2"'] & P'P' \end{tikzcd} \qquad (1)$$

where $P \longrightarrow P'$ and $Q \longrightarrow Q'$ are two β-redexes and $\Delta = (\lambda x.xx)$.

The standardisation axioms are easily verified on $(\mathcal{G}_\lambda, \rhd_\lambda)$ which therefore is an ARS.

STRUCTURE OF THE PAPER. Axiomatic Rewriting Systems are presented in Section 1. Section 2 presents the fundamental theorem in the framework of Axiomatic Rewriting Systems (ARSs). A series of concrete examples is given at the end of the section. Section 3 describes quickly the state of the art in Axiomatic Rewriting Theory. The fundamental theorem is proved in Section 4. Section 5 reformulates the theorem in more categorical terms. Section 6 presents open problems and related works.

[2] For a definition of residual and development, see [Ba].

2 The fundamental theorem

DISJOINT VS NESTED. One distinctive feature of ARSs compared to asynchronous transition systems [WN] or concurrent transition systems [S] is that every permutation $f \rhd g$ is "oriented". This accounts for the so-called *standardisation* procedure in Rewriting Theory, see [Ba, HL, Bo, GLM, CK] for details. In particular, a permutation $f \rhd g$ is declared *disjoint* when $g \rhd f$ and *standardising* when $\neg(g \rhd f)$.

The concepts of *nested* and *disjoint* redexes can be derived as follows: two redexes u and v are declared disjoint, $u \| v$, when there exists a disjoint permutation $u; f \rhd v; g$; a redex u nests (or contains) another redex v, $u \prec v$, when there exists a standardising permutation $v; g \rhd u; f$. The axiomatics of ARSs insures that $u \prec v$, $u \| v$ and $v \prec u$ are exclusive.

Observe that the definitions of nested and disjoint redex correspond to the usual notions in the case of $(\mathcal{G}_\lambda, \rhd_\lambda)$. Check in diagram (1) that the first permutation $v; u' \rhd u; v'$ is disjoint and that the second one $v; u' \rhd u; v_1; v_2$ is standardising.

COMPATIBILITY. By definition, two coinitial redexes u and v can be permuted when $v \prec u$, $u \| v$ or $u \prec v$. We say in any of the three cases that u and v are *compatible*, otherwise that they are *incompatible* or *conflicting*. In $(\mathcal{G}_\lambda, \rhd_\lambda)$ in particular, every two coinitial redexes are compatible.

LÉVY PERMUTATION EQUIVALENCE. The Lévy permutation equivalence $\equiv$ on the paths of $(\mathcal{G}, \rhd)$ is defined as the least equivalence relation (reflexive, symmetric, transitive) such that $d_1; f; d_2 \equiv d_1; g; d_2$ whenever there are paths $d_1 : M \longrightarrow P$, $d_2 : Q \longrightarrow N$ and $f, g : P \longrightarrow Q$ such that $f \rhd g$.

OPEN-STABLE SETS. We follow the spirit of [GK] and declare that a set $\mathcal{V}$ of vertices is *open-stable* in $(\mathcal{G}, \rhd)$ when the three properties below are verified by every permutation diagram $v; u' \rhd u; f$:

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Leftarrow"] & Q \arrow[d, "u'"] \\ P \arrow[r, "f"'] & N \end{tikzcd}$$

[head]	if $u \prec v$ and $Q \in \mathcal{V}$, then $M \in \mathcal{V}$,
[stable]	if $u \| v$ and $P, Q \in \mathcal{V}$, then $M \in \mathcal{V}$,
[closed]	if $u \| v$ and $Q \in \mathcal{V}$ then $M \in \mathcal{V}$ or $N \in \mathcal{V}$.

Observe that the condition [closed] is automatically verified when the set $\mathcal{V}$ is reduction-closed because $Q \in \mathcal{V}$ and $Q \longrightarrow N$ implies $N \in \mathcal{V}$.

THE FUNDAMENTAL THEOREM. Let $\mathcal{V}$ be an open-stable set in the ARS $(\mathcal{G}, \rhd)$. The following *stability* theorem is established in section 4:

> For every term M, there exists a cone $(e_i : M \longrightarrow V_i)_{i \in I}$ of paths with each $V_i \in \mathcal{V}$ such that for every path $f : M \longrightarrow W$ with $W \in \mathcal{V}$, there exists one and only one $i \in I$ such that f factors as $f \equiv e_i ; h$ for some path h.
>
> $$\begin{tikzcd} M \arrow[r, "e_i"] \arrow[dr, "f"'] & V_i \arrow[d, dotted, "h"] \\ & W \end{tikzcd}$$
>
> Moreover, the path h is unique up to $\equiv$.

The theorem formalises the idea that there exists from every term M a cone of *minimal* computations to the set of *head-normal forms* $\mathcal{V}$.

A series of examples follows.

EXAMPLE 1. The set $\mathcal{V}_\lambda$ of head-normal forms is open-stable in the λ-calculus ARS $(\mathcal{G}_\lambda, \rhd_\lambda)$. The [head] and [stable] conditions derive from the fact that a β-redex $u : M \longrightarrow V$ from $M \notin \mathcal{V}_\lambda$ to $V \in \mathcal{V}_\lambda$ is necessarily M's *head* β-redex. The [closed] condition is a consequence of reduction-closedness of $\mathcal{V}_\lambda$. So, the fundamental theorem applies on $\mathcal{V}_\lambda$ and repeats Lévy's interpretation of Wadsworth's theorem, see [Ba] Theorem 8.3.11. Observe that the cone $(e_i : M \longrightarrow V_i)_{i \in I}$ is empty or singleton because the λ-calculus admits no conflict (orthogonality). This path is precisely the head-reduction described syntactically by Wadsworth.

EXAMPLE 2. Another instance of the theorem was described by Plotkin in the λ_v-calculus [Pl]. Recall that a *value* is a variable or a λ-term of the form $\lambda x.M$ and that the β_v-reduction $(\lambda x.M)V \to M[V/x]$ is the β-rule restricted to value arguments V. Since the λ_v-calculus verifies the axioms in [1] and the set of values is open-stable (w.r.t. the λ_v-calculus), the fundamental theorem implies that there is a head-reduction in the λ_v-calculus. In fact, this non-trivial reduction was described by Plotkin in [Pl] as the strategy which implements Landin's SECD machine. See also [FH].

EXAMPLE 3. The fundamental theorem has been used by the author to prove that the weak evaluation strategy defined by Abadi and Cardelli on their object ς-calculus is complete, see [AC] pages 60-65. More generally, the fundamental theorem bridges the rewriting (one-step) and operational (big-step) semantics of programming languages.

EXAMPLE 4. Examples 1,2 and 3 consider non conflicting systems where the theorem characterises at most one reduction path which, for correctness reasons, is generally chosen by implementers. But the distinctive feature of the fundamental theorem (and its conceptual novelty) is that it also applies to Rewriting Systems with conflicts (critical pairs). In that case the cone $(e_i : M \longrightarrow V_i)_{i \in I}$ may contain several paths, even be infinite. A simple example is the first order Rewriting System $A \longrightarrow A$ and $A \longrightarrow B$ where $\mathcal{V} = \{B\}$. The set $\{B\}$ is open-stable because B is a normal form. The minimal cone $(e_i : A \longrightarrow B)_{i \in I}$ contains every path $A \longrightarrow \cdots \longrightarrow A \longrightarrow B$ and is therefore infinite.

