MPRI 2–2 TD 1 (with solutions)

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1) The goal of this exercise is to study the properties of the objects of the Eilenberg Moore category $\mathbf{Rel}^!$ of $\mathbf{Rel}$, the relational model of LL.

Let P be an object of $\mathbf{Rel}^!$ (the category of coalgebras of $!\_$). Remember that $P = (\underline{P}, \mathsf{h}_P)$ where $\underline{P}$ is an object of $\mathbf{Rel}$ (a set) and $\mathsf{h}_P \in \mathbf{Rel}(\underline{P}, !\underline{P})$ satisfies the following commutations:

$$
\begin{tikzcd}
\underline{P} \arrow[r, "\mathsf{h}_P"] \arrow[dr, "\underline{P}"'] & !\underline{P} \arrow[d, "\mathsf{der}_{\underline{P}}"] \\
& \underline{P}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "\mathsf{h}_P"] \arrow[d, "\mathsf{h}_P"'] & !\underline{P} \arrow[d, "\mathsf{dig}_{\underline{P}}"] \\
!\underline{P} \arrow[r, "!\mathsf{h}_P"] & !!\underline{P}
\end{tikzcd}
$$

1.1) Check that these commutations mean:

- for all $a, a' \in \underline{P}$, one has $(a, [a']) \in \mathsf{h}_P$ iff $a = a'$
- and for all $a \in \underline{P}$ and $m_1, \dots, m_k \in !\underline{P}$, one has $(a, m_1 + \cdots + m_k) \in \mathsf{h}_P$ iff there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, m_i) \in \mathsf{h}_P$ for $i = 1, \dots, k$.

Intuitively, $(a, [a_1, \dots, a_k])$ means that a can be decomposed into "$a_1 + \cdots + a_k$" where the "+" is the decomposition operation associated with P.

Solution. *The first commutation means that, for all $(a, a') \in \underline{P}^2$, one has $a = a'$ iff there is $m \in !\underline{P}$ such that $(a, m) \in \mathsf{h}_P$ and $(m, a') \in \mathsf{der}_{\underline{P}}$. This latter condition means $m = [a']$. Hence $(a, [a]) \in \mathsf{h}_P$ for all $a \in \underline{P}$ and conversely if $(a, [a']) \in \mathsf{h}_P$ then $a = a'$.*

Let now $a \in \underline{P}$ and $m_1, \dots, m_k \in !\underline{P}$.

- *$(a, [m_1, \dots, m_k]) \in !\mathsf{h}_P\, \mathsf{h}_P$ means that there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, m_i) \in \mathsf{h}_P$ for $i = 1, \dots, k$.*
- *And $(a, [m_1, \dots, m_k]) \in \mathsf{dig}_{\underline{P}}\, \mathsf{h}_P$ means that $(a, m_1 + \cdots + m_k) \in \mathsf{h}_P$.*

Whence the announced statement expressing this commutation.

1.2) Prove that if P is an object of $\mathbf{Rel}^!$ such that $\underline{P} \neq \emptyset$ then there is at least one element e of $\underline{P}$ such that $(e, [\,]) \in \mathsf{h}_P$. Explain why such an e could be called a "coneutral element of P".

Solution. *We apply the statements above. Let $a \in \underline{P}$. We know that $(a, [a]) \in \mathsf{h}_P$ and since $[a] = [a] + [\,]$ there are $a', e \in \underline{P}$ such that $(a, [a', e]) \in \mathsf{h}_P$, $(a', [a]) \in \mathsf{h}_P$ and $(e, []) \in \mathsf{h}_P$. Therefore we must have $a = a'$. So we have shown that there must be $e \in \underline{P}$ such that $(e, [\,]) \in \mathsf{h}_P$ and $(a, [a, e]) \in \mathsf{h}_P$. This latter property means that e is coneutral for a (since a can be decomposer in a and e).*

If P and Q are objects of $\mathbf{Rel}^!$, remember that an $f \in \mathbf{Rel}^!(P, Q)$ (morphism of coalgebras) is an $f \in \mathbf{Rel}(\underline{P}, \underline{Q})$ such that the following diagram commutes

$$
\begin{tikzcd}
\underline{P} \arrow[r, "f"] \arrow[d, "\mathsf{h}_P"'] & \underline{Q} \arrow[d, "\mathsf{h}_Q"] \\
!\underline{P} \arrow[r, "!f"] & !\underline{Q}
\end{tikzcd}
$$

1.3) Check that this commutation means that for all $a \in \underline{P}$ and $b_1, \dots, b_k \in \underline{Q}$, the two following properties are equivalent

- there is $b \in \underline{Q}$ such that $(a, b) \in f$ and $(b, [b_1, \dots, b_k]) \in \mathsf{h}_Q$
- there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, b_i) \in f$ for $i = 1, \dots, k$.

