Ieke Moerdijk   Dendroidal topology IV

Quillen model categories

Simplicial sets

Dendroidal sets

Tomorrow — comparison to Lurie's ∞-operads.

<u>Definition</u>: A Quillen model structure on a category $\mathcal{M}$ is given by three classes of maps:

fibrations, cofibrations, weak equivalences

satisfying axioms CM1–5.

<u>CM1</u> $\mathcal{M}$ has finite $\varprojlim$ and $\varinjlim$

<u>CM2</u> all three classes are closed under retract.

<u>CM3</u> if two out three maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{g\circ f} Z$ are weak equivalences, so is the third.

<u>CM4</u> in a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & X \end{tikzcd}$$

with u a cofibration and p a fibration, a diagonal h with $ph = g$ & $hu = f$ exists as soon as one of p or u is a weak equivalence

CM5: Every map $X \xrightarrow{f} Y$ can be factored as a fibration $Z \xrightarrow{p} Y$ preceded by a cofibration $X \xrightarrow{u} Z$ in two ways, one in which u is also a weak eq. and one in which p is also a we.

Remarks: (a) for CM2, say $A \xrightarrow{f} B$ is a retract of $C \xrightarrow{g} D$ if there is a diagonal

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"'] & C \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] & ri = 1_A \\ B \arrow[r, hook, "j"'] & D \arrow[r, "s"'] & B & sj = 1_B \end{tikzcd}$$

(b) A (co)fibration which is also a we is called a trivial (co) fibration

(c) An object X is called fibrant if $X \longrightarrow 1$ is a fibration (here, 1 is the terminal object) and cofibrant if $0 \rightarrow X$ is a cofibration.

(d) for an arbitrary object X we can factor the maps

$$0 \longrightarrow X \longrightarrow 1$$

as in CM5

$$\begin{tikzcd} & & X_f \arrow[dr, "\text{fib}"] & \\ 0 \arrow[r] \arrow[dr] & X \arrow[ur, tail, "\text{trivial cof}"] \arrow[rr] & & 1 \\ & X_c \arrow[u, "\text{trivial fib}"'] & & \end{tikzcd}$$

to get a fibrant and cofibrant resolution (resp. X_f and X_c) of the object X.

(e) the "category of model categories" has much better closure conditions if one strengthens the axioms a bit, giving the axioms of a <u>cofibrantly generated</u> model category.

In general, a map $Y \xrightarrow{p} X$ is a (trivial) fibration iff it has the RLP (Right Lifting Property) wrt all)trivial(cofibrations. $\mathcal{M}$ is called <u>cofibrantly generated</u> when there are two <u>sets</u> I and J of cofibrations and trivial cofibrations which already detect the trivial fib's resp. the fib's.

(f) By the iff just mentioned, the class of fibrations (or trivial fibrations is closed under pullbacks & composition.
Dually, the class of cofibrations (or trivial cofibrations) is closed under pushout & composition.

(g) the closure property becomes even better if $\mathcal{M}$ is left proper, meaning that We's are closed under pushout along any cof.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\sim"'] & C \arrow[d, "\sim"] \\ B \arrow[r, tail] & D \end{tikzcd}$$

The classical ("Kan-Quillen") model structure on sSets

- The cofibrations are the monomorphisms (every object is cofibrant)
- the weak equivalences $X \xrightarrow{f} Y$ are the maps with the property that

$$\pi_n(|X|, x) \xrightarrow{\sim} \pi_n(|Y|, fx)$$

is an iso for every $n \geqslant 0$ and every vertex $x \in X_0$.

Here, $|-|$ is the geometric realization.

- A map $Y \xrightarrow{f} X$ is a fibration iff it has the RLP with respect to all inclusions of the form

$$\Lambda^k[n] \longrightarrow \Delta[n] \qquad 0 \leq k \leq n$$

($\Lambda^k[n]$ is the union of all the faces containing the k-th vertex)

these are called Kan fibrations.

Not every object is fibrant

e.g. for a small category $\mathbb{C}$

its nerve $N(\mathbb{C})$ $\qquad N(\mathbb{C})_n = \{ x_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} x_n \text{ in } \mathbb{C} \}$

is fibrant iff $\mathbb{C}$ is a groupoid.

In particular, none of the representables $\Delta[n]$ is fibrant.

How to make an object fibrant?

