Definition: A Quillen model structure on a category $\mathcal{M}$ is given by three classes of maps:
- fibrations,
- cofibrations,
- weak equivalences
satisfying axioms CM1–5.

CM1 \( \mathcal{M} \) has finite \( \lim \) and \( \lim \)

CM2 all three classes are closed under retract.

CM3 if two out of three maps $X \xrightarrow{f} Y \xrightarrow{g} Z$
and $X \xrightarrow{g \circ f} Z$ are weak equivalences,
so is the third.

CM4 in a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{p} \\
B & \xrightarrow{g} & X
\end{array}
\]

with \( u \) a cofibration and \( p \) a fibration,
a diagonal \( h \) with \( ph = g \) & \( hu = f \) exists
as soon as one of \( p \) or \( u \) is a weak equivalence.
CM5: Every map $X \xrightarrow{f} Y$ can be factored as a fibration $Z \xrightarrow{p} Y$ preceded by a cofibration $X \xrightarrow{u} Z$ in two ways, one in which $u$ is also a weak eq and one in which $p$ is also a we.

Remarks: (a) for CM2, say $A \xrightarrow{f} B$ is a retract of $C \xrightarrow{g} D$ if there is a diagonal

\[
\begin{array}{ccc}
A & \xrightarrow{i} & C \\
\downarrow{f} & & \downarrow{g} \\
B & \xrightarrow{j} & D \\
\downarrow{s} & & \downarrow{f} \\
& B & \xrightarrow{s} B
\end{array}
\]

$ri = 1_A$ and $sj = 1_B$

(b) A (co)fibration which is also a we is called a trivial (co)fibration.

(c) An object $X$ is called fibrant if $X \rightarrow 1$ is a fibration (here, 1 is the terminal object) and cofibrant if $0 \rightarrow X$ is a cofibration.

(d) for an arbitrary object $X$ we can factor the maps $0 \rightarrow X \rightarrow 1$ as in CM5

[Diagram]

\[
\begin{array}{ccc}
0 & \rightarrow & X \\
\downarrow{Xe} & & \downarrow{Xf} \\
& X & \rightarrow 1 \\
& \xrightarrow{fib} & \xrightarrow{trivial cof}
\end{array}
\]
to get a fibrant and cofibrant resolution (resp. Xf and Xc) of the object X.

(e) The "category of model categories" has much better closure conditions if one strengthens the axiom a bit, giving the axioms of a cofibrantly generated model category.

In general, a map $Y \to X$ is a (trivial) fibration iff it has the RLP (Right Lifting Property) wrt all (trivial) cofibrations. M is called cofibrantly generated when there are two sets $I$ and $J$ of cofibrations and trivial cofibrations which already detect the trivial fib's resp. the fib's.

(f) By the iff just mentioned, the class of fibrations (or trivial fibrations) is closed under pullbacks & composition. Dually, the class of cofibrations (or trivial cofibrations) is closed under pushout & composition.

(g) The closure property becomes even better if M is left proper, meaning that we's are closed under pushout along any cof.

\[
\begin{array}{ccc}
A & \to C \\
\downarrow^2 & & \downarrow^2 \\
B & \to D
\end{array}
\]
The classical ("Kan-Quillen") model structure on sSets

- The cofibrations are the monomorphisms (every object is cofibrant)
- The weak equivalences \( X \xrightarrow{f} Y \) are the maps with the property that
  \[
  \pi_n \left( |X|, x \right) \cong \pi_n \left( |Y|, fx \right)
  \]
  is an iso for every \( n \geq 0 \) and every vertex \( x \in X_0 \).
  Here, \(|-|\) is the geometric realization.
- A map \( Y \xrightarrow{f} X \) is a fibration iff it has the RLP with respect to all inclusions of the form
  \[
  \Lambda^k [n] \rightarrow \Delta [n] \quad 0 \leq k \leq n
  \]
  \( \Lambda^k [n] \) is the union of all the faces containing the \( k \)-th vertex.
  These are called Kan fibrations.

Not every object is fibrant

- e.g. for a small category \( C \), its nerve \( N(C) \)
  \[
  N(C) = \bigoplus \text{in } C
  \]
  is fibrant iff \( C \) is a groupoid.

In particular, none of the representables \( \Delta [n] \) is fibrant.
How to make an object fibrant?