EXAMPLE 5. A well-known calculus with conflict is the $\lambda\sigma$-calculus, a λ-calculus with explicit substitutions, see [ACCL] for details. Its eleven rules and eleven critical pairs combine to describe rich and subtle substitution mechanisms. Many traditional techniques were adapted to understand its rewriting dynamics but the whole picture was still unclear until the fundamental theorem. In fact, the theorem offers the first picture of the various reduction paths from a $\lambda\sigma$-term to its values (the $\lambda\sigma$-head-normal forms).

To see why the fundamental theorem applies let us say that the $\lambda\sigma$-calculus is a left-linear first-order rewriting system, therefore an ARS, and that the set $\mathcal{V}_{\lambda\sigma}$ of $\lambda\sigma$-head-normal forms is open-stable for the same reasons as $\mathcal{V}_\lambda$ is open-stable in example 1. Consequently, there exists from every $\lambda\sigma$-term M a canonical cone $(e_i : M \longrightarrow V_i)_{i \in I}$ of head-reductions to $\mathcal{V}_{\lambda\sigma}$.

As an aside, it should be noted that this cone is *finite*. This result, proved syntactically in [M] and axiomatically in [2], has the very important consequence that every *needed* strategy[3] normalises — a property which would be very difficult (and depressing) to establish with traditional techniques.

Example 6. Petri nets form another interesting class of Rewriting Systems with conflict. A Petri net is a quadruple $N = (P, T, pre, post)$ where P is the set of *places* and T is the set of *transitions* of the net. A *state* of the net N is a multiset of elements of P. To every transition t the functions $pre, post : T \longrightarrow S$ associate a *pre-condition* $pre(t)$ and a *post-condition* $post(t)$ in the set of states S. The multiset $pre(t)$ is supposed here to be non empty. We follow the spirit of [S] and construct the ARS $(\mathcal{G}, \rhd)$ associated to the net N. Its transition graph $\mathcal{G}$ is defined as follows:

1. its vertices are the states of N,
2. its arrows $p \longrightarrow q$ are the triples (p, t, q) such that $p = p_0 \uplus pre(t)$ and $q = p_0 \uplus post(t)$ for some state p_0, where $\uplus$ is the multiset addition.

Two paths $u; v' \rhd v; u'$ are related precisely when:

1. u, v, u' and v' are redexes $u = (p, t_1, p_1)$, $v = (p, t_2, p_2)$, $u' = (p_2, t_1, p_{21})$ and $v' = (p_1, t_2, p_{12})$,
2. both $pre(t_1) \cap pre(t_2)$ and $post(t_1) \cap post(t_2)$ are empty multisets (this implies that $p_{12} = p_{21}$).

The ARS $(\mathcal{G}, \rhd)$ is "linear" in the sense that every two compatible redexes are disjoint. Let us fix a place p and define $\mathcal{V}_p$ as the set of states containing at least one occurrence of p. $\mathcal{V}_p$ then is open-closed: it verifies the [head] condition by linearity, the [stable] condition because the post-conditions of disjoint redexes are disjoint, and the [closed] condition because the preconditions of u and u' are equal whenever $v; u' \rhd u; v'$. So, the fundamental theorem applies here although $\mathcal{V}_p$ is not necessarily reduction-closed.

Example 7. One interesting and maybe unexpected point about our stability theorem is that it applies also to non sequential Rewriting Systems. Take for instance the *parallel or* operator defined on the signature $\{\perp, \mathrm{true}\}$ by the rules $\mathrm{por}(\mathrm{true}, x) \to \mathrm{true}$, $\mathrm{por}(x, \mathrm{true}) \to \mathrm{true}$ and $\perp \to \mathrm{true}$. The set $\mathcal{V} = \{\mathrm{true}\}$ is open-stable and the minimal cone $(e_i : \mathrm{por}(\perp, \perp) \longrightarrow \mathrm{true})_{i \in I}$ contains the *two* following computations:

$$e_1 : \mathrm{por}(\perp, \perp) \longrightarrow \mathrm{por}(\mathrm{true}, \perp) \longrightarrow \mathrm{true}$$

$$e_2 : \mathrm{por}(\perp, \perp) \longrightarrow \mathrm{por}(\perp, \mathrm{true}) \longrightarrow \mathrm{true}$$

The traditional classification of $\mathrm{por}(-,-)$ as a non stable operator can be interpreted as the effect of projecting the two incompatible computations e_1 and e_2 to *compatible* morphisms $(\perp, \perp) \longrightarrow (\mathrm{true}, \perp)$ and $(\perp, \perp) \longrightarrow (\perp, \mathrm{true})$ in the denotational model.

Example 8. Many symbolic constraint problems can be expressed and solved in the framework of Conditional Rewriting Systems [BK], see [CDJK]. The fundamental theorem should help to design lazy evaluation techniques for them.

3 An axiomatic toolbox

This section recalls the various tools developped in [GLM, M, 1, 2, 3].

Standardisation theorem. In the following, we write $d \Rightarrow e$ when there is a sequence of oriented permutations operating from d to e, and $d \simeq e$ when all the permutations are disjoint in the sequence[4]. In particular, $d \simeq e$ implies $d \Rightarrow e$ and $e \Rightarrow d$. Notice that the Lévy permutation equivalence $\equiv$ is simply the symmetric transitive closure of $\Rightarrow$.

An example is $v; u' \Rightarrow u; v_2; v_1'$ in the ARS $(\mathcal{G}_\lambda, \rhd_\lambda)$:

$$\begin{tikzcd}
\Delta(Ia) \arrow[rr, "v"] \arrow[dd, "u"'] \arrow[ddrr, phantom, "\Leftarrow"] & & \Delta a \arrow[dd, "u'"] \\
& a(Ia) \arrow[dr, "v_2'"] & \\
(Ia)(Ia) \arrow[ur, bend left, "v_1"] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\simeq"] & & aa \\
& (Ia)a \arrow[ur, bend right, "v_1'"'] &
\end{tikzcd}
\qquad
\begin{array}{rcl}
\Delta & = & \lambda x.xx \\
I & = & \lambda x.x \\
v; u' & \rhd & u; v_1; v_2' \\
v_1; v_2' & \rhd & v_2; v_1' \\
v_2; v_1' & \rhd & v_1; v_2'
\end{array}$$

A path d is *standard* when no standardising permuta-

[3] See [M, 2] for a definition of needed strategies in Rewriting Systems with conflicts.

[4] In fact [1, 2, 3] interprets every ARS $(\mathcal{G}, \rhd)$ as a 2-category $\mathcal{C}$ whose cells $\alpha : d \Rightarrow e$ between paths $d : M \longrightarrow P$ and $e : M \longrightarrow P$ are (equivalence classes of) sequences of standardising and disjoint permutations. Here we will not present the construction of $\mathcal{C}$, only keep some notations. In particular, $d \Rightarrow e$ (resp. $d \simeq e$) corresponds to a cell (resp. an isomorphic cell) in $\mathcal{C}$.

tion can operate on d after a sequence of disjoint permutations. In particular, $d \Rightarrow e$ implies $d \simeq e$. The *axiomatic* standardisation theorem established in [GLM, M, 1] states that there exists a standard path in every Lévy permutation class and that this path is unique up to disjoint permutations: If we associate to each path d a standard path $\Downarrow_d$ such that $d \Rightarrow \Downarrow_d$, the uniqueness property can be expressed as (2):

$$\text{For every paths } d \text{ and } e, \quad d \equiv e \Rightarrow \Downarrow_d \simeq \Downarrow_e \qquad (2)$$

External path. A path $e : M \longrightarrow P$ is external, see [M, 2], when for every standard path $f : P \longrightarrow Q$, the path $e; f : M \longrightarrow Q$ is standard. Externality of e can also be expressed this way:

$$\forall Q \in \mathcal{T},\ \forall f : P \longrightarrow Q, \qquad e; \Downarrow_f \simeq \Downarrow_{e;f}$$

For example, any head-redex $(\lambda x.M)N \longrightarrow M[N/x]$ is external in $(\mathcal{G}_\lambda, \rhd_\lambda)$, whereas the β-redex $\Delta(Ia) \xrightarrow{v} \Delta a$ is not external. Observe also that any normalising and standard path is external by definition.

