Ieke Moerdijk - Dendroidal Topology I

Dendroidal Topology

operads

∞-operads

Simplicial Topology

category theory

classical Kan Quillen

∞-categories (Joyal Lurie)

Today: review of operads

We will work in the context of topological spaces (or even sets)

Def: an operad is a sequence $P = \{P(n)\}_{n \geq 0}$ (of spaces) where $P(n)$ is to be thought of the space of operations of n variables. Structure:

- Σ_n acts on $P(n)$, ("permute the variables")
- a substitution map $P(n) \times P(k_1) \times \dots \times P(k_n) \to P(k)$
  $k = k_1 + \dots + k_n$
- a unit $1 \in P(1)$ ("the identity operation")

Axioms: substitution is associative, 1 is 2-sided unit for substitution, and substitution is compatible with the Σ_n-actions

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Given an operad P, a P-algebra is a space A equipped with maps $P(n) \times A^n \longrightarrow A$ $(p, a_1, \dots, a_n) \mapsto p(a_1, \dots, a_n)$ compatible with the operad structure. Eg for the substituti[on]

$p(q_1, \dots, q_n) = \gamma(p, q_1, \dots, q_n)$ we have

$$p(q_1, \dots, q_n)(\vec{a}_1, \dots, \vec{a}_n) = p(q_1(\vec{a}_1), \dots, q_n(\vec{a}_n)) \quad \vec{a}_i \in A^{k_i}$$

So we have a category $|P\text{-Alg}|$ of P-algebras, with as morphisms continuous maps respecting the action of P.

Operads also form a category, where a morphism $P \xrightarrow{\varphi} Q$ is given by $P(n) \xrightarrow{\varphi_n} Q(n)$ compatible with the structure

Such a φ induces an obvious functor

$$\varphi^*: (Q\text{-Alg}) \longrightarrow (P\text{-Alg})$$

which has a left adjoint $\varphi_!$

<u>Notation</u>: for $p \in P(n)$, $q \in P(k)$, we write

$$p \circ_i q = p(\overset{1}{1}, \overset{2}{1}, \dots, \overset{i}{q}, 1, \dots, \overset{n}{1})$$

Obviously, everything can also be axiomatized in terms of these "circle-i operations".

<u>Examples.</u>

1) $P(n) = pt$ for all n, this is the operad Comm
its algebras are the commutative topological monoids.

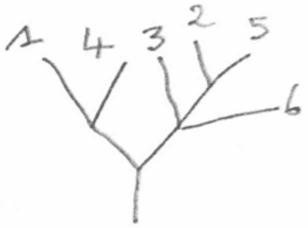
2) $P(n) = \Sigma_n$ for all n, this is the operad Ass
its algebras are the topological monoids.

3) $P(n) = \mathbb{R}[x_1, \dots, x_n]$ polynomials
or $P(n) = C^\infty(\mathbb{R}^n)$

<u>Remark.</u> If we leave out the Σ_n-actions we get the notion of a non-Σ operad
(the earlier ones are sometimes called <u>symmetric</u> <u>operads</u> for emphasis).

To every non-Σ operad P we can associate a symmetric one $\mathrm{Sym}(P)$ by freely adjoining the actions (left adjoint to the forgetful functor).

4) $P(n)$ = planar trees with n leaves numbered by $1, \dots, n$



$\in P(6)$

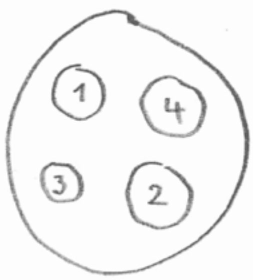
Σ_n acts by changing the numbering of the leaves

Substitution is defined by grafting.

5) D_d = "little d-dimensional disks" - operad

$D_d(n)$ = space of all configurations of n numbered closed balls in the d-dimensional unit balls with disjoint interiors.

Substitution: "rescale and plug in the pictures".



This operad (or its cubical version) has as algebras d-fold loop spaces, and lies at the origin of the theory of operads (Boardman-Vogt, Segal, May, ...)