Solution. *Soit $a \in \underline{P}$ and $b_1, \dots, b_k \in \underline{Q}$.*

- $(a, [b_1, \ldots, b_k]) \in\, !f\, \mathsf{h}_P$ *means that there are* $a_1, \ldots, a_k \in \underline{P}$ *such that* $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_P$ *and* $(a_i, b_i) \in f$ *for* $i = 1, \ldots, k$.
- *And* $(a, [b_1, \ldots, b_k]) \in \mathsf{h}_Q\, f$ *means that there is* $b \in \underline{Q}$ *such that* $(a, b) \in f$ *and* $(b, [b_1, \ldots, b_k]) \in \mathsf{h}_Q$.

Whence the announced statement.

1.4) Remember that 1 (the set $\{*\}$) can be equipped with a structure of coalgebra (still denoted 1) with $\mathsf{h}_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$. Prove that the elements of $\mathbf{Rel}^!(1, P)$ can be identified with the subsets x of $\underline{P}$ such that: for all $a_1, \ldots, a_k \in \underline{P}$, one has $a_1, \ldots, a_k \in x$ iff there exists $a \in x$ such that $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_P$. We call *values* of P these subsets of $\underline{P}$ and denote as $\mathsf{val}(P)$ the set of these values.

Prove that an element of $\mathsf{val}(P)$ is never empty and that $\mathsf{val}(P)$, equipped with inclusion, is a complete partially ordered set (cpo), that is: the union of a set of values which is directed (with respect to $\subseteq$) is still a value.

Solution. *Let* $x \subseteq \underline{P}$*, considered as an element of* $\mathbf{Rel}(1, \underline{P})$ *(that is, we identify* x *with* $\{(*, a) \mid a \in x\} \in \mathbf{Rel}(1, \underline{P})$*). Then applying the previous question to* $f = x$ *we get that* x *is a value iff for all* $a_1, \ldots, a_k \in \underline{P}$*, one has* $a_1, \ldots, a_k \in x$ *iff there is* $a \in x$ *such that* $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_P$.

Let $x \in \mathsf{val}(P)$*. Applying the characterization above to the case* $k = 0$ *we get that there is* $e \in x$ *such that* $(e, [\,]) \in \mathsf{h}_P$ *(that is e is a coneutral element).*

Let $D \subseteq \mathsf{val}(P)$ *be directed and let* $x = \cup D$*. Let* $a_1, \ldots, a_k \in \underline{P}$*. Assume first that there is* $a \in x$ *such that* $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_P$*. Let* $y \in D$ *be such that* $a \in y$*. Since* $y \in \mathsf{val}(P)$ *we must have* $a_1, \ldots, a_k \in y$ *and hence* $a_1, \ldots, a_k \in x$*. Conversely assume that* $a_1, \ldots, a_k \in x$*. Since* D *is directed there is* $y \in D$ *such that* $a_1, \ldots, a_k \in y$ *(we use crucially the fact that the set* $\{a_1, \ldots, a_k\}$ *is finite). Since* $y \in \mathsf{val}(P)$ *there must be* $a \in y$ *such that* $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_P$*. Since* $a \in y$ *we have* $a \in x$ *and this ends the proof that* x *is a value.*

1.5) Remember that if E is an object of $\mathbf{Rel}$ then $(!E, \mathsf{dig}_E)$ is an object of $\mathbf{Rel}^!$ (the free coalgebra generated by E, that we can identify with an object of the Kleisli category $\mathbf{Rel}_!$). Prove that, as a partially ordered set, $\mathsf{val}(!E, \mathsf{dig}_E)$ is isomorphic to $\mathcal{P}(E)$.