Given $X \in sSet$, look at all the maps

$$\Lambda^k [n] \xrightarrow{\alpha} X \quad (\text{all } 0 \leq k \leq n)$$

If $X$ were fibrant, these would extend to $\Delta [n]$

Force this by writing $X'$ to be the pushout

$$\begin{array}{ccc}
\Lambda^k [n] & \xrightarrow{\alpha} & X' \\
\downarrow & & \downarrow \\
\Delta [n] & \xrightarrow{\gamma} & X'
\end{array}$$

Now $X'$ looks a little like a fibrant object because we have an extension

$$\Lambda^k [n] \xrightarrow{f} X'$$

Whenever $f$ lands in $X \subseteq X'$. Now repeat, to get $X' \xrightarrow{} X'' \xrightarrow{} \ldots$

and let $X_\infty$ to be the colimit.

This is a fibrant resolution of $X$ because any map

$$\Lambda^k [n] \xrightarrow{} X_\infty$$

lands in some $X^{(p)} = X''' \ldots$
and thus extends into $X^{(p+1)} \subseteq X_\infty$.

The map $X \to X_\infty$ is a trivial cofibration because it is obtained from inclusion of the form

$$\Lambda^k [n] \to \Delta [n]$$

by coproducts, pushouts, and (infinite) composition

"the small object argument".

Remark: another way to get a fibrant replacement of $X$ is to take Kan's $\text{Ex}_\infty(X)$.

This has the advantage of preserving products:

$$\text{Ex}_\infty(X \times Y) \cong \text{Ex}_\infty(X) \times \text{Ex}_\infty(Y)$$

The other construction simply gives a weak equivalence

$$(X \times Y)_\infty \to X_\infty \times Y_\infty$$

How to compare model categories?

If $M$ is a model category, it has an associated homotopy category $\text{Ho}(M)$

$$\text{Ho}(M) := M [(\text{we})^{-1}]$$

$\cong M_{cf} [(\text{we})^{-1}]$

$M_{cf}$ = full subcategory of objects which are both cofibrant and fibrant
the last one can be concretely described in terms of cofibrant & fibrant objects & "homotopy classes of maps

A pair of adjoint functors \( M \xleftarrow{f^*} N \xrightarrow{f_!} \)

between model categories is called a Quillen pair

if \( f_! \) preserves cofibrations & trivial cof's

equivalently, \( f^* \) preserves trivial fib's & fib's

Such a Quillen pair induces an adjoint pair

\[
\begin{align*}
  \text{Ho}(M) & \xleftarrow{\text{LF}_!} \text{Ho}(N) \\
  \text{RF}^* & \xrightarrow{\text{RF}_!} \text{Ho}(N)
\end{align*}
\]

\((\text{LF}_!)(M) = f_!(M_c) \) for a cof. replacement \( M_c \rightarrow M \)

\((\text{RF}^*)(N) = f^*(M_f) \) for a fibrant repl. \( N \rightarrow N_f \)

Remark: This model structure is left proper
and cofibrantly generated with

\[ J = \{ \exists \Delta[n] \rightarrow \Delta[n], \quad n \geq 0 \} \]

\[ J = \{ \Lambda^k[n] \rightarrow \Delta[n], \quad 0 \leq k \leq n \} \]
The Joyal model structure on sSets
- cofibrations are again all the monos
- the fibrant objects are the "\(\alpha\)-categories"
  ie those \(X\) for which extension i

\[
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Delta[n] & \leftarrow & I
\end{array}
\]

always exists if \(0 < k < n\).

Now \(N(C)\) is always fibrant!

Remarks: (a) if \(A\) is arbitrary and \(X\) is fibrant
then \(X^A\) is fibrant.

(apply the fact that

\[
\Lambda^k[n] \times \Delta[m] \cup \Delta[n] \times \Delta[m] \longrightarrow \Delta[n] \times \Delta[m]
\]

is a composition of pushouts of "inner horns"

\[
\Lambda^i[p] \longrightarrow \Delta[p] \quad 0 < i < p
\]

b) We'll use a special category \(0 \overset{\sim}{\leftarrow} 1\)

and its nerve \(\mathbf{J}\)
\[
\Delta[0] \times X \xrightarrow{1 \times X} J \times X \\
\| \quad f \quad \downarrow H \\
X \xrightarrow{g} Y
\]

Now, say that a map \( A \rightarrow B \) is a weak equivalence if \( X^B \rightarrow X^A \) is a homotopy equivalence for every fibrant \( X \).