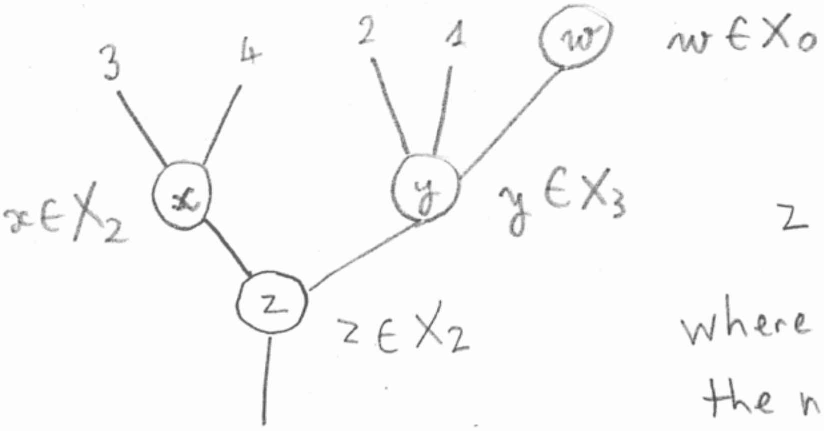
Rainer Vogt

6) if one is given a "collection", a sequence of spaces $X = \{X_n\}_{n \geq 0}$ where X is equipped with a Σ_n-action one can form the free operad $\mathrm{Free}(X)$

(left adjoint to the forgetful functor

$$\text{Operads} \longrightarrow \text{Collections} \;)$$

The points in $\mathrm{Free}(X)(n)$ are represented by planar trees with n numbered leaves, and vertices labeled by elements of $X(k)$ where $k = \#$ inputs of the vertex

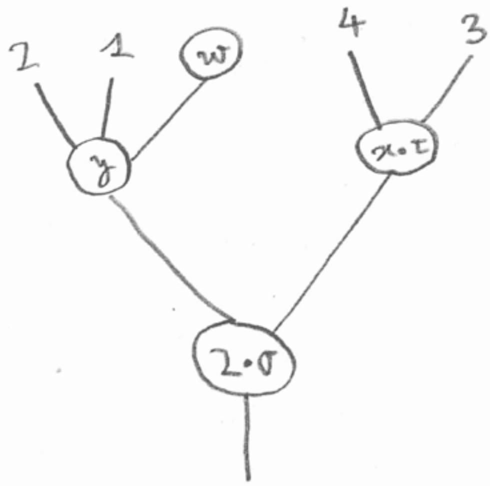


$z(x, y \circ_3 w) \cdot \sigma$

where σ codes the numbers of the leaves.

The points are equivalence classes of such, where the equivalence relation is given by (non-planar) isomorphisms of trees and the Σ_n-actions on the X_n

eg. the element in $\mathrm{Free}(X)(4)$ just described can also be represented by



etc...

the topology is the obvious one.

Operadic composition is grafting.

Identity is $|$.

A <u>coloured</u> <u>operad</u> is a pair (P, C)
where C is a <u>set</u> (of "colours")
and for each sequence $c_1, \dots, c_n, c$ of elements in C $(n \geq 0)$
P provides a space

$P(c_1, \dots, c_n ; c)$ "of operations taking n inputs of type $c_1, \dots, c_n$ to an output of type c".

<u>Structure maps</u>: Σ_n acts, substitution, units

- $1_c \in P(c;c)$
- if $p \in P(c_1, \dots c_n; c)$ then $p \cdot \sigma \in P(c_{\sigma^{-1}(1)} \dots c_{\sigma^{-1}(n)}; c)$
- substitution

$$P(c_1, \dots, c_n; c) \times P(\vec{a_1}, c_1) \times \dots \times P(\vec{a_n}, c_n) \longrightarrow P(\vec{a_1}, \dots, \vec{a_n}, c)$$

<u>axioms</u> as before.

<u>An algebra</u> A for such a coloured operad is a sequence of spaces $A = \{A_c\}_{c \in C}$
together with maps

$$P(c_1, \dots, c_n, c) \times A_{c_1} \times \dots \times A_{c_n} \longrightarrow A_c$$

compatible with the structure of P.

