# Braids described as an orthogonal rewriting system

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**Abstract.** We show that braids can be described as an orthogonal system in the sense of [1][2]. Their computation is therefore confluent, which was already shown by Garside [4].

## 1 An addition on binary relation

We propose a notion of addition on binary relations. Let $A$ be a binary relation on $X$. We note $A^{\neg}$ the negation of $A$, $A^{op}$ its reverse and $A^{\star} = (A^{op})^{\neg}$ its dual. Let $A$ and $B$ two binary relations on $X$. We define the addition $A + B$ as the relation:

$$A + B = (A \cap B^{\star}) \cup B$$

**Lemma 1 (associativity of $A + B$)** *Let $A$, $B$ and $C$ be three binary relations on $X$. Then $(A + B) + C = A + (B + C)$.*

Proof:

$(A + B) + C$
$= ((A + B) \cap C^{\star}) \cup C$
$= (((A \cap B^{\star}) \cup B) \cap C^{\star}) \cup C$
$= (A \cap B^{\star} \cap C^{\star}) \cup (B \cap C^{\star}) \cup C$
$= (A \cap B^{\star} \cap C^{\star}) \cup (B + C)$
$A + (B + C)$
$= (A \cap (B + C)^{\star}) \cup (B + C)$
$= (A \cap \ ((B \cap C^{\star}) \cup C)^{\star}\ ) \cup (B + C)$
$= (A \cap (B \cap C^{\star})^{\star} \cap C^{\star}) \cup (B + C)$
$= (A \cap (B^{\star} \cup C) \cap C^{\star}) \cup (B + C)$
$= (A \cap B^{\star} \cap C^{\star}) \cup (A \cap C \cap C^{\star}) \cup (B + C)$

We have to check that $(A \cap C \cap C^{\star}) \subset (A \cap B^{\star} \cap C^{\star}) \cup (B + C)$ which comes from $C \subset (B + C)$. $\square$

We remark that $A + B$ may be computed to $(A \cup B) \cap (A \cup A^{\star})$. We define $\widehat{A} = A \cup A^{\star}$.

**Lemma 2 (commutativity)** *Let $A$ and $B$ be two binary relations on $X$. Then: $A + B = B + A$ if and only if ($A \subset \widehat{B}$ and $B \subset \widehat{A}$).*

Let $A$ and $B$ be two binary relations on $X$. We know that $A \subset B + A$ and $B \subset A + B$. More, $A + B$ can be computed in $(A \cup B) \cap (B \cup B^{\star})$, which implies that $A+B \subset A \cup B$ and by symetry: $B+A \subset A \cup B$. The four relations imply that $A+B = B+A$ if and only if $A+B = B+A = A \cup B$. This last property may be translated as follow: $A+B = (A \cup B) \cap \widehat{B} = A \cup B$ and $B+A = (A \cup B) \cap \widehat{A} = A \cup B$, which is equivalent as saying that $A \cup B \subset \widehat{A}$ and $A \cup B \subset \widehat{B}$. We have $A \subset \widehat{A}$ and $B \subset \widehat{B}$ in any case. We obtain:

$$A+B = B+A \iff (A \subset \widehat{B} \text{ and } B \subset \widehat{A})$$

□

**Corollary 1.** *We give some properties of $A+B$:*

- $A+A = A$,
- $B+A+B = A+B$,
- $A+A^{op} = A^{op}$,
- $(A \cup B)+C = (A+C) \cup (B+C)$,
- $(A \cap B)+C = (A+C) \cap (B+C)$.

Proof:

- $A+A = (A \cup A) \cap \widehat{A}$ with $A \subset \widehat{A}$, which shows that $A+A = A$,
- we show that $(A+B)$ and $B$ commute. First, $B \subset (A+B) \subset (\widehat{A+B})$. Second, $(A+B) = (A \cup B) \cap \widehat{B} \subset \widehat{B}$. The commutativity lemma shows that $B+(A+B) = (A+B)+B$. The associativity and $B+B = B$ terminate the proof: $B+(A+B) = A+B+B = A+B$,
- $A+A^{op} = (A \cup (A^{op})^{\star}) \cup A^{op} = (A \cup A^{\neg}) \cup A^{op} = A^{op}$.

