Towards a Quantum Model of Linear Logic

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A “quantum” model of LL?

- Linear logic introduced the idea that logic is in some way *actions* on *ressources*. We would like to go a bit further: interpret logic as *quantum* actions on *quantum* ressources.

- The linear implication “consumes” its hypotheses, similarly in quantum information, one can’t duplicate objects (non-cloning property).

- The quantum coherence spaces model is related to a denotational model of P. Selinger for quantum computation. It uses the langage of quantum information.
1. Quantum Information
2. Quantum Coherence Spaces
3. Connectives
4. Additives and $\eta$-expansion
5. Conclusion: physics
1 Quantum Information
   - Density operators and superpositive applications
   - A categorical point of view
   - The fundamental adjunction

2 Quantum Coherence Spaces

3 Connectives

4 Additives and $\eta$-expansion

5 Conclusion: physics
From principles to properties

- Superposition principle $\rightarrow$ operators,

- Measures $O = \sum \alpha P_\alpha$, $p_\alpha = Tr(\rho P_\alpha) \rightarrow$ positivity,

- Union of two systems $\rightarrow$ existence of a tensor product.
To represent a system state: let $H$ be a Hilbert space

**Definition**

A density matrix or operator is a positive matrix (or operator) on a Hilbert space $H$ that satisfies:

$$\text{Tr} (\rho) \leq 1.$$  

Why $\leq 1$ instead of $= 1$? A density operator can both represent a state or the result of an algorithm. In the latter case, the algorithm terminates with probability $\text{Tr}(\rho)$. 
Morphisms

Definition

Let $A$ be a linear application from the operators on $\mathcal{L}(H_1)$ to $\mathcal{L}(H_2)$. $A$ is said to be **superpositive** if, for any Hilbert space $H$, the linear application

$$\text{Id}_{\mathcal{L}(H)} \otimes A : \mathcal{L}(H \otimes H_1) \longrightarrow \mathcal{L}(H \otimes H_2)$$

satisfies that for all $\rho$ a positive operator on $H \otimes H_1$, its image through $\text{Id}_{\mathcal{L}(H)} \otimes A$ is a positive operator on $H \otimes H_2$.

It means that not only $A$ has to transform positive operators into positive operators but also that whichever space we choose to couple to the first one, the corresponding application must conserve positiveness too.
To begin with

The main idea is to build a category whose objects are sets of positive operators and morphisms are superpositive applications.

One sees that we really need here superpositiveness since we’ll use the tensor product as the product of the category.

The sets will need to satisfy certain properties, and we’ll call them later quantum coherent spaces, on account of their closeness to J.-Y. Girard’s classical coherent spaces.
There are several connectives that will have to be represented, the first one being polarity, or to phrase it in a counter-intuitive way, negation:

- \( \sim \) is the afore mentioned negation, considering we use linear logics it has to be involutive (for each object \( A \), we must have \( \sim \sim A = A \)),

- the multiplicative connector \( \otimes \) and its dual \( \oslash \), which is merely represented by the tensor product,

- the additive connector \( \oplus \) and its dual \( \& \), which will turn out to be a bit more tricky,

- ideally, the modality \( ! \), about which we may have some prospects that we will discuss in the conclusion.
We see that we’ll also need to represent the implication connector $\rightarrow$ not only as $A \rightarrow B = \sim A \vDash B$, but also as a set of morphisms which transform $A$ into $B$.

It means that we will what is called an adjunction, that is to say a way to transform the set of homomorphisms

$$\text{Hom}(A \otimes B, C)$$

into the set

$$\text{Hom}(A, B \rightarrow C).$$

More generally, we’ll give a transformation from $\text{Hom}(A, B)$ to $A \rightarrow B$ defined as $\sim A \vDash B$. 
Let $A$ be a set of positive operators on a Hilbert space $H_1$ and $B$ on $H_2$. We are looking for a general transformation from $A \rightarrow B$ set of positive operators on the Hilbert space $H_1^* \otimes H_2$ and $\text{Hom}(A, B)$ set of superpositive applications from $\mathcal{L}(H_1)$ to $\mathcal{L}(H_2)$.

What we'll exhibit will be a general reversible transformation $\chi$ from $\mathcal{L}(H_1) \rightarrow \mathcal{L}(H_2)$ to $\mathcal{L}(H_1 \otimes H_2)$ that will satisfy:

For all $F \in \mathcal{L}(H_1) \rightarrow \mathcal{L}(H_2)$, $\chi_F$ is positive $\iff$ $F$ is superpositive.

