Ieke Moerdijk — Dendroidal topology V

Model structure on dendroidal sets

Recall the notion of ∞-operad (dendroidal "inner Kan complex

Recall also that if $A \rightarrowtail B$ is a normal mono, then

$$\Lambda^e[T] \otimes B \cup \Omega[T] \otimes A \rightarrowtail \Omega[T] \otimes B$$

is a composition of pushouts of inner horn inclusions,
for any inner edge e in a tree T.

As a consequence, $\underline{\mathrm{Hom}}(A, X)$ is an ∞-operad whenever X is, for any normal object A.
So, its restriction

$$i^* \underline{\mathrm{Hom}}(A, X) = \underline{\mathrm{hom}}(A, X) \text{ is an } \infty\text{-category.}$$

Definition of the classes in dSets

- the cofibrations are the normal monos
- a map between normal objects $A \to B$ is a we
  iff $\quad \underline{\mathrm{hom}}(B, X) \longrightarrow \underline{\mathrm{hom}}(A, X)$
  is a Joyal equivalence between ∞-categories
  for any ∞-operad X.

How to define we between arbitrary objects?
For any A, we already know how to construct a cofibrant resolution $A' \twoheadrightarrow A$, ie A' is normal & $A' \twoheadrightarrow A$ has the RLP wrt <u>all</u> normal monos

or equivalently, wrt all $\partial\Omega[T] \to \Omega[T]$.
Use the small object argument.

We call A' a normalisation of A.

Any map $A \to B$ can be "normalized" by a commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, two heads] & B' \arrow[d, two heads] \\ A \arrow[r] & B \end{tikzcd}$$

where $A' \twoheadrightarrow A$, $B' \twoheadrightarrow B$ are normalizations.

(we can even do this in a way to make $A' \to B'$ a cofibration).

Now say $A \to B$ is a we iff $\underline{\mathrm{hom}}(B', X) \to \underline{\mathrm{hom}}(A', X)$ is a Joyal weak equivalence.

(if $A' \rightarrowtail B'$ is a cof, it will be a trivial fibration)

Finally, say a map is a <u>fibration</u> if it has the RLP wrt all the trivial cof's.

<u>Thm</u>. these classes define a Quillen model structure on dSets. Moreover, it has additional properties like the Joyal one:

- left proper
- cofibrantly generated

– the fibrant objects are exactly the ∞-operads

– the fibrations between fibrant objects are the maps having the RLP wrt

ⓐ $\Lambda^e[T] \rightarrowtail \Omega[T]$ e inner in a tree T

ⓑ $i_!\{0\} \otimes \Omega[T] \cup i_! J \otimes \partial\Omega[T] \longrightarrow i_! J \otimes \Omega[T]$
(homotopy lifting & extension)

<u>Remark</u>: for a map $Y \to X$, RLP wrt ⓑ is equivalent to the "homotopy extension & lifting property": for any normal mono $A \rightarrowtail B$

$$\begin{tikzcd}
(i_!(J) \otimes A) \cup (i_!\{0\} \otimes B) \arrow[r] \arrow[d, tail] & Y \arrow[d] \\
i_!(J) \otimes B \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

Continuation of the theorem:

– the induced model structure on $dSets/\eta$
(remember $\eta = \Omega[1] = i_!(\Delta[0])$)
coincides with the Joyal structure under the identification $sSets = dSets/\eta$.

The pair $i_! : sSets \rightleftarrows dSets : i^*$

is a Quillen pair, and moreover the induced adjunction

$$\mathrm{Ho}(sSets) \underset{R i^*}{\overset{L i_!}{\rightleftarrows}} \mathrm{Ho}(dSets)$$

represents the Joyal homotopy category as a full (and coreflective) subcategory of that of dendroidal sets

(easy because $i_!$ preserves fibrant objects as well).

– the model structure is also <u>monoidal</u> meaning that if $A \rightarrowtail B$ is a cof, and $Y \longrightarrow X$ is a fib then

$$\underline{\mathrm{Hom}}(B,Y) \longrightarrow \underline{\mathrm{Hom}}(A,Y) \underset{\underline{\mathrm{Hom}}(A,X)}{\times} \underline{\mathrm{Hom}}(B,X)$$

is again a fibration, which is trivial if one of $A \to B$ or $Y \to X$ is.

