MPRI 2–2 Models of programming languages: domains, categories, games
TD1

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The signs **(*)** and **(**)** try to indicate more difficult and interesting questions. These are of course completely subjective indications!

1. This exercise develops a somehow degenerate model of Linear Logic which does not satisfy $*$-autonomy but satisfies all the other requirements. A pointed set is a structure $X = (\underline{X}, 0_X)$ where $\underline{X}$ is a set and $0_X \in \underline{X}$. Given pointed sets X, X_1, X_2 and Y,

   - a morphism of pointed sets from X to Y is a function $f : \underline{X} \to \underline{Y}$ such that $f(0_X) = 0_Y$
   - and a bimorphism of pointed sets from X_1, X_2 to Y is a function $f : \underline{X_1} \times \underline{X_2} \to \underline{Y}$ such that $f(0_{X_1}, x_2) = f(x_1, 0_{X_2}) = 0_Y$ for each $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$.

   (a) Prove that pointed sets together with morphisms of pointed sets form a category $\mathbf{Set}_0$. What are the isos in that category?

   > **Solution:** It suffices to observe that morphisms are closed under composition for the ordinary composition of functions and that the identity functions are morphisms. Since composition of morphisms in $\mathbf{Set}_0$ is the ordinary composition of functions, an iso from X to Y is a bijection $f : \underline{X} \to \underline{Y}$ such that $f(0_X) = 0_Y$. Indeed this latter condition implies $f^{-1}(0_Y) = 0_X$.

   One sets $1 = (\{0_1, *\})$ where $*$ and 0_1 are are distinct chosen elements (for instance 0_1 is the integer 0 and $*$ is the integer 1). Given pointed sets X_1 and X_2 one defines $X_1 \otimes X_2$ as follows:

   $$\underline{X_1 \otimes X_2} = \{(x_1, x_2) \in \underline{X_1} \times \underline{X_2} \mid x_1 = 0_{X_1} \Leftrightarrow x_2 = 0_{X_2}\} \quad \text{and} \quad 0_{X_1 \otimes X_2} = (0_{X_1}, 0_{X_2})\,.$$

   Given $x_i \in \underline{X_i}$ for $i = 1, 2$, one defines

   $$x_1 \otimes x_2 = \begin{cases} (0_{X_1}, 0_{X_2}) & \text{if } x_1 = 0_{X_1} \text{ or } x_2 = 0_{X_2} \\ (x_1, x_2) & \text{otherwise.} \end{cases}$$

   (b) Prove that the function $(x_1, x_2) \mapsto x_1 \otimes x_2$ is a bimorphism from X_1, X_2 to $X_1 \otimes X_2$ which is surjective as a function $\underline{X_1} \times \underline{X_2} \to \underline{X_1 \otimes X_2}$ and that for any bimorphism f from X_1, X_2 to Y there is exactly one morphism $\widetilde{f} \in \mathbf{Set}_0(X_1 \otimes X_2, Y)$ such that $f(x_1, x_2) = \widetilde{f}(x_1 \otimes x_2)$ for all $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$.

   > **Solution:** The first statement results immediately from the definition of $x_1 \otimes x_2$ and of $\underline{X_1 \otimes X_2}$, let us prove the second so let f be as stipulated. Given $(x_1, x_2) \in \underline{X_1 \otimes X_2}$ we set $\widetilde{f}(x_1, x_2) = f(x_1, x_2)$. The map so defined belongs to $\mathbf{Set}_0(X_1 \otimes X_2, Y)$ because $\widetilde{f}(0_{X_1 \otimes X_2}) = f(0_{X_1}, 0_{X_1}) = 0_Y$ since f is a bimorphism. Let now $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$,

we have

$$\widetilde{f}(x_1 \otimes x_2) = \begin{cases} \widetilde{f}(0_{X_1}, 0_{X_2}) = 0_Y & \text{if } x_1 = 0_{X_1} \text{ or } x_2 = 0_{X_2} \\ f(x_1, x_2) & \text{otherwise.} \end{cases} \qquad = f(x_1, x_2)$$

since f is a bimorphism. Uniqueness of $\widetilde{f}$ follows from the observation that any element of $\underline{X_1 \otimes X_2}$ is of shape $x_1 \otimes x_2$.

(c) Given $f_i \in \mathbf{Set}_0(X_i, Y_i)$ for $i = 1, 2$, deduce from the above that there is exactly one morphism $f_1 \otimes f_2 \in \mathbf{Set}_0(X_1 \otimes X_2, Y_1 \otimes Y_2)$ such that

$$\forall x_1 \in \underline{X_1}\, \forall x_2 \in \underline{X_2} \quad (f_1 \otimes f_2)(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2)\,.$$

Solution: Let $g : \underline{X_1} \times \underline{X_2} \to \underline{Y_1 \otimes Y_2}$ be defined by $g(x_1, x_2) = f_1(x_1) \otimes f_2(x_2)$. This function is clearly a bimorphism from X_1, X_2 to Y since f_1 and f_2 are morphisms and $\_ \otimes \_$ is a bimorphism. Hence there is exactly one morphism $f_1 \otimes f_2$ satisfying the required conditions.

(d) Using again the universal property of Question (b) prove that the operation on morphisms defined in Question (c) is a functor.

Solution: Let $f_i \in \mathbf{Set}_0(X_i, Y_i)$ and $g_i \in \mathbf{Set}_0(Y_i, Z_i)$ for $i = 1, 2$. We know that there is exactly one morphism $(g_1\, f_1) \otimes (g_2\, f_2) \in \mathbf{Set}_0(X_1 \otimes X_2, Z_1 \otimes Z_2)$ such that $((g_1\, f_1) \otimes (g_2\, f_2))(x_1 \otimes x_2) = g_1(f_1(x_1)) \otimes g_2(f_2(x_2))$ for all $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$. Observe that $(g_1 \otimes g_2)\,(f_1 \otimes f_2) \in \mathbf{Set}_0(X_1 \otimes X_2, Z_1 \otimes Z_2)$ and that

$$\begin{aligned} ((g_1 \otimes g_2)\,(f_1 \otimes f_2))(x_1 \otimes x_2) &= (g_1 \otimes g_2)(f_1(x_1) \otimes f_2(x_2)) \\ &= g_1(f_1(x_1)) \otimes g_2(f_2(x_2)) \end{aligned}$$

and hence $(g_1 \otimes g_2)\,(f_1 \otimes f_2) = (g_1\, f_1) \otimes (g_2\, f_2)$. One has similarly $\mathsf{Id} \otimes \mathsf{Id} = \mathsf{Id}$ because $(\mathsf{Id} \otimes \mathsf{Id})(x_1 \otimes x_2) = x_1 \otimes x_2$.