□

## 2 Anti-symetric binary relations

Let $\Delta$ be the reflexive relation: $x\Delta x$. Then $\Delta^{op} = \Delta$. A binary relation $A$ is anti-when symetric $A \cap A^{op} \subset \Delta$, and strict when $A \cap \Delta = \emptyset$.

We define $[A,B] = B \cap A^{op}$, called the distance from $A$ to $B$.

**Theorem 1 (on distance and strict anti-symetric relations)** *Let $A$ and $U$ be two strict and anti-symetric binary relations. Then $A+U$ is also strict and anti-symetric, and $[A, A+U] = U \cap A^{op}$.*

Proof:

- $A+U = (A \cap U^{\star}) \cup U$ is strict because $U$ and $A$ are strict,
- to prove anti-symetry requires some computation. We use the definitions of $A^{\neg}$, $A^{\star}$, and the fact that $U \cap U^{op} = A \cap A^{op} = \emptyset$.
  $(A+U) \cap (A+U)^{op}$
  $= (A \cup U) \cap (A^{op} \cup U^{op}) \cap (U \cup U^{\star}) \cap (U \cup U^{\star})^{op}$
  $= ((A \cap A^{op}) \cup (A \cap U^{op}) \cup (U \cap A^{op}) \cup (U \cap U^{op})) \cap ((U \cup U^{\star}) \cap (U^{op} \cup U^{\neg}))$
  $= ((A \cap U^{op}) \cup (U \cap A^{op})) \cap ((U \cap U^{op}) \cup (U \cap U^{\neg}) \cup (U^{\star} \cap U^{op}) \cup (U^{\star} \cap U^{\neg}))$
  $= ((A \cap U^{op}) \cup (U \cap A^{op})) \cap (U^{\star} \cap U^{\neg})$
  $= ((A \cap U^{op}) \cup (U \cap A^{op})) \cap (U \cup U^{op})^{\neg}$

We know that $((A \cap U^{op}) \cup (U \cap A^{op})) \subset (U \cup U^{op})$, which means that $(A+U) \cap (A+U)^{op} = \emptyset$. The binary relation $(A+U)$ is anti-symetric.
- we compute $[A, A+U]$ into $U \cap A^{op}$:
  $[A, A+U]$
  $= (A+U) \cap A^{op}$
  $= ((A \cap U^{\star}) \cup U) \cap A^{op}$
  $= (A \cap U^{\star} \cap A^{op}) \cup (U \cap A^{op})$
  $= U \cap A^{op}$

□

## 3 total binary relations

We call $\top = X \times X$. $\top$ verifies $\top^{op} = \top$. A binary relation $A$ is total when $A \cup A^{op} \cup \Delta = \top$. A binary relation $A$ is an order when it is transitive and anti-symetric.

**Theorem 2 (on total binary relations)** *Let $A$ and $U$ be two binary relations, and $A$ total. Then $A+U$ is total.*

Proof:
$(A+U) \cup (A+U)^{op} \cup \Delta$
$= ((A \cup U) \cap \widehat{U}) \cup ((A \cup U) \cap \widehat{U})^{op} \cup \Delta$
$= ((A \cup U) \cap \widehat{U}) \cup ((A^{op} \cup U^{op}) \cap (U^{op} \cup U^{\neg})) \cup \Delta$
$= \{(A \cup U \cup A^{op} \cup U^{op}) \cap (A \cup U \cup U^{op} \cup U^{\neg})$
$\cap(\widehat{U} \cup A^{op} \cup U^{op}) \cap (\widehat{U} \cup U^{op} \cup U^{\neg})\} \cup \Delta$
$= \{(\widehat{U} \cup A^{op} \cup U^{op}) \cap (\widehat{U} \cup U^{op} \cup U^{\neg})\} \cup \Delta$
$= \{(U \cup U^{\star} \cup A^{op} \cup U^{op}) \cap (U \cup U^{\star} \cup U^{op} \cup U^{\neg})\} \cup \Delta$
$= \top$

**Lemma 3** *Let $A$ be a strict anti-symetric and total binary relation.*

- $A^{\neg} = A^{op} \cup \Delta$,
- $\widehat{A} = A^{\star} = A \cup \Delta$,

We can give a reverse property to the theorem 1 in the case of total binary relations.