We say that a path $m : M \longrightarrow Q$ is *internal* when there is no external path $e : M \longrightarrow P$ (except the identity) such that $m \equiv e; f$.

A factorisation theorem. In the following, $\mathcal{C}$ denotes the free category on $\mathcal{G}$ (its morphisms are the paths $d : M \longrightarrow P$) and $[\mathcal{C}]$ the category $\mathcal{C}$ quotiented by the permutation equivalence $\equiv$ (its morphisms are the equivalence classes of paths wrt. $\equiv$). We write $[-] : \mathcal{C} \longrightarrow [\mathcal{C}]$ the functor which associates to every path d its equivalence class $[d]$.

Write $\mathcal{E}$ and $\mathcal{M}$ for the classes of external and internal paths in $\mathcal{C}$, and $[\mathcal{E}]$ and $[\mathcal{M}]$ for their image in $[\mathcal{C}]$. A factorisation theorem established in [3] states that $([\mathcal{E}], [\mathcal{M}])$ is a factorisation system of $[\mathcal{C}]$ in the sense of Freyd and Kelly [FK]. This means that the three following properties are verified in every Axiomatic Rewriting System:

1. $[\mathcal{E}]$ and $[\mathcal{M}]$ are categories,
2. every morphism f can be factored as $f = e; m$ with $e \in [\mathcal{E}]$ and $m \in [\mathcal{M}]$,
3. the factorisation is functorial: if $(e; m); v = u; (f; n)$ where $e, f \in [\mathcal{E}]$ and $m, n \in [\mathcal{M}]$, there is a unique w rendering commutative the diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "u"'] & N \arrow[r, "m"] \arrow[d, dotted, "w"] & P \arrow[d, "v"] \\ M' \arrow[r, "f"] & N' \arrow[r, "n"] & P' \end{tikzcd}$$

Conceptually, factorisation means that the efficient (external) part of a computation can always be separated from its junk. Technically, it permits to focus one's attention to the space of external paths.

4 Proof of the fundamental theorem

Methodology. We prove the fundamental theorem in three steps:

1. we prove that every internal path m is Lévy equivalent to a sequence of internal redexes,
2. we prove that every internal path $m : M \longrightarrow P$ is useless in the sense that $M \notin \mathcal{V}$ implies $P \notin \mathcal{V}$,
3. we compare the various external paths from M to $\mathcal{V}$ and establish that one of them is *minimal* in every computational direction. This exhibits a cone $(e_i : M \longrightarrow V_i)_{i \in I}$ of minimal external paths; we then conclude with factorisation and step 2.

Each step i is tackled in section 4.i.

We will not introduce the standardisation axioms here, see [GLM, M, 1] for details. Instead we present in Figure 1 the seven properties of every ARS $(\mathcal{G}, \rhd)$ that we use in the course of the proof.

4.1 The segmentation theorem

By factorisation, $\mathcal{M}$ is a subcategory of $\mathcal{C}$. In particular, every path $\xrightarrow{r_1} \cdots \xrightarrow{r_k}$ consisting of internal redexes r_i's is internal. We establish the converse property here that every internal path $m : P \longrightarrow Q$ is Lévy equivalent to a sequence $\xrightarrow{r_1} \cdots \xrightarrow{r_k}$ of internal redexes. The discovery that this *segmentation* property holds in every ARS was a big, big surprise, and in fact the starting point of this paper.

The situation differs notably in the case of external paths. A reduction path like

$$((\lambda a)b)[s][t] \xrightarrow{Beta} a[b \cdot \mathrm{id}][s][t] \xrightarrow{Clos} a[(b \cdot \mathrm{id}) \circ s][t]$$

A.	a redex is either external or internal
B.	If $e_1; e_2$ is external then e_2 is external
C.	If $m_1; m_2$ is internal then m_1 is internal
D.	If $u; f; v \simeq u; g; v$, then $f \simeq g$.
E.	Lemma 4.1
F.	Lemma 4.4
G.	Lemma 7.1.
H.	Lemma 7.2.

Figure 1. Seven properties of Axiomatic Rewriting Systems

is external in the $\lambda\sigma$-calculus but certainly not a sequence of external redexes, even up to Lévy equivalence $\equiv$, see [M, 2].

The proof of segmentation goes as follows.

STARTS-STOPS. Let $d : M \longrightarrow Q$ be a path. A redex $u : M \longrightarrow P$ *starts* d when $d \simeq u; e$ for some path $e : P \longrightarrow Q$. A redex $v : P \longrightarrow Q$ *stops* d when $d \simeq e; v$ for some path $e : M \longrightarrow P$.

Lemma 4.1 ([1]) *A path $d; e$ is standard if and only if*

- *the path d is standard, and*
- *the path $u; e$ is standard whenever u stops d.*

Corollary 4.2 *A standard path e is external if and only if every redex which stops e is external.*

The corollary is straightforward and leads to the main result of this section.

Theorem 4.3 (segmentation) *Every internal path $m \in \mathcal{M}$ is Lévy equivalent to a path $r_1; \cdots; r_k$ consisting of internal redexes r_i's.*

PROOF By induction on the length of $\Downarrow_m$. The case 0 is trivial. Otherwise, the standard path $\Downarrow_m$ is not external, so by lemma 4.1 one non external redex r_k stops $\Downarrow_m$. This means that $\Downarrow_m \simeq n; r_k$ for some path n which by construction is standard: $n \simeq \Downarrow_n$, and by figure 1(C) internal. r_k is also internal by figure 1(A), and we conclude by induction hypothesis on $\Downarrow_n$. ■

4.2 Internal paths are "junk"

We prove that an internal redex is useless by first characterising standard paths as the paths which do not contain any anti-standard path. Remark: this characterisation was our original definition of standard path in [GLM].