(e) Exhibit isomorphisms $\lambda_X \in \mathbf{Set}_0(1 \otimes X, X)$ and $\alpha_{X_1,X_2,X_3} \in \mathbf{Set}_0((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$.

Solution: We have $\underline{1 \otimes X} = \{(i, x) \in \{0_1, *\} \times \underline{X} \mid i = 0_1 \Leftrightarrow x = 0_X\} = \{(0_1, 0_X)\} \cup \{(*, x) \mid x \in \underline{X} \setminus \{0_X\}\}$ and the second projection $\underline{1 \otimes X} \to \underline{X}$ is the sought isomorphism. Given $x_i \in \underline{X_i}$ for $i = 1, 2, 3$, there are two possibilites:

- either $x_i = 0_{X_i}$ for some $i \in \{1, 2, 3\}$ and in that case $(x_1 \otimes x_2) \otimes x_3 = 0_{(X_1 \otimes X_2) \otimes X_3}$ and $x_1 \otimes (x_2 \otimes x_3) = 0_{X_1 \otimes (X_2 \otimes X_3)}$
- or $x_i \in \underline{X_i} \setminus \{0_{X_i}\}$ for $i = 1, 2, 3$ and in that case $(x_1 \otimes x_2) \otimes x_3 = ((x_1, x_2), x_3)$ and $x_1 \otimes (x_2 \otimes x_3) = (x_1, (x_2, x_3))$

and we can define α as follows:

$$\begin{aligned} \alpha(0_{(X_1 \otimes X_2) \otimes X_3}) &= 0_{X_1 \otimes (X_2 \otimes X_3)} \\ \text{and } \alpha((x_1, x_2), x_3) &= (x_1, (x_2, x_3)) \text{ if } x_i \in \underline{X_i} \setminus \{0_{X_i}\} \text{ for } i = 1, 2, 3 \end{aligned}$$

which a bijection which preserves 0 and hence an isomorphism.

So $\mathbf{Set}_0$ is an SMC (there is a symmetry iso $\gamma_{X_1,X_2} \in \mathbf{Set}_0(X_1 \otimes X_2, X_2 \otimes X_1)$ such that $\gamma_{X_1,X_2}(x_1 \otimes x_2) = x_2 \otimes x_1$ which is quite easy to define, and the McLane coherence diagrams commute).

(f) One defines $X \multimap Y$ by $\underline{X \multimap Y} = \mathbf{Set}_0(X, Y)$ and for $0_{X \multimap Y}$ we take the function such that $0_{X \multimap Y}(x) = 0_Y$ for all $x \in \underline{X}$. Let $e : \underline{X \multimap Y} \times \underline{X} \to \underline{Y}$ be defined by $e(f, x) = f(x)$. Prove that e is a bimorphism and that the SMC $\mathbf{Set}_0$ is closed.

Solution: For all $x \in \underline{X}$ we have $e(0_{X \multimap Y}, x) = 0_Y$ by definition of $0_{X \multimap Y}$ and for all $f \in \mathbf{Set}_0(X, Y)$ we have $e(f, 0_X) = 0_Y$ by definition of morphisms. Therefore e is a bimorphism and there is a unique $\mathsf{ev} = \widetilde{e} \in \mathbf{Set}_0((X \multimap Y) \otimes X, Y)$ such that $\mathsf{ev}(f \otimes x) = e(f, x) = f(x)$ for all $f \in \mathbf{Set}_0(X, Y)$ and $x \in \underline{X}$. Let $f \in \mathbf{Set}_0(Z \otimes X, Y)$ so that the function $g : \underline{X} \times \underline{Y} \to \underline{Z}$ defined by $g(z, x) = f(z \otimes x)$ is a bimorphism. Therefore for each $z \in \underline{Z}$ the function $f_z : \underline{X} \to \underline{Y}$ defined by $f_z(x) = g(z, x) = f(z \otimes x)$ is a morphism, that is $f_z \in \underline{X \multimap Y}$. Moreover $f_{0_Z} = 0_{X \multimap Y}$ and hence the map $\mathsf{cur}\, f = z \mapsto f_z$ belongs to $\mathbf{Set}_0(Z, X \multimap Y)$. By definition $\mathsf{ev}(\mathsf{cur}\, f \otimes \mathsf{Id}_X)(z \otimes x) = \mathsf{ev}(f_z \otimes x) = f_z(x) = f(z \otimes x)$ for all $z \in \underline{Z}$ and $x \in \underline{X}$ and hence $\mathsf{ev}(\mathsf{cur}\, f \otimes \mathsf{Id}_X) = f$. It remains to check that $\mathsf{cur}\, f$ is the unique $g \in \mathbf{Set}_0(Z, X \multimap Y)$ such that $\mathsf{ev}(g \otimes \mathsf{Id}_X) = f$ but this condition implies $g(z)(x) = f(z \otimes x)$ for all $z \in \underline{Z}$ and $x \in \underline{X}$, that is $g(z) = f_z = (\mathsf{cur}\, f)(z)$.

(g) Prove that there is no object $\perp$ of $\mathbf{Set}_0$ which turns this symmetric monoidal closed category into a $*$-autonomous category.

Solution: $\perp$ most contain at least $0_\perp$. If $\perp = \{0_\perp\}$ then $\perp$ is the terminal object and hence $\underline{(X \multimap \perp) \multimap \perp}$ is a singleton so that X cannot always be isomorphic to $\underline{(X \multimap \perp) \multimap \perp}$. Otherwise let $z \in \underline{\perp}$ with $z \neq 0_\perp$. Then for each subset I of $\underline{X} \setminus \{0_X\}$ we can define a function $f_I : \underline{X} \to \underline{\perp}$ by

$$f_I(x) = \begin{cases} z & \text{if } x \in I \\ 0_\perp & \text{otherwise} \end{cases}$$

and one has $f_I \in \mathbf{Set}_0(X, \perp)$. Moreover $f_I = f_J \Rightarrow I = J$ so that $\#\underline{(X \multimap \perp)} > 2^{\#(\underline{X} \setminus \{0_X\})} > \#\underline{X}$ as soon as $\#\underline{X} > 1$. Therefore when $\#X > 1$ we have $\#\underline{((X \multimap \perp) \multimap \perp)} > \#\underline{X}$.

