Parallel and serial hypercoherences

Thomas Ehrhard
Institut de Mathématiques de Luminy
C.N.R.S. U.P.R. 9016
ehrhard@iml.univ-mrs.fr

Abstract

It is known that the strongly stable functions which arise in the semantics of PCF can be realized by sequential algorithms, which can be considered as deterministic strategies in games associated to PCF types. Studying the connection between strongly stable functions and sequential algorithms, two dual classes of hypercoherences naturally arise: the parallel and serial hypercoherences. The objects belonging to the intersection of these two classes are in bijective correspondence with the so-called "serial-parallel" graphs, that can essentially be considered as games.

We show how to associate to any hypercoherence a parallel hypercoherence together with a projection onto the given hypercoherence and present some properties of this construction. Intuitively, it makes explicit the computational time of a hypercoherence.

Notice: This is a preliminary version of the paper [Ehr00] entitled "Parallel and serial hypercoherences", published in *Theoretical Computer Science*, North Holland, volume 247, pages 39-81, 2000.

Introduction

In [Ehr99], we proved that the hypercoherence model of PCF is the extensional collapse of the sequential algorithm model. J. van Oosten and J.R. Longley proved recently similar results [vO97, Lon98] in a realizability setting where realizers are deterministic strategies encoded as partial functions from the set of natural numbers to itself.

In all these works, a relation is established between a world of deterministic intensional realizers (sequential algorithms, or strategies encoded as partial functions on natural numbers) and strongly stable functions on hypercoherences: a realizer is related to a function if they "compute the same thing" (this is expressed as a logical relation, or as a realizability predicate, the latter being roughly speaking an untyped version of the former). It is shown that strongly stable functions admit an intensional realizer, which clearly means that all strongly stable functions are sequentially computable, if "sequentially" means "deterministically": for instance, all finite sequential algorithms are definable in a language which is an extension of PCF by a "catch and throw" operator (see [CCF94]), a perfectly deterministic primitive (in sharp contrast with the "parallel or" function for instance).

A hypercoherence X is just a set $|X|$ equipped with a set $\Gamma(X)$ of finite and non-empty subsets of $|X|$ containing all singletons (it is a "reflexive" and "symmetric" unlabeled hypergraph, just like coherence spaces are reflexive and symmetric unlabeled graphs). The elements of $\Gamma(X)$ are called "coherent", and they can have a complicated structure: coherent sets can contain incoherent sets, which themselves can contain coherent sets..., and moreover, these sets overlap. We would like

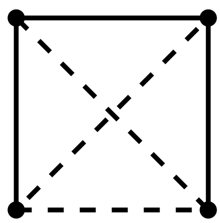

Figure 1: a P_4

to understand better the computational meaning of this structure. Our intuition is that there is a correspondence between the coherent sets of a hypercoherence and Player's positions (that is, the positions where the last move has been played by Player) of the corresponding game, and between the incoherent sets and Opponent's positions. From this viewpoint, the inclusion relation should be considered as a kind of game-theoretic accessibility relation, a position u being accessible from v if $u \subseteq v$. However, hypercoherences are not games, as in the strongly stable semantics, one identifies strategies that perform the same elementary operations, but in a different order. It is a much more "implicit" semantics than game semantics: the extensional collapse result mentioned above means that any strongly stable function (in the PCF types hierarchy) can be scheduled into *some* deterministic strategy, but the strongly stable function itself does not contain any explicit description of such a strategy. In some sense, both game semantics and strongly stable semantics deal with a fundamental notion of "computational time", the former in an explicit way and the latter in an implicit way. The extensional collapse result means precisely that, for a given PCF type, all informations required for describing the possible temporal computational behaviors at that type are present in the hypercoherence interpretation of that type. We would like to develop a *purely graphical* (that is, in some sense, geometrical) theory of the process of making explicit the temporal informations contained in the hypergraphical structure of a hypercoherence. Such a theory, we hope, might shed some new light on the notion of computational time.