**Theorem 3 (on distance and strict orders)** *Let $A$ and $B$ be two strict, anti-symetric and total binary relations. Then $[A, B] \subset A^{op}$ and $A + [A, B] = B$*

Proof:

- $[A, B] = B \cap A^{op}$,
- we compute $A + [A, B]$:
  $A + [A, B]$
  $= (A \cap (B \cap A^{op})^{\star}) \cup (B \cap A^{op})$
  $= (A \cap (B^{\star} \cup A^{\neg})) \cup (B \cap A^{op})$
  $= (A \cap B^{\star}) \cup (B \cap A^{op})$

The relation $B$ is strict and anti-symetric, and therefore $B^\star = B \cup \Delta$.
$A + [A, B]$
$= (A \cap B) \cup (A \cap \Delta) \cup (B \cap A^{op})$
$= (B \cap A) \cup (B \cap A^{op}) \cup (B \cap \Delta)$
$= B$
since $A \cap \Delta = B \cap \Delta = \emptyset$ and $A \cup A^{op} \cup \Delta = \top$.

$\square$

## 4 Transitive binary relations

We write $A \cdot B$ the product of $A$ and $B$. A binary relation $A$ is transitive when $A \cdot A \subset A$. A couple $(A, B)$ is $\cup$-transitive when:

$$\begin{aligned} A \cdot A &\subset A \cup B \\ B \cdot B &\subset A \cup B \\ A \cdot B &\subset A \cup B \\ B \cdot A &\subset A \cup B \end{aligned}$$

A couple $(A, B)$ is $\cap$-transitive when:

$$\begin{aligned} (A \cap B) \cdot (A \cap B) &\subset A \\ (A \cap B) \cdot (A \cap B) &\subset B \end{aligned}$$

**Lemma 4** *$A \cup B$ is transitive if and only if $(A, B)$ is $\cup$-transitive. $A \cap B$ is transitive if and only if $(A, B)$ is $\cap$-transitive.*

**Lemma 5** *Let $A$ be transitive. Then $A \cdot A^\star \subset A^\star$ and $A^\star \cdot A \subset A^\star$. Moreover, $A \cdot \widehat{A} \subset \widehat{A}$ and $\widehat{A} \cdot A \subset \widehat{A}$.*

Proof:

- transitivity gives ($zAx$ and $xAy \Rightarrow zAy$), which means that ($xAy$ and $zA^\neg y \Rightarrow zA^\neg x$), or ($xAy$ and $yA^\star z \Rightarrow xA^\star z$).
- we prove the second relation the same way: ($yAz$ and $zAx \Rightarrow yAx$) means ($yAz$ and $yA^\neg x \Rightarrow zA^\neg x$), or ($yAz$ and $xA^\star y \Rightarrow xA^\star z$).

$\square$

**Theorem 4 (on transitive relations)** *Let $A$ be a strict total order and $B$ be a transitive binary relation. Then $A + B$ is transitive if and only if $A \cap B^\star$ is transitive.*

Proof: the relation $A + B = (A \cap B^\star) \cup B$ is transitive if and only if:

$$\begin{cases} B \cdot B \subset (A \cap B^\star) \cup B \\ (A \cap B^\star) \cdot (A \cap B^\star) \subset (A \cap B^\star) \cup B \\ B \cdot (A \cap B^\star) \subset (A \cap B^\star) \cup B = (A \cup B) \cap \widehat{B} \\ (A \cap B^\star) \cdot B \subset (A \cap B^\star) \cup B = (A \cup B) \cap \widehat{B} \end{cases}$$

The first assertion is always true. We use the fact that $B \cdot B^\star \subset \widehat{B}$ and $B^\star \cdot B \subset \widehat{B}$ to show that the three other assertions are equivalent to the following ones:

$$\begin{cases} (A \cap B^\star) \cdot (A \cap B^\star) \subset (A \cap B^\star) \cup B \\ B \cdot (A \cap B^\star) \subset A \cup B \\ (A \cap B^\star) \cdot B \subset A \cup B \end{cases}$$

We translate the second relation in:

$$xBy \text{ and } yAz \Rightarrow xAz \text{ or } xBz \text{ or } zBy$$

which is equivalent to:

$$xBy \text{ and } yAz \text{ and } xA^\neg z \Rightarrow xBz \text{ or } zBy$$

The relation $A$ is total and strict and anti-reflexive. The assertion above is then equivalent to:

$$xBy \text{ and } yAz \text{ and } (zAx \text{ or } z\Delta x) \Rightarrow xBz \text{ or } zBy$$