(h) Given a family $(X_i)_{i \in I}$ of objects of $\mathbf{Set}_0$ we define an object X as follows: $\underline{X} = \prod_{i \in I} \underline{X_i}$ and $0_X = (0_{X_i})_{i \in I} \in \underline{X}$ so that the the projections $\pi_i : \underline{X} \to \underline{X_i}$ ar obviously morphisms of $\mathbf{Set}_0$. Prove that X, together with these projections, is the cartesian product of the family $(X_i)_{i \in I}$ that we denote as $\&_{i \in I} X_i$.

Solution: Let $(f_i)_{i \in I}$ with $f_i \in \mathbf{Set}_0(Y, X_i)$. The function $g : \underline{Y} \to \underline{X}$ given by $g(y) = (f_i(y))_{i \in I}$ is a morphism since $g(0_Y) = (0_{X_i})_{i \in I}$ since each f_i is a morphism. By definition $\pi_i\, g = f_i$ for all $i \in I$ and g is the unique function satisfying these equations so the universal property of the product is satisfied.

Notice that the terminal object (which is the product of an empty family of objects) is $\top = (\{0_\top\}, 0_\top)$.

Contrarily to **Rel**, the category $\mathbf{Set}_0$ has all (projective) limits. It seems rather difficult to build $*$-autonomous categories which are at the same type complete. A noticeable exception is the category of complete lattices.

Given an object X of $\mathbf{Set}_0$, we define $!X$ by $\underline{!X} = \{(0, 0_!)\} \cup \{1\} \times \underline{X}$ where $0_!$ is a chosen element (for instance, a given integer) and $0_{!X} = (0, 0_!)$. Notice that $(1, 0_X) \in \underline{!X}$ but $0_{!X} \neq (1, 0_X)$.

Given $f \in \mathbf{Set}_0(X, Y)$, we define $!f \in \mathbf{Set}_0(!X, !Y)$ by $!f(0_{!X}) = 0_{!Y}$ and $!f(1, x) = (1, f(x))$. This obviously defines a functor $\mathbf{Set}_0 \to \mathbf{Set}_0$.

(i) We define $\mathsf{der}_X : \mathbf{Set}_0(!X, X)$ by $\mathsf{der}_X(0_{!X}) = 0_X$ and $\mathsf{der}_X(1, x) = x$. Prove that this is a natural transformation.

Solution: Let $f \in \mathbf{Set}_0(X,Y)$. We have $(\mathsf{der}_Y\ !f)(0,0_!) = \mathsf{der}_Y(0,0_!) = 0_Y$ and $(f\ \mathsf{der}_X)(0,0_!) = f(0_X) = 0_Y$. And if $x \in \underline{X}$ we have $(\mathsf{der}_Y\ !f)(1,x) = \mathsf{der}_Y(1,f(x)) = f(x)$ and $(f\ \mathsf{der}_X)(1,x) = f(x)$. We have proven that $\mathsf{der}_Y\ !f = f\ \mathsf{der}_X$.

(j) We define $\mathsf{dig}_X \in \mathbf{Set}_0(!X, !!X)$ by $\mathsf{dig}_X(0,0_!) = (0,0_!)$, that is $\mathsf{dig}_X(0_{!X}) = 0_{!!X}$, and $\mathsf{dig}_X(1,x) = (1,(1,x))$ which is easily seen to be a natural transformation. Prove that equipped with the natural transformations der and dig the functor $!\_$ is a comonad.

Solution: We have

$$\begin{aligned}
\mathsf{der}_{!X}(\mathsf{dig}_X(0_{!X})) &= \mathsf{der}_{!X}(0_{!!X}) = 0_{!X}\\
\mathsf{der}_{!X}(\mathsf{dig}_X(1,x)) &= \mathsf{der}_{!X}(1,(1,x)) = (1,x)\\
!\mathsf{der}_X(\mathsf{dig}_X(0_{!X})) &= !\mathsf{der}_X(0_{!!X}) = 0_{!X}\\
!\mathsf{der}_X(\mathsf{dig}_X(1,x)) &= !\mathsf{der}_X(1,(1,x)) = (1,\mathsf{der}_X(1,x)) = (1,x)\\
\mathsf{dig}_{!X}(\mathsf{dig}_X(0_{!X})) &= \mathsf{dig}_{!X}(0_{!!X}) = 0_{!!!X}\\
!\mathsf{dig}_X(\mathsf{dig}_X(0_{!X})) &= !\mathsf{dig}_X(0_{!!X}) = 0_{!!!X}\\
\mathsf{dig}_{!X}(\mathsf{dig}_X(1,x)) &= \mathsf{dig}_{!X}(1,(1,x)) = (1,(1,(1,x)))\\
!\mathsf{dig}_X(\mathsf{dig}_X(1,x)) &= !\mathsf{dig}_X(1,(1,x)) = (1,\mathsf{dig}_X(1,x)) = (1,(1,(1,x)))
\end{aligned}$$

(k) Given two objects X and Y of $\mathbf{Set}_0$, exhibit an isomorphism between $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$.

Solution: We define $\mathsf{m}^2_{X,Y} : \underline{!X \otimes !Y} \to \underline{!(X \mathbin{\&} Y)}$ by

$$\begin{aligned}
\mathsf{m}^2_{X,Y}(0_{!X}, 0_{!Y}) &= 0_{!(X\&Y)}\\
\mathsf{m}^2_{X,Y}((1,x),(1,y)) &= (1,(x,y)) \quad \text{where } x \in \underline{X} \text{ and } y \in \underline{Y}.
\end{aligned}$$

These are the only cases we have to consider due to the definition of the operation $\otimes$ on pointed sets. Since $0_{!X\otimes !Y} = (0_{!X}, 0_{!Y})$ this function $\mathsf{m}^2_{X,Y}$ is a morphism of $\mathbf{Set}_0$. Moreover it is an injective function because $(1,(x,y)) \neq 0_{!(X\&Y)}$ for each x, y. It is surjective because an element of $\underline{!(X \mathbin{\&} Y)}$ is either $0_{!(X\&Y)}$ or of shape $(1,(x,y))$ and hence $\mathsf{m}^2_{X,Y}$ is also surjective. So it is a bijective morphism and hence an isomorphism.

