Ieke Moerdijk – Dendroidal Topology III

Tensor products & Shuffle of trees

1) there is a tensor product of operads $P \otimes Q$ with the property that $P \otimes Q$-algebras are P-algebras in the category of Q-algebras (or vice versa).

▷ If P has set of colours C and Q has set of colours D then the set of colours of $P \otimes Q$ is $C \times D$

▷ the operations of $P \otimes Q$ are <u>generated</u> by two kinds

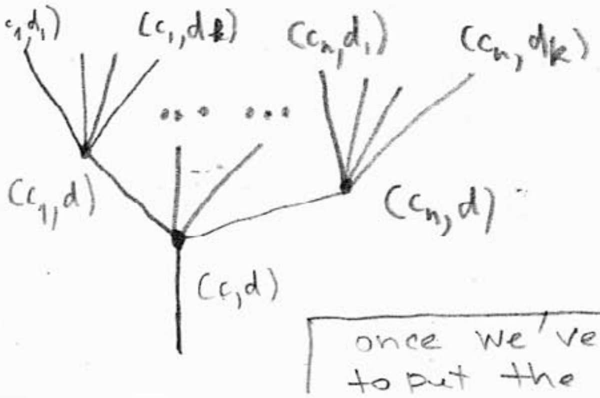
if $p \in P(c_1, \dots, c_n; c)$ and $d \in D$

then $p \otimes d \in P \otimes Q\big((c_1, d), \dots, (c_n, d); (c, d)\big)$

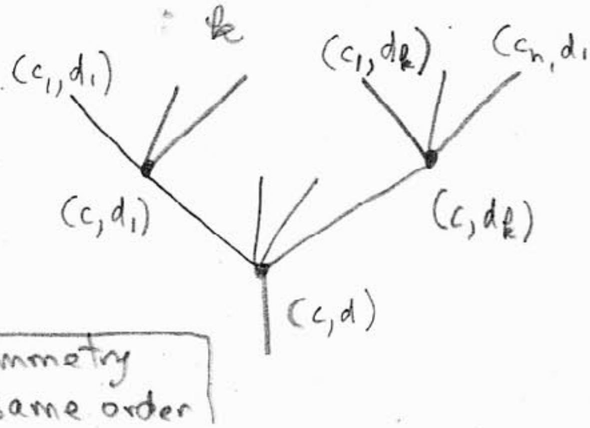
if $q \in Q(d_1, \dots, d_k, d)$ and $c \in C$

then $c \otimes q \in P \otimes Q\big((c, d_1), \dots, (c, d_k); (c, d)\big)$

▷ Relations "$p(\underbrace{q, \dots, q}_{n}) = q(\underbrace{p, \dots, p}_{k})$"



is the same as this



$\boxed{\text{once we've acted by a symmetry to put the entries in the same order}}$

NB. this only makes sense for symmetric operads.

In particular, we get a functor:

$$\Omega \times \Omega \longrightarrow \begin{pmatrix}\text{operads}\\ \text{in sets}\end{pmatrix} \xrightarrow{N} d\,Sets$$

$$(S, T) \longmapsto \Omega(S) \otimes \Omega(T) \longmapsto N(\Omega(S) \otimes \Omega(T$$

By (left) Kan extension, this gives a tensor product:

$$d\,Sets \times d\,Sets \xrightarrow{\otimes} d\,Sets$$

completely determined by the requirements:

- it preserves colimits in each variable separately
- for representables, $\Omega[S] \otimes \Omega[T] = N(\Omega(S) \otimes \Omega(T.$

Remarks

1) the functor $\tau : d\,Sets \longrightarrow$ Operads $\begin{pmatrix}\text{left adjoi}\\ \text{to the nerve}\end{pmatrix}$

is strictly monoidal in the sense that

$$\tau(X \otimes Y) \cong \tau(X) \underset{\substack{\uparrow\\ \text{tensor}\\ \text{of operads}}}{\otimes} \tau(Y)$$

(clear from the fact

that N is full and faithful

hence $\tau \circ N \cong Id$)

2) the embedding $i_! : sSets \hookrightarrow dSets$

is also strictly monoidal:

$$i_!(M \times N) = i_!(M) \times i_!(N)$$

(this can be checked directly on representables)

3) the unit is the representable

$$\eta = \Omega[1] = i_!(\Delta[0]).$$

How is $\Omega[S] \otimes \Omega[T]$ a colimit of representables?