We consider that the results reported in the present paper indicate that such a theory might be based on the notions of *parallel* and *serial* hypercoherences, and on a general way of converting a hypercoherence into a parallel one, the *rigid parallel unfolding.*

Our main methodological *a priori* concerning games is to consider them as coherence spaces of a very simple kind, corresponding to the standard notion of "serial-parallel graph[1]". A finite graph is serial-parallel if it contains[2] no "P_4". A P_4 is a graph which has four pairwise distinct vertices a_1, a_2, a_3, a_4 with an edge between a_i and a_j iff $j = i + 1$ or $i = j$ or $i = j + 1$. This configuration is pictured in figure 1 (in our graphical pictures, two points are related by a continuous line if they are related in the graph, that is, if they are "coherent" in the coherence space terminology, and by a dashed line if they are not related in the graph, that is, if they are "incoherent").

The serial-parallel finite coherence spaces are the elements of the smallest class of coherence spaces containing the one-vertex graphs and closed under the "&" and the "$\oplus$" operations on coherence spaces (which correspond respectively to serial and parallel composition of graphs). Moreover, the decomposition of a serial-parallel graph in terms of these two operations (up to associativity and commutativity of & and of $\oplus$) is unique.

In the infinite case, things are more complicated, and a coherence space can perfectly well not contain P_4 without being in a non-trivial way of the shape $E \,\&\, F$ or $E \oplus F$. For instance, the

[1]By "graph", we always mean reflexive and symmetric unlabeled graph, that is, coherence space.

[2]In that context, by "contain", we always mean "contain as an induced subgraph".

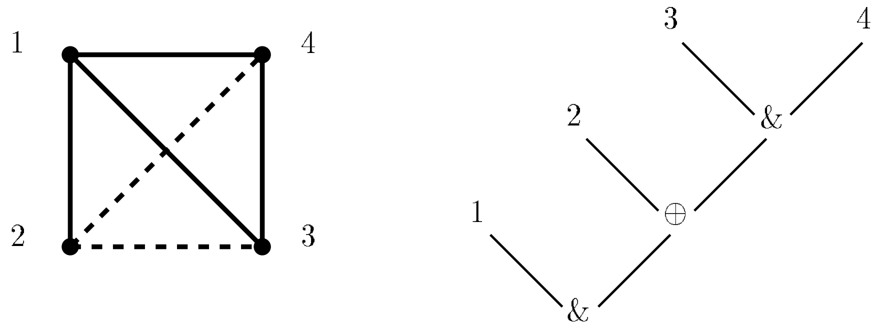

Figure 2: a serial-parallel graph and the corresponding tree

graph which has the natural numbers as vertices and where i is related to j (for $i < j$) iff i is even, contains no P_4, but cannot be decomposed.

Nevertheless, "serial-parallel" will basically mean for us "containing no P_4".

A (finite) serial-parallel coherence space can essentially be seen as a tree, vertices of the coherence space corresponding to leaves of the tree, and two vertices being related by an edge if the longest common prefix of the two corresponding paths (starting at the root) in the tree is of even length (this of course is conventional: observe that the complementary graph of a serial-parallel graph is also serial-parallel). This tree describes the unique decomposition of the coherence space in terms of the (multi-ary) & and $\oplus$ operations (see figure 2). The notion of serial-parallel graph is standard in graph theory (see for instance [BBS99]).

In this paradigm, we can see a serial-parallel coherence space as a game (see [Cur94] for a game-theoretic account of sequential algorithms), "Player's positions" corresponding to &-nodes and "Opponent's positions" to $\oplus$-nodes. Observe then that taking the orthogonal of the coherence space corresponds exactly to exchanging Opponent and Player in the corresponding game, which is the standard notion of duality in game models. The points of the coherence space are "extremal" positions in the game, that is positions closing the game. Observe that they do not belong to Player (&) or to Opponent ($\oplus$), they are in some sense neutral (this corresponds to the fact that in coherence spaces or hypercoherences, a singleton is both coherent and incoherent). This is of course very different from the standard game-theoretic situation. A similar notion of neutral extremal position appears in [Joy95].

Then a clique in the corresponding coherence space essentially corresponds to a deterministic partial strategy for Player. Whence the idea of studying the connection between hypercoherences and serial parallel graphs.