Given $X \in sSets$, look at all the maps

$$\Lambda^k[n] \xrightarrow{\alpha} X \quad \left(\begin{array}{ll} \text{all} & 0 \leq k \leq n \\ n = n_\alpha & k = k_\alpha \end{array} \right)$$

If X were fibrant, these would extend to $\Delta[n]$

Force this by writing X' to be the pushout

$$\begin{tikzcd} \coprod_\alpha \Lambda^{k_\alpha}[n_\alpha] \arrow[r] \arrow[d, tail] & X \arrow[d, tail, dashed] \\ \coprod_\alpha \Delta[n_\alpha] \arrow[r, dashed] & X' \end{tikzcd}$$

Now X' looks a little like a fibrant object

because we have an extension

$$\begin{tikzcd} \Lambda^k[n] \arrow[r, "f"] \arrow[d] & X' \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

whenever f lands in $X \subseteq X'$. Now repeat,

to get $X' \rightarrowtail X'' \rightarrowtail \cdots$

and let X_∞ to be the colimit.

This is a fibrant resolution of X because any map

$\Lambda^k[n] \longrightarrow X_\infty$ lands in some $X^{(p)} = X^{'''\cdots'''}$ (p power and thus extends into $X^{(p+1)} \subseteq X_\infty$.

The map $X \longrightarrow X_\infty$ is a trivial cofibration because it is obtained from inclusion of the form

$$\Lambda^k[n] \hookrightarrow \Delta[n]$$

by coproducts, pushouts, and (infinite) composition.

"<u>the small object argument</u>".

<u>Remark</u>: another way to get a fibrant replacement of X is to take Kan's $Ex_\infty(X)$

this has the advantage of preserving products:

$$Ex_\infty(X \times Y) \xrightarrow{\cong} Ex_\infty(X) \times Ex_\infty(Y)$$

the other construction simply gives a we

$$(X \times Y)_\infty \longrightarrow X_\infty \times Y_\infty.$$

<u>How to compare model categories?</u>

If $\mathcal{M}$ is a model category, it has an associated homotopy category $Ho(\mathcal{M})$

$$Ho(\mathcal{M}) := \mathcal{M}\left[(we)^{-1}\right]$$
$$\cong \mathcal{M}_{cf}\left[(we)^{-1}\right]$$

$\mathcal{M}_{cf}$ = full subcategory of objects which are both cofibrant & fibrant

The last one can be concretely described in terms of cofibrant & fibrant objects & "homotopy classes of maps

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A pair of adjoint functors $\mathcal{M} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{N}$ between model categories is called a <u>Quillen pair</u> if $f_!$ preserves cofibrations & trivial cof's (equivalently, f^* preserves trivial fib's & fib's).

Such a Quillen pair induces an adjoint pair

$$Ho(\mathcal{M}) \underset{Rf^*}{\overset{Lf_!}{\rightleftarrows}} Ho(\mathcal{N}).$$

$(Lf_!)(M) = f_!(M_c)$ for a cof. replacement $M_c \overset{\sim}{\twoheadrightarrow} M$

$(Rf^*)(N) = f^*(M_f)$ for a fibrant repl. $N \overset{\sim}{\rightarrowtail} N_f$

<u>Remark</u>: this model structure is left proper and cofibrantly generated with

$$I = \{\partial\Delta[n] \rightarrowtail \Delta[n],\ n \geq 0\}$$

$$J = \{\Lambda^k[n] \rightarrowtail \Delta[n],\ 0 \leq k \leq n\}$$

The Joyal model structure on sSets

- cofibrations are again all the monos
- the fibrant objects are the "∞-categories"

ie those X for which extension i

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, tail] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

always exists if $0 < k < n$.

Now $N(\mathbb{C})$ is <u>always</u> <u>fibrant</u>!

<u>Remarks</u>: (a) if A is arbitrary and X is fibrant then X^A is fibrant.

(apply the fact that

$$\Lambda^k[n] \times \Delta[m] \cup \Delta[n] \times \partial\Delta[m] \rightarrowtail \Delta[n] \times \Delta[m]$$

is a composition of pushouts of "inner horns"

$$\Lambda^i[p] \rightarrowtail \Delta[p] \qquad 0 < i < p).$$

b) We'll use a special category $0 \overset{\sim}{\longleftrightarrow} 1$

and its nerve J

$$\begin{tikzcd} \Delta[0] = 1 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \Delta[1] \arrow[d, tail] \\ & J \end{tikzcd}$$

($\Leftarrow$ fibrant replacement of $\Delta[1]$ in the classical structure)

We can use $1 \rightrightarrows J$ to define homotopy of maps and homotopy equivalence.

Two maps $f, g : X \rightrightarrows Y$ are homotopic if there is a diagram

$$\begin{tikzcd} \Delta[0] \times X \arrow[r, shift left, "0 \times X"] \arrow[r, shift right, "1 \times X"'] \arrow[d, equal] & J \times X \arrow[d, "H"] \\ X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

Now, say that a map $A \longrightarrow B$ is a weak equivalence if $X^B \longrightarrow X^A$ is a homotopy equivalence for every fibrant X.

Now define fibration as the maps with the RLP wrt. the trivial cofibrations.