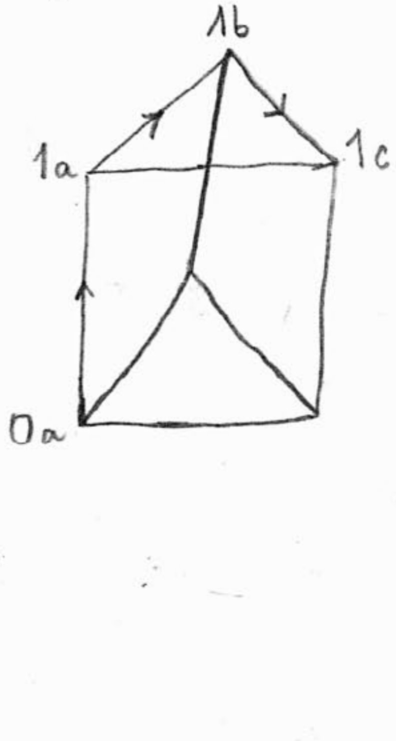
Special case of simplices:

recall $\Delta[n] \times \Delta[m] = \bigcup_{\sigma} \Delta[n+m]$

the union over all shuffles of the orders

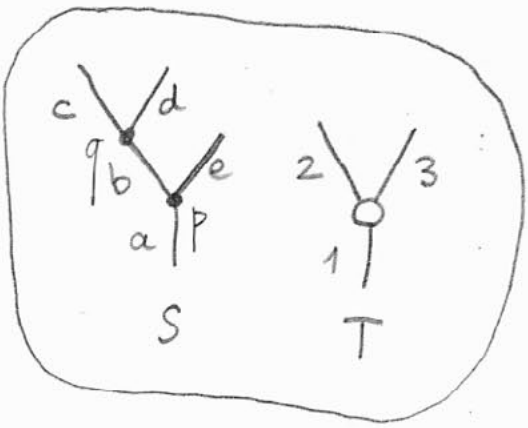
$\{0, \dots, n\}$ and $\{0, \dots, m\}$

$\Delta[1] \times \Delta[2]$

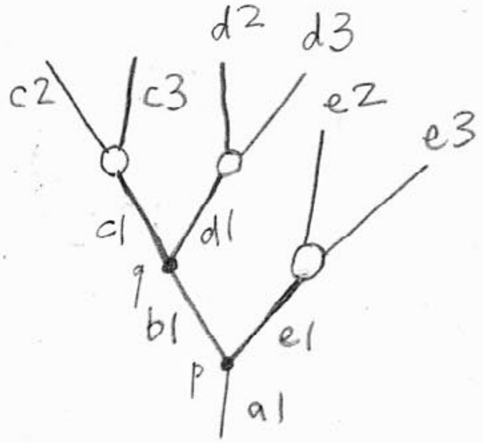


It works in this way for trees.