Then, we’ll define $A \rightarrow B$ as

$$\{\chi_F \geq 0 \mid F(A) \subseteq B\}.$$
1 Quantum Information

2 Quantum Coherence Spaces
   • Basics
   • An important exemple

3 Connectives

4 Additives and $\eta$-expansion

5 Conclusion : physics
Writing classical coherence spaces in terms of diagonal square matrix:

$X \subseteq \{ a_1, a_2, \ldots, a_n \}$ can be seen as

$$M_X := \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_n \end{pmatrix}$$

where $\varepsilon_i$ is 1 if $a_i \in X$ and 0 otherwise.

From that point of view, $X \perp Y$ (that is, $\#|X \cap Y| \leq 1$) can be rewritten:

$$Tr(M_X M_Y) \leq 1$$
Writing probabilistic coherence spaces in terms of diagonal square matrix:

$X$, a “probabilistic subset” of $\{ a_1, a_2, ..., a_n \}$ can be seen as

$$M_x := \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_2 & \lambda_1 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_n & \lambda_n & \cdots & \lambda_1 \end{pmatrix} \quad \text{where } \lambda_i = p( a_i \in X )$$

Again, we can rewrite polarity as:

$$Tr(M_x M_y) \leq 1$$
We will use the following isomorphism between $\mathcal{L}(E)$ and $\mathcal{L}(E^*)$, called *transposition*:

**Definition**

If $E$ is a Hilbert space and $f \in \mathcal{L}(E)$, we define the *transposed* of $f$ (notation $t_f$) by:

$$\text{For all } \varphi \in E^* : \quad t_f(\varphi) = \varphi \circ f$$

Transposition enjoys all the properties we need: preservation of positivity, of the trace, of the norm...
We now consider positive hermitians, a natural generalisation of the diagonal matrix presentation we saw.

**Definition**

Two positive operators $f \in \mathcal{H}^+(E)$ and $g \in \mathcal{H}^+(E^*)$ are polar (notation $f \perp g$) whenever:

$$Tr(f^*g) \leq 1$$

The polar of $X \subseteq \mathcal{H}^+(E)$: $\sim X := \{ y \mid y \perp x, \text{ for all } x \in X \}$
Quantum Coherence Space

Definition

We call a Quantum Coherence Space (QCS), a subset $X \subseteq \mathcal{H}^+(E)$ such that $X$ is equal* to its bipolar, i.e.:

$$X = \sim \sim X$$

* : As we use Hilbert spaces, we identify $E$ and its bidual, $E^{**}$

Definition

If $X \subseteq \mathcal{H}^+(E)$ we will call $E$ the support of $X$, notation $|X|$. 
We have an isomorphism between $\mathcal{L}(E_1^*) \otimes \mathcal{L}(E_2)$ and $\mathcal{L}(E_1) \rightarrow \mathcal{L}(E_2)$. It is defined by:

**Definition**

If $F \in \mathcal{L}(E_1) \rightarrow \mathcal{L}(E_2)$, $\chi_F$ is the only operator on $E_1^* \otimes E_1$ satisfying:

For all $x \in \mathcal{L}(E_1)$ and $y \in \mathcal{L}(E_2)$: $Tr(F(x).y) = Tr(\chi_F. t_x \otimes y)$

**Remark** : the polarity is compatible with this adjunction, i.e.,

$$A \rightarrow \downarrow \simeq \sim A$$
Theorem

\[ F \text{ is superpositive if and only if } \chi_F \text{ is positive.} \]

With the matrix representation in some base, there is an easier way to compute \( \chi_F \):

\[
\chi_F := \begin{pmatrix}
F(E_{1,1}) & \cdots & F(E_{1,n}) \\
\vdots & \ddots & \vdots \\
F(E_{n,1}) & \cdots & F(E_{n,n})
\end{pmatrix}
\]

where \( E_{i,j} \) is the square matrix with 1 in position \((i,j)\) and 0 elsewhere.
This theorem characterises the geometry of QCS:

**Theorem**

A set of positive hermitian $X \subseteq \mathcal{H}^+(E)$, is equal to its bipolar $\sim\sim X$ if and only if:

- $0$ is in $X$
- $X$ is convex and closed
- if $x \in X$ and $y \leq x$, then $y \in X$

Proof uses convex projection theorem and linear algebra.
Quantum Booleans and density operators

The set of density operators on a Hilbert $E$ (which are used to represent the state of a quantum system) is a QCS:

$$\text{Dens}_E := \{ h \in \mathcal{H}^+(E) \mid \text{Tr}(h) \leq 1 \}$$

contains 0, is convex and closed, and downward closed for $\leq$ (order on positives)

We will call this set “quantum booleans” or “canonical positive” over $E$, notation $P_E$.