ANTI-STANDARD PATH. *Anti-standard* paths[5] are defined by induction (see figure 2) as follows:

1. a path $M \xrightarrow{u} P \xrightarrow{\mathbf{id}_P} P \xrightarrow{v} N$ is anti-standard when u and v are redexes and $u; v$ is not standard,
2. a path $M \xrightarrow{u} P \xrightarrow{r;e} Q \xrightarrow{v} N$ is anti-standard when u and v and r are redexes, $\xrightarrow{u}\xrightarrow{r'} \simeq \xrightarrow{r}\xrightarrow{u'}$ and $M' \xrightarrow{u'} P' \xrightarrow{e} Q \xrightarrow{v} N$ is anti-standard.

$$\begin{tikzcd}
M \arrow[r, "r'"] \arrow[d, "u"'] \arrow[dr, phantom, "\simeq"] & M' \arrow[d, "u'"] & & \\
P \arrow[r, "r"'] & P' \arrow[r, "e"] & Q \arrow[r, "v"] & N
\end{tikzcd}$$

Figure 2. The inductive step in the definition of anti-standard paths

Lemma 4.4 ([1]) *A path d is standard if and only if it cannot be factored as $d = d_1; f; d_2$ where f is an anti-standard path.*

Corollary 4.5 *If a redex u is internal, then there exists an anti-standard path $M \xrightarrow{u} P \xrightarrow{e} Q \xrightarrow{v} N$.*

PROOF By definition, an internal redex is non external. Hence there is a standard path $f : P \longrightarrow Q$ such that $u; f : M \longrightarrow M'$ is not standard. By lemma 4.4, there is an anti-standard path in $u; f$, but not in f. We conclude. ■

Lemma 4.6 *Suppose that $\mathcal{V}$ is open-stable and that $u : M \longrightarrow V$ is an internal redex. If $V \in \mathcal{V}$, then $M \in \mathcal{V}$.*

[5] Anti-standard paths were called conflicts in [GLM]. The terminology here is inspired by [K]'s anti-standard pairs.

Proof By induction on the length of anti-standard paths $M \xrightarrow{u} V \xrightarrow{e} Q \xrightarrow{v} N$ such that $V \in \mathcal{V}$. If e is empty, then $u;v$ is not standard, hence $u;v \rhd w;h$ for some redex w which contains u, $w \prec u$, so that $M \in \mathcal{V}$ by [head]. If $e = r;e'$, then $M \xrightarrow{u} V \xrightarrow{r} P \simeq M \xrightarrow{r'} M' \xrightarrow{u'} P$ and the path $M' \xrightarrow{u'} P \xrightarrow{e} Q \xrightarrow{v} N$ is anti-standard.

$$\begin{tikzcd}
M \arrow[r, "r'"] \arrow[d, "u"'] \arrow[dr, phantom, "\simeq"] & M' \arrow[d, "u'"] & & \\
V \arrow[r, "r"'] & P \arrow[rr, "e"] & & Q \arrow[r, "v"] & N
\end{tikzcd}$$

By [closed], either $M \in \mathcal{V}$, and we conclude, or $P \in \mathcal{V}$. In that case, by induction, $M' \in \mathcal{V}$ and by [stable], $M \in \mathcal{V}$. We conclude. ■

The main result of the subsection is a consequence of segmentation (theorem 4.3) and lemma 4.6.

Lemma 4.7 *Suppose that $\mathcal{V}$ is open-stable. If $m : M \longrightarrow V$, $m \in \mathcal{M}$ and $V \in \mathcal{V}$, then $M \in \mathcal{V}$.*

4.3 A cone of minimal external paths

An important consequence of factorisation and Lemma 4.7 is that every path $d : M \longrightarrow W$ to $W \in \mathcal{V}$ factors as $d \equiv e;m$ where $e : M \longrightarrow V$ is external and $V \in \mathcal{V}$ is a value. Armed with this property we can focus our attention to the space of external path whose structure is much simpler than the global reduction space.

Lemma 4.8 *Let $\mathcal{V}$ be an open-stable set. Suppose that $e_1 : M \longrightarrow V_1$ to $V_1 \in \mathcal{V}$ and $e_2 : M \longrightarrow V_2$ to $V_2 \in \mathcal{V}$ are external paths, and that there are paths $h_1 : V_1 \longrightarrow N$ and $h_2 : V_2 \longrightarrow N$ such that $e_1;h_1 \equiv e_2;h_2$. Then there exists a path $e : M \longrightarrow V$ to $V \in \mathcal{V}$ and paths $i_1 : V \longrightarrow V_1$, $i_2 : V \longrightarrow V_2$ such that $e_1 \simeq e;i_1$ and $e_2 \simeq e;i_2$.*

Proof See appendix. ■

From lemma 4.8 we obtain a cone of *minimal* paths from M to terms $V_i \in \mathcal{V}$. Theorem 4.9 shows that this cone is the poly-reflection cone we are looking for.

Minimal. An external path $e : M \longrightarrow V$ is *minimal* when $V \in \mathcal{V}$, and for every $M \xrightarrow{f} W \xrightarrow{g} V$ with $e \simeq f;g$ and $W \in \mathcal{V}$ we have $g = \mathbf{id}_V$.

Theorem 4.9 (fundamental theorem) *Let $\mathcal{V}$ be an open-stable set of the ARS $(\mathcal{G}, \rhd)$. For every term M, there exists a cone $(e_i : M \longrightarrow V_i)_{i \in I}$ of paths with each $V_i \in \mathcal{V}$ such that for every path $f : M \longrightarrow W$ with $W \in \mathcal{V}$, there exists one and only one $i \in I$ such that $f \equiv e_i;h$ for some path h.*

$$\begin{tikzcd}
M \arrow[r, "e_i"] \arrow[dr, "f"'] & V_i \arrow[d, dotted, "h"] \\
& W
\end{tikzcd}$$

Moreover, this path h is unique up to $\equiv$.

Proof See appendix. ■

5 A categorical approach

We have just established a very useful rewriting theorem but we would like to go one step further and connect our work with mainstream denotational semantics. Our motto is that the fundamental theorem is *really* a stability theorem in the spirit of Berry [Be].

To understand this, assume that a functor $[\![-]\!] : [\mathcal{C}] \longrightarrow \mathcal{S}$ interprets the terms of the ARS $(\mathcal{G}, \rhd)$ in a small category $\mathcal{S}$. Assume now that the functor verifies a series of conditions corresponding in fact to open-stability of Section 2. We prove (see theorem 5.4) that the restriction of $[\![-]\!]$ to each slice-category $M \downarrow [\mathcal{C}]$ is a *stable* functor

Stable functors introduced in [La, T] generalise the well-known concept of a functor with a left adjoint and extend Berry's definition of stability on dI-domains [Be]. A functor $\mathcal{X} \longrightarrow \mathcal{Y}$ is *stable* when its restriction to each slice $\mathcal{X} \downarrow X$ has a left adjoint, see [T] for alternative definitions.

Notation. If $\mathcal{X}$ denotes a category and M an object of $\mathcal{X}$, then $\mathcal{X}^M$ denotes the slice-category usually noted $M \downarrow \mathcal{X}$. Similarly, if $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor, $F^M : \mathcal{X}^M \longrightarrow \mathcal{Y}^{FM}$ denotes the functor restricting F to the slice $\mathcal{X}^M$.

We introduce the conditions we need on $[\![-]\!]$ to prove stability, theorem 5.4.

Interpretation functor. A functor $[\![-]\!] : [\mathcal{C}] \longrightarrow \mathcal{S}$ is an interpretation functor when, for every two *compatible* redexes u and v:

[head] if $v \prec u$, then $[\![u]\!]$ is iso in $\mathcal{S}$,

[stable] if $u \| v$, then the diagram below is a *pullback*, for $u; v' \simeq v; u'$:

$$\begin{tikzcd} [\![M]\!] \arrow[r, "[\![u]\!]"] \arrow[d, "[\![v]\!]"'] & [\![P]\!] \arrow[d, "[\![v']\!]"] \\ [\![Q]\!] \arrow[r, "[\![u']\!]"] & [\![N]\!] \end{tikzcd}$$

Note that we identify every redex u or path $u; v'$ in $\mathcal{C}$ with the corresponding morphism in $[\mathcal{C}]$. The identification is harmless and we will repeat it throughout this section.