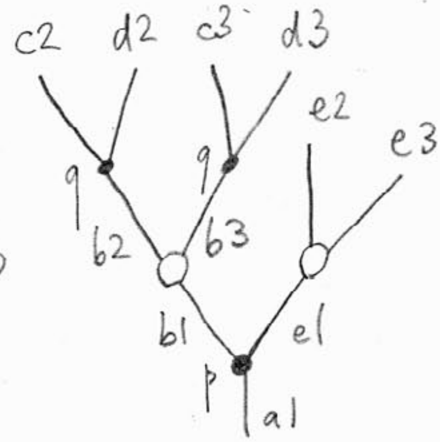
Let us compute $\Omega[S] \otimes \Omega[T]$ for



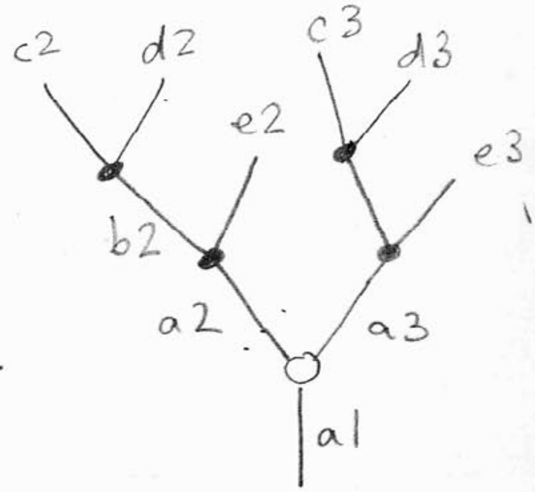
A



shuffles:



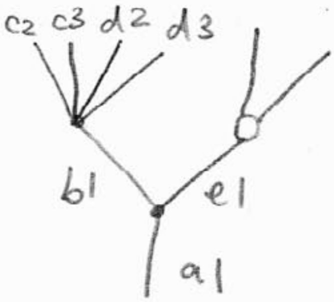
B



C

Remark:

A and B are glued along



which is a "codim 2 face" of A

and maps via a symmetry to a codim 2 face of B.

This is the general pattern for arbitrary trees S and T: start with copies of T on top of S and percolate vertices of T down through vertices of T, to finally arrive at "copies of S on top of T".

Normal objects and skeletal filtration

Every dendroidal set X can be written as

$$X = \bigcup X^{(n)}$$

where $X^{(n)}$ is the subpresheaf generated by $x \in X(T)$ where T has $\leq n$ vertices.

So $X^{(0)}$ is generated by $X(|) = \hom(\eta, X)$.

So all the non-degenerate elements of $X^{(n)}$ come from trees T with $\leq n$ vertices.

We get $X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X^{(n)} \subseteq \ldots$

Call X normal if $\mathrm{Aut}(T)$ acts freely on $X(T)$ for every tree T

<u>Prop</u> If X is normal, then for each $n \geqslant 1$ there is a pushout

$$\begin{tikzcd}
\coprod_{[e]} \partial\Omega[T_e] \arrow[r] \arrow[d, tail] & X^{(n-1)} \arrow[d] \\
\coprod_{[e]} \Omega[T_e] \arrow[r] & X^{(n)}
\end{tikzcd}$$

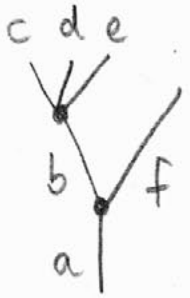
where e ranges over isomorphism classes of non degenerate elements $e \in X(T_e)$ where T_e has exactly n vertices, and

$$\partial\Omega[S] = \bigcup \Omega[F]$$

for all faces $F \rightarrowtail S$ for any tree S

("its boundary")

<u>Example:</u>

for $S =$  its boundary is "  "

Similarly, we call a mono $X \rightarrowtail Y$ <u>normal</u> if $\mathrm{Aut}(T)$ acts freely on $Y(T) \setminus X(T)$ for any tree T.

In that case, we can again write Y as the union of relative skeletons

$$X = Y_X^{(0)} \subseteq Y_X^{(1)} \subseteq \dots$$

where each $Y_X^{(n-1)} \longrightarrow Y_X^{(n)}$ is a pushout of a coproduct of boundary inclusions indexed by isomorphism classes of non degenerate elements $y \in Y(T) \setminus X(T)$ for T with n vertices.