With this respect, a fundamental property of hypercoherences is that they allow to split the notion of "serial-parallel graph" in two dual notions: "serial hypercoherences" and "parallel hypercoherences". We shall say that a hypercoherence $X = (|X|, \Gamma(X))$ is parallel if, whenever two elements of $\Gamma(X)$ have a non-empty intersection, their union belongs to $\Gamma(X)$, and that X is serial if $X^\perp$ is parallel.

There is a bijective correspondence between serial-parallel coherence spaces and hypercoherences which are both parallel and serial.

These notions are presented in section 4, and in section 5, we make precise the connection between serial-parallel coherence spaces and games, in the finite case.

The present paper describes a general "parallel unfolding" construction that associates to any hypercoherence X a parallel hypercoherence $\widehat{X}$ together with a linear map $p_X : \widehat{X} \to X$ (of a special kind: its trace is a function). As far as we know, this pair $(\widehat{X}, p_X)$ has no universal property with respect to X. It has however a categorical "rigidity" property, presented in section 3, that guarantees its uniqueness up to unique isomorphism. Our intuition here is that the implicit character of time informations in hypercoherences leads to situations where a point of the web of a hypercoherence is contained in coherent sets which are "incompatible" in the sense that their union is not coherent (at such a point, the hypercoherence is not parallel). This intuition is developed on a simple example in section 2. The parallel unfolding of X has thus to be understood as a process of making computational time explicit. It basically consists in splitting each point where X is not parallel in as many points as there are maximal sets of coherent subsets of $|X|$ which contain the given point and are closed under union. So it looks like an "ultrafilter" construction, and the map $p_X : \widehat{X} \to X$ is the operation which forgets this splitting of the elements of the web. This unfolding is presented in section 6. We also describe it in a more intuitive way in the particular case of finite and serial hypercoherences in section 7. We present some basic properties of this unfolding construction:

- In section 8, we show that it does not cause an explosion of the cardinality of the webs of hypercoherences, as soon as one deals with hypercoherences satisfying a "local finiteness" condition which is preserved by all connectives of linear logic, and by the rigid parallel unfolding itself.
- In section 9, we show that the rigid parallel unfolding satisfies many commutation properties with respect to the connectives of linear logic: it has a good "logical social life".

Last, in section 10, we show how the rigid parallel unfolding can be used for associating to any formula of propositional linear logic a serial-parallel coherence space which is likely to be related to the game-theoretic interpretation of the formula. We prove that the main isomorphisms of linear logic are satisfied by this interpretation of formulae.

1 Preliminaries

If A is a set, we denote by $\#A$ its cardinality.

We first recall some basic definitions on coherence spaces and hypercoherences. For more informations on these topics, we refer to [Gir95, Ehr93].

Definition 1 A *coherence space* is a symmetric and reflexive graph. More precisely, it is a pair $E = (|E|, \coh_E)$ where $|E|$ is a set (the *web* of E, its elements are called *atoms* or *vertices*) and $\coh_E$ is a symmetric and reflexive binary relation on $|E|$. Two elements of $|E|$ which are related by this relation are said to be *coherent*.

A *clique* of E is a subset x of $|E|$ such that for all $a, a' \in x$, $a \coh_E a'$.

We denote by $\frown_E$ and call *strict* coherence relation of E the relation obtained from $\coh_E$ by removing the diagonal. Of course, a coherence space E can as well be defined by giving the anti-reflexive relation $\frown_E$.

If E is a coherence space, a *subspace* of E is a coherence space F such that $|F| \subseteq |E|$, and, for all $a, b \in |F|$, $a \coh_F b$ iff $a \coh_E b$.

We recall how linear negation and the additive connectives & and $\oplus$ (which are De Morgan dual of each other) are defined. Let E, E_1 and E_2 be coherence spaces.

- *Linear negation.* $E^\perp$ is defined by $|E^\perp| = |E|$ and $a \coh_{E^\perp} a'$ iff it does not hold that $a \frown_E a'$.

- *With.* $E_1 \mathbin{\&} E_2$ is defined by $|E_1 \mathbin{\&} E_2| = (\{1\} \times |E_1|) \cup (\{2\} \times |E_2|)$, and $(i,a) \coh_{E_1 \& E_2} (j,b)$ if $i = j \Rightarrow a \coh_{E_i} b$.