<u>Examples</u>.

1) there is a coloured operad on $C = \{m, e\}$ whose algebras A are a monoid $M = A_m$ together with a space $E = A_e$ on which M acts.

Exercise: Spell out $P(c_1, \dots, c_n, c)$ for all sequences $c_1, \dots, c_n, c$.

2) there is a coloured operad Cat_S for any set S whose set of colours is $S \times S$ and whose algebras are categories with S as set of objects:

$$\mathrm{Cat}_S(-;(s,s)) = *$$

$$\mathrm{Cat}_S\left((s_1,s_1'),\dots,(s_n,s_n');(t,t')\right) = \begin{cases} \phi \text{ otherwise} \\ * \text{ if } s_1 = t \quad s_n' = t' \\ \qquad s_1' = s_2,\ s_2' = s_3, \dots \end{cases}$$

(this $*$ is the operation of composition).

<u>Remark</u>: this is a <u>non-Σ coloured operad</u>.

3) if $\mathbb{C}$ is a category we can view it as a coloured operad $P_{\mathbb{C}}$

$$P_{\mathbb{C}}(c_1,\dots,c_n;c) = \phi \text{ if } n \neq 1$$

$$P_{\mathbb{C}}(c_1;c) = \mathbb{C}(c_1,c)$$

the algebras are covariant functors $\mathbb{C} \longrightarrow$ Top spaces

4) if $\{X_{c_1,\dots,c_n;c}\}_{(c_1,\dots,c_n,c)\in C^{n+1}}$ is a "coloured collection"

(with $\sigma \in \Sigma_n$ acting on $\coprod_{\tau \in \Sigma_n} X_{c_{\tau^{-1}(1)},\dots,c_{\tau^{-1}(n)};c}$)

one can construct the free coloured operad Free(X) much like before but with trees whose edges are coloured by elements of C).

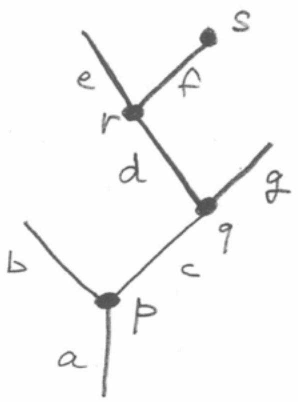
5) if T is a tree (finite, rooted, not necessarily planar)

then it freely generates an operad $\Omega(T)$

whose colours are the edges of T

and whose operations are generated by the vertices of T.

E.g.



edges a b c d e f g

vertices p q r s

- Colours of $\Omega(T) = \{a, b, \dots, g\}$
- generating operations

$p \in \Omega(T)(b, c; a)$

$q \in \Omega(T)(d, g; c)$

$r \in \Omega(T)(e, f; d)$

$s \in \Omega(T)(-; f)$

From these we get generated operations like

- $1_a \; 1_b \; 1_c$
- $p \circ_2 q \in \Omega(T)(b, d, g; a)$
- $q \cdot \sigma \in \Omega(T)(g, d; c)$ etc, etc...

Nota Bene: this operad has generators defined in terms of a planar structure of the tree.

Another planar structure would give different generators but the same operad (a coloured operad in Sets)

Remark: Just like before if P is a coloured operad then its P-algebras form a category.

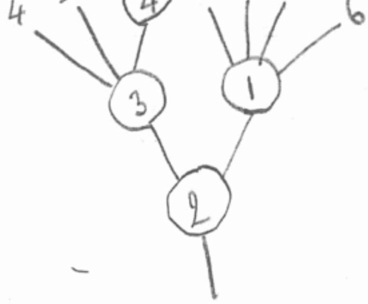
And the collection of all coloured operads forms a category: a map $(P,C) \xrightarrow{(f,\varphi)} (Q,D)$ is a pair consisting of a function $f: C \longrightarrow D$ and for each $c_1,\dots,c_n \in C$ a map

$$\varphi = \varphi_{c_1,\dots,c_n;c} : P(c_1,\dots,c_n;c) \longrightarrow Q(fc_1,\dots,fc_n;fc)$$

respecting the structure. We again have a pullback operation $(f,\varphi)^*$ on the algebras with a left adjoint $(f,\varphi)_!$