<u>Another way of saying</u>:

the maps $\partial\Omega[T] \rightarrowtail \Omega[T]$

<u>generate</u> all the normal monos in dSets,

in exactly the same way $\partial\Delta[n] \to \Delta[n]$ generate all the monos in sSets

<u>Remarks</u>: ① every representable is normal

② if $Y \to X$, X normal $\Rightarrow$ Y normal

③ the presheaf P defined by

$$P(T) = \text{planar structures on } T$$

is normal.

④ $\Omega[S] \otimes \Omega[T]$ is normal.

⑤ $$\left(\partial\Omega[S] \otimes \Omega[T]\right) \bigcup_{\substack{\partial\Omega[S] \\ \otimes \\ \partial\Omega[T]}} \left(\Omega[S] \otimes \partial\Omega[T]\right) \to \Omega[S] \otimes \Omega[T]$$

mono, hence a normal mono.

From this we conclude by "induction" that

the maps $A \rightarrowtail X$, $B \rightarrowtail Y$ are normal monos $\Longrightarrow$ the map $A \otimes Y \bigcup_{A \otimes B} X \otimes B \rightarrowtail X \otimes Y$ is normal mono

Inner horns

Recall that in sSets, X is the nerve of a category if and only if any map

$$\Lambda^k[n] \longrightarrow X$$

extends <u>uniquely</u> to $\Delta[n]$,

or if and only if any map

$$Sc[n] \longrightarrow X$$

extends <u>uniquely</u> to $\Delta[n]$

Here $0 < k < n$ and $\Lambda^k[n]$ is the union of the faces of $\Delta[n]$ except the one opposite the vertex k, and $Sc[n]$ the <u>Segal core</u>

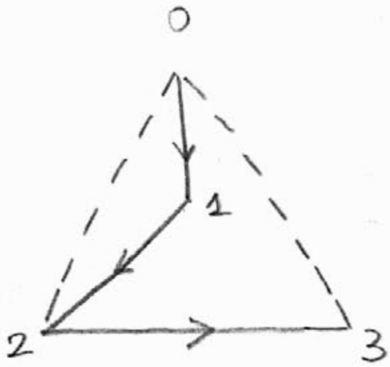
of $\Delta[n]$ is the union of the ribbons

(1-simplices $\Delta(i, i+1) \subset \Delta[n]$ $\quad i=0,\dots,n-1$)

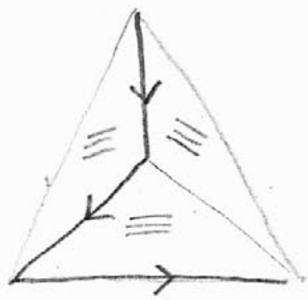
$n=2$

$\Lambda^1[2] = Sc[2]$

$n=3$

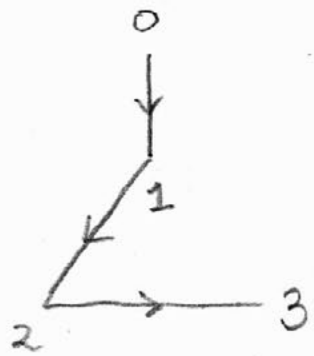


$\Delta[3]$



all faces except front

$\Lambda^1[3]$



$Sc[3]$

Recall that X is called an ∞-category if any $\Lambda^k[n] \longrightarrow X$ extends to $\Delta[n] \longrightarrow X$ not necessarily uniquely, for $0<k<n$.

We will say the same for dendroidal sets: if T is a tree and e is an inner edge in T, then $\Lambda^e[T] \subseteq \partial\Omega[T]$ is the union of all the faces except the one given by contracting e.

We call $\Lambda^e[T]$ an inner horn.

$Sc[T]$ is the union of all maps $\Omega[S] \rightarrowtail \Omega[T]$ where S has <u>at most</u> one vertex.