Thm (Joyal) this defines a Quillen model structure on sSets, whose fibrant objects are indeed the ∞-categories)

- it is cofibrantly generated
- it is left proper
- for a functor $f: \mathbb{C} \to \mathbb{D}$ the map

  $N\mathbb{C} \longrightarrow N\mathbb{D}$ is a we iff

  $\mathbb{C} \longrightarrow \mathbb{D}$ is an equivalence of categories.

In contrast, for the classical structure, we have that a map $f: U \to V$ of spaces is a homotopy eq. iff $\mathrm{Sing}(U) \longrightarrow \mathrm{Sing}(V)$ is a we of simplicial sets)

singular complexes

- for a map $f: X \to Y$ between ∞-categories, the following are equivalent:

  1) f is a fibration

  2) f has the RLP (a) wrt $\Lambda^k[n] \rightarrowtail \Delta[n]$ $0 < k < n$

  (f is an inner Kan fibration)

  and (b) wrt $\Delta[0] \xrightarrow{i} J$ $(i = 0, 1)$.

  3) like 2) with (b) strengthened as

$$A \times J \cup B \times \Delta[0] \hookrightarrow B \times J$$

for any $A \rightarrowtail_{\text{mono}} B$

("homotopy lifting & extension")

<u>Remark</u>: so we have an explicit small set of trivial cof's to test whether a map between ∞-categories is a fibration. But although the model structure is cofibrantly generated, there is no such explicit J known which detects fibrations between arbitrary objects.

- For a map $f : X \to Y$ between ∞-categories tfae
  - f is we
  - f is a J-homotopy equivalence
  - f is essentially surjective & fully faithful in the following sense.

If $x, y \in X_0$ are two objects we can define $X(x,y)$ by the pullback

$$\begin{tikzcd}
X(x,y) \arrow[r] \arrow[d] & X^{\Delta[1]} \arrow[d, "(ev_0, ev_1)"] \\
1 \arrow[r, "(x,y)"] & X \times X
\end{tikzcd}$$

<u>Prop:</u> $X(x,y)$ is a Kan complex.

Say f is <u>f&f</u> iff $X(x,y) \longrightarrow Y(fx, fy)$ is a weak equivalence (in the classical model structur

It remains to define "essentially surjective".

For a 1-simplex, $\Delta[1] \xrightarrow{\alpha} X$

an "arrow" $\alpha : x \longrightarrow y$

$$x = d_1\alpha, \qquad y = d_0\alpha$$

Say α is an equivalence if there are 2-simplices

$$\Delta[2] \overset{H}{\underset{K}{\rightrightarrows}} X$$

which looks like

$$\begin{tikzcd}
0 \arrow[r, "\alpha"] \arrow[dr, equal, "\text{degenerate } x"'] & 1 \arrow[d, dashed, "\beta"] \arrow[dl, phantom, "H"] \\
& 2
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, dashed, "y"] \arrow[dr, equal, "\text{degenerate } y"'] & 1 \arrow[d, "\alpha"] \arrow[dl, phantom, "K"] \\
& 2
\end{tikzcd}$$

this is equivalent to the condition that

$\Delta[1] \xrightarrow{\alpha} X$ extends to $J \longrightarrow X$.

(harder, see Joyal, JPAA)

Now, $X \xrightarrow{f} Y$ is called <u>essentially surjective</u> if for any $y \in Y_0$ there exists an $x \in X_0$ and an equivalence $f(x) \xrightarrow{\alpha} y$.

<u>Rk</u>: An ∞-category is a Kan complex (ie fibrant in the classical structure) iff every 1-simplex is an equivalence

(cf $N(\mathbb{C})$ is Kan iff $\mathbb{C}$ is a groupoid).

	Joyal	classical
cof	same	same
we's	less	more
fibrant objects	more	less

So we have a Quillen pair

$$(sSets)_{Joyal} \underset{id}{\overset{id}{\rightleftarrows}} (sSets)_{classical}$$

We'll look at

$$\mathrm{Ho}(\mathrm{sSets}_{\mathrm{Joyal}}) \underset{\mathrm{Rid}}{\overset{\mathrm{Lid}}{\rightleftarrows}} \mathrm{Ho}(\mathrm{sSets}_{\mathrm{class}})$$

$(\mathrm{Rid})\circ(\mathrm{Lid}) \longleftarrow X$ is a we for every X

hence an iso in $\mathrm{Ho}(\mathrm{sSets}_{\mathrm{classical}})$.

So we have a full reflective subcategory

$$\mathrm{Ho}(\mathrm{sSets}_{\mathrm{classical}}) \hookrightarrow \mathrm{Ho}(\mathrm{sSets}_{\mathrm{Joyal}})$$

this is an example of "left Bousfield localization".

(somewhat analogous to sheaves & presheaves).