Its polar is: $\sim P = N_E := \{ h \in \mathcal{H}^+(E) \mid \|\|h\|\| \leq 1 \}$ called “canonical negative” over $E^*$.

$\|\|h\|\| \text{ is the operator norm of } h : \|\|h\|\| := \text{Max}_{v \in E} \frac{\|h(v)\|}{\|v\|}$
1. Quantum Information

2. Quantum Coherence Spaces

3. Connectives
   - Multiplicatives
   - $\otimes$ and separable states
   - The good news about coupled systems
   - Additives
   - Exemplars, interpretation in physics

4. Additives and $\eta$-expansion

5. Conclusion: physics
Definitions

**Definition**

The QCS corresponding to linear implication, $X \vdash Y$, is defined on the support $|X|^* \otimes |Y|$, by:

$$X \vdash Y := \{ \chi_F \mid F \in X \rightarrow^+ Y \}$$

That is, the set of the $\chi_F$ corresponding to the $F$ superpositive such that $F(X) \subseteq Y$.

From this, we derive definitions of $\otimes$ and $\nabla$, by duality:

$$X \nabla Y = \sim X \vdash Y \quad \text{and} \quad X \otimes Y = \sim (\sim X \nabla \sim Y)$$

**Proposition**

$$X \otimes Y = \sim \sim \{ x \otimes y \mid x \in X, y \in Y \}$$
Another point of view for $\otimes$

Proposition

Let $A, B$ be two coherent spaces that we suppose bounded. We denote $A_m$ (resp. $B_m$) the set of maximal elements of $A$ (resp. $B$), and $C_m$ the convex hull of $\{a \otimes b \mid a \in A_m \text{ and } b \in B_m\}$. The QCS $A \otimes B$ is equal to the set:

$$\{\rho \mid \exists \rho_m \in C_m \text{ such that } \rho_m \geq \rho\}.$$

Hence, the set of maxima in $A \otimes B$ is equal to $C_m$.

Since the traces of states in quantum physics are supposed to be less than 1, the quantum coherent spaces formed by physically meaningful operators are bounded.
Suppose now that $A_m$ contains the operators that represents admissible states for a first system $S_A$ and $B_m$ for a second one $S_B$.

Then, $(A \otimes B)_m$ is the convex hull of $A_m$ and $B_m$. That is to say, $(A \otimes B)_m$ is the set of the separable states one can obtain by combining $S_A$ and $S_B$.

Being separable for two states means you can act on one without touching to the other one. States that are not separable are called entangled.

Entanglement is the property of quantum states that gives a special interest to quantum computation.
Consequently, even though we can not represent separable states with the simple operation of $\otimes$, we have this property onto the set of maxima.

What’s more, one can deduce from the precedent property that $A \otimes B$ is the QCS generated by the separable states obtained by combining the systems $S_A$ and $S_B$.

Regarding the canonical exemple of QCS $P_H$ (which represents all possible states on a Hilbert space $H$), we have that the tensor connective $P_{H_1} \otimes P_{H_2}$ is the QCS generated by all possible separable states on $H_1 \otimes H_2$, that is to say:

$$P_{H_1} \otimes P_{H_2} = \sim\sim \{\text{separable states over} H_1 \otimes H_2\}.$$
Remember that $P_H = \{ \rho \geq 0 \mid \text{Tr}(\rho) \leq 1 \}$.