Now, $X \in dSets$ is the nerve of an operad iff for any tree T and inner edge e any map

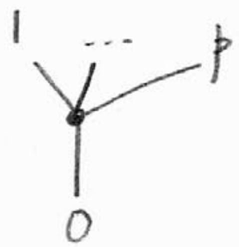
$$\Lambda^e[T] \longrightarrow X$$

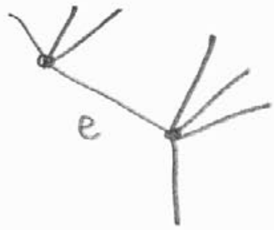
extends <u>uniquely</u> to $\Omega[T] \longrightarrow X$.

or iff $Sc[T] \longrightarrow X$ extends <u>uniquely</u> to a map $\Omega[T] \longrightarrow X$.

<u>Definition</u>: X is called an <u>∞-operad</u> if any map $\Lambda^e[T] \rightarrow X$ extends (not nec'ly uniquely) to a map $\Omega[T] \longrightarrow X$, for any inner edge e in any tree T.

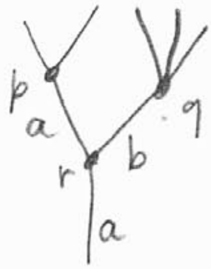
$T = |$ $\qquad Sc(T) = T$

$T = C_p =$  $\qquad Sc[T] = T$

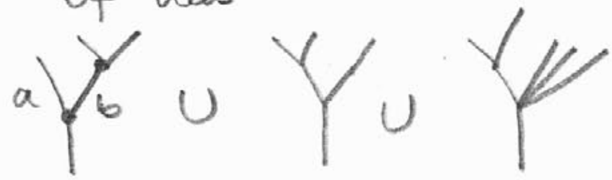
$T =$ 

$$\Lambda^e[T] = S_c[T]$$
$$= \Omega[C_3] \cup_\eta \Omega[C_4]$$

= "  "

$T =$ 

$\Lambda^a[T] =$ union of representables of trees



$\partial_p T \qquad \partial_q T \qquad \partial_b T$

$$S_c[T] = (C_p \underset{a}{\cup} C_r) \underset{b}{\cup} C_q$$

union of three corollas.

<u>Prop.</u> ① if A is normal
and X is an ∞-operad
then <u>Hom</u>(A, X) is again an ∞-operad

(here, <u>Hom</u> is the internal hom wrt $\otimes$, ie
$$\underline{Hom}(A, X)(T) = Hom_{dSets}(A \otimes \Omega[T], X).)$$

② in fact, a more general statement is true:
if $A \rightarrowtail B$ is a normal mono
and if $Y \to X$ has the Right Lifting Property
wrt all inner horn inclusions, then so does

$$\underline{\mathrm{Hom}}(B,Y) \longrightarrow \underline{\mathrm{Hom}}(A,Y) \times_{\underline{\mathrm{Hom}}(A,X)} \underline{\mathrm{Hom}}(B,X)$$

$$\begin{tikzcd}
\Lambda^e[T] \arrow[r] \arrow[d] & Y \arrow[d] \\
\Omega[T] \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

Right Lifting Property.

The proof boils down, by the usual induction on skeletons, to:

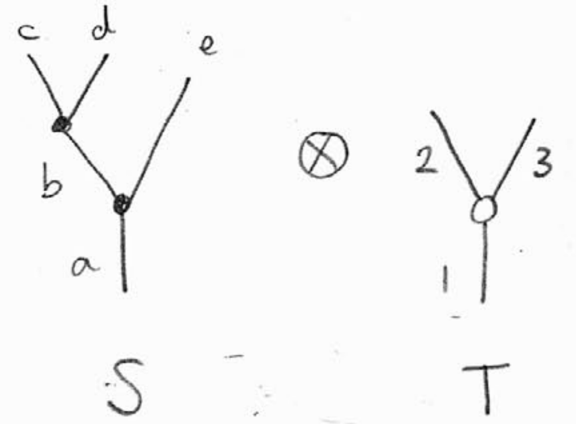
<u>Lemma</u>. For any two trees S and T and any inner edge $e \in S$ the map

$$\left(\Lambda^e[S] \otimes \Omega[T]\right) \cup \left(\Omega[S] \otimes \partial\Omega[T]\right) \rightarrowtail \Omega[S] \otimes \Omega[T]$$