By the factorisation theorem, each forgetful functor $U^M : [\mathcal{E}]^M \longrightarrow [\mathcal{C}]^M$ has a right adjoint $F^M : [\mathcal{C}]^M \longrightarrow [\mathcal{E}]^M$ which extracts the external path corresponding to a morphism in $[\mathcal{C}]$. We use the lemma below to factor $[\![-]\!]^M$ as F^M followed by $[\![-]\!]^M_{\mathcal{E}} = [\![-]\!]^M \circ U^M$, up to natural isomorphism.

Lemma 5.1 *Assume that $[\![-]\!]$ is an interpretation functor and that $m : M \longrightarrow P$ is an internal path. Then $[\![m]\!]$ is iso.*

Proof We prove the property for any internal redex m and then conclude with segmentation, theorem 4.3. Because m is an internal redex, there exists by corollary 4.5 an anti-standard path $\xrightarrow{m}\xrightarrow{f}\xrightarrow{v}$. We prove that $[\![m]\!]$ is iso by induction on the length of f. It is a consequence of [head] if f is empty. If $f = u; g$, then $m; u' \simeq u; n$ for some redexes u' and n. By induction hypothesis, $[\![n]\!]$ is iso. We use [stable] on $m \| u$, and the fact that pullbacks preserve isos to deduce that $[\![m]\!]$ is iso. We conclude. ■

Thanks to the natural isomorphism

$$[\![-]\!]^M \cong [\![-]\!]^M_{\mathcal{E}} \circ F^M$$

and the stability of F^M, we have reduced the problem of proving the stability of $[\![-]\!]^M$ to proving the stability of $[\![-]\!]^M_{\mathcal{E}} : [\mathcal{E}]^M \longrightarrow \mathcal{S}^{[\![M]\!]}$. This should be easier because $[\mathcal{E}]$ is "linear" in the sense that two paths equal up to $\equiv$ in $\mathcal{E}$ are also equal up to $\simeq$.

Guided by this intuition we consider the category $\mathcal{C}_{\simeq}$ result of quotienting $\mathcal{C}$ by $\simeq$. Precomposing $[\![-]\!]$ with the forgetful functor $V : \mathcal{C}_{\simeq} \longrightarrow [\mathcal{C}]$ induces the functor

$$[\![-]\!]_{\simeq} : \mathcal{C}_{\simeq} \longrightarrow \mathcal{S}$$

To prove the stability of $[\![-]\!]^M_{\mathcal{E}}$ we plan to prove the stability of $[\![-]\!]^M_{\simeq}$ and then use a lifting technique due to Michael Barr [BW, Po], see lemma 5.3.

Lemma 5.2 *The functor $[\![-]\!]^M_{\simeq} : \mathcal{C}^M_{\simeq} \longrightarrow \mathcal{S}^{[\![M]\!]}$ is stable when $\mathcal{S}$ is small.*

Proof (sketch) $\mathcal{C}^M_{\simeq}$ has pullbacks and cofiltered colimits, hence all small wide pullbacks, which are moreover preserved by $[\![-]\!]^M_{\simeq}$ thanks to the [stable] condition. We conclude that $[\![-]\!]^M_{\simeq}$ is stable with the smallness of $\mathcal{S}$ and the solution set condition, see [T], theorem 1.3.2. ■

Lemma 5.3 (Barr) *Given*

$$\begin{tikzcd} A' \arrow[r, "V'"] \arrow[d, shift left] & B' \arrow[d, shift left] \\ A \arrow[r, "V"'] \arrow[u, shift left] \arrow[u, phantom, "\vdash"] & B \arrow[u, shift left] \arrow[u, phantom, "\vdash"] \end{tikzcd}$$

exhibiting A' as a coreflective subcategory of A and B' as a coreflective subcategory of B, and supposing that both functors from A to B' are the same up to natural isomorphism: If V has a left adjoint, so has V'.

Theorem 5.4 (stability) *Assume that $[\![-]\!]$ is an interpretation functor from $\mathcal{C}$ to a small category $\mathcal{S}$. Then every functor $[\![-]\!]^M : [\mathcal{C}]^M \longrightarrow \mathcal{S}^M$ is stable.*

Proof We prove that $[\![-]\!]^M_{\mathcal{E}} : [\mathcal{E}]^M \longrightarrow \mathcal{S}^{[\![M]\!]}$ is stable. This means that for every external path $e : M \longrightarrow N$, the "slice" functor $[\![-]\!]^e_{\mathcal{E}} : [\mathcal{E}]^e \longrightarrow \mathcal{S}^{[\![e]\!]}$ is a right adjoint — where $[\mathcal{E}]^e$ and $\mathcal{S}^{[\![e]\!]}$ are the slices $([\mathcal{E}]^M) \downarrow e$ and $\mathcal{S}^{[\![M]\!]} \downarrow [\![e]\!]$ of $[\mathcal{E}]^M$ and $\mathcal{S}^{[\![M]\!]}$.

We use the following diagram with the obvious denotations for $\mathcal{C}^e_{\simeq} = (\mathcal{C}^M_{\simeq}) \downarrow e$ and $[\![-]\!]^e_{\simeq}$:

$$\begin{tikzcd} [\mathcal{E}]^e \arrow[r, "[\![-]\!]^e_{\mathcal{E}}"] \arrow[d, shift left, "U^e_{\simeq}"] & \mathcal{S}^{[\![e]\!]} \arrow[d, shift left, "\mathbf{id}"] \\ \mathcal{C}^e_{\simeq} \arrow[r, "[\![-]\!]^e_{\simeq}"'] \arrow[u, shift left, "G^e"] \arrow[u, phantom, "\vdash"] & \mathcal{S}^{[\![e]\!]} \arrow[u, shift left] \arrow[u, phantom, "\vdash"] \end{tikzcd}$$

The functors $U^e_{\simeq}$ and G^e are the slices of the forgetful functor $U_{\simeq} : [\mathcal{E}] \longrightarrow \mathcal{C}_{\simeq}$ and the functor $G : \mathcal{C}_{\simeq} \longrightarrow [\mathcal{E}]$ obtained by precomposing $F : [\mathcal{C}] \longrightarrow [\mathcal{E}]$ with the forgetful functor $V : \mathcal{C}_{\simeq} \longrightarrow [\mathcal{C}]$. We explicit the adjunction $U^e_{\simeq} \dashv G^e$ as follows: letting (f_1, f_2) be a couple of $[\mathcal{E}]$-morphisms $M \xrightarrow{f_1} P \xrightarrow{f_2} N$ and (g_1, g_2) of $\mathcal{C}_{\simeq}$-morphisms $M \xrightarrow{g_1} Q \xrightarrow{g_2} N$ such that $U_{\simeq}(f_1; f_2) = g_1; g_2 = e$ in $\mathcal{C}_{\simeq}$, there are two bijections, both natural in (f_1, f_2) and (g_1, g_2):

$$\mathcal{C}^e_{\simeq}(U^e_{\simeq}(f_1, f_2), (g_1, g_2)) \cong [\mathcal{C}]^e(U^e(f_1, f_2), V^e(g_1, g_2)) \qquad (3)$$

and

$$[\mathcal{C}]^e(U^e(f_1, f_2), V^e(g_1, g_2)) \cong [\mathcal{E}]^e((f_1, f_2), G^e(g_1, g_2)) \tag{4}$$

Bijection (3) is coming from the fact that f_1 is external and g_1 is standard, using figure 1(D). Bijection (4) is the adjunction $U^M \dashv F^M$ projected on the slices $[\mathcal{C}]^e$ and $[\mathcal{E}]^e$. Composing the two bijections, we obtain $U^e_{\simeq} \dashv G^e$.