**Proposition**

Let $H_1$ and $H_2$ be two Hilbet spaces. The QCS $P_{H_1} \otimes P_{H_2}$ is merely $P_{H_1 \otimes H_2}$. That is to say, $\otimes$ combines the sets of all possible states on $H_1$ and $H_2$ into the set of all possible states on $H_1 \otimes H_2$.

i.e.

$$P_{H_1 \otimes H_2} = P_{H_1} \otimes P_{H_2}$$
Definitions

We define the $\oplus$ and $\&$ connectives, on the support $|X| \oplus |Y|$, by:

**Definition**

$$X \oplus Y := \sim \sim \left( \{x \oplus 0 | x \in X\} \cup \{0 \oplus y | y \in Y\} \right)$$

**Definition**

$$X \& Y := \{ h | p_{|X|} hp_{|X|} \in X \text{ and } p_{|Y|} hp_{|Y|} \in Y \}$$

*By $p_{|X|}$, we mean the orthogonal projection on $|X|$ in the hilbert $|X| \oplus |Y|$.*

These two definitions are dual:

**Proposition**

$$X \oplus Y = \sim (\sim X \& \sim Y)$$
As expected, $\otimes$ distributes over $\oplus$:

(And dually, $\ominus$ distributes over $\&$)

**Proposition**

For all $A$, $B$, $C$,

$$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$$

The proof only consist in lifting the isomorphism

$$|A| \otimes (|B| \oplus |C|) \simeq (|A| \otimes |B|) \oplus (|A| \otimes |C|) \text{ on supports.}$$
Let $\Omega_A$ and $\Omega_B$ be two regions of $\mathbb{R}^3$ such that $\Omega_A \cap \Omega_B = \emptyset$ and $V$ a potential such that

\[ V \begin{cases} = \infty & \text{outside } \Omega_A \cup \Omega_B \\ \text{is bounded} & \text{in } \Omega_A \cup \Omega_B \end{cases} \]

Let $V_A = \infty$ on $\Omega_B$, $V_A = V$ outside.
Some ideal experiment II

One can see that a particle submitted to $V_A$ (resp. $V_B$) stays in $\Omega_A$. Let’s say that initially we have a particle in $\Omega_A$ or $\Omega_B$ and submitted to $V$. Then, we cut the potential (we suppose we can do it in no time, abruptly and perfectly). The particle delocalizes itself and occupies the whole space $\mathbb{R}^3$, meaning also both $\Omega_A$ and $\Omega_B$.

After what, we plug the potential again and we found out that the particle is restricted to $\Omega_A \cup \Omega_B$ and that the information contained in $\Omega_A$ is independent from the one in $\Omega_B$, since the two regions are separated from infinite potential (they evolve independently).
Interpretation of $\Theta$

Then imagine that you allow a system to be represented by a certain set of states $A$ or $B$, linearly independant from each other. The states resulting in a stochastic combination of states of $A$ and of $B$ are the convex hull of $A \cup B$.

**Proposition**

Let $A$ and $B$ two bounded QCS and $A_m, B_m$ the sets of their maxima, and $C_m$ the convex hull of $A_m \oplus 0_{|B|} \cup 0_{|A|} \oplus B_m$. Then $A \oplus B$ can be rewritten as:

$$A \oplus B = \{ \rho \mid \exists \rho_m \in C_n \text{ such that } \rho \leq \rho_m \}.$$  

Therefore, the sets of stochastic combination of $A_m$ and $B_m$ is the set of maxima of $A \oplus B$. Once more, there’s no direct result on $A \oplus B$ but on its maxima.
1. Quantum Information
2. Quantum Coherence Spaces
3. Connectives
4. Additives and \( \eta \)-expansion
   - Interpreting LL additive rules
   - Another construction
   - Cut-elimination
5. Conclusion: physics
Interpreting LL additive rules

The proof:

\[
\begin{align*}
A \vdash A & \quad \text{(Ax)} \\
A \vdash A \oplus B & \quad \text{(⊕-r)} \\
B \vdash B & \quad \text{(Ax)} \\
B \vdash A \oplus B & \quad \text{(⊕-l)} \\
A \oplus B & \vdash A \oplus B \quad \text{(&)}
\end{align*}
\]

is interpreted by: \( \begin{pmatrix} U & W \\ W^\dagger & V \end{pmatrix} \rightarrow \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \) i.e. \( \text{Id}_\mathcal{L}(|A|) \oplus \text{Id}_\mathcal{L}(|B|) \)

But the proof:

\[
A \oplus B \vdash A \oplus B \quad \text{(Ax)}
\]

is interpreted by the identity, \( \text{Id}_\mathcal{L}(|A| \oplus |B|) \)
Combining superpositive operators