If ($xBy$ and $yAz$ and $z\Delta x$) then $zBy$ by $\Delta \cdot B = B$. The relation is therefore equivalent to:

$$yAzAx \text{ and } xBy \Rightarrow xBz \text{ or } zBy \qquad (*)$$

We use this result to compute the third assertion:

$$(A \cap B^\star) \cdot B \subset A \cup B$$

is equivalent to

$$B^{op} \cdot (A \cap B^\star)^{op} \subset A^{op} \cup B^{op}$$

which we compute to:

$$B^{op} \cdot (A^{op} \cap (B^{op})^\star) \subset A^{op} \cup B^{op}$$

The relation $A^{op}$ is total and strict. The assertion above is thus equivalent to:

$$yA^{op}zA^{op}x \text{ and } xB^{op}y \Rightarrow xB^{op}z \text{ or } zB^{op}y$$

or equivalently to:

$$xAzAy \text{ and } yBx \Rightarrow yBz \text{ or } zBx$$

which is the *same* assertion as $(*)$ with $x$ and $y$ inter-changed.

The three assertions above are therefore equivalent to:

$$\begin{cases} (A \cap B^\star) \cdot (A \cap B^\star) \subset (A \cap B^\star) \cup B \\ \forall (x, y, z) \in X^3,\ (xAyAz \text{ and } zBx \Rightarrow zBy \text{ or } yBx) \end{cases}$$

The second assertion here is equivalent for $(x, y, z)$ to:

$$xAyAz \text{ and } xB^\star yB^\star z \Rightarrow xB^\star z$$

which means that $(A\cap B^{\star})\cdot(A\cap B^{\star})\subset B^{\star}$ and is equivalent to $A\cap B^{\star}$ transitive, since $A$ is transitive:

$$(A\cap B^{\star})\cdot(A\cap B^{\star})\subset(A\cap B^{\star})$$

This second assertion clearly implies the first one. This shows that $A+B$ is transitive if and only if $A\cap B^{\star}$ is transitive. □

We write $xA\otimes Bz$ when $\forall y\in X, xAy$ or $yBz$. A relation $A$ is scopic when $A\otimes A\supset A$.

**Lemma 6** *$A$ is scopic if and only if $A^{\neg}$ is transitive.*

**Corollary 2.** *Let $A$ be a total and strict order and $B$ transitive. Then $A+B$ is transitive if and only if $A\cup B\cup\Delta$ is scopic.*

Proof: we give a sequence of equivalent assertions:
$A+B$ is transitive
$A\cap B^{\star}$ is transitive
$A^{\neg}\cup B^{op}$ is scopic
$A^{op}\cup\Delta\cup B^{op}$ is scopic
$A\cup B\cup\Delta$ is scopic
□

**Corollary 3.** *Let $A$ be a strict total order and $B$ any binary relation. The following assertions are equivalent:*

- *$A+B$ is transitive*
- *$A\cap B^{\star}$ is transitive*
- *$A\cup B\cup\Delta$ is scopic*
- $(A\cap B^{\star})\cup\Delta\subset(A\cup B\cup\Delta)\otimes B^{\star}$
- $(A\cap B^{\star})\cup\Delta\subset B^{\star}\otimes(A\cup B\cup\Delta)$
- $B\subset(A\cup B\cup\Delta)\otimes(A\cup B\cup\Delta)$

Proof: everything comes from $A^{\star}=A\cup\Delta$ and a careful study of the theorem 4.
□

**Lemma 7** *Let $A$ and $B$ be two scopic relations. Then $A\cup B$ is scopic.*

**Lemma 8** *Let $A$ be a strict order and $U$, $V$ two binary relations such that $U\subset A^{op}$. If $A\cup U\cup\Delta$ and $A\cup V\cup\Delta$ are scopic then $A\cup(U+V)\cup\Delta$ is scopic.*

Proof: we use the fact that $U\subset A^{op}\subset V^{\star}$:
$A\cup(U+V)\cup\Delta$
$=A\cup V\cup(U\cap V^{\star}\cup\Delta$
$=A\cup V\cup U\cup\Delta$
$=(A\cup V\cup\Delta)\cup(A\cup U\cup\Delta)$
□

**Corollary 4.** *Let $A$ be a strict order and $U$, $V$ two binary relations such that $U\subset A^{op}$. If $A+U$ and $A+V$ are transitive then $(A+U+V)$ is transitive.*

Proof: immediate.