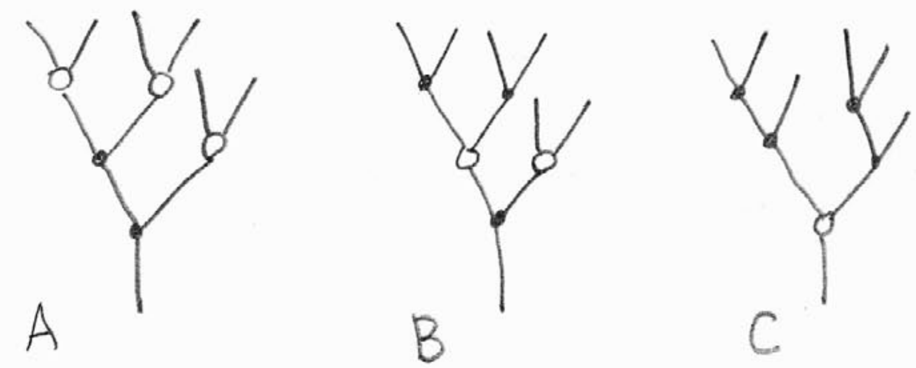
is a composition of pushouts of inner horn inclusions

<u>Proof</u> is hard.

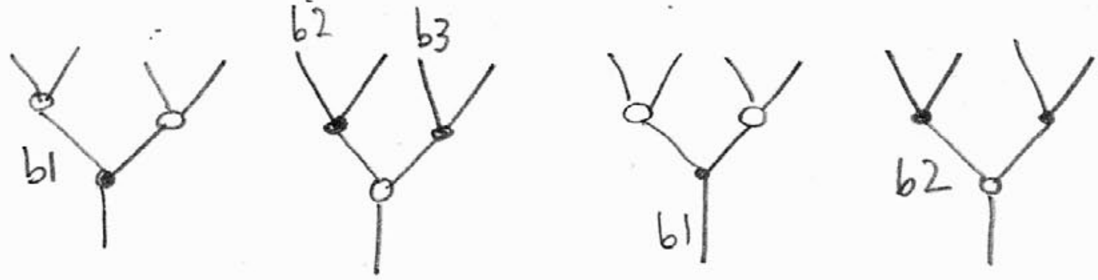
<u>Example</u>:



is the union of



$\Lambda^b[S] \otimes T$ is the union of

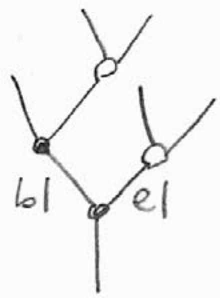


$S \otimes \partial T$ is the union of three copies of S labelled by 1, 2, 3 respectively.

How do we glue eg A to $\Lambda^b[S] \otimes T \cup S \otimes \partial T$?

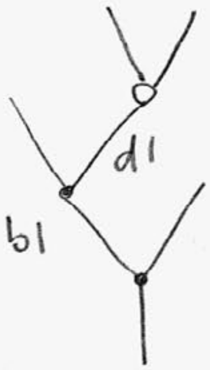
First, glue all the faces of A <u>except</u> the one given by contraction of b

Eg, glue



We do this by first glueing on all <u>its</u> faces except the one contracting b_1.

Eg



we do this by first glueing all <u>its</u> faces except the one for b1.

the two outer faces are already there, so we only need ∂_{d1}.

this has a unique inner horn.