2. In this exercise we study a model of linear logic which is based on complete sup-semilattices and linear maps. A complete sup-semilattice is a partially ordered set S (the order relation will always be denoted as $\leq$ or $\leq_S$ if required) such that any subset A of S has a least upper bound $\bigvee A \in S$. Remember that this means

- $\forall x \in A\ x \leq \bigvee A$
- $\forall x \in S\ (\forall y \in A\ y \leq x) \Rightarrow \bigvee A \leq x$.

In particular we have two elements $0 = \bigvee \emptyset$ which is the least element of S and $1 = \bigvee S$ which is the greatest element of S.

A subset A of S is *down-closed* if for all $x \in A$ and all $y \in S$, if $y \leq x$ then $y \in A$. Given $x \in S$ we set $\downarrow x = \{y \in S \mid y \leq x\}$.

A linear morphism of sup-semilattices from S to T is a function $f : S \to T$ such that for all $A \subseteq S$ $f(\bigvee A) = \bigvee f(A)$ where we define as usual $f(A) = \{f(x) \mid x \in A\}$. Notice that this implies that f is monotone: given $x \leq y$ in S we have $f(y) = f(\bigvee\{x,y\}) = f(x) \vee f(y)$, that is $f(x) \leq f(y)$. Let Slat be the category whose objects are the sup-semilattices and morphisms are the linear maps of sup-semilattices. We set $\perp = \{0 < 1\}$ for the object of Slat which has exactly two elements.

It is important to remember that any inf-semilattice, partially ordered set S where each $A \subseteq S$ has an inf (greatest lower bound) $\bigwedge A$, is also a sup-semilattice: $\bigvee A = \bigwedge\{x \in S \mid \forall y \in A\ y \leq x\}$.

It is easy to check that Slat is cartesian. The product of a family $(S_j)_{j\in J}$ of objects of Slat is the usual cartesian product $\prod_{j\in J} S_j$ equipped with the product order and projection defined in the usual way. We also use $S = \&_{j\in J} S_j$ for this product and $\pi_j \in \mathsf{Slat}(S, S_j)$ for the projections. The terminal object is $\top = \{0\}$.

(a) Show that the isomorphisms of Slat are the linear morphisms which are bijections.

> **Solution:** The condition is clearly necessary since if f is an iso of inverse f^{-1} then f^{-1} is also the inverse of f in **Set** and hence f is a bijection. If f is a linear morphism which is a bijection it suffices to prove that $\bigwedge f$ is linear, so let $B \subseteq T$. We must prove that $f^{-1}(\bigvee B) = \bigvee f^{-1}(B)$ and for this, since f is bijective, it suffices to prove that $f(f^{-1}(\bigvee B)) = f(\bigvee f^{-1}(B))$ which results from the linearity of f.

(b) Given a set X we denote as $\mathcal{P}(X)$ its powerset (that is, the set of all of its subsets) ordered under inclusion, so that $\mathcal{P}(X)$ is a sup-semilattice for $\bigvee A = \bigcup A$ for any $A \subseteq \mathcal{P}(X)$. Given $t \in \mathbf{Rel}(X, Y)$ we define $\widehat{t} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\widehat{t}(x) = t \cdot x = \{b \in Y \mid \exists a \in x\ (a, b) \in t\}$. Prove that $\widehat{t} \in \mathsf{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ and that, for any $f \in \mathsf{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ there is exactly one $t = \mathsf{tr} f \in \mathbf{Rel}(X, Y)$ such that $f = \widehat{t}$. In other words, the functor $L : \mathbf{Rel} \to \mathsf{Slat}$ which maps X to $\mathcal{P}(X)$ and t to $\widehat{t}$ is full and faithful.

> **Solution:** The fact that $\widehat{t}$ commutes with unions (and hence belongs to $\mathsf{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$) follows immediately from the definition. Let now $f \in \mathsf{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ and let us set $t = \mathsf{tr} f = \{(a, b) \mid b \in f(\{a\})\} \in \mathbf{Rel}(X, Y)$. Let $x \in \mathcal{P}(X)$, we have
>
> $$\begin{aligned}\widehat{t}(x) &= \{b \in Y \mid \exists a \in x\ (a, b) \in t\}\\ &= \bigcup_{a\in x} f(\{a\})\\ &= f(x)\end{aligned}$$
>
> since f commutes with unions. This shows that $t \mapsto \widehat{t}$ is surjective. On the other hand given $t \in \mathbf{Rel}(X, Y)$ we have
>
> $$\mathsf{tr}\,\widehat{t} = \{(a, b) \in X \times Y \mid b \in \widehat{t}(\{a\})\} = t$$
>
> by definition of $\widehat{t}$ which shows that $t \mapsto \widehat{t}$ is injective.

(c) Prove that the category Slat has all equalizers, in other words: given objects S and T of Slat and $f, g \in \mathsf{Slat}(S, T)$ there is an object E of Slat and a morphism $e \in \mathsf{Slat}(E, S)$ such that $f\,e = g\,e$ and, for any object V of Slat and any morphism $h \in \mathsf{Slat}(V, S)$ such that $f\,h = g\,h$, there is exactly one morphism $h_0 \in \mathsf{Slat}(V, E)$ such that $h = e\,h_0$.

> **Solution:** We take $E = \{x \in S \mid f(x) = g(x)\}$, equipped with the induced order relation (that is $x \leq_E y$ is $x \leq_S y$). Given $A \subseteq E$ we have $A \subseteq S$ so let x_0 be the sup of A in S. Since f and g are linear we have
>
> $$\begin{aligned}f(x_0) &= \bigvee f(A)\\ &= \bigvee g(A) \quad \text{since } A \subseteq E\\ &= g(x_0)\end{aligned}$$
>
> and hence $x_0 \in E$. Next one proves that x_0 is the sup of A *in* E. First given $x \in A$ one has $x \leq_S x_0$ and hence $x \leq_E x_0$ since $x, x_0 \in E$. Next let $y \in E$ be such that $\forall x \in A\ x \leq_E y$, we have $\forall x \in A\ x \leq_S y$ and hence $x_0 \leq_S y$, that is $x_0 \leq_E y$ since $x_0, y \in E$.
>
> The inclusion map $e : E \to S$ (that is $e(x) = x$) is linear since we have seen that the sups are computed in E exactly as in S.