## 5 transitive closures

We write $\overline{A}$ the transitive closure of $A$.

**Lemma 9** *– if $A$ and $B$ are strict then $A \otimes B \subset A \cap B$*
*– if $\Delta \subset A$ and $\Delta \subset B$ then $A \cup B \subset A \cdot B$*

Proof: let $x(A \otimes B)y$: then $xAy$ or $yBy$ from the definition of $A \otimes B$ used on $y$. Therefore, $xAy$ since $B$ is strict. Similarly, $xAx$ or $xBy$, which means $xBy$ since $A$ is strict. The second assertion comes from the strictness of $A^{\neg}$ and $B^{\neg}$ if $\Delta \subset A$ and $\Delta \subset B$. We use the first assertion and obtain: $A^{\neg} \otimes B^{\neg} \subset A^{\neg} \cap B^{\neg}$, which means that $(A \cdot B)^{\neg} \subset (A \cap B)^{\neg}$. We obtain: $A \cdot B \supset A \cap B$. □

**Lemma 10** *– let $B$ be transitive. Then $(A \otimes B) \cdot B \subset (A \otimes B)$,*
*– let $A$ be transitive. Then $A \cdot (A \otimes B) \subset (A \otimes B)$.*

Proof: suppose that $B$ is transitive and $x(A \otimes B)yBz$. Then

$$\forall u \in X, \ xAu \text{ or } uBy$$

which leads to:

$$\forall u \in X, \ xAu \text{ or } uBz$$

with $yBz$. We obtain that $xA \otimes Bz$, which means that $(A \otimes B) \cdot B \subset (A \otimes B)$. This assertion applied to $B^{op}$ then gives the second assertion. □

**Lemma 11** *Let $A$ and $B$ be strict and transitive. Then $A \otimes B$ is strict and transitive.*

Proof: the relations $A$ and $B$ are strict, which gives $A \otimes B \subset A \cap B$. Then, $(A \otimes B) \cdot (A \otimes B) \subset (A \otimes B) \cdot B$ and $(A \otimes B) \cdot B \subset A \otimes B$ terminate the proof. The strictness comes from $A \otimes B \subset A \cup B$. □

**Corollary 5.** *Let $A$ and $B$ be two scopic relations such that $\Delta \subset A$ and $\Delta \subset B$. Then $A \cdot B$ is scopic. Furthermore, $\overline{A}$ is scopic.*

The lemma 11 can in fact be generalised:

**Lemma 12** *Let $A$ be anti-symetric and strict. If $A \cup B$ and $A \cup C$ are scopic with $\Delta \subset B$ and $\Delta \subset C$ then $A \cup (B \cdot C)$ is scopic.*

Proof: we want to show that $A \cup (B \cdot C) \subset (A \cup (B \cdot C) \otimes (A \cup (B \cdot C))$. First, $A \subset (A \cup B) \subset ((A \cup B) \otimes (A \cup B))$ and $((A \cup B) \otimes (A \cup B)) \subset ((A \cup (B \cdot C) \otimes (A \cup (B \cdot C)))$ since $A \cup B \subset A \cup (B \cdot C)$ from $\Delta \subset C$. Second, let $xByCz$. Then by hypothesis:

$$xAu \text{ or } xBu \text{ or } uAy \text{ or } uBy$$

and

$$yAu \text{ or } yCu \text{ or } uAz \text{ or } uCz$$

The two assertions induce the two following:

$$xAu \text{ or } xBuCu \text{ or } uAy \text{ or } uByCz$$

and

$$yAu \text{ or } xByCu \text{ or } uAz \text{ or } uBuCz$$

All cases except one fall under the assertion:

$$x(A \cup (B \cdot C))u \text{ or } u(A \cup (B \cdot C)z$$

The only remaininig case is $uAy$ in the first assertion and $yAu$ in the second one, which cannot happen at the same time since $A$ is a strict and anti-symetric relation. We conclude that $A \cup (B \cdot C)$ is scopic. $\square$

**Corollary 6.** *Let $A$ be anti-reflexive and strict. Let $B$ be a binary relation with $\Delta \subset B$. If $A \cup B$ is scopic then $A \cup \overline{B}$ is also scopic.*