<u>Remark</u>: the proof is formal to a large extent, except that we have to do some combinatorial work to show that for a tree T which looks like

T  with a unary vertex near the root, we can lift in a diagram

$$\begin{tikzcd}
\Lambda^v[T] \arrow[r, "f"] \arrow[d, tail] & Y \arrow[d] \\
\Omega[T] \arrow[r, "g"] \arrow[ur, dashed] & X
\end{tikzcd}$$

whenever ① X and Y are fibrant,

$\Lambda^v[T] =$ union of all the faces except the one cutting away v,

and ② $f\left(\begin{smallmatrix}|\\ \bullet\\ |\end{smallmatrix}\, v\right)$ is an equivalence in $i^*(X)$

the latter being an ∞-category.

By way of example, let us check that trivial cof's are closed under pushout

$$\begin{tikzcd}[row sep=small, column sep=small]
A' \arrow[rr] \arrow[dr] \arrow[ddd, dash] & & B' \arrow[dr] & \\
& C' \arrow[rr] \arrow[ddd] & & D' \arrow[ddd] \\
& & & \\
A \arrow[rr] \arrow[dr, "\sim"'] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

the top is still a pushout so

$$\begin{tikzcd}
\underline{\mathrm{hom}}(D', X) \arrow[r] \arrow[d] & \mathrm{hom}(B', X) \arrow[d] \\
\underline{\mathrm{hom}}(C', X) \arrow[r, "\sim"] & \mathrm{hom}(A', X)
\end{tikzcd}$$

is a pullback.

Lurie's theory of ∞-operads

Let Γ^o be the category of finite sets

and partial maps.

Any mono $A \hookrightarrow B$ of finite sets

defines a map $i^{-1} : B \longrightarrow A$ in Γ^o.

(defined on $i(A) \subseteq B$)

The maps of this kind are called inert.

If P is a coloured operad, we can construct

a category $P^{\otimes}$ over Γ^o,

with the property that inerts have cocartesian lifts

(in particular $P^{\otimes} \to \Gamma^o$ restricts to a

cofibered category over the inerts).

Fix an object $\underline{n} = \{1, \dots, n\}$ in Γ^o,

- the objects of $P^{\otimes}$ are sequences of

  colours $c_1, \dots, c_n$ of P

- for an arrow $\underline{n} \xrightarrow{\alpha} \underline{m}$ in Γ^o, the arrows

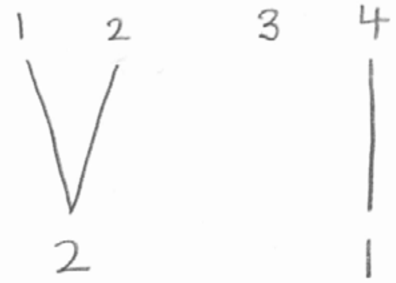
  of $P^{\otimes}$ from an object $\bar{c}$ over $\underline{n}$ to an object $\underline{d}$ over $\underline{m}$

  are sequences of operations $p_1, \dots, p_m$ of P

  $p_i \in P(\bar{c}_{\alpha(i)}, d_i)$

eg.

$$\underline{4} \xrightarrow{\alpha} \underline{2}$$



a map $c_1 \cdots c_4 \longrightarrow d_1, d_2$

is a pair $p_1 \in P(c_1, c_2; d)$ $p_2 \in P(c_4; d_1)$

<u>Observe</u> if $P^{\otimes}_{\underline{n}}$ denotes the fiber over $\underline{m}$

then $P^{\otimes}_{\underline{n}} \xrightarrow{\sim} \prod_{i=1}^{n} P^{\otimes}_{1}$

($P^{\otimes}_1$ is the underlying category of P, made up out of the unary operations called $j^*(P)$ in the first lecture).

- the cocartesian arrow over an inert $\underline{n} \longrightarrow A$ where $A \subset \{1, \dots, n\}$ with colours $\vec{c} = c_1, \dots, c_n$ is the map $(c_1 \dots c_n) \longrightarrow \{c_i : i \in A\}$

- $$\mathrm{Hom}^{\alpha}_{P^{\otimes}}(\vec{c}, \vec{d}) = \prod_{i=1}^{m} \mathrm{Hom}_{P^{\otimes}}(c_{\alpha^{-1}(i)}, d_i)$$

for $\alpha : \underline{n} \longrightarrow \underline{m}$ and $\vec{c} \longrightarrow \bar{c}_{\alpha^{-1}(i)}$ is cocartesian over the inert map $n \longrightarrow \alpha^{-1}(i)$

From this, we get

$$N(P^{\otimes}) \longrightarrow N(\Gamma^{o}) \quad \text{in } sSets_{/N\Gamma^{o}}$$

the ∞-operads will be objects in $sSets_{/N\Gamma^{o}}$ with these properties "up to homotopy".