Let now V be a sup-semilattice and $h \in \mathsf{Slat}(V, S)$ be such that $f\, h = g\, h$. This means that actually $\forall v \in V\ h(v) \in E$. So we can define $h_0 : V \to E$ by $h_0(v) = h(v)$. Again, the linearity of h_0 results from the fact that the sups in E are computed exactly as in S so that $h_0 \in \mathsf{Slat}(V, E)$. Last the uniqueness of h_0 results from the fact that e is injective.

Remember that the Cantor space is the set $\{0,1\}^\omega$ of all infinites sequences α of 0 and 1 equipped with the following topology (which is the product topology of the discrete space $\{0,1\}$): a subset U of $\{0,1\}^\omega$ is open iff for any $\alpha \in U$ there is a finite prefix w of α such that, for any $\beta \in \{0,1\}^\omega$, if w is a prefix of β then $\beta \in U$. In other words, a subset F of $\{0,1\}^\omega$ is closed iff it has the following property: if $\alpha \in \{0,1\}^\omega$ is such that, for any finite prefix w of α there exists $\beta \in F$ such that w is a prefix of β, then $\alpha \in F$. As in any topological spaces, if $\mathcal{F}$ is a set of closed subsets then $\bigcap \mathcal{F}$ is closed (you are advised to check this directly using the characterization above of closed subsets).

So the set of closed subsets of $\{0,1\}^\omega$ is an inf-semilattice and hence also a sup-semilattice: the sup of a set of closed sets is the closure of its union (the intersectin of all closed sets which contain this union).

(d) **(**)** Let $W = \{0,1\}^*$ be the set of all finite sequences of 0 and 1. If $w = \langle a_1, \dots, a_n \rangle \in W$ is such a sequence and $a \in \{0,1\}$ let $wa = \langle a_1, \dots, a_n, a \rangle$. Let $\theta = \{(wa, w) \mid w \in W \text{ and } a \in \{0,1\}\} \in \mathbf{Rel}(W, W)$. Let (C, c) be the equalizer of $\mathsf{Id}, \widehat{\theta} \in \mathsf{Slat}(\mathcal{P}(W), \mathcal{P}(X))$ (so that C is a sup-semilattice and $c \in \mathsf{Slat}(C, \mathcal{P}(W))$). Exhibit an order isomorphism between C and the set of all closed subsets of the Cantor space ordered under inclusion.

Solution: We know that $C = \{x \subseteq W \mid \theta \cdot x = x\}$. So for $x \subseteq W$, the condition $x \in C$ means:

- $\theta \cdot x \subseteq x$, that is, if $wa \in x$ then $w \in x$, that is, x is prefix-closed
- and $x \subseteq \theta \cdot x$ that is, if $w \in x$ then there is $a \in \{0,1\}$ such that $wa \in x$: any element of x has an extension in x.

So we see such an x as the set of prefixes of a set of elements of $\{0,1\}^\omega$. More precisely let $\varphi(x)$ be the set of all $\alpha \in \{0,1\}^\omega$ such that, for any $w \in W$ which is a prefix of α, one has $w \in x$.

Then $\varphi(x)$ is a closed subset of $\{0,1\}^\omega$. Let indeed $\alpha \in \{0,1\}^\omega$ be such that for all $w < \alpha$ (meaning that $w \in W$ and w is a prefix of α) there is $\beta \in \{0,1\}^\omega$ such that $\beta \in \varphi(x)$ and $w < \beta$. This implies $\forall w \in W\ w < \alpha \Rightarrow w \in x$ and hence $\alpha \in \varphi(x)$, so $\varphi(x)$ is closed. Notice that the map φ is monotone (with respect to set inclusion).

Conversely given a closed $F \subseteq \{0,1\}^\omega$ let $\psi(F) = \{w \in W \mid \exists \alpha \in F\ w < \alpha\}$. Then we clearly have $\psi(F) \in C$ and it is also clear that ψ is monotone. Let us prove that $\varphi(\psi(F)) = F$. We first prove that $F \subseteq \varphi(\psi(F))$ so let $\alpha \in F$. For all $w < \alpha$ we have $w \in \psi(F)$ by definition of ψ, and hence $\alpha \in \varphi(\psi(F))$ by definition of φ. Conversely let $\alpha \in \varphi(\psi(F))$. This means $\forall w \in W\ w < \alpha \Rightarrow w \in \psi(F)$ that is $\forall w \in W\ w < \alpha \Rightarrow \exists \beta \in F\ w < \beta$ which implies $\alpha \in F$ because F is closed. So we have proven that $\varphi \circ \psi = \mathsf{Id}$, we prove now that $\psi \circ \varphi = \mathsf{Id}$.

Let $x \in C$, we prove first that $x \subseteq \psi(\varphi(x))$. Let $w \in x$. Using the assumption that $x \in C$ we can build a sequence $a_1, a_2, \dots$ of elements of $\{0,1\}$ such that, for all $n \in \mathbb{N}$, one has $wa_1 \dots a_n \in x$ for all $n \in \mathbb{N}$. So let $\alpha = wa_1a_2 \cdots \in \{0,1\}^\omega$. If $w' < \alpha$ we have either $w' = wa_1 \dots a_n$ for some n or w' is a prefix of w. Hence $w' \in x$. This shows that $\alpha \in \varphi(x)$. Since $w < \alpha$ it follows that $w \in \psi(\varphi(x))$. Conversely let $w \in \psi(\varphi(x))$. Let $\alpha \in \varphi(x)$ be such that $w < \alpha$. By definition of $\varphi(x)$, we have $w \in x$.

Given a lattice S, we say that $x \in S$ is *prime* if

$$\forall A \subseteq S \quad x \leq \bigvee A \Rightarrow \exists y \in A \; x \leq y$$

(e) **(*)** Prove that, for a set X, the prime elements of $\mathcal{P}(X) \in \mathsf{Slat}$ are exactly the singletons. Prove that C, in sharp contrast with the previous case, has no prime elements.