Solution. *First let* $u \subseteq E$*, then we have* $\varphi(u) = \mathcal{M}_{\mathrm{fin}}(u) \in \mathsf{val}(!E, \mathsf{dig}_E)$ *by the very definition of* dig_E*. Conversely given* $x \in \mathsf{val}(!E, \mathsf{dig}_E)$ *let* $\psi(x) = \{a \mid [a] \in x\} \subseteq E$*. Both functions* φ *and* ψ *are obviously monotone.*

Given $u \subseteq E$ *we have* $\psi(\varphi(u)) = \{a \mid [a] \in \mathcal{M}_{\mathrm{fin}}(u)\} = u$.

Conversely let $x \in \mathsf{val}(!E, \mathsf{dig}_E)$*, we prove that* $\varphi(\psi(x)) = x$*. So let* $m = [a_1, \ldots, a_k] \in \varphi(\psi(x))$*, that is* $a_1, \ldots, a_k \in \psi(x)$ *which means that* $[a_i] \in x$ *for* $i = 1, \ldots, k$*. Since* $(m, [[a_1], \ldots, [a_k]]) \in \mathsf{dig}_E$ *we must have* $m \in x$*. Conversely let* $m = [a_1, \ldots, a_k] \in x$*. Since* $(m, [[a_1], \ldots, [a_k]]) \in \mathsf{dig}_E$ *we must have* $[a_i] \in x$ *for* $i = 1, \ldots, k$*, that is* $a_1, \ldots, a_k \in \psi(x)$ *so that* $m \in \varphi(\psi(x))$.

1.6) Is it always true that if $x_1, x_2 \in \mathsf{val}(P)$ then $x_1 \cup x_2 \in \mathsf{val}(P)$?

Solution. *Of course not. Take for instance* $P = 1 \oplus 1$ *which is a coalgebra (see question 1.9 below). Then the values of* P *are* $\{(1, *)\}$ *and* $\{(2, *)\}$ *and* $\{(1, *), (2, *)\}$ *is not a value since there is no* a *such that* $(a, [(1, *), (2, *)]) \in \mathsf{h}_{1 \oplus 1} = \{((i, *), k[(i, *)]) \mid k \in \mathbb{N}$ *and* $i \in \{1, 2\}\}$.

1.7) We have seen (without proof) that $\mathbf{Rel}^!$ is cartesian. Remember that the product of P_1 and P_2 is $P_1 \otimes P_2$, the coalgebra defined by $\underline{P_1 \otimes P_2} = \underline{P_1} \otimes \underline{P_2}$ and $\mathsf{h}_{P_1 \otimes P_2}$ is the following composition of morphisms in $\mathbf{Rel}$:

$$\underline{P_1} \otimes \underline{P_2} \xrightarrow{\mathsf{h}_{P_1} \otimes \mathsf{h}_{P_2}} !\underline{P_1} \otimes !\underline{P_2} \xrightarrow{\mu^2_{\underline{P_1}, \underline{P_2}}} !(\underline{P_1} \otimes \underline{P_2})$$

where $\mu^2_{E_1, E_2} \in \mathbf{Rel}(!E_1 \otimes !E_2, !(E_1 \otimes E_2))$ is the lax monoidality natural transformation of $!\_$, remember that in $\mathbf{Rel}$ we have

$$\mu^2_{E_1, E_2} = \{(([a_1, \ldots, a_k], [b_1, \ldots, b_k]), [(a_1, b_1), \ldots, (a_k, b_k)]) \mid k \in \mathbb{N} \text{ and } (a_1, b_1), \ldots, (a_k, b_k) \in E_1 \times E_2\}\ .$$

Concretely, we have simply that $((a_1, a_2), [(a_1^1, a_2^1), \ldots, (a_1^k, a_2^k)]) \in \mathsf{h}_{P_1 \otimes P_2}$ iff $(a_i, [a_i^1, \ldots, a_i^k]) \in \mathsf{h}_{P_i}$ for $i = 1, 2$.

Prove that $P_1 \otimes P_2$, equipped with suitable projections, is the cartesian product of P_1 and P_2 in $\mathbf{Rel}^!$. Prove also that 1 is the terminal object of $\mathbf{Rel}^!$. Warning: $\mathcal{L}^!$ is always cartesian when $\mathcal{L}$ is a model of LL; I'm not asking for a general proof, just for a verification that this is true in $\mathbf{Rel}^!$.

1.8) Check directly that the partially ordered sets $\mathsf{val}(P_1 \otimes P_2)$ and $\mathsf{val}(P_1) \times \mathsf{val}(P_2)$ are isomorphic.

