Ieke Moerdijk - Dendroidal Topology I
- coloured operad in $\text{Sets}$, in $\text{Top}$.
- for any finite rooted tree $T$ we have introduced the operad $\Omega(T)$, "freely" generated by $T$. Today, we will discuss a category of presheaves $\Omega^\text{op}$ on a small category $\Omega$, made up from these trees.

Definition of $\Omega$: objects are finite rooted trees $T$

arrows are maps of coloured operads $\Omega(S) \to \Omega(T)$

What does this category look like?

Note that there is a fully faithful embedding

$\Delta \xrightarrow{i} \Omega$

from the category $\Delta$ of finite linear orders

$[n] = \{0, \ldots, n\}$ ($n \geq 0$)

$$i[n] = \begin{array}{c}
\vdots \\
\vdots \\
0
\end{array}$$
Examples of arrows in $\Omega$

Termiology:
- $\sigma_v$ is a degeneracy
- $\partial_r, \partial_p$ are "external" faces
- $\partial_c$ is an "internal" face
- there is an isomorphism

Remark: there is an external face "cutting away" the root vertex only if the root vertex has only one internal face attached to it.
2) for a corolla $C_n$ there are external faces $\rightarrow$

Proposition: Every arrow $S \rightarrow T$ in $\Omega$
can be written essentially uniquely as a composition

$S \rightarrow S' \overset{\alpha}{\rightarrow} T' \overset{\beta}{\rightarrow} T$

where $\alpha$ is an iso, $\sigma$ is a composition of degeneracies
$\tau$ is a composition of faces.

So in this sense $\Omega$ is generated by faces
and degeneracies as well as symmetries
(in $\Delta$ we only have faces & degeneracies).

One could (but I will not) write a list of relations
similar to the simplicial identities, eg

$\partial_e \partial_f = \partial_f \partial_e$ if $e, f$ are inner edges in a tr.
\[ \partial_{\nu} \partial_{e} = \partial_{\nu_{w} e} \partial_{e} \]

if \( e \) relates an external vertex \( \nu \) to a vertex \( w \) so that \( e \) is the only input edge of \( w \) which is internal.

**Proposition.** The classifying space of \( \Omega \) is contractible.

**Proof:** Let \( \Omega \) be the full subcategory of "closed" trees, i.e., trees without leaves. The inclusion has a left adjoint

\[
\begin{align*}
\Omega & \xrightarrow{\text{cl}} \Omega \\
\Omega & \xleftarrow{\text{cl}} \Omega
\end{align*}
\]

\[ \text{cl}(\begin{array}{c} y \\
\end{array}) = \begin{array}{c} y \\
\end{array} \]

Now on \( \Omega \) we have the following functors

\[
\begin{array}{ccc}
\Omega & \xrightarrow{\lambda} & \Omega \\
\Downarrow \alpha & & \Downarrow \beta \\
\Omega & \xrightarrow{id} & \Omega
\end{array}
\]

\[ \lambda(T) = \begin{array}{c} \nu_T \\
\end{array} \text{new root vertex} \]

\( \alpha : T \rightarrow \lambda(T) \) is the obvious embedding

\( \beta : \begin{array}{c} 1 \\
\end{array} \rightarrow \lambda(T) \) sends the vertex to the new \( \nu_T \) in \( \lambda(T) \).
Remark: \( \lambda \) is a functor on \( \overline{\Omega} \) but not on \( \Omega \).

Presheaves on \( \Omega \)

Definition: the category of presheaves on \( \Omega \) will be called that of dendroidal sets, and denoted \( dSets = \Omega^{op} \).

Let's see a few examples:

1. every simplicial set can be viewed as a dendroidal set since \( \Delta \overset{i}{\to} \Omega \) induces adjoint functors

\[
\begin{array}{ccc}
\text{s Sets} & \xleftarrow{i^*} & \text{d Sets} \\
\text{i}_* & \xrightarrow{i^*} & \end{array}
\]

Since \( i \) is full and faithful, so are \( i^* \) and \( i_* \).

\( i^*_! (M) = \text{"extension by zero"} \).

i.e. \( i^*_! (M)_T = \begin{cases} M_T & \text{if } T = i [n] \\ \emptyset & \text{otherwise} \end{cases} \)

In fact, \( sSets \) is an open subtopos of \( dSets \).

there is an isomorphism

\[ \Omega / \mathcal{I} = \Delta \]

which induces an equivalence of categories

\[ dSets / \Omega (-,-,1) = sSets. \]
Under this iso,

\[ i! \] corresponds to "forget"

\[ i^* \] corresponds to "pullback"

Note that \[ \Omega(-,1) = i_!(\Delta[0]) \].

2. every object \( T \) in \( \mathcal{S} \) defines a representable presheaf
\[ \Omega[T] := \Omega(-,T) \]

note \[ i_!(\Delta[n]) = \Omega[i[n]] \]

3. every coloured operad \( P \) in Sets defines a dendroidal set called its nerve.
\[ N(P)_T = \text{Hom}(\Omega(T), P) \]

Remark: an element of \( N(P)_T \) looks like a labelling of \( T \):
- its edges by colours of \( P \)
- its vertices by operations of \( P \).
More formally, an element of $N(P)^T$ is an equivalence class given by a planar structure on $T$ and such a labelling.