[*Hint:* prove first that if F is prime, it must be a singleton $\{\alpha\}$ and then prove that no such singleton is prime. For this notice that, for a collection $\mathcal{F}$ of closed subsets of $\{0,1\}^\omega$, the closed set $\bigvee \mathcal{F}$ is the closure of $\bigcup \mathcal{F}$ (the intersection of all closed sets which contain $\bigcup \mathcal{F}$). So consider a set $\mathcal{F}$ of shape $\mathcal{F} = \{\{\alpha(n)\} \mid n \in \mathbb{N}\}$ where $\alpha(n) \to_{n\to\infty} \alpha$ and $\forall n \in \mathbb{N} \; \alpha(n) \neq \alpha$.]

Solution: For the first part observe that for any $x \subseteq X$ one has $x = \bigcup_{a \in x} \{a\}$. So if x is prime we must have $x \subseteq \{a\}$ for some $a \in X$. We cannot have $x = \emptyset$ since $\emptyset = \bigcup \emptyset$. Conversely it is obvious that if a is a singleton then $\{a\}$ is prime.

Concerning C notice first that each singleton $\{\alpha\}$ is closed and hence $\{\alpha\} \in C$. Now let F be closed and assume that F is not a singleton. If $F = \emptyset$ then F is not prime because $\bigvee \emptyset = \emptyset$. So let $\alpha, \beta \in F$ with $\alpha \neq \beta$. Let $w < \alpha$ be such that $w \not< \beta$. Let $G = \{\gamma \in \{0,1\}^\omega \mid w < \gamma\}$. This set is closed and open as easily checked. Hence $F \cap G$ and $F \setminus G$ are both closed and satisfy $(F \cap G) \vee (F \setminus G) = F$, so F is not prime since $\alpha, \beta \in F$, $\beta \notin F \cap G$ and $\alpha \notin F \setminus G$.

Now we prove that $\{\alpha\}$ is never prime, whetever be $\alpha = \langle a_1, a_2, \dots \rangle$. For each $n \in \mathbb{N}$ let $\alpha(n) \in \{0,1\}^\omega$ be defined (for instance) by

$$\alpha(n) = \langle a_1, \dots, a_{n-1}, 1 - a_n, 0, 0, \dots \rangle$$

so that $\alpha(n) \to_{n\to\infty} \alpha$. It follows that

$$\alpha \in \bigvee_{n=1}^{\infty} \{\alpha(n)\}$$

but by construction $\alpha \neq \alpha(n)$ for all n. It follows that $\{\alpha\}$ is not prime and hence C has no prime elements.

This example is a concrete illustration of the fact that the category $\mathbf{Rel}$ is not complete, indeed it has no equalizer for the two maps $\theta, \mathsf{Id} \in \mathbf{Rel}(W, W)$ because the equalizer of $\widehat{\theta}$ and Id in Slat is not an object of $\mathbf{Rel}$ (one would need a further proof to make this argument completely rigorous!).

(f) Prove that the set of linear morphisms $S \to T$, equipped with the pointwise order (that is $f \leq g$ if $\forall x \in S \; f(x) \leq g(x)$), is a sup-semilattice. We denote it as $S \multimap T$.

Solution: Let F be a set of linear functions $S \to T$. Let $g : S \to T$ be defined by

$$g(x) = \bigvee_{f \in F} f(x)\,.$$

Let $A \subseteq S$, we have

$$\begin{aligned} g(\bigvee A) &= \bigvee_{f \in F} f(\bigvee A) \\ &= \bigvee_{f \in F} \bigvee_{x \in A} f(x) \quad \text{by linearity of the elements of } A \\ &= \bigvee_{x \in A} \bigvee_{f \in F} f(x) \quad \text{easy property of } \bigvee \\ &= \bigvee g(A) \end{aligned}$$

which shows that $g \in S \multimap T$. By definition we have $f \leq g$ for each $f \in F$. Let now $h \in S \multimap T$ such that $f \leq h$ for each $f \in F$. This means that, for each $x \in S$, we have $f(x) \leq h(x)$ for all $f \in F$, and hence $g(x) \leq h(x)$. Therefore $g \leq h$ and we have shown that g is the sup of the set F.

(g) Given $x \in S$ define a function $x^* : S \to \perp$ by

$$x^*(y) = \begin{cases} 1 & \text{if } y \not\leq x \\ 0 & \text{if } y \leq x \end{cases}$$

Prove that $x^* \in S \multimap \perp$.

Solution: Let $A \subseteq S$. We have

$$\begin{aligned} x^*(\bigvee A) = 1 &\Leftrightarrow \bigvee A \not\leq x \\ &\Leftrightarrow \exists y \in A \; y \not\leq x \\ &\Leftrightarrow \exists y \in A \; x^*(y) = 1 \\ &\Leftrightarrow \bigvee x^*(y) = 1 \end{aligned}$$

(h) Given a sup-semilattice S, we use S^{op} for the same set S equipped with the reverse order: $x \leq_{S^{\mathsf{op}}} y$ if $y \leq_S x$. Prove that the map $x \mapsto x^*$ is an order isomorphism from the poset S^{op} to $S \multimap \perp$. Warning: one must prove that it is monotone in both directions because a monotone bijection is not necessarily an order isomorphism! Call $\mathsf{k} : (S \multimap \perp) \to S^{\mathsf{op}}$ the inverse isomorphism.

Solution: Let $x_1, x_2 \in S$ with $x_1 \leq x_2$. Given $y \in S$, one has $x_1{}^*(y) = 0 \Leftrightarrow y \leq x_1 \Rightarrow y \leq x_2 \Leftrightarrow x_2{}^*(y) = 0$ and hence $x_2{}^* \leq x_1{}^*$. Hence the map $x \mapsto x^*$ is monotone. Given $x' \in S \multimap \perp$ let $\mathsf{k}(x') = \bigvee\{x \in S \mid x'(x) = 0\} \in S$. If $x'_1 \leq x'_2$ then $x'_2(x) = 0 \Rightarrow x'_1(x) = 0$ and hence $\mathsf{k}(x'_2) \leq_S \mathsf{k}(x'_1)$ so k is monotone $(S \multimap \perp) \to S^{\mathsf{op}}$. Now let $x \in S$, we have

$$\mathsf{k}(x^*) = \bigvee\{y \in S \mid x^*(y) = 0\} = \bigvee\{y \in S \mid y \leq x\} = x$$

and let $x' \in S \multimap \perp$, for each $y \in S$ we have

$$\mathsf{k}(x')^*(y) = 0 \Leftrightarrow y \leq \mathsf{k}(x') \Leftrightarrow x'(y) = 0$$

because $x'(\mathsf{k}(x')) = 0$ by linearity of x'.