The functor $[\![-]\!]^e_{\simeq}$ has a left adjoint by lemma 5.2, so we use Barr's lemma and deduce that $[\![-]\!]^e_{\mathcal{E}}$ has a left adjoint too. The functor $[\![-]\!]^M_{\mathcal{E}}$ is therefore stable, and so is $[\![-]\!]^M$. ∎

Example 1. Take the ARS $(\mathcal{G}_\lambda, \rhd_\lambda)$ and the domain $\mathcal{S}$ of Böhm trees. The functor $[\![-]\!]$ which associates to every λ-term its "immediate" Böhm tree is an interpretation functor. Each restriction $[\![-]\!]^M$ is therefore stable.

Example 2. Qualitative domains can be seen as simple forms of ARSs. A qualitative domain [G] is a couple (E, X) consisting of a set X called the *web* and a set E of subsets of X such that:

1. if $e \in E$ and $e' \subset e$, then $e' \in E$,
2. E is closed under directed unions.

The definition of the ARS $(E, X)_{\mathrm{ARS}} = (\mathcal{G}, \rhd)$ corresponding to a qualitative domain (E, X) is straightforward:

1. $\mathcal{G}$'s vertices are the finite subsets of X elements of E ; $\mathcal{G}$'s arrows $e \longrightarrow e'$ are the triples (e, x, e') where $x \in X$, and e' is the disjoint union of e and the singleton $\{x\}$,
2. $f \rhd g$ when $f = v; u'$ and $g = u; v'$ with $u = (e, x, e_1)$, $v = (e, y, e_2)$, $u' = (e_2, x, e')$, $v' = (e_1, y, e')$ and $x \neq y$.

The category $[\mathcal{C}]$ corresponding to $(E, X)_{\mathrm{ARS}}$ is an order category isomorphic to the slice category $[\mathcal{C}]^\emptyset$. So, the theorem 5.4 tells here that a continuous function $[\![-]\!]$ from (E, X) to (F, Y) is stable (in the usual sense) if and only if $[\![-]\!]$ verifies [stable] as a functor on $(E, X)_{\mathrm{ARS}}$, or put another way, if and only if

$$[\![e]\!] = [\![e \cup x]\!] \cap [\![e \cup y]\!]$$

for every $e \in E$ and every two elements $x \neq y$ of $X \setminus e$ such that $e \cup \{x, y\} \in E$.

Example 3. Let $\Sigma = \{\bot \longrightarrow \top\}$ be the Sierpinsky category. Interpretation functors $[\![-]\!] : \mathcal{C} \longrightarrow \Sigma$ are in one-to-one correspondence with reduction-closed open-stable sets on $(\mathcal{G}, \rhd)$. In fact, the stability of $[\![-]\!]^M$ implies the existence of a poly-reflectivity cone $(e_i : M \longrightarrow V_i)_{i \in I}$ from M to the inverse image of $\top$ — hence the fundamental theorem 4.9 in the case of reduction-closed open-stable sets.

6 Related works and open problems

A result related to ours appears in [PSS] where a class of monotone input/output automata is studied and shown to compute stable functions when it verifies a *local* stability property. The result can be interpreted in the light of theorem 4.9 by *projecting* the universal cone $(e_i : M \longrightarrow V_i)_{i \in I}$ to the subautomata of non-input events. The projected cone is in general not universal (see example 7 in Section 2) but in the case of deterministic systems like [PSS]'s, it is.

Other "linear" systems with intrinsic stability are Mazurkiewicz trace languages and asynchronous transition systems, see [WN]. In fact, stability of such true concurrency models is behind their interpretation as event structures.

To conclude, let us mention [GK] where a theory of needed computations is developed in a conflict-free duplicating abstract framework. [GK] contains the interesting idea of describing sets of head-normal forms from their frontier property, and we reuse the idea in Section 2 with our notion of open-stable set.

We list a few open problems:

1. the fundamental theorem offers a new perspective for the analysis of normalisation properties, see for instance [M, 2] and its treatment of the $\lambda\sigma$-calculus. A connection with the existing normalisation results on well-balanced [To] and almost orthogonal [R] rewriting systems would clarify the field.
2. our axiomatics is at an early stage of development. In particular graph rewriting systems like DAGs (directed acyclic graph) are not described directly but using their translation as trees, see [CK, M] for details. Also, associative-

commutative rules are not modelled as they should be, as isomorphisms in the Lévy category.

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References

[AC] M. Abadi, L. Cardelli, "*A Theory of Objects*", Monographs in Computer Science, David Gries and Fred B. Schneider Editors, Springer-Verlag, 1996.

[ACCL] M. Abadi, L. Cardelli, P.-L. Curien, J.-J. Lévy. "Explicit substitutions". *Journal of Functional Programming*, 1(4):375–416, 1991.

[Ba] H. Barendregt, "*The Lambda Calculus: Its Syntax and Semantics*". North Holland, 1985.

[Be] G. Berry, "Stable models of typed λ-calculi", in C. Böhm, G. Ausiello, eds, Third colloquium on automata, languages and programming (Udine, 1978). Lecture Notes in Computer Science 62, Springer, Berlin, 1978.

[BK] J. Bergstra, J. W. Klop, "Conditional rewrite rules", *Journal of Computer and System Science*, 32:323-362, 1986.

[Bo] G. Boudol, "*Computational semantics of term rewriting systems*". Algebraic methods in Semantics, Maurice Nivat and John C. Reynolds (eds), Cambridge University Press, 1985.

[BW] M. Barr, C. Wells, "*Toposes, Triples and Theories*", Grundlehren der mathematischen Wissenschaft 278, Springer-Verlag, 1985.

[CDJK] H. Comon, M. Dincbas, J.-P. Jouannaud, C. Kirchner. "A Methodological View of Constraint Solving", unpublished, 1995.

[CK] D. Clark and R. Kennaway, "Event structures and non-orthogonal term graph rewriting". in *Mathematical Structures in Computer Science*, vol. 6, pp. 545-578, 1996.

[D] Y. Diers, "Multimonads and multimonadic categories", *Journal of Pure and Applied Algebra 17*, pp. 153-170, 1980.

[FH] M. Felleisen, R. Hieb, "The revised report on the syntactic theories of sequential control and state". *Theoretical Computer Science*, 1992.

[FK] P.J. Freyd, G.M. Kelly, "Categories of continuous functors, I". *Journal of Pure and Applied Algebra* 2, pp 169—191, 1972.

[G] J.-Y. Girard, "The system F of variable types, fifteen years later", *Theoretical Computer Science 45*, 1985.

[GK] J. Glauert, Z. Khasidashvili, "Relative normalisation in deterministic residual structures", Proceedings of CAAP'96, 1996.

[GLM] G. Gonthier, J.-J. Lévy, P.-A. Melliès, "An abstract standardisation theorem". Symposium on Logic In Computer Science, August 1992.

[HL] G. Huet, J.-J. Lévy, "*Computations in orthogonal rewriting systems*". In J.-L. Lassez and G. D. Plotkin, editors, *Computational Logic; Essays in Honor of Alan Robinson*, pages 394–443. MIT Press, 1991.

[K] J.W. Klop, "*Combinatory Reduction Systems*". Volume 127 of *Mathematical Centre Tracts*, CWI, Amsterdam, 1980. PhD thesis.