Using the properties of superpositive operators, one can define an operation, noted $\oplus$, such that:

If $F \in \Gamma \otimes A \rightarrow^+ C$ and $G \in \Gamma \otimes B \rightarrow^+ D$
Then $F \oplus G \in \Gamma \otimes (A \oplus B) \rightarrow^+ C \oplus D$

Remark: $\text{Id}_\mathcal{L}(E) \oplus \text{Id}_\mathcal{L}(F) = \text{Id}_\mathcal{L}(E \oplus F)$

This operation on morphisms would correspond to a rule of the form:

\[
\Gamma, A \vdash C \quad \Gamma, B \vdash D \quad (\text{Add}) \quad \text{or equivalently}
\]
\[
\Gamma, A \oplus B \vdash C \oplus D
\]

\[
\vdash \Gamma, A, C \quad \vdash \Gamma, B, D \quad (\text{Add})
\]
\[
\vdash \Gamma, A \& B, C \oplus D
\]
The corresponding rule

The (Add) rule can be derived in LL:

\[
\frac{\Gamma, A \vdash C}{\Gamma, A \vdash C \oplus D} \quad \frac{\Gamma, B \vdash D}{\Gamma, B \vdash C \oplus D} \quad \frac{\Gamma, A \vdash C \oplus D \quad \Gamma, B \vdash C \oplus D}{\Gamma, A \oplus B \vdash C \oplus D} \quad (\ominus-1)
\]

But the interpretation differs in the QCS model, for instance:

\[
\frac{A \vdash A}{A \oplus B \vdash A \oplus B} \quad (\text{Ax}) \quad \frac{B \vdash B}{A \oplus B \vdash A \oplus B} \quad (\text{Ax}) \quad \text{is interpreted by :}
\]

\[
Id_{\mathcal{L}}(|A|) + Id_{\mathcal{L}}(|B|) \left( = Id_{\mathcal{L}}(|A| \oplus |B|) \right) \neq Id_{\mathcal{L}}(|A|) \oplus Id_{\mathcal{L}}(|B|)
\]
A (Cut)-elimination case for (Add)

Let's look at a case of (Cut)-elimination for the (Add) rule:

Work in progress...
The same elimination case for the derived rule

Work in progress...
1. Quantum Information
2. Quantum Coherence Spaces
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5. Conclusion: physics
   - Quantum additives?
   - Fock space and exponentials?
Quantum additives?

- What happens if we “switch” the definitions of additives, that is to say, if we write:

  **Definition**

  \[ X \oplus_q Y := \{ h \geq 0 \mid \exists x \in X, y \in Y, \lambda, \mu \in \mathbb{R}^+, \text{ such that } \lambda + \mu \leq 1, p_{|X|} hp_{|X|} = \lambda x \text{ and } p_{|Y|} hp_{|Y|} = \mu y \} \]

  and

  **Definition**

  \[ X \&_q Y := \sim\sim\{ x \oplus y \mid x \in X, y \in Y \} \]

- \( \oplus_q \) corresponds to a natural operation on spaces in quantum mechanics.
- We obtain odd commutation relations (\( \oplus_q \gamma \) and \( \&_q \otimes \)).
- But this might be of some interest to interpret exponentials.
Fock spaces

- Fock spaces are based on operators of anihilation and creation: $a_k^\dagger$ et $a_k$

- That is to say, a basis of this space is given by $\prod a_k^\dagger |0\rangle$.

- There are two kinds of Fock spaces: if the $a_k$ commute, we have a symmetric space, if they anticommute, an anti symmetric one.
In the latter case, Fock space corresponds to the exterior algebra.

**Proposition**

Let $H_1, H_2$ be two vector spaces and $\Lambda H_1, \Lambda H_2$ their respective exterior algebra. There’s a canonical isometry between $\Lambda(H_1 \oplus H_2)$ and $\Lambda H_1 \otimes \Lambda H_2$.

Which leads us to think there might be a glint of hope as to express $!A$ as a QCS with support $\Lambda|A|$.

By lifting the isometry on Hilbert spaces up to a transformation onto their operators, we expect to find an isomorphism between $!(X \& Y)$ and $(!X) \otimes (!Y)$. 
Other rules

- Operators of creation and annihilation provides us intuitive transformations.

- Creation from $X \otimes !X$ to $!X$

- Anihilation from $\sim X \otimes !X$ to $!X$
Questions?