(i) **(*)** Given $f \in (S \multimap T)$ define $f^* : (T \multimap \perp) \to (S \multimap \perp)$ by $f^*(y') = y' f$. Prove that $f^* \in \mathsf{Slat}(T \multimap \perp, S \multimap \perp)$. Let $f^\perp \in \mathsf{Slat}(T^{\mathsf{op}}, S^{\mathsf{op}})$ be the associated morphism (through the iso k defined above, that is $f^\perp(y) = \mathsf{k}(f^*(y^*))$). Prove that

$$\forall x \in S \, \forall y \in T \quad f(x) \leq y \Leftrightarrow x \leq f^\perp(y)\,.$$

One says that f and $f^\perp$ define a Galois connection between S and T. Last prove that $f^{\perp\perp} = f$.

Solution: Let $B' \subseteq T \multimap \perp$, we have that $f^*(\bigvee B') \in S \multimap \perp$ satisfies $f^*(\bigvee B')(x) = (\bigvee B')(f(x)) = \bigvee_{y' \in B'} y'(f(x)) = \bigvee_{y' \in B'} f^*(y')(x) = (\bigvee_{y' \in B'} f^*(y'))(x)$ which proves that f^* is linear.

Let $x \in S$ and $y \in T$, one has

$$\begin{aligned} x \leq f^\perp(y) &\Leftrightarrow x \leq \mathsf{k}(f^*(y^*)) \\ &\Leftrightarrow x \leq \bigvee\{x_1 \in S \mid f^*(y^*)(x_1) = 0\} \\ &\Leftrightarrow x \leq \bigvee\{x_1 \in S \mid y^*(f(x_1)) = 0\} \\ &\Leftrightarrow x \leq \bigvee\{x_1 \in S \mid f(x_1) \leq y\} \\ &\Leftrightarrow f(x) \leq y \end{aligned}$$

In the last equivalence we have $f(x) \leq y \Rightarrow x \leq \bigvee\{x_1 \in S \mid f(x_1) \leq y\}$ because if $f(x) \leq y$ then $x \in \{x_1 \in S \mid f(x_1) \leq y\}$ and conversely if $x \leq \bigvee\{x_1 \in S \mid f(x_1) \leq y\}$ then $f(x) \leq f(\bigvee\{x_1 \in S \mid f(x_1) \leq y\}) = \bigvee\{f(x_1) \in S \mid f(x_1) \leq y\} \leq y$.
Let $f \in \mathsf{Slat}(S,T)$ so that $f^\perp \in \mathsf{Slat}(T^{\mathsf{op}}, S^{\mathsf{op}})$ and $f^{\perp\perp} \in \mathsf{Slat}(S,T)$. We have $x \leq_S f^\perp(y) \Leftrightarrow f(x) \leq_T y$ and $y \leq_{T^{\mathsf{op}}} f^{\perp\perp}(x) \Leftrightarrow f^\perp(y) \leq_{S^{\mathsf{op}}} x$ that is $y \geq_T f^{\perp\perp}(x) \Leftrightarrow f^\perp(y) \geq_S x$. So we get $\forall x \in S, y \in T$ $f(x) \leq_T y \Leftrightarrow f^{\perp\perp}(x) \leq_T y$. Taking $y = f(x)$ and then $y = f^{\perp\perp}(x)$ we get $f^{\perp\perp}(x) = f(x)$.

(j) Given sup-semilattices S and T we define $S \otimes T$ as the set of all $I \subseteq S \times T$ such that

- I is down-closed
- and, for all $A \subseteq S$ and $B \subseteq T$, if A and B satisfy $A \times B \subseteq I$ then $(\bigvee A, \bigvee B) \in I$.

Prove that $(S \otimes T, \subseteq)$ is an inf-semilattice (that is, is closed under arbitrary intersections). As a consequence, it is also a sup-semilattice: if $\mathcal{I} \subseteq S \otimes T$ then $\bigvee \mathcal{I} = \bigcap\{I \in S \otimes T \mid \bigcup \mathcal{I} \subseteq I\}$. But notice that in this sup-semilattice, the sups are not defined as unions in general.

Solution: Let $\mathcal{I} \subseteq S \otimes T$ and let $I = \bigcap \mathcal{I}$. Let $A \subseteq S$ and $B \subseteq T$ be such that $A \times B \subseteq I$. For each $J \in S \otimes T$ we have $A \times B \subseteq J$ and hence $(\bigvee A, \bigvee B) \in J$. It follows that $(\bigvee A, \bigvee B) \in I$.

(k) Prove that the least element of $S \otimes T$ is $0_{S \otimes T} = S \times \{0\} \cup \{0\} \times T$.

Solution: Notice first that $0_{S\otimes T}$ is down-closed. Let $A \subseteq S$ and $B \subseteq T$ be such that $A \times B \subseteq 0_{S\otimes T}$. Notice that we must have $A \subseteq \{0\}$ or $B \subseteq \{0\}$. Indeed otherwise we can find $x \in A \setminus \{0\}$ and $y \in B \setminus \{0\}$, but then we have $(x,y) \in A \times B \subseteq 0_{S\otimes T}$ which is not possible. It follows that $(\bigvee A, \bigvee B) \in 0_{S\otimes T}$. Let now $I \in S \otimes T$, then we have $\emptyset = \emptyset \times T \subseteq I$ and hence $(0,1) = (\bigvee \emptyset, \bigvee T) \in I$. Similarly $(1,0) \in I$, which shows that $0_{S\otimes T} \subseteq I$ since I is down-closed.