To keep track of inerts & cocartesian arrows, we move to "marked simplicial sets"

$$\begin{tikzcd} sSets^{+} \arrow[r] & sSets \arrow[l, shift right=2ex, "\flat"'] \arrow[l, shift left=2ex, "\#"] \end{tikzcd}$$

an object of sSets is a pair (X, M) where $s_0(X_0) \subseteq M \subseteq X_1$ is a set of 1-simplices containing all the degenerate ones

$\flat(X) = (X, s_0(X_0))$ "only mark the degenerate 1-simplices"

$\#(X) = (X, X_1)$ "mark everything".

There is a model structure on $sSets^{+}$ for which the fibrant objects are exactly the ∞-categories with their equivalences marked, and the cofibrations the monos.

It is Quillen equivalent to sSets with the Joyal structure — that is related by a Quillen pair inducing an equivalence of categories

$$\mathrm{Ho}(sSets^+) \cong \mathrm{Ho}(sSets)$$

(there are going to be more generating trivial cofibrations, eg.

$$\Lambda^1[2]^\# \underset{\Lambda^1[2]^\flat}{\cup} \Delta[2] \rightarrowtail \Delta[2]^\#$$

"the composition of equivalences is an equivalence again"),

If $Y \longrightarrow X$ is a map between ∞-categories (in sSets) what would it mean for an arrow $y_0 \xrightarrow{\alpha} y_1$ (a 1-simplex in Y) to be cocartesian?

It means we have lifting for

$$\begin{tikzcd}
\Delta[1] \arrow[d, "0,1"] \arrow[dr, "\alpha"] & \\
\Lambda^0[n] \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

$n \geq 2$

perhaps ask this for p which already have the RLP wrt inner horns, or are even <u>fibrations</u> between ∞-categories)

Def: an ∞-operad is a marked simplicial set X

over $N(\Gamma^\circ)^\natural$ where $\natural$ means "inerts are marked"

- $X \to N(\Gamma^\circ)^\natural$ is a fibration between fibrant objects in $sSet^+$
- for any vertex x in $X_{\underline{n}} = p^{-1}(\underline{n})$

and any inert $\alpha : \underline{n} \longrightarrow \underline{m}$ there exists

a cocartesian 1-simplex over α

with vertex x, denoted $x \longrightarrow \alpha_!(x)$

Using this, one can construct a map

$$X_{\underline{n}} \xrightarrow{\alpha_!} X_{\underline{m}}$$

by induction on the skeleton.

Conditions:

- the map $X_{\underline{n}} \xrightarrow{(p_i)_!} \prod_{i=1}^{n} X_{\underline{1}}$

induced by the inerts $p_i : \underline{n} \longrightarrow \underline{1}$

corresponding to $\{i\} \subseteq \underline{n}$;

is an equivalence of ∞-categories

- for x a vertex in $X_{\underline{n}}$, eg one in $X_{\underline{m}}$

we had defined $X(x,y)$ (a Kan complex)

which decomposes as

$$X(x,y) = \coprod_{\alpha : \underline{n} \to \underline{m}} X_\alpha(x,y)$$

by the maps $X \to N(\Gamma^{o})$

and we ask

$$X_\alpha(x, y) \longrightarrow \prod_{i=1}^{n} X_{\alpha^{-1}(i) \xrightarrow{\alpha} \{i\}}\Big((\alpha^{-1}(i))_!\,(x)\ , (f_i)_!\, y \Big)$$

to be a homotopy equivalence of Kan complexes.

__Thm__. there is a model structure on $sSets^{+}/N(\Gamma^{o})^{\natural}$ with monos as cofibrations, and ∞-operads as fibrant objects.

__Conjecture__: there is a zig-zag of Quillen equivalences between this one and $dSets$.

There is a proof by Henks - Hinich - IM, details still to be written.