Proof: $\overline{B} = \bigcup_{n>0}(B^n)$ where $B^n = B \cdots B$ $n$ times. We show using the lemma 12 that $A \cup B^n$ is scopic. Then $A \cup \overline{B} = \bigcup_{n>0}(A \cup B^n)$, which is a infinite union of scopic relations. If $x(\bigcup_{n>0}(A \cup B^n))y$, there exists a $k > 0$ such that $x(A \cup B^k)y$, which implies that for all $u \in X$, $x(A \cup B^k)u$ or $u(A \cup B^k)y$, and therefore: $x(\bigcup_{n>0}(A \cup B^n))u$ or $u(\bigcup_{n>0}(A \cup B^n))y$. Which means that $A \cup \overline{B}$ is scopic. $\square$

**Theorem 5 (on strict total orders)** *Let $A$ be a strict total order and $U$, $V$ two strict orders. If $A+U$ and $A+V$ are transitive then $A+(\overline{U+V})$ is transitive.*

Proof: we will show that if $A \cup U \cup \Delta$ and $A \cup V \cup \Delta$ are scopic then $A \cup (\overline{U \cup V}) \cup \Delta$ is also scopic. We use the fact that $\overline{U \cup V \cup \Delta} = \overline{U \cup V} \cup \Delta$. We need then to prove that $A \cup (U \cup V \cup \Delta)$ is scopic, and use the lemma 12. Let us write

$$A \cup (U \cup V \cup \Delta) = (A \cup U \cup \Delta) \cup (A \cup V \cup \Delta)$$

which shows that $A \cup (U \cup V \cup \Delta)$ is the union of the two scopic relations: $A \cup U \cup \Delta$ and $A \cup V \cup \Delta$, and is therefore scopic. $\square$

## 6 the rewriting system

The terms are the strict total orders $A$. A redex $U$ from $A$ is any couple $(U, A)$ such that:

- $U$ is transitive,
- $U \subset A^{op}$,
- $A \cup U \cup \Delta$ is scopic.

We write $U_A$ for $(U, A)$. The redex $U_A$ reduces $A$ into $A+U$, which is a strict total order.

We define the residual $U[V]$ by

$$U_A[V_A] = (\ (\overline{U+V}) \cap (A+V)^{op}\ ,\ A+V\ )$$

**Lemma 13 (permutation A)** *Let $U_A$ and $V_A$ be two redexes. Then $U + V = V + U = U \cup V$.*

Proof: $U \subset A^{op}$ and therefore $U \subset A^{op} \subset A^{\neg} \subset V^{\star}$. We have $U + V = (U \cap V^{\star}) \cup V = U \cup V$, which induces that $U + V = U \cup V = V \cup U = V + U$. $\square$

**Lemma 14 (permutation B)** $U + ((\overline{V + U}) \cap (A + U)^{op}) = V + ((\overline{U + V}) \cap (A + V)^{op}) = \overline{U \cup V}$

Proof:

$U + ((\overline{V + U}) \cap (A + U)^{op})$
$= (U \cap ((\overline{U \cup V})^{\star} \cup (A + U)^{\neg})) \cup (\overline{U \cup V} \cap (A + U)^{op})$
$= (U \cup (U \cap (A + U)^{\neg})) \cup (\overline{U \cup V} \cap (A + U)^{op})$
$= U \cup (\overline{U \cup V} \cap (A + U)^{op})$
$= \overline{U \cup V} \cap (U \cup (A + U)^{op})$
$= \overline{U \cup V} \cap (U \cup (A^{op} \cap U^{\neg}) \cup U^{op})$
$= \overline{U \cup V} \cap (U \cup A^{op} \cup U^{op})$
$= \overline{U \cup V}$

**Lemma 15** *$(\overline{U \cup V}, A)$ is a redex, and therefore $A + (\overline{U \cup V})$ is a total order.*

Proof: $(\overline{U \cup V})$ is an order and is contained in $A^{op}$. We use the theorem 5 to show that $A \cup (\overline{U \cup V}) \cup \Delta$ is scopic, from $A \cup U \cup \Delta$ and $A \cup V \cup \Delta$ scopic. $\square$

**Lemma 16** *The residual $U_A[V_A]$ is a redex.*

Proof: the relation $(\overline{U \cup V}) \cap (A + V)^{op}$ is transitive because $\overline{U \cup V}$ and $A + V$ are transitive. It is also a subset of $(A + V)^{op}$. We know that

$$(A + V) + ((\overline{U \cup V}) \cap (A + V)^{op})$$

is $A + (\overline{U \cup V})$, which is an order.