Solution. *First let $z \in \mathsf{val}(P_1 \otimes P_2)$ and let $x_1 = \{a^1 \in \underline{P_1} \mid \exists a^2 \in \underline{P_2}\ (a^1, a^2) \in z\}$. We define $x_2 \subseteq \underline{P_2}$ similarly. We prove that $x_1 \in \mathsf{val}(P_1)$ and that $z = x_1 \times x_2$.*

Let $a_1^1, \dots, a_k^1 \in \underline{P_1}$. Assume first that $a_1^1, \dots, a_k^1 \in x_1$. Let $a_1^2, \dots, a_k^2 \in x_2$ be such that

$$(a_1^1, a_1^2), \dots, (a_k^1, a_k^2) \in z\,.$$

Since $z \in \mathsf{val}(P_1 \otimes P_2)$ there is $(a^1, a^2) \in z$ such that

$$((a^1, a^2), [(a_1^1, a_1^2), \dots, (a_k^1, a_k^2)]) \in \mathsf{h}_{P_1 \otimes P_2}$$

that is $(a^i, [a_1^i, \dots, a_k^i]) \in \mathsf{h}_{P_i}$ for $i = 1, 2$ and we have $a^i \in x_i$, in particular $a^1 \in x_1$. Conversely assume that $(a^1, [a_1^1, \dots, a_k^1]) \in \mathsf{h}_{P_1}$ and that $a^1 \in x_1$. Let $a^2 \in x_2$ be such that $(a^1, a^2) \in z$. Then we can find $a_1^2, \dots, a_k^2$ such that $(a^2, [a_1^2, \dots, a_k^2]) \in \mathsf{h}_{P_2}$: for instance, we can take $a_1^2 = a^2$ and $a_i^2 = e^2$ for $i = 2, \dots, k$ where e^2 is neutral for a^2 in P_2. Then we have $((a^1, a^2), [(a_1^1, a_1^2), \dots, (a_k^1, a_k^2)]) \in \mathsf{h}_{P_1 \otimes P_2}$ and hence $(a_1^1, a_1^2), \dots, (a_k^1, a_k^2) \in z$ since $z \in \mathsf{val}(P_1 \otimes P_2)$. It follows that $a_1^1, \dots, a_k^1 \in x_1$ and we have proven that $x_1 \in \mathsf{val}(P_1)$. Similarly $x_2 \in \mathsf{val}(P_2)$. We prove now that $z = x_1 \times x_2$. The inclusion $z \subseteq x_1 \times x_2$ is obvious. Let $(a^1, a^2) \in x_1 \times x_2$. We can find $b^1 \in x_1$ and $b^2 \in x_2$ such that $(a^1, b^2), (b^1, a^2) \in z$ so since z is a value there is $(c^1, c^2) \in z$ such that $((c^1, c^2), [(a^1, b^2), (b^1, a^2)]) \in \mathsf{h}_{P_1 \otimes P_2}$. It follows that $(c^i, [a^i, b^i]) \in \mathsf{h}_{P_i}$ for $i = 1, 2$ and therefore $((c^1, c^2), [(a^1, a^2), (b^1, b^2)]) \in \mathsf{h}_{P_1 \otimes P_2}$ and hence $(a^1, a^2) \in z$.

This shows that there is an order isomorphism between $\mathsf{val}(P_1) \times \mathsf{val}(P_1)$ and $\mathsf{val}(P_1 \otimes P_2)$ which maps (x^1, x^2) to $x^1 \times x^2$.

1.9) Remember also that we have defined $P_1 \oplus P_2 = (\underline{P_1} \oplus \underline{P_2}, \mathsf{h}_{P_1 \oplus P_2})$ where $\mathsf{h}_{P_1 \oplus P_2}$ is the unique element of $\mathbf{Rel}(\underline{P_1} \oplus \underline{P_2}, !(\underline{P_1} \oplus \underline{P_2}))$ such that, for $i = 1, 2$, the morphism $\mathsf{h}_{P_1 \oplus P_2}\, \overline{\pi}_i$ coincides with the following composition of morphisms in $\mathbf{Rel}$:

$$\underline{P_i} \xrightarrow{\mathsf{h}_{P_i}} !\underline{P_i} \xrightarrow{!\overline{\pi}_i} !(\underline{P_1} \oplus \underline{P_2})$$

Describe $\mathsf{h}_{P_1 \oplus P_2}$ as simply as possible and prove that, equipped with suitable injections, $P_1 \oplus P_2$ is the coproduct of P_1 and P_2 in $\mathbf{Rel}^!$.