6) there is a coloured operad $\underline{\mathrm{Oper}}$ whose algebras are (non-coloured or ordinary or monochromati[c]) operads: its set of colours is $\mathbb{N}$

$\mathrm{Oper}(n_1,\dots,n_k;n)$ = equivalence classes of planar trees with n leaves numbered by $1,\dots,n$ and k vertices numbered $1,\dots,k$ such that the vertex numbered i has n_i input edges



represents an element T of

$$T \in \underline{\mathrm{Oper}}(4,2,3,0;6)$$

If P is an operad, $p \in P(4)$, $q \in P(2)$
$r \in P(3)$, $s \in P(0)$

we have to produce

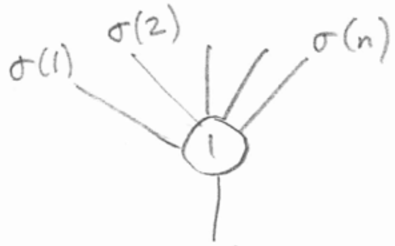
$$T(p,q,r,s) \in P(6)$$

solution:

$\boxed{q(r \circ_3 s, p) \cdot \tau}$ where τ is the element in Σ_6 given by the numbering of the leaves (wrt. their planar order).

This defines symmetric operads

$\sigma \in \Sigma_n$ acts on $P(n)$ as the action of the tree



or perhaps $(\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$

The Boardman-Vogt resolution of an operad

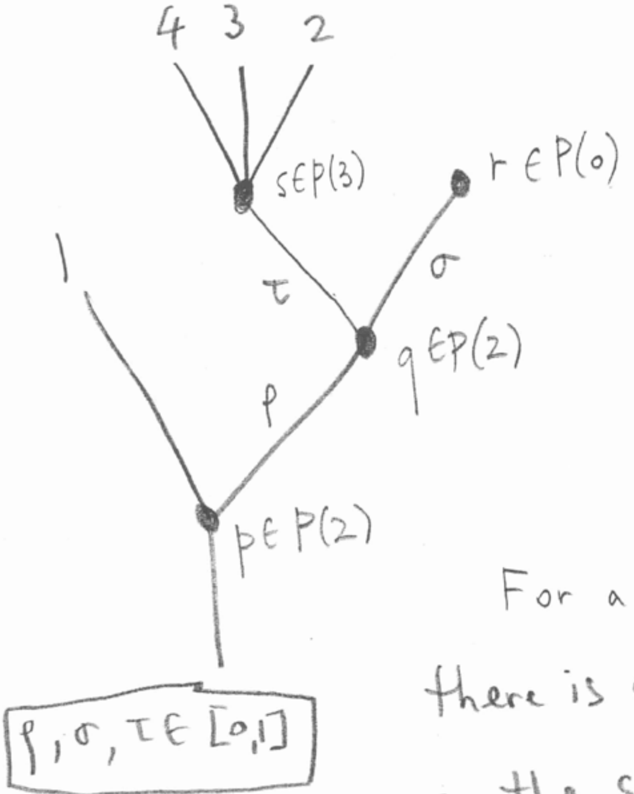
It assigns to an operad P a new operad $W(P)$ and a map. $S: W(P) \longrightarrow P$

We will see that $W(P)$ is a cofibrant resolution of P, and that "having a $W(P)$-algebra structure" is invariant under homotopy equivalence of spaces]

We will do the monochromatic case.

$W(P)(n)$ will be a space, whose elements are represented by labelled planar trees

- the leaves are labelled $1, \dots, n$
- the vertices are labelled by points of P.
- the inner edges will be labelled by a length $\tau \in [0,1]$



represents an element in $W(P)(4)$.