4 Every coloured operad $P$ in $\text{Top}$ (the operations form spaces but there is still a set of colours) defines a dendroidal set $\mathcal{D}(P)$ defined as

$$\mathcal{D}(P)^T = \text{Hom}(W_\Omega(T), P) / W(T)$$

Recall from Lecture 1 that $\Omega(T)$ is an operad where colours are the edges of $T$ and with exactly one operation $a_1, \ldots, a_n \to b$ if there is a subtree $S$ of $T$ with leaves $a_1, \ldots, a_n$ and root $b$.

Hence elements of $W(T)(a_1, \ldots, a_n, b)$ are such subtrees $S$ with lengths $\Theta \in [0,1]$ assigned to the internal edges of $S$ and these are composed by taking the union as subtrees of $T$ and by assigning length 1 to the edges connecting these subtrees.
$D(P)$ is called the homotopy coherent nerve of $P$
this defines a functor

\[ \mathcal{D} : \text{(top. operads)} \rightarrow \text{dSets} \]

It has a left adjoint:

\[ \mathcal{W} : \text{dSets} \rightarrow \text{(top. operads)} \]

\[ \mathcal{W}(\Omega[T]) = \mathcal{W}(T) \]

$\mathcal{W}$ preserves colimits

$\mathcal{W}$ is the left Kan extension of the BV resolution applied to the operads $\Omega(T)$ given by representables $\Omega[T].$

In fact, for an operad $P$ in Sets

\[ \mathcal{W}N(P) = \mathcal{W}(P). \]

\[ \begin{array}{ccc}
\Omega & \xrightarrow{\Omega[T]} & \text{d Sets} \\
\mathcal{W} & \downarrow & \\
\text{dSets} & \xrightarrow{\mathcal{W}} & \text{(top. operads)}
\end{array} \]
The ordinary nerve

\[ N : \text{operads in sets} \rightarrow d\text{Sets} \]

also has a left adjoint

\[ \Gamma : d\text{Sets} \rightarrow \text{operads} \]

completely determined on representables by

\[ \Gamma(\Omega[T]) = \Omega(\Gamma). \]

\[
\begin{array}{ccc}
\text{dSets} & \xrightarrow{\Gamma} & (\text{operads in sets}) \\
\downarrow{N} & & \downarrow{i} \\
\text{sSets} & \xleftarrow{N} & (\text{categories})
\end{array}
\]

\[
\begin{array}{ccc}
(\text{top. operads}) & \xrightarrow{W} & \text{dSets} \\
\downarrow{D} & & \downarrow{i} \\
(\text{top. cats}) & \xleftarrow{W} & \text{sSets}
\end{array}
\]

\[
\begin{array}{ccc}
\text{classical} \\
\text{homotopy} \\
\text{coherent nerve}
\end{array}
\]

\[
\begin{array}{ccc}
(\text{topological}) \\
\text{categories} \\
W & \xrightarrow{\Delta} & \text{sSets}
\end{array}
\]
Precise description of $W(C)$ for a category $C$, the Boardman-Vogt resolution of $C$ viewed as an operad $P_C$. It is a topological category (cf. lecture 1).

Its objects are the same as of $C$.

An arrow $c \to d$ is an equivalence class of sequences:

$$c = c_n \overset{f_n}{\longrightarrow} c_{n-1} \overset{\Theta_{n-1}}{\longrightarrow} c_{n-2} \to \ldots \to c_2 \overset{\Theta_1}{\longrightarrow} c_1 \overset{f_1}{\longrightarrow} c_0 = d$$

$\Theta_i \in [0,1]$ are "waiting times".

If $\Theta_i = 0$, the sequence represents the same arrow as

$$c = c_n \overset{f_n}{\longrightarrow} c_{n-1} \to \ldots \overset{\Theta_i+1}{\longrightarrow} c_i+1 \overset{f_i \circ f_{i-1}}{\longrightarrow} c_{i-1} \to \ldots \to c_0$$

Composition is defined by writing length 1 on the new intermediate object.

Eq: if $C = [n] = n \to n-1 \to \ldots \to 0$ an arrow $i \to j$ can always be represented by a full sequence

$$i \to i-1 \to \ldots \to j+1 \to j$$

as long as we put 0's.
\[(W[n])(i, j) = [0, 1]^{j-1-i}\]

This relates to the classical description of the homotopy coherent nerve of a topological category \( \mathcal{C} \) as

\[\mathrm{HcN}(\mathcal{C})_n = \mathrm{Hom}(W[n], \mathcal{C}).\]

Tomorrow's lecture:

There is a symmetric monoidal closed structure \( \otimes, \mathrm{Hom} \) on \( d\text{Sets} \), extending the cartesian structure on \( s\text{Sets} \)

\[s\text{Sets} \xrightarrow{i!} d\text{Sets}\]

\[i!(M \times N) = i!(M) \otimes i!(N)\]

Its unit is \( \eta = \Omega [1] \).