2) The goal of this exercise is to illustrate the fact that $\mathbf{Rel}$, the relational model of LL, can be equipped with additional structures of various kinds *without modifying the interpretation of proofs and programs.* As an example we shall study the notion of *non-uniform coherence space* (NUCS). A NUCS is a triple $X = (|X|, \frown_X, \smile_X)$ where

- $|X|$ is a set (the web of X)
- and $\frown_X$ and $\smile_X$ are two symmetric relations on $|X|$ such that $\frown_X \cap \smile_X = \emptyset$. In other words, for any $a, a' \in |X|$, one never has $a \frown_X a'$ and $a \smile_X a'$.

So we can consider an ordinary coherence space (in the sense of the first part of thise series of lectures) as a NUCS X which satisfies moreover:

$$\forall a, a' \in |X| \quad (a \frown_X a' \text{ or } a \smile_X a') \Leftrightarrow a \neq a'\,.$$

It is then possible to introduce three other natural symmetric relations on the elements of $|X|$:

- $a \equiv_X a'$ if it is not true that $a \frown_X a'$ or $a \smile_X a'$.
- $a \frown\!\!\!\!\smile_X a'$ if $a \frown_X a'$ or $a \equiv_X a'$.
- $a \asymp_X a'$ if $a \smile_X a'$ or $a \equiv_X a'$.

A *clique* of a NUCS X is a subset x of $|X|$ such that $\forall a, a' \in |X|\ a \coh_X a'$, we use $\mathsf{Cl}(X)$ for the set of cliques of X.

We say that a NUCS X satisfies the Boudes' Condition[1] (or simply that X is Boudes) if

$$\forall a, a' \in |X|\ a \equiv_X a' \Rightarrow a = a' .$$

We shall show that the class of NUCS's can be turned into a categorical model of LL in such a way that all the operations on objects coincide with the corresponding operations on objects in **Rel**. For instance we shall define $!X$ in such a way that $|!X| = !|X| = \mathcal{M}_{\mathrm{fin}}(|X|)$. Moreover, all the "structure morphisms" of this model *will be defined exactly as in* **Rel**. For instance, the digging morphism from $!X$ to $!!X$ will simply be $\mathsf{dig}_{|X|}$. Important: such definitions are impossible with ordinary coherence spaces. When defining $|!E|$ in ordinary coherence spaces one *needs* to restrict to the finite multisets (or finite sets) of elements of $|E|$ which *are cliques of* E. It is exactly for that reason that, in *NUCS*'s, the relation $\equiv_X$ is not required to coincide with equality. Nevertheless, the weaker Boudes' condition will be preserved by all of our constructions.

2.1) Check that a NUCS can be specified by $|X|$ together with any of the following seven pairs of relations.

- Two symmetric relations $\coh_X$ and $\frown_X$ on $|X|$ such that $\frown_X \subseteq \coh_X$. Then setting $\smile_X = (|X| \times |X|) \setminus \coh_X$, the relation $\coh_X$ is the one canonically associated with the NUCS $(|X|, \frown_X, \smile_X)$.
- Two symmetric relations $\asymp_X$ and $\smile_X$ on $|X|$ such that $\smile_X \subseteq \asymp_X$. How should we define $\frown_X$ in that case?
- Two symmetric relations $\coh_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \coh_X$. How should we define $\frown_X$ and $\smile_X$ in that case?
- Two symmetric relations $\asymp_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \asymp_X$. How should we define $\frown_X$ and $\smile_X$ in that case?
- Two symmetric relations $\frown_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \frown_X = \emptyset$. How should we define $\smile_X$ in that case?
- Two symmetric relations $\smile_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \smile_X = \emptyset$. How should we define $\frown_X$ in that case?
- Two symmetric relation $\coh_X$ and $\asymp_X$ such that $\coh_X \cup \asymp_X = |X| \times |X|$. How should we define $\frown_X$ and $\smile_X$ in that case?