- *Plus.* $E_1 \oplus E_2$ is defined by $|E_1 \oplus E_2| = (\{1\} \times |E_1|) \cup (\{2\} \times |E_2|)$, and $(i,a) \coh_{E_1 \oplus E_2} (j,b)$ if $i = j$ and $a \coh_{E_i} b$.

Definition 2 A *hypercoherence* is a symmetric and reflexive hypergraph. More precisely, it is a pair $X = (|X|, \Gamma(X))$ where $|X|$ is a set (the *web* of X, its elements are called *atoms* or *vertices*) and $\Gamma(X)$ is a set of finite and non-empty subsets of $|X|$ which contains all singletons (the *coherence* of X, its elements are called *coherent sets* or *hyperedges*).

A *clique* of X is a subset x of $|X|$ such that all finite and non-empty subsets of x lie in $\Gamma(X)$. We denote by $\mathrm{qD}(X)$ the poset whose elements are the cliques of X ordered under inclusion[3].

We denote by $\Gamma^*(X)$ the set of all non-singleton elements of $\Gamma(X)$. A hypercoherence X can as well be defined by giving its *strict* coherence $\Gamma^*(X)$.

If X is a hypercoherence, a *subspace* of X is a hypercoherence Y such that $|Y|$ is subset of $|X|$, and $\Gamma(Y) = \Gamma(X) \cap \mathcal{P}(|Y|)$.

One says that X is *finite* if the set $|X|$ is finite.

If u and U are two sets, we say that u is a *section* of U and write $u \lhd U$ if

$$\forall a \in u\ \exists x \in U\ a \in x \quad \text{and} \quad \forall x \in U\ \exists a \in u\ a \in x\ .$$

Let us recall the interpretation of the connectives of linear logic in hypercoherences. Let X, X_1 and X_2 be hypercoherences.

- *Linear negation.* $X^\perp$ is defined by $|X^\perp| = |X|$ and $u \in \Gamma(X^\perp)$ if $u \notin \Gamma^*(X)$.

- *With.* $X_1 \mathbin{\&} X_2$ is defined by $|X_1 \mathbin{\&} X_2| = (\{1\}\times|X_1|)\cup(\{2\}\times|X_2|)$, and $(\{1\}\times u_1)\cup(\{2\}\times u_2) \in \Gamma(X_1 \mathbin{\&} X_2)$ if
$$u_2 = \emptyset \Rightarrow u_1 \in \Gamma(X_1) \quad \text{and} \quad u_1 = \emptyset \Rightarrow u_2 \in \Gamma(X_2)\ .$$
Let us also spell out the n-ary version of this construction, as it plays a central role in the paper. Let $X_1, \dots, X_n$ be hypercoherences. The hypercoherence $X_1 \mathbin{\&} \cdots \mathbin{\&} X_n$ has $(\{1\} \times |X_1|) \cup \cdots \cup (\{n\} \times |X_n|)$ as web, and a subset $u = (\{1\} \times u_1) \cup \cdots \cup (\{n\} \times u_n)$ of this web belongs to $\Gamma(X_1 \mathbin{\&} \cdots \mathbin{\&} X_n)$ iff u is finite and non-empty and, if u is contained in a unique component of the disjoint sum of the $|X_i|$'s, that is, if there exists $i \in \{1, \dots, n\}$ such that $u_j = \emptyset$ for all $j \neq i$, then u is coherent in that component, that is, $u_i \in \Gamma(X_i)$. In particular, if u_i and u_j are non-empty for two distinct indexes i and j, then u is always coherent. Last, let us quote that, when $x \in \mathrm{qD}(X_1 \mathbin{\&} \cdots \mathbin{\&} X_n)$ (so that $x = (\{1\} \times x_1) \cup \cdots \cup (\{n\} \times x_n)$ with $x_i \subseteq |X_i|$), one has $x_i \in \mathrm{qD}(X_i)$, and the map $x \mapsto (x_1, \dots, x_n)$ establishes a bijective order-preserving correspondence between $\mathrm{qD}(X_1 \mathbin{\&} \cdots \mathbin{\&} X_n)$ and $\mathrm{qD}(X_1) \times \cdots \times \mathrm{qD}(X_n)$, endowed with the product order.