$$(A + V) \cup ((\overline{U \cup V}) \cap (A + V)^{op}) \cup \Delta$$

is therefore scopic, see the theorem 4. Nevertheless, we will prove that result directly:

$(A + V) \cup ((\overline{U \cup V}) \cap (A + V)^{op}) \cup \Delta$
$= (A + V \cup (\overline{U \cup V}) \cup \Delta) \cap (A + V \cup (A + V)^{op} \cup \Delta)$
$= A + V \cup (\overline{U \cup V}) \cup \Delta$
$= (A \cap V^{\star}) \cup (\overline{U \cup V \cup \Delta})$
$= (A \cup \overline{U \cup V \cup \Delta}) \cap V^{\star}$

We want to show that:

$$(A \cup \overline{U \cup V \cup \Delta}) \cap V^{\star} \qquad (\bullet)$$

is scopic. The relation $V^{\star} = (V^{op})^{\neg}$ is scopic. $A \cup \overline{U \cup V \cup \Delta}$ is also scopic because $A \cup (U \cup V \cup \Delta)$ is scopic and theorem 5. Therefore, the relation $(\bullet)$ is transitive if and only if:

$$\begin{cases} (A \cup \overline{U \cup V \cup \Delta}) \cap V^\star \subset (A \cup \overline{U \cup V \cup \Delta}) \otimes V^\star \\ (A \cup \overline{U \cup V \cup \Delta}) \cap V^\star \subset V^\star \otimes (A \cup \overline{U \cup V \cup \Delta}) \end{cases}$$

which may be translated in:

$$\begin{cases} A \cap V^\star \subset (A \cup \overline{U \cup V \cup \Delta}) \otimes V^\star & (1) \\ (\overline{U \cup V \cup \Delta}) \cap V^\star \subset (A \cup \overline{U \cup V \cup \Delta}) \otimes V^\star & (2) \\ A \cap V^\star \subset V^\star \otimes (A \cup \overline{U \cup V \cup \Delta}) & (3) \\ (\overline{U \cup V \cup \Delta}) \cap V^\star \subset V^\star \otimes (A \cup \overline{U \cup V \cup \Delta}) & (4) \end{cases}$$

The assertions (1) and (3) come from the corollary 3 since $(V \cup \Delta) \subset (\overline{U \cup V \cup \Delta})$.

We show by transitivity that

$$(\overline{U \cup V \cup \Delta}) \subset (\overline{U \cup V \cup \Delta}) \otimes V^\star$$

is true. The assertion means the following:

$$x(\overline{U \cup V \cup \Delta})z \text{ and } zVy \Rightarrow x(\overline{U \cup V \cup \Delta})y$$

The property implies the assertion (2).

Similarly, the assertion (3) is deduced from:

$$(\overline{U \cup V \cup \Delta}) \subset V^\star \otimes (\overline{U \cup V \cup \Delta})$$

which means

$$x(\overline{U \cup V \cup \Delta})z \text{ and } yVx \Rightarrow y(\overline{U \cup V \cup \Delta})z$$

□

**Lemma 17** $U_A[V_A] = (\ (\overline{U \cup V}) \cap U^\neg\ ,\ A + U)$

Proof:
$(\overline{U + V}) \cap (A + V)^{op}$
$= (\overline{U + V}) \cap (A^{op} \cup V^{op}) \cap (\widehat{V})^{op}$
$= (\overline{U + V}) \cap (V^{op} \cup V^\neg)$
$= (\overline{U + V}) \cap V^\neg$
because $(\overline{U + V}) \subset A^{op}$ and $V^{op} \subset A$. □

**Theorem 6 (orthogonality)** *Let $U_A$, $V_A$ be two redexes of $A$. Then $U_A \cdot V_A[U_A]$ and $V_A \cdot U_A[V_A]$ terminate on the same term and induce the same residual relation.*