Now define fibration as the maps with the RLP wrt. the trivial cofibrations.
Thm (Joyal) this defines a Quillen model structure
on $sSets$, whose fibrant objects are the $\infty$-categories

- it is cofibrantly generated
- it is left proper
- for a functor $f : C \to D$ the map
  
  $NC \to ND$ is a we iff
  
  $C \to D$ is an equivalence of categories.

In contrast, for the classical structure, we have that
a map $f : U \to V$ of spaces is a homotopy eq.
iff $\text{Sing}(U) \to \text{Sing}(V)$ is a we of simplicial sets

- for a map $f : X \to Y$ between $\infty$-categories,
  the following are equivalent:

  1) $f$ is a fibration
  2) $f$ has the RLP wrt $\Lambda^k[n] \to \Delta[n]
     \quad (0 < k < n)
     \quad (f$ is an inner Kan fibration)

     and $\circ$ wrt $\Delta[0] \to J \quad (i = 0, 1)$.

  3) like 2) with $\circ$ strengthened as
$A \times J \cup B \times \Delta[0] \longrightarrow B \times J$

for any $A \xrightarrow{\text{mono}} B$

(\text{"homotopy lifting \& extension"})

Remark: so we have an explicit small set of trivial cof's to test whether a map between $\infty$-categories is a fibration. But although the model structure is cofibrantly generated, there is no such explicit J known which detects fibrations between arbitrary objects.

- For a map $f: X \to Y$ between $\infty$-categories $f$ has
  - $f$ is we
  - $f$ is a $J$-homotopy equivalence
  - $f$ is essentially surjective \& fully faithful
  in the following sense.

If $x, y \in X_0$ are two objects
we can define $X(x,y)$ by the pullback
Prop: \( X(z, y) \) is a Kan complex.

Say \( f \) is \( f \& f \) iff \( X(z, y) \rightarrow Y(fx, fy) \) is a weak equivalence (in the classical model structure).

It remains to define "essentially surjective".

For a 1-simplex, \( \Delta[1] \xrightarrow{\alpha} X \)

an "arrow" \( \alpha : x \rightarrow y \)

\( d\alpha \quad d\sigma \alpha \)

Say \( \alpha \) is an equivalence if there are 2-simplices \( \Delta[2] \xrightarrow{H} X \)

\( \Delta[2] \xrightarrow{K} X \)

Which looks like

\[
\begin{array}{ccc}
0 & \xrightarrow{\alpha} & 1 \\
\downarrow{H} & & \downarrow{1} \\
\delta & \xrightarrow{\beta} & 2
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{\gamma} & 1 \\
\downarrow{K} & & \downarrow{1} \\
\delta & \xrightarrow{\sigma \gamma} & 2
\end{array}
\]
This is equivalent to the condition that
\[ \Delta[1] \xrightarrow{\alpha} X \text{ extends to } J \xrightarrow{\gamma} X. \]

(harder, see Joyal, JPAA)

Now, \( X \xrightarrow{f} Y \) is called **essentially surjective** if for any \( y \in Y_0 \), there exist an \( x \in X_0 \) and an equivalence \( f(x) \xrightarrow{\alpha} y \).

**Rk:** An \( \infty \)-category is a Kan complex (ie fibrant in the classical structure)
iff every 1-simplex is an equivalence
\( (\text{cf } N(C) \text{ is Kan iff } C \text{ is a groupoid}) \).

<table>
<thead>
<tr>
<th>cof</th>
<th>Joyal</th>
<th>classical</th>
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</thead>
<tbody>
<tr>
<td>we's</td>
<td>same</td>
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<tr>
<td>fibrant objects</td>
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So we have a Quillen pair
\[
\begin{array}{ccc}
(sSets) & \xrightarrow{id} & (sSets)_{\text{classical}} \\
\xleftarrow{\text{Joyal}} & & \\
\end{array}
\]
We'll look at

\[ \text{Ho (sSets Joyal)} \xrightarrow{\text{Lid}} \text{Ho (sSets classical)} \xleftarrow{\text{Rid}} \]

\[(\text{Rid}) \circ (\text{Lid}) \xleftarrow{X} \text{ is a we for every } X\]

hence an iso in \( \text{Ho (sSets classical)} \).

So we have a full reflective subcategory

\[ \text{Ho (sSets classical)} \hookrightarrow \text{Ho (sSets Joyal)} \]

this is an example of "left Bousfield localization".

(somewhat analogous to sheaves & presheaves).