Ieke Moerdijk – Dendroidal Topology II

- coloured operad in $\underline{\text{Sets}}$, in $\underline{\text{Top}}$.
- for any finite rooted tree T we have introduced the operad $\Omega(T)$, "freely" generated by T.

Today, we will discuss a category of presheaves

$$\text{Sets}^{\Omega^{op}}$$

on a small category Ω, made up from these trees.

Definition of Ω: objects are finite rooted trees T

arrows are maps of coloured operads $\Omega(S) \longrightarrow \Omega(T)$

What does this category look like?

Note that there is a fully faithful embedding

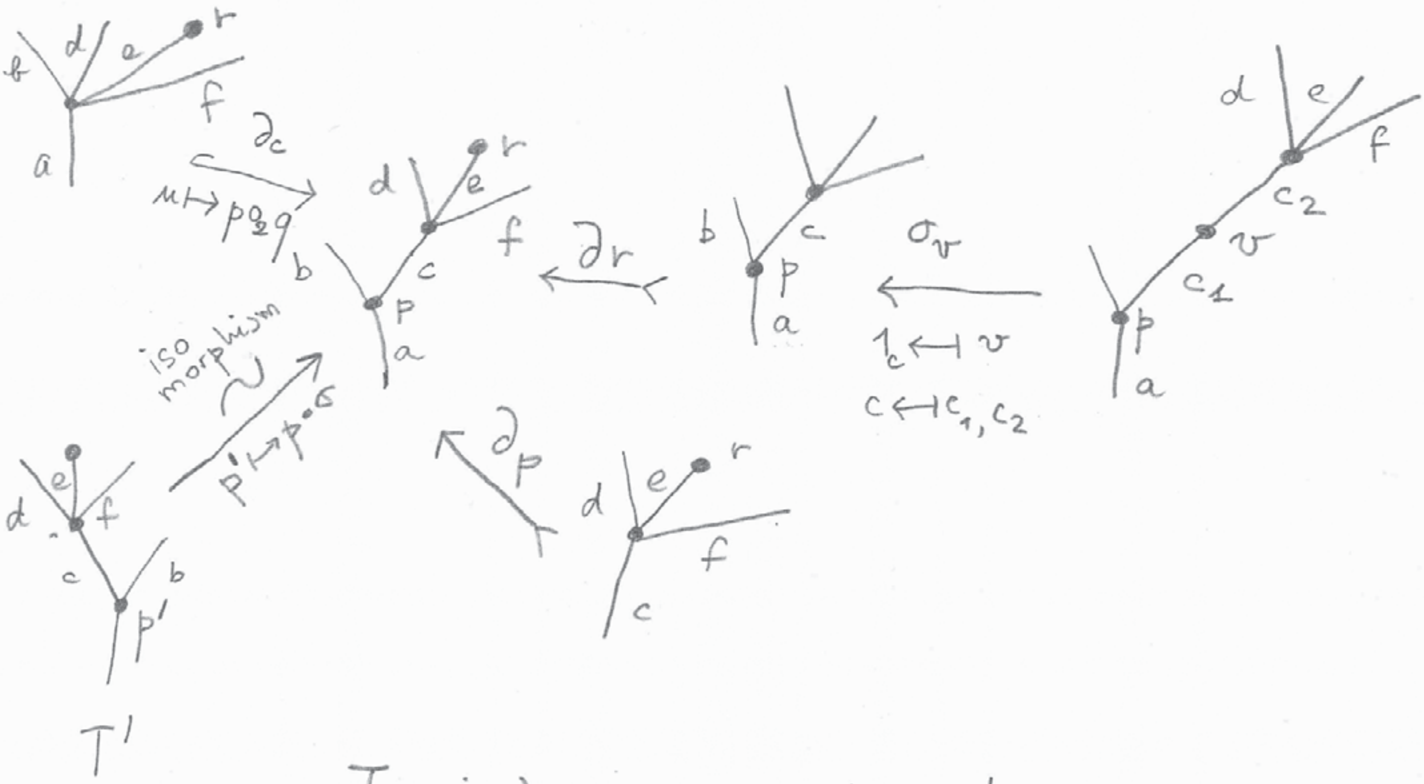
$$\Delta \xrightarrow{i} \Omega$$

from the category Δ of finite linear orders

$$[n] = \{0, \ldots, n\} \qquad (n \geq 0)$$

$i[n] =$ 

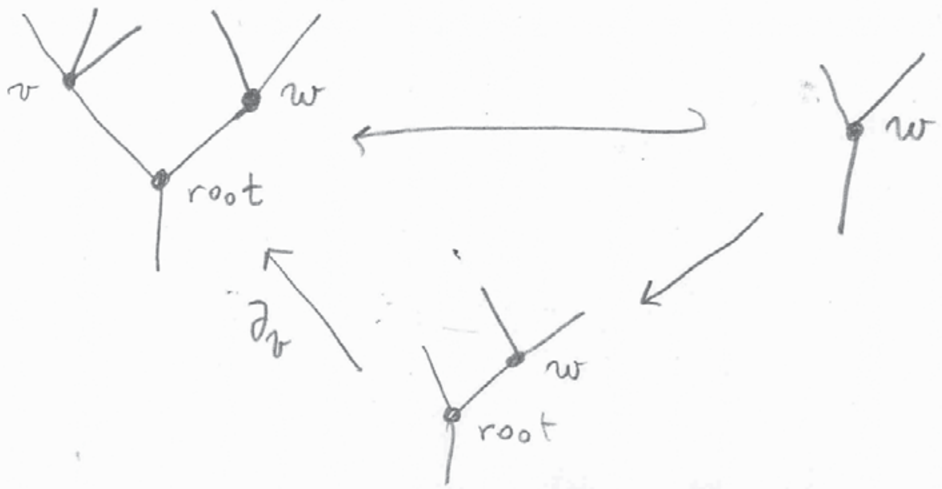
Examples of arrows in Ω

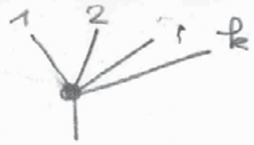


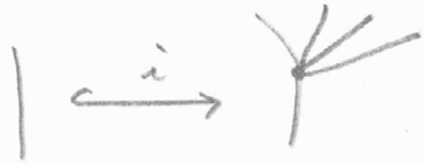
Terminology:

- σ_v is a degeneracy
- ∂_r, ∂_p are "external" faces