For a fixed tree T there is an obvious topology on the set of labellings

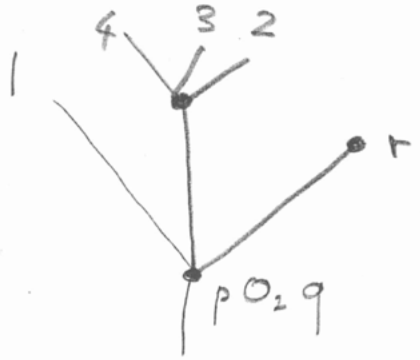
$$P(2) \times P(2) \times P(0) \times P(3) \times [0,1]^3$$

Equivalence relation is generated by three ingredients

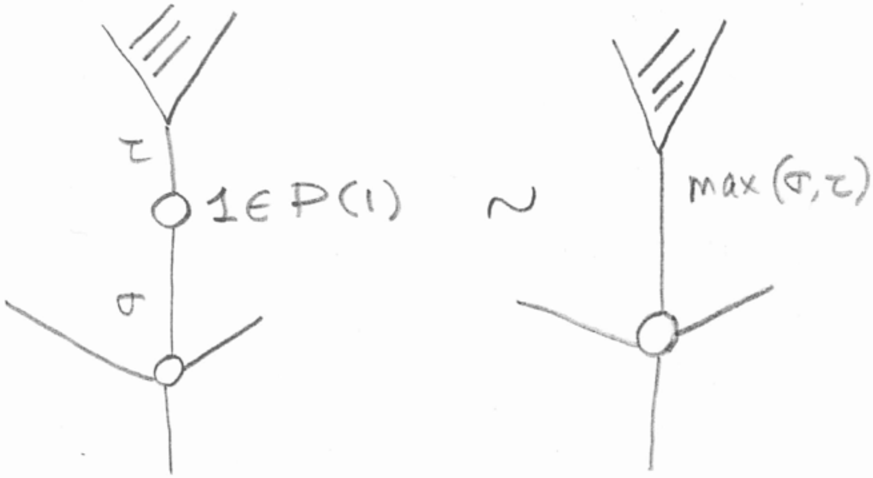
① – non planar isomorphisms (as in the free operad)

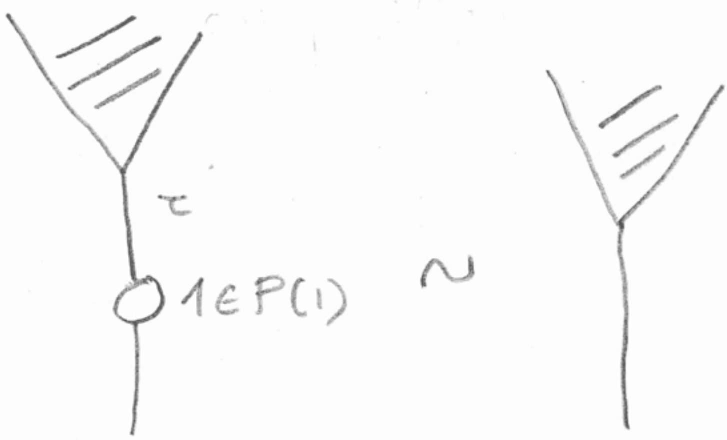
② – if a length is 0, contract the edge and compose in P

Eg. if $p=0$ then this labelled tree represents the same point in $W(P)(4)$ as



③ if a vertex is labelled by $1 \in P(1)$ erase the vertex and take the max of the lengths (if the resulting edge is inner)





(forget τ)

Operadic composition again by grafting with newly arising internal edges of length 1.

There are maps

$$P \xrightarrow{\eta} W(P) \xrightarrow{\varepsilon} P$$

ε takes all the lengths to 0 : this is an operad map.

$\eta(P) :=$  is <u>not</u> an operad map

but only a map of collections.

So we do get a diagram in the category of operads:

$$\begin{tikzcd} \mathrm{Free}(\mathrm{forget}(P)) \arrow[rr] \arrow[dr, "\bar{\eta}"'] & & P \\ & W(P) \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

this is a good resolution

if Σ_n acts freely on $P(n$

($P(n)$ should be a Σ_n-CW-complex