- *Plus.* $X_1 \oplus X_2$ is defined by $|X_1 \oplus X_2| = (\{1\}\times|X_1|)\cup(\{2\}\times|X_2|)$, and $(\{1\}\times u_1)\cup(\{2\}\times u_2) \in \Gamma(X_1 \oplus X_2)$ if
$$u_2 = \emptyset \quad \text{and} \quad u_1 \in \Gamma(X_1)\ ,$$
or
$$u_1 = \emptyset \quad \text{and} \quad u_2 \in \Gamma(X_2)\ .$$

[3]The poset so defined belongs to the class of *qualitative domains* introduced by Girard in [Gir86]. Qualitative domains can equivalently be considered as dI-domains where all prime elements are atomic.

It is the De Morgan dual of with. Using the same notations as in the description above of the n-ary version of the with, u is coherent in $X_1 \oplus \cdots \oplus X_n$ iff u is contained in a unique component of the disjoint sum of the $|X_i|$'s, and coherent in that component. That is: there exists $i \in \{1, \ldots, n\}$ such that $u_j = \emptyset$ for all $j \neq i$, and $u_i \in \Gamma(X_i)$.

- *Tensor.* $X_1 \otimes X_2$ is defined by $|X_1 \otimes X_2| = |X_1| \times |X_2|$ and $w \in \Gamma(X_1 \otimes X_2)$ if

$$\pi_i(w) \in \Gamma(X_i) \quad \text{for} \quad i = 1, 2 .$$

- *Par.* It is the De Morgan dual of tensor. More explicitly, $|X_1 \parr X_2| = |X_1| \times |X_2|$ and $w \in \Gamma^*(X_1 \parr X_2)$ iff w is a finite and non-empty subset of $|X_1 \parr X_2|$ satisfying

$$\pi_i(w) \in \Gamma^*(X_i) \quad \text{for} \quad i = 1 \quad \text{or} \quad i = 2 .$$

- *Linear implication.* $X_1 \multimap X_2$ is defined by

$$X_1 \multimap X_2 = X_1{}^\perp \parr X_2 .$$

In other words, a subset w of $|X_1 \multimap X_2|$ belongs to $\Gamma(X_1 \multimap X_2)$ iff w is finite, non-empty, and satisfies

$$\pi_1(w) \in \Gamma(X_1) \Rightarrow (\pi_2(w) \in \Gamma(X_2) \text{ and } (\#\pi_2(w) = 1 \Rightarrow \#\pi_1(w) = 1)) .$$

A *linear strongly stable morphism* (or simply linear morphism) from X_1 to X_2 is a clique of $X_1 \multimap X_2$ (and so is a relation on $|X_1| \times |X_2|$), and composition of morphisms is defined as the composition of the corresponding relations. The identity morphism from X_1 to X_1 is the diagonal subset of $|X_1| \times |X_1|$. A linear morphism from X_1 to X_2 can also be seen as a function from $\mathrm{qD}(X_1)$ to $\mathrm{qD}(X_2)$ which commutes to the unions of arbitrary bounded families, maps coherent families of cliques[4] to coherent families, and commutes to the intersections of these families.

- *Exclamation mark.* We consider here the set version. There is also a multiset version. The web of $!X$ is the set of all finite cliques of X. A family U of finite cliques of X is in $\Gamma(!X)$ if it is finite and non-empty, and if

$$\forall u \lhd U \quad u \in \Gamma(X) .$$

- *Question mark.* It is the dual of exclamation mark. The web of $?X$ is the set of all finite cliques of $X^\perp$. A family U of finite cliques of $X^\perp$ is in $\Gamma^*(?X)$ if it is finite and non-empty, and if

$$\exists u \lhd U \quad u \in \Gamma^*(X) .$$

This entails that U is not a singleton, otherwise, $U = \{y\}$ where $y \in \mathrm{qD}(X^\perp)$, and then all finite sections of U belong to $\Gamma(X^\perp)$, as in that case all sections of U are subsets of y.

For more details on hypercoherences and the hypercoherent semantics of linear logic, we refer to [Ehr93].

[