- ∂_c is an "internal" face
- there is an isomorphism

Remark: there is an external face "cutting away" the root vertex only if the root vertex has <u>only</u> <u>one</u> internal face attached to it.



2) for a corolla $C_n =$ 

there are external faces 

$i = 0, \dots, k$

Proposition: Every arrow $S \to T$ in Ω can be written essentially uniquely as a composition

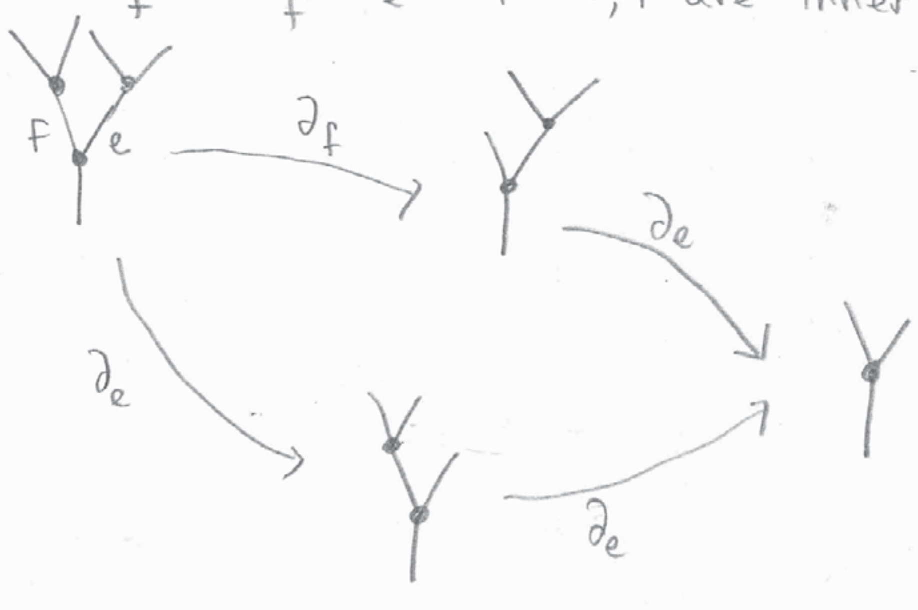
$$S \overset{\sigma}{\twoheadrightarrow} S' \underset{\alpha}{\xrightarrow{\sim}} T' \overset{\partial}{\rightarrowtail} T$$

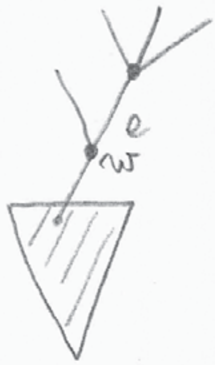
where α is an iso, σ is a composition of degeneracies, ∂ is a composition of faces.

So in this sense Ω is generated by faces and degeneracies as well as symmetries (in Δ we only have faces & degeneracies).

One could (but I will not) write a list of relations similar to the simplicial identities, eg

$\partial_e \partial_f = \partial_f \partial_e$ if e, f are inner edges in a tr.