[La] F. Lamarche, "*Modelling polymorphisms with categories*", Ph.D thesis, McGill University, 1988.

[Lé] J.-J. Lévy, "Réductions correctes et optimales dans le λ-calcul". Thèse de Doctorat d'Etat, Université Paris VII, 1978.

[Mc] S. Mac Lane, "*Categories for the working mathematician*", Springer Verlag, 1971.

[M] P.-A. Melliès, "Description abstraite des systèmes de réécriture", Thèse de l'Université Paris VII, December 1996.

[1] P.-A. Melliès, "Axiomatic Rewriting Theory I: An axiomatic standardisation theorem", in preparation.

[2] P.-A. Melliès, "Axiomatic Rewriting Theory II: The lambda-sigma-calculus has the finite cone property", presented at the School on Rewriting and Type Theory, Glasgow, September 1996.

[3] P.-A. Melliès, "Axiomatic Rewriting Theory III: A factorisation theorem in Rewriting Theory", Symposium on Category Theory and Computer Science, September 1997.

[Pl] G. Plotkin, "Call-by-name, call-by-value, and the λ-calculus", *Theoretical Computer Science 1*, 1975.

[Po] A. J. Power, "A unified approach to the lifting of adjoints", *Cahiers de topologie et géométrie différentielle catégoriques*, vol. XXIX-1, 1988.

[PSS] P. Panangaden, V. Shanbhogue, E. W. Stark, "Stability and sequentiality in dataflow networks", *Proceedings of the 17th International Colloquium on Automata Languages and Programming*, Warwick, 1990, pp 181-194, Lecture Notes in Computer Science 443.

[R] F. van Raamsdonk, "*Confluence and Normalisation for Higher-Order Rewriting*", PhD thesis, Vrije Universiteit van Amsterdam, 1996.

[S] E.W. Stark, "Concurrent transition systems", *Theoretical Computer Science 64*, May 1989.

[T] P. Taylor, "The trace factorisation of stable functors", paper available at `http://www.dcs.qmw.ac.uk/ pt/`, 1988.

[To] Y. Toyama, "Strong sequentiality of left-linear overlapping term rewriting systems", Symposium on Logic In Computer Science, August 1992.

[WN] G. Winskel, M. Nielsen, "Models for concurrency", in S. Abramsky et al, eds, *Handbook of Logic in Computer Science*, vol. 4, Semantic modelling, Oxford University Press, 1995.

7 Appendix

Proof of lemma 4.8

We first characterise starts and stops as crossing and co-crossing redexes.

CROSS. A redex $u : M \longrightarrow P$ crosses a path $e : M \longrightarrow N$ to a redex $v : N \longrightarrow P$ when:

1. $u = v$ and $e = \mathbf{id}_M$,
2. $e = r; e'$ and $\xrightarrow{u}\xrightarrow{r'} \simeq \xrightarrow{r}\xrightarrow{u'}$ and u' crosses e' to v.

CO-CROSS. A redex $u : Q \longrightarrow N$ co-crosses a path $e : M \longrightarrow N$ to a redex $v : P \longrightarrow M$ when:

1. $u = v$ and $e = \mathbf{id}_P$,
2. $e = e'; r$ and $\xrightarrow{r'}\xrightarrow{v} \simeq \xrightarrow{v'}\xrightarrow{r}$ and v' co-crosses e' to u.

Lemma 7.1 ([1]) *A redex $u : M \longrightarrow P$ starts a path $d : M \longrightarrow N$ if and only if d factors as $d = d_1; v; d_2$ and u crosses d_1 to v.*

Lemma 7.2 ([1]) *A redex $u : Q \longrightarrow N$ stops a path $d : M \longrightarrow N$ if and only if d factors as $d = d_1; v; d_2$ and u co-crosses d_2 to v.*

Lemma 7.3 *Suppose that V_1, V_2 and W are $\mathcal{V}$-values and that a path $e : V_1 \longrightarrow V_2$ and a redex $v : W \longrightarrow V_2$. If v co-crosses e to a redex $u : M \longrightarrow V_1$, then $M \in \mathcal{V}$.*

PROOF Proceed as in lemma 4.6, by induction on the length of e. ■

PROOF **(of lemma 4.8)** By induction on the couple (n_1, n_2) lexicographically ordered, where n_1 and n_2 are the lengths of e_1 and e_2. Let u be the first reduction step of e_1 so that $e_1 = u; E_1$ for some path E_1. Note that h_1 and h_2 can be chosen standard (take $\Downarrow_{h_1}$ and $\Downarrow_{h_2}$) so that $e_1; h_1 \simeq e_2; h_2$. With this choice u starts $e_2; h_2$, and by lemma 7.1 one of two following cases occurs:

$$
\begin{tikzcd}
M \arrow[r, "u"] \arrow[dr, bend right, "e_2"'] & M' \arrow[rr, "E_1"] \arrow[d, "E_2"] \arrow[dl, phantom, "\simeq"] & & V_1 \arrow[d, "h_1"] \\
& V_2 \arrow[rr, "h_2"] & \simeq & N
\end{tikzcd}
$$

Figure 3. Case A. in lemma 4.8.

A. either u starts e_2, hence $M \xrightarrow{u} M' \xrightarrow{E_2} V_2 \simeq M \xrightarrow{e_2} V_2$ for some path E_2,

B. or u crosses e_2 to a redex v which starts h_2. In particular $M \xrightarrow{u} M' \xrightarrow{E_2} W_2 \simeq M \xrightarrow{e_2} V_2 \xrightarrow{v} W_2$ and $V_2 \xrightarrow{v} W_2 \xrightarrow{H_2} N \simeq V_2 \xrightarrow{h_2} N$ for some paths E_2 and H_2.

If case A. occurs, $E_1 : M' \longrightarrow V_1$ and $E_2 : M' \longrightarrow V_2$ are external by right-decomposition of external paths, see figure 1(B). Moreover, $E_1; h_1 \simeq E_2; h_2$ by figure 1(D) and $u; E_1; h_1 \simeq u; E_2; h_2$. By induction hypothesis, there is a path $e' : M' \longrightarrow V$ to some $V \in \mathcal{V}$ and paths $i_1 : V \longrightarrow V_1$ and $i_2 : V \longrightarrow V_2$ such that $e'; i_1 \simeq E_1$ and $e'; i_2 \simeq E_2$. We conclude that $e = u; e'$ verifies the properties $e; i_1 \simeq e_1$ and $e; i_2 \simeq e_2$.