Solution. *This is a simple logical verification. For instance, if we are given two symmetric relations $\asymp_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \asymp_X$, we say that $a \frown_X b$ it is not true that $a \asymp_X b$ and we say that $a \smile_X b$ if $a \asymp_X b$ and it is not true that $a \equiv_X b$.*

2.2) Given *NUCS*'s X and Y, we define a NUCS $X \multimap Y$ by $|X \multimap Y| = |X| \times |Y|$ and

- $(a, b) \equiv_{X \multimap Y} (a', b')$ if $a \equiv_X a'$ and $b \equiv_Y b'$
- and $(a, b) \frown_{X \multimap Y} (a', b')$ if $a \smile_X a'$ or $b \frown_Y b'$.

Check that we have defined in that way a NUCS. Prove that $\mathsf{Id}_{|X|} = \{(a, a) \mid a \in |X|\} \in \mathsf{Cl}(X \multimap X)$. Prove that if X and Y are Boudes then $X \multimap Y$ is Boudes.

Solution. *To check that we have defined a NUCS, it suffices to check that we cannot have at the same time $(a, b) \equiv_{X \multimap Y} (a', b')$ and $(a, b) \frown_{X \multimap Y} (a', b')$. This is clear because we cannot have $a \equiv_X a'$ and $a \smile_X a'$, and we cannot have $b \equiv_Y b'$ and $b \frown_Y b'$.*

To check that the identity is a clique, take $a, a' \in |X|$ and observe that if $a \equiv_X a'$ then $(a, a) \equiv_{X \multimap X} (a', a')$, and if $a \frown_X a'$ or $a \smile_X a'$ then $(a, a) \frown_{X \multimap X} (a', a')$.

2.3) Prove that, if $s \in \mathsf{Cl}(X \multimap Y)$ and $t \in \mathsf{Cl}(Y \multimap Z)$ then $t\,s \in \mathsf{Cl}(X \multimap Z)$. So we define a category **Nucs** by taking the NUCS's as object and by setting $\mathbf{Nucs}(X, Y) = \mathsf{Cl}(X \multimap Y)$.

[1] From Pierre Boudes who discovered this condition and the nice properties of these objects.

Solution. *First notice that if $(a,b),(a',b') \in |X \multimap Y|$ one has $(a,b) \coh_{X\multimap Y} (a',b')$ if $a \coh_X a' \Rightarrow b \coh_Y b'$ and $a \frown_X a' \Rightarrow b \frown_Y b'$.*

Let $(a,c),(a',c') \in t\,s$. There are b,b' such that $(a,b),(a',b') \in s$ and $(b,c),(b',c') \in t$. If $a \coh_X a'$ then $b \coh_Y b'$ since s is a clique, and hence $c \coh_Y c'$ since t is a clique. Similarly $a \frown_X a' \Rightarrow c \frown_Z c'$.

2.4) We define $X^\perp$ by $|X^\perp| = |X|$, $\frown_{X^\perp} = \smile_X$ and $\smile_{X^\perp} = \frown_X$. Then we set $X \otimes Y = \left(X \multimap Y^\perp\right)^\perp$. Define as simply as possible the NUCS structure of $X \otimes Y$. We set $1 = (\{*\}, \emptyset, \emptyset)$ (in other words $* \equiv_1 *$). Prove that if X and Y are Boudes then $X^\perp$ and $X \otimes Y$ is Boudes.

Solution. *Assume that X is Boudes. If $a \equiv_{X^\perp} a'$ then $a \equiv_X a'$ and hence $a = a'$ since X is Boudes.*

Observe that $(a,b) \equiv_{X\otimes Y} (a',b')$ iff $a \equiv_X a'$ and $b \equiv_Y b'$ and that $(a,b) \coh_{X\otimes Y} (a',b')$ iff $a \coh_X a'$ and $b \coh_Y b'$. So assuming that X and Y are Boudes, if $(a,b) \equiv_{X\otimes Y} (a',b')$ then $a = a'$ and $b = b'$ and hence $X \otimes Y$ is Boudes.

2.5) Given $s_i \in \mathbf{Nucs}(X_i, Y_i)$ for $i = 1,2$, prove that $s_1 \otimes s_2 \in \mathbf{Rel}(|X_1| \otimes |X_2|, |Y_1| \otimes |Y_2|)$ (defined as in **Rel**) does actually belong to $\mathbf{Nucs}(X_1 \otimes X_2, Y_1 \otimes Y_2)$.