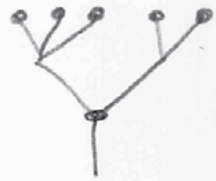
$$\partial_w \partial_v = \partial_{w \circ_1 v} \partial_e$$

if e relates an external vertex v to a vertex w so that e is the only input edge of w which is internal.

<u>Proposition</u>. The classifying space of Ω is contractible.

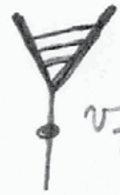
<u>Proof</u>: let $\overline{\Omega}$ be the full subcategory of "closed" trees, ie trees without leaves. The inclusion has a left adjoint

$$\Omega \underset{\longleftarrow}{\xrightarrow{cl}} \overline{\Omega}.$$

$cl($  $)=$ 

Now on $\overline{\Omega}$ we have the following functors

$$\begin{tikzcd} \overline{\Omega} \arrow[r, bend left=50, "\lambda"] \arrow[r, "id" description] \arrow[r, bend right=50, "\text{constant}"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Uparrow\beta"] & \overline{\Omega} \end{tikzcd}$$

$\lambda(T) =$  v_T new root vertex

$\alpha : T \hookrightarrow \lambda(T)$ is the obvious embedding

$\beta :$ | $\hookrightarrow \lambda(T)$ sends the vertex to the new v_T in $\lambda(T)$.

Remark: λ is a functor on $\overline{\Omega}^{\cdot}$ but not on Ω.

Presheaves on Ω

Definition. the category of presheaves on Ω will be called that of dendroidal sets, and denoted $dSets = Sets^{\Omega^{op}}$.

Let's see a few examples:

1. every simplicial set can be viewed as a dendroidal set since $\Delta \xhookrightarrow{i} \Omega$ induces adjoint functors

$$\begin{tikzcd} sSets \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & dSets \arrow[l, "i^*"'] \end{tikzcd}$$

Since i is full and faithful, so are $i_!$ and i_*.

$i_!(M)$ = "extension by zero".

i.e. $i_!(M)_T = \begin{cases} M_n & \text{if } T = i[n] \\ \phi & \text{otherwise} \end{cases}$

In fact, sSets is an open subtopos of dSets.

there is an isomorphism

$$\Omega/1 = \Delta$$

which induces an equivalence of categories

$$dSets/_{\Omega(-,1)} = sSets.$$

Under this iso,

$i_!$ corresponds to "forget"

i^* corresponds to "pullback"

Note that $\Omega(-,1) = i_!(\Delta[0])$.

(2) every object T in Ω defines a representable preshea

$$\Omega[T] := \Omega(-,T)$$

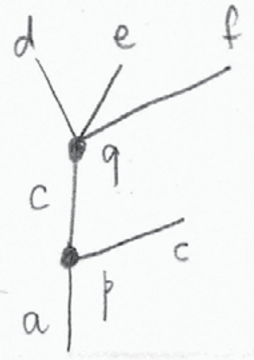
note $\quad i_!(\Delta[n]) = \Omega[i[n]]$

(3) every coloured operad P in Sets defines a dendroidal set called its nerve.

$$N(P)_T = \mathrm{Hom}(\Omega(T), P).$$

<u>Remark</u>: an element of $N(P)_T$ looks like a labelling of T:

- its edges by colours of P
- its vertices by operations of P.



More formally, an element of $N(P)_T$ is an equivalence class given by a planar structure on T and such a labelling.

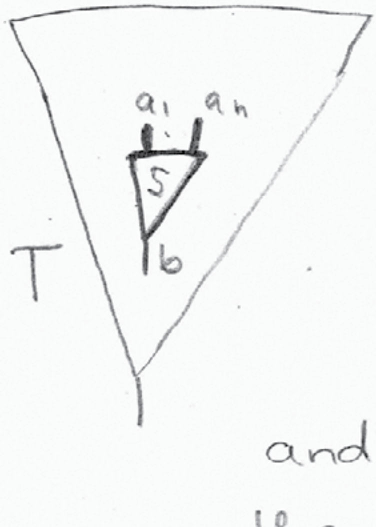
④ Every coloured operad P in Top

(the operations form spaces but there is still a set of colours)

defines a dendroidal set $\mathcal{D}(P)$ defined as

$$\mathcal{D}(P)_T = \mathrm{Hom}(\underbrace{W\Omega(T)}_{W(T)}, \mathcal{P})$$

Recall from Lecture 1 that $\Omega(T)$ is an operad where colours are the edges of T and with exactly one operation $a_1, \dots, a_n \to b$ if there is a subtree S of T with leave $a_1, \dots, a_n$ and root b.



Hence elements of $W(T)(a_1, \dots, a_n; b)$ are such subtrees S with lengths $\theta \in [0,1]$ assigned to the internal edges of S and these are composed by taking the union as subtrees of T and by assigning length 1 to the edges connecting these subtrees.