In case B., observe before anything else that $u; E_2; H_2 \simeq u; E_1; h_1$ implies with figure 1(D) that $E_2; H_2 \simeq E_1; h_1$. Now, suppose that there exists a term $W \in \mathcal{V}$ so that e_2 factors as $e_2 \simeq f; g$ where $M \xrightarrow{f} W \xrightarrow{g} V_2$ and g is non empty. We call this case B1. By the factorisation theorem, f factors as $f \equiv e_f; m_f$, with e_f external and m_f internal. Note that the path f is standard because e_2 is standard. The series of equivalence

$$f \simeq \Downarrow_f \simeq \Downarrow_{e_f; m_f} \simeq e_f; \Downarrow_{m_f}$$

shows that the length of e_f is less than the length of f, hence strictly less than the length of e_2. Moreover, the target of $e_f : M \longrightarrow W_2$ is a term $W_2 \in \mathcal{V}$, and if h_f denotes $\Downarrow_{m_f}; g; h_2$, then $e_f; h_f$ is $\simeq$-equivalent to $e_2; h_2$, hence to $e_1; h_1$. By induction hypothesis applied to $e_1 : M \longrightarrow V_1$ and $e_f : M \longrightarrow W_2$, there

$$\begin{tikzcd}
M \arrow[r, "u"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\simeq"] & M' \arrow[r, "E_1"] \arrow[d, "E_2"] \arrow[dr, phantom, "\simeq"] & V_1 \arrow[d, "h_1"] \\
V_2 \arrow[r, "v"] & W_2 \arrow[r, "H_2"] & N
\end{tikzcd}$$

Figure 4. Case B. in lemma 4.8.

$$\begin{tikzcd}
 & M \arrow[r, "u"] \arrow[dl, bend right, "F"'] \arrow[dd, "e_2"] & M' \arrow[dr, bend left, "E"] \arrow[dd, "E_2"] & \\
W_F \arrow[dr, bend right, "G"'] \arrow[r, phantom, "\simeq"] & {} \arrow[r, phantom, "\simeq"] & {} \arrow[r, phantom, "\simeq"] & W \arrow[dl, bend left, "j_2"] \\
 & V_2 \arrow[r, "v"] & W_2 &
\end{tikzcd}$$

Figure 5. Case B2. in lemma 4.8.

exists a path $E_f : M \longrightarrow W_f$ to $W_f \in \mathcal{V}$ and two paths $j_1 : W_f \longrightarrow V_1$ and $j_2 : W_f \longrightarrow W_2$ such that $E_f; j_1 = e_1$ and $E_f; j_2 = e_f$. We conclude that $e = E_f$, $i_1 = j_1$ and $i_2 = j_2; \Downarrow_{m_f}; g$ verify the required properties $e; i_1 \simeq e_1$ and $e; i_2 \simeq e_2$.

We still have to consider case B2. when $f; g \simeq e_2$ and g non empty implies that f's target is not in $\mathcal{V}$. Now, let $w : V_3 \longrightarrow V_2$ be the last reduction step of e_2. By [closed], either $V_3 \in \mathcal{V}$ or $W_2 \in \mathcal{V}$. The first case has been treated in case B1 and we treat the second case here. First of all, the path E_2 is standard because $u; E_2; H_2 \simeq e_2; h_2$ and $e_2; h_2$ is standard, but E_2 is not necessarily external. By the factorisation theorem E_2 factors as $E_2 \equiv E_3; M_3$ where $E_3 : M' \longrightarrow W_3$ is external and $M_3 : W_3 \longrightarrow W_2$ is internal. By lemma 4.7, $W_3 \in \mathcal{V}$. By the series of equivalence

$$\begin{array}{lll}
 & E_3; \Downarrow_{M_3; H_2} & \\
\simeq & \Downarrow_{E_3; M_3; H_2} & \text{because } E_3 \text{ is external,} \\
\simeq & \Downarrow_{E_2; H_2} & \text{because } E_3; M_3 \equiv E_2, \\
\simeq & \Downarrow_{E_1; h_1} & \text{because } E_2; H_2 \equiv E_1; h_1, \\
\simeq & E_1; \Downarrow_{h_1} & \text{because } E_1 \text{ is external.}
\end{array}$$

the induction hypothesis can be applied on $E_1 : M' \longrightarrow V_1$ and $E_3 : M' \longrightarrow W_3$ to show that there exists a path $E : M' \longrightarrow W$ to $W \in \mathcal{V}$ and paths $J_1 : W \longrightarrow V_1$ and $J_2 : W \longrightarrow W_3$ such that $E; J_1 \simeq E_1$ and $E; J_2 \simeq E_3$. If $j_1 = J_1$ and $j_2 = J_2; \Downarrow_{M_3}$, we deduce that $E; j_1 \simeq E_1$ and $E; j_2 \simeq E_2$ with the series of equivalence:

$$\begin{array}{lll}
 & E; j_2 & \\
= & E_3; \Downarrow_{M_3} & \text{because } E; J_2 = E_3, \\
\simeq & \Downarrow_{E_3; M_3} & \text{because } E_3 \text{ is external,} \\
\simeq & E_2 & \text{because } E_2 \text{ is standard.}
\end{array}$$

Note that the redex v stops $u; E_2 \simeq u; E; j_2$, hence by lemma 7.2 v co-crosses j_2 to some redex $w : W_F \longrightarrow W$ and this redex w co-crosses E to the redex u. The definition of co-crossing implies the existence of paths F and G such that u crosses F to w and w crosses G to v — see figure 5. Here, $W_F \in \mathcal{V}$ because $V_2 \in \mathcal{V}$, $W_2 \in \mathcal{V}$ and $W \in \mathcal{V}$, see lemma 7.3. By figure 1(D) and the series of equivalence

$$F; G; v \simeq F; w; j_2 \simeq u; E; j_2 \simeq u; E_2 \simeq e_2; v$$

we deduce $F; G \simeq e_2$. We are in case B2. and $W_F \in \mathcal{V}$, hence G must be empty, therefore j_2 is empty too, and consequently $E \simeq E_2$. We conclude that $e = e_2$, $i_1 = v; j_1$ and $i_2 = \mathbf{id}_{V_2}$ verify $e; i_1 \simeq e_1$ and $e; i_2 \simeq e_2$. This concludes the proof. ■

Proof of theorem 4.9

Proof We show that the cone $(e_i : M \longrightarrow V_i)_{i \in I}$ of all minimal paths from M to $\mathcal{V}$ verifies the enunciated property. Suppose that $f : M \longrightarrow V$ is a path to a term $V \in \mathcal{V}$. We apply the factorisation theorem to factor f as $f \equiv e; m$ so that, by lemma 4.7, $e : M \longrightarrow W$ transforms M into a term $W \in \mathcal{V}$. The path e is external. Among the external paths $E : M \longrightarrow P$ such that $E; F \simeq e$ with a target P in $\mathcal{V}$, there is a path whose length is minimal. By construction, this path is minimal and therefore a member of the cone. Let $e_i : M \longrightarrow V_i$ be this path. The path f factors as $f \equiv e_i; h$ for some path h.

To show the unicity of i let $j \in I$ be another indice such that f factors as $f \equiv e_j; h'$. Lemma 4.8 tells us that there exists a path $E : M \longrightarrow W$ to $W \in \mathcal{V}$, and paths $k_1 : W \longrightarrow V_i$, $k_2 : W \longrightarrow V_j$ such that $E; k_1 \simeq e_i$ and $E; k_2 \simeq e_j$. By factorisation, E factors as $e_E; m_E$ where e_E is external and m_E is internal. By lemma 4.7, e_E's target is in $\mathcal{V}$. Also, $e_E; m_E; k_1 \equiv e_i$ and $e_E; m_E; k_2 \equiv e_j$ implies that $e_E; \Downarrow_{m_E; k_1} \simeq e_i$ and $e_E; \Downarrow_{m_E; k_2} \simeq e_j$. By minimality of e_i and e_j, $e_E \simeq e_i$ and $e_E \simeq e_j$. Hence $i = j$.

We only have to check that h is unique up to $\equiv$. Suppose that $e_i; h \equiv e_i; h'$. Then $e_i; \Downarrow_h \simeq e_i; \Downarrow_{h'}$ because e_i is external, hence $\Downarrow_h \simeq \Downarrow_{h'}$ by figure 1(D). We conclude that $h \equiv \Downarrow_h \equiv \Downarrow_{h'} \equiv h'$. ■