Solution. *Use the characterizations above of $\coh$ in tensor products and linear function spaces.*

2.6) Check quickly that **Nucs** (equipped with the $\otimes$ defined above and 1 as tensor unit, and $\perp = 1$ as dualizing object) is a $*$-autonomous category.

2.7) Prove that the category **Nucs** is cartesian and cocartesian, with $X = \&_{i\in I} X_i$ given by $|X| = \bigcup_{i\in I} \{i\} \times |X_i|$, and

- $(i,a) \equiv_X (i',a')$ if $i = i'$ and $a \equiv_{X_i} a'$
- $(i,a) \smile_X (i',a')$ if $i = i'$ and $a \smile_{X_i} a'$.

and the associated operations (projections, tupling of morphisms) defined as in **Rel**.

Prove that if all X_i's are Boudes then $\&_{i\in I} X_i$ is Boudes.

2.8) We define $!X$ as follows. We take $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and, given $m, m' \in |!X|$

- we have $m \coh_{!X} m'$ if for all $a \in \mathsf{supp}(m)$ and $a' \in \mathsf{supp}(m')$ one has $a \coh_X a'$
- and $m \equiv_{!X} m'$ if $m \coh_{!X} m'$ and $m = [a_1, \dots, a_k]$, $m' = [a'_1, \dots, a'_k]$ with $a_i \equiv_X a'_i$ for each $i \in \{1, \dots, k\}$.

Notice that $m \smile_{!X} m'$ iff there is $a \in \mathsf{supp}(m)$ and $a' \in \mathsf{supp}(m')$ such that $a \smile_X a'$. Remember that $\mathsf{supp}(m) = \{a \in |X| \mid m(a) \neq 0\}$.

Let $s \in \mathbf{Nucs}(X,Y)$. Prove that $!s \in \mathbf{Rel}(!|X|, !|Y|)$ actually belongs to $\mathbf{Nucs}(!X, !Y)$.

Solution. *Let $(m,p),(m',p') \in\, !s$ so that we can write $m = [a_1, \dots, a_l]$, $p = [b_1, \dots, b_l]$, $m' = [a'_1, \dots, a'_r]$ and $p' = [b'_1, \dots, b'_r]$ with $(a_i, b_i), (a'_j, b'_j) \in s$ for $i = 1, \dots, l$ and $j = 1, \dots, r$. Assume that $m \coh_{!X} m'$, that is, for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, r\}$ one has $a_i \coh_X a'_j$ and therefore $b_i \coh_X b'_j$ since s is a clique. If follows that $p \equiv_{!Y} p'$. Assume moreover that $p \equiv_{!Y} p'$ so that $l = r$ and we can assume that for all $i \in \{1, \dots, l\}$ we have $b_i \equiv_Y b'_i$. Since $a_i \equiv_X a'_i$, it follows that $a_i \equiv_X a'_i$ since s is a clique.*

Notice that we have used the following characterization of $\coh_{X\multimap Y}$: $(a,b) \coh_{X\multimap Y} (a',b')$ iff

$$a \coh_X a' \Rightarrow (b \coh_Y b' \text{ and } b \equiv_Y b' \Rightarrow a \equiv_X a')\,.$$

2.9) Prove that $\mathsf{der}_{|X|} = \{([a], a) \mid a \in |X|\}$ belongs to $\mathbf{Nucs}(!X, X)$.

2.10) Prove that $\mathsf{dig}_X = \{(m_1 + \cdots + m_k, [m_1, \dots, m_k]) \mid m_1, \dots, m_k \in \mathcal{M}_{\mathrm{fin}}(|X|)\}$ is an element of $\mathbf{Nucs}(!X, !!X)$.

Solution. *Let* $(m, M), (m', M') \in \mathsf{dig}_{|X|}$ *so that* $M = [m_1, \dots, m_l]$, $M' = [m'_1, \dots, m'_r]$ *with* $m = \sum_{i=1}^{l} m_i$ *and* $m' = \sum_{j=1}^{r} m'_j$. *Assume that* $m \frown_{!X} m'$. *This implies that for all* $i \in \{1, \dots, l\}$ *and* $j \in \{1, \dots, r\}$ *and for all* $a \in \mathsf{supp}(m)$ *and* $a' \in \mathsf{supp}(m'_j)$ *one has* $a \frown_X a'$ *and hence* $m_i \frown_{!X} m'_j$. *Therefore* $M \frown_{!!X} M'$. *Assume moreover that* $M \equiv_{!!X} M'$. *So we have* $l = r$ *and we can assume that for all* $i \in \{1, \dots, n\}$ *one has* $m_i \equiv_{!X} m'_i$. *So for each* i *we can write* $m_i = [a^i_1, \dots, a^i_{k(i)}]$ *and* $m'_i = [b^i_1, \dots, b^i_{k(i)}]$ *with* $a^i_j \equiv_X b^i_j$ *for* $j = 1, \dots, k(i)$. *Since* $m = \sum_{i=1}^{l} m_i$ *and* $m' = \sum_{j=1}^{r} m'_j$ *we have* $m \equiv_X m'$.