(l) We say that a map $f : S \times T \to U$ (where S, T, U are sup-semilattices) is bilinear if for all $A \subseteq S$ and $B \subseteq T$ we have $\bigvee f(A \times B) = f(\bigvee(A \times B)) = f(\bigvee A, \bigvee B)$. Prove that this condition is equivalent to the following:

- for all $x \in S$ and $B \subseteq T$, one has $f(x, \bigvee B) = \bigvee_{y \in B} f(x,y)$
- and for all $y \in T$ and $A \subseteq S$, one has $f(\bigvee A, y) = \bigvee_{x \in A} f(x,y)$

that is, f is separately linear in both variables.

Solution: Assume first that f is bilinear, then with these notations we have $f(x, \bigvee B) = f(\bigvee \{x\} \times B) = \bigvee f(\{x\} \times B) = \bigvee_{y \in B} f(x,y)$. Conversely assume that f is separately linear, given $A \subseteq S$ and $B \subseteq T$, we have $f(\bigvee A, \bigvee B) = \bigvee_{b \in B} f(\bigvee A, y) = \bigvee_{y \in B} \bigvee_{x \in A} f(x,y) = \bigvee f(A \times B)$.

(m) **(*)** Given $x \in S$ and $y \in T$ let $x \otimes y = \downarrow(x,y) \cup 0_{S\otimes T} \subseteq S \times T$. Prove that $x \otimes y \in S \otimes T$ and that the function $\tau : (x,y) \mapsto x \otimes y$ is a bilinear map $S \times T \to S \otimes T$.

Solution: First $x \otimes y$ is down-closed as a union of down-closed sets. Next let $A \subseteq S$ and $B \subseteq T$ be such that $A \times B \subseteq x \otimes y$. For any $x_1 \in A \setminus \{0\}$ and $y_1 \in B \setminus \{0\}$ we must have $x_1 \leq x$ and $y_1 \leq y$ since $(x_1, y_1) \in (A \times B) \setminus 0_{S \otimes T}$, it follows that $(\bigvee A, \bigvee B) \leq (x, y)$. This shows that $x \otimes y \in S \otimes T$.

To prove the bilinearity of τ we must show that $\bigvee \tau(A \times B) = \bigvee A \otimes \bigvee B$. We have $\bigvee \tau(A \times B) \subseteq \bigvee A \otimes \bigvee B$ because the map τ is clearly monotone so it suffices to prove the converse inclusion $\bigvee A \otimes \bigvee B \subseteq \bigvee \tau(A \times B)$. This amounts to proving that for any $I \in S \otimes T$, if $\bigcup \tau(A \times B) \subseteq I$ then $\bigvee A \otimes \bigvee B \subseteq I$. Since we already know that $0_{S \otimes T} \subseteq I$, it suffices to see that $(\bigvee A, \bigvee B) \in I$. We know that $\bigcup \tau(A \times B) \subseteq I$, that is $\bigcup_{(x,y) \in A \times B} (\downarrow x \times \downarrow y) \cup 0_{S \otimes T} \subseteq I$ and hence $A \times B \subseteq I$ so that $(\bigvee A, \bigvee B) \in I$ since $I \in S \otimes T$.

(n) Let $(S, T) \multimap U$ be the set of all bilinear maps $S \times T \to U$ ordered pointwise (that is $f \leq g$ if $\forall (x, y) \in S \times T \; f(x, y) \leq g(x, y)$). Prove that $(S, T) \multimap U \simeq (S \multimap (T \multimap U))$. Deduce from this fact that $(S, T) \multimap U$ is a sup-semilattice.

Solution: Given $f \in (S, T) \multimap U$ let $\lambda(f) : S \to U^T$ be defined by $\lambda(f)(x)(y) = f(x, y)$. By bilinearity of f, for each x the function $\lambda(f)(x) : T \to U$ is linear, and the map $\lambda(f)$ itself is linear because $T \multimap U$ is ordered pointwise. The fact that λ is an order isomorphism is an easy verification.

(o) Given $I \in X \otimes Y$ let $f^I : S \times T \to \perp$ be given by

$$f^I(x, y) = \begin{cases} 0 & \text{if } (x, y) \in I \\ 1 & \text{otherwise.} \end{cases}$$

Prove that f^I is bilinear. Conversely given $f \in (S, T) \multimap \perp$ prove that $\mathsf{ker}_2 f = \{(x, y) \in S \times T \mid f(x, y) = 0\}$ belongs to $S \otimes T$. Prove that these operations define an order isomorphism between $S \otimes T$ and $((S, T) \multimap \perp)^{\mathsf{op}}$.

Solution: Let first $I \in S \otimes T$ and let use prove that $f^I \in (S, T) \multimap \perp$. Observe that f^I is monotone because I is down-closed. Let $A \subseteq S$ and $B \subseteq T$, it suffices to prove that $f^I(\bigvee A, \bigvee B) \leq \bigvee f^I(A \times B)$ so assume that $\bigvee f^I(A \times B) = 0$. This means that $\forall (x, y) \in A \times B \; f^I(x, y) = 0$, that is $A \times B \subseteq I$. So we have $(\bigvee A, \bigvee B) \in I$, that is $f^I(\bigvee A, \bigvee B) = 0$ and hence f^I is bilinear. If $I, J \in S \otimes T$ are such that $I \subseteq J$, we have $f^I(x, y) = 0 \Rightarrow (x, y) \in I \Rightarrow (x, y) \in J \Rightarrow f^J(x, y) = 0$ so that $f^J \leq f^I$ in $(S, T) \multimap \perp$.

Conversely if $f \in (S, T) \multimap \perp$ we have $\mathsf{ker}_2 f \in S \otimes T$ by bilinearity of f. If $f \leq g$ we clearly have $\mathsf{ker}_2 g \subseteq \mathsf{ker}_2 f$.

Let $I \in S \otimes T$, we have $(x, y) \in \mathsf{ker}_2 f^I \Leftrightarrow f^I(x, y) = 0 \Leftrightarrow (x, y) \in I$ so that $\mathsf{ker}_2 f^I = I$. And given $f \in (S, T) \multimap \perp$ we have $f^{\mathsf{ker}_2 f}(x, y) = 0 \Leftrightarrow (x, y) \in \mathsf{ker}_2 f \Leftrightarrow f(x, y) = 0$ so that $f^{\mathsf{ker}_2 f} = f$. So we have exhibited the required iso between $((S, T) \multimap \perp)^{\mathsf{op}}$ and $S \otimes T$.

To be continued…