$\mathcal{D}(P)$ is called the homotopy coherent nerve of P

this defines a functor

$$\mathcal{D} : (\text{top. operads}) \longrightarrow \text{dSets}$$

It has a left adjoint:

$$\mathcal{W} : \text{dSets} \longrightarrow (\text{top. operads})$$

$$\mathcal{W}(\Omega[T]) = W(T)$$

$\mathcal{W}$ preserves colimits

$\mathcal{W}$ is the left Kan extension of the BV resolution applied to the operads $\Omega(T)$ given by representables $\Omega[T]$.

In fact, for an operad P in Sets

$$\mathcal{W}N(P) = W(P).$$

$$\begin{tikzcd}
 & (\text{Top. operads}) & \\
\Omega \arrow[ur, "W"] \arrow[rr, hook, "{\Omega[-]}"] & & \text{d Sets} \arrow[ul, "\mathcal{W}"']
\end{tikzcd}$$

the ordinary nerve

$$N : (\text{operads in sets}) \longrightarrow \text{dSets}$$

also has a left adjoint

$$\tau : \text{dSets} \longrightarrow \text{operads}$$

completely determined on representables by

$$\tau(\Omega[T]) = \Omega(T).$$

$$\begin{tikzcd}
\text{dSets} \arrow[r, shift left, "\tau"] \arrow[d, shift left, "i^*"] & (\text{operads in sets}) \arrow[l, shift left, "N"] \arrow[d, shift left, "j^*"] \\
\text{sSets} \arrow[r, shift left, "\tau"] \arrow[u, shift left, "i_!"] & (\text{categories}) \arrow[l, shift left, "N"] \arrow[u, shift left, "j_!"]
\end{tikzcd}$$

$$\begin{tikzcd}
(\text{top. operads}) \arrow[r, shift right, "\mathcal{D}"'] \arrow[d, shift left] & \text{dSets} \arrow[l, shift right, "\mathcal{W}"'] \arrow[d, shift left, "i^*"] \\
(\text{top cats}) \arrow[r, shift right, "\text{classical homotopy coherent nerve}"'] \arrow[u, shift left] & \text{sSets} \arrow[l, shift right, "\mathcal{W}"'] \arrow[u, shift left, "i_!"]
\end{tikzcd}$$

$$\begin{tikzcd}
& (\text{topological categories}) & \\
\Delta \arrow[ur, "W"] \arrow[rr, hook, "\Delta[-]"] & & \text{sSets} \arrow[ul, "\mathcal{W}"']
\end{tikzcd}$$

Precise description of $W(\mathbb{C})$ for a category $\mathbb{C}$, the Boardman-Vogt resolution of $\mathbb{C}$ viewed as an operad $P_{\mathbb{C}}$.
It is a topological category (cf. lecture 1)

Its objects are the same as of $\mathbb{C}$

An arrow $c \longrightarrow d$ is an equivalence class

of sequences:

$$c = c_n \xrightarrow{f_n} c_{n-1} \xrightarrow[\theta_{n-1}]{f_{n-1}} c_{n-2} \xrightarrow{\theta_{n-2}} \cdots \xrightarrow{f_2} c_1 \xrightarrow[\theta_1]{f_1} c_0 = d$$

$\theta_i \in [0,1]$ are "waiting times".

If $\theta_i = 0$, the sequence represents the same arrow

as

$$c = c_n \xrightarrow{f_n} c_{n-1} \xrightarrow{\theta_{n-1}} \cdots c_{i+1} \xrightarrow[\theta_{i+1}]{f_i \circ f_{i-1}} c_{i-1} \xrightarrow{\theta_{i-1}} \cdots \to c_0$$

Composition is defined by writing length 1 on the new intermediate object.

Eg: if $\mathbb{C} = [n] = n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 0$

an arrow $i \longrightarrow j$ can always be represented

by a full sequence

$$i \longrightarrow i-1 \longrightarrow \cdots \longrightarrow j+1 \longrightarrow j$$

as long as we put 0's.

$$(W[n])(i,j) = [0,1]^{j-i-1}$$

this relates to the classical description of the homotopy coherent nerve of a topological category $\mathcal{C}$ as

$$HcN(\mathcal{C})_n = \mathrm{Hom}(W[n], \mathcal{C}).$$

Tomorrow's lecture:

there is a symmetric monoidal closed structure $\otimes$, $\underline{\mathrm{Hom}}$ on dSets,

extending the cartesian structure on sSets

$$\mathrm{sSets} \xrightarrow{i_!} \mathrm{dSets}$$

$$i_!(M \times N) = i_!(M) \otimes i_!(N)$$

its unit is $\eta = \Omega[1]$.