2.11) Prove that if X is Boudes then $!X$ is Boudes.

2.12) Let $X = 1 \oplus 1$, and let $\mathbf{t}, \mathbf{f}$ be the two elements of $|X|$ (X is the "type of booleans"). Let $s \in \mathbf{Rel}(|X| \otimes |X|, |X|)$ by $s = \{((\mathbf{t}, \mathbf{f}), \mathbf{t}), ((\mathbf{f}, \mathbf{t}), \mathbf{f})\}$. Prove that $s \in \mathbf{Nucs}(X \otimes X, X)$. Let then $t \in \mathbf{Nucs}(!X, X)$ be defined by the following composition of morphisms in **Nucs**:

$$!X \xrightarrow{\mathsf{c}_X} !X \otimes !X \xrightarrow{\mathsf{der}_X \otimes \mathsf{der}_X} X \otimes X \xrightarrow{s} X$$

We recall that contraction $\mathsf{c}_X \in \mathbf{Nucs}(!X, !X \otimes !X)$ is given by $\mathsf{c}_X = \{m_1 + m_2, (m_1, m_2) \mid m_1, m_2 \in !|X|\}$ and dereliction $\mathsf{der}_X \in \mathbf{Nucs}(!X, X)$ is given by $\mathsf{der}_X = \{([a], a) \mid a \in |X|\}$.

Prove that $([\mathbf{t}, \mathbf{f}], \mathbf{t}), ([\mathbf{t}, \mathbf{f}], \mathbf{f}) \in t$. So any notion of coherence on $!|X|$ *must* satisfy $[\mathbf{t}, \mathbf{f}] \smile_{!X} [\mathbf{t}, \mathbf{f}]$ since we have $\mathbf{t} \smile_X \mathbf{f}$ by the definition of the NUCS $1 \oplus 1$ since we *must have* $([\mathbf{t}, \mathbf{f}], \mathbf{t}) \frown_{!X \multimap X} ([\mathbf{t}, \mathbf{f}], \mathbf{f})$ because t is a clique. In particular it is impossible to endow $!|X|$ with a notion of Girard's coherence space since in such a coherence space we would have $[\mathbf{t}, \mathbf{f}] \frown_{!X} [\mathbf{t}, \mathbf{f}]$ and hence $([\mathbf{t}, \mathbf{f}], \mathbf{t}) \smile_{!X \multimap X} ([\mathbf{t}, \mathbf{f}], \mathbf{f})$.

Solution. *We have* $((\mathbf{t}, \mathbf{f}), \mathbf{t}) \frown_{X \otimes X \multimap X} ((\mathbf{f}, \mathbf{t}), \mathbf{f})$ *because* $(\mathbf{t}, \mathbf{f}) \smile_{X \otimes X} (\mathbf{f}, \mathbf{t})$.

We have $([\mathbf{t}, \mathbf{f}], ([\mathbf{t}], [\mathbf{f}])) \in \mathsf{c}_X$, *hence* $([\mathbf{t}, \mathbf{f}], (\mathbf{t}, \mathbf{f})) \in (\mathsf{der}_X \otimes \mathsf{der}_X)\, \mathsf{c}_X$ *so that* $([\mathbf{t}, \mathbf{f}], \mathbf{t}) \in s\, (\mathsf{der}_X \otimes \mathsf{der}_X)\, \mathsf{c}_X$. *Similarly* $([\mathbf{f}, \mathbf{t}], \mathbf{f}) \in s\, (\mathsf{der}_X \otimes \mathsf{der}_X)\, \mathsf{c}_X$, *and notice that* $[\mathbf{t}, \mathbf{f}] = [\mathbf{f}, \mathbf{t}]$.