Proof: Let $W_A$ be a redex of $A$. Then
$W_A[U_A][V_A[U_A]]$
$= ((\overline{W \cup U}) \cap U^\neg, A + U)[(\overline{V \cup U}) \cap U^\neg, A + U]$
$= ((\overline{((\overline{W \cup U}) \cap U^\neg) \cup (\overline{V \cup U} \cap U^\neg)}) \cap (\overline{U \cup V} \cap U^\neg)^\neg, A + (\overline{U \cup V}))$
$= ((\overline{((\overline{W \cup U}) \cup (\overline{V \cup U})) \cap U^\neg}) \cap ((\overline{U \cup V})^\neg \cup U), A + (\overline{U \cup V}))$

$((\overline{W \cup U}) \cup (\overline{V \cup U})) \cap U^{\neg}$ is a subset of $A^{op} \cap U^{\neg}$, which is $(A \cap U^{\star})^{op}$ and is therefore transitive. We deduce that its clositive closure

$$\overline{(((\overline{W \cup U}) \cup (\overline{V \cup U})) \cap U^{\neg})}$$

is also a subset of $A^{op} \cap U^{\neg}$, and therefore:

$$\begin{aligned} &\overline{(((\overline{W \cup U}) \cup (\overline{V \cup U})) \cap U^{\neg})} \cap ((\overline{U \cup V})^{\neg} \cup U) \\ &= \overline{(((\overline{W \cup U}) \cup (\overline{V \cup U})) \cap U^{\neg})} \cap (\overline{U \cup V})^{\neg} \qquad (1) \end{aligned}$$

We prove now that this relation is equal to:

$$(\overline{U \cup V \cup W}) \cap (\overline{U \cup V})^{\neg} \qquad (2)$$

First, (1) $\subset$ (2) because

$$((\overline{W \cup U}) \cup (\overline{V \cup U})) \cap U^{\neg} \subset (\overline{U \cup V \cup W})$$

Second, let

$$x(\overline{U \cup V \cup W})y$$

Let

$$x = x_1 U_1 x_2 U_2 \cdots x_{n-1} U_{n-1} x_n = y$$

the sequence of relations $x_i U_i x_{i+1}$ where $\forall i > 0, U_i \in \{U, V, W\}$ between $x$ and $y$ with the least number of $U_i$ equal to $W$ or $V$. $x(2)y$ if and only if the number of $i$ such that $U_i = W$ is not null: $\exists i > 0, U_i = W$. We can define in that case $1 = 1 < l_2 \cdots l_{k-1} < l_k = n$ and $\Sigma_j$, for $j$, $1 \leq j \leq k$, such that:

- $\forall j > 0,\ (\Sigma_j = \{U, V\}$ or $\Sigma_j = \{U, W\})$,
- $\forall j > 0,\ (l_j \leq i < l_{j+1} \Rightarrow U_i \in \Sigma_j)$,
- $\forall j > 0,\ \exists i, (l_j \leq i < l_{j+1})$ and $U_i \neq U$

Then

$$x_{l_j} U_{l_j} \cdots x_{l_{j+1}-1} U_{(l_{j+1}-1)} x_{l_{j+1}}$$

implies that $x_{l_j}(\overline{U \cup V})x_{l_{j+1}}$ if $\Sigma_j = \{U, V\}$ and $x_{l_j}(\overline{U \cup W})x_{l_{j+1}}$ if $\Sigma_j = \{U, W\}$. Suppose that $x_{l_j} U x_{l_{j+1}}$. The subsequence

$$x_{l_j} U_{l_j} \cdots x_{l_{j+1}-1} U_{l_{j+1}-1} x_{l_{j+1}}$$

may then be replaced by $x_{l_j} U x_{l_{j+1}}$ in

$$x = x_1 U_1 x_2 U_2 \cdots x_{n-1} U_{n-1} x_n = y$$

which contradicts the minimality of the number of $W$ and then $V$ in the sequence of $U_i$. We obtain thus that:

$$\forall j,\ x_{l_j}((\overline{U \cup V} \cup \overline{U \cup W}) \cap U^{\neg}) x_{l_{j+1}}$$

which implies that

$$x(\overline{(\overline{U \cup V} \cup \overline{U \cup W}) \cap U^{\neg}})y$$

We thus obtain that $(2) \subset (1)$, and therefore that:

$$W_A[U_A][V_A[U]A] = ((\overline{U \cup V \cup W}) \cap (\overline{U \cup V})^{\neg}, A \cup (\overline{U \cup V}))$$

This equation is symetric in $U$ and $V$ which implies that the system is orthogonal:

$$[U_A][V_A[U_A]] = [V_A][U_A[V_A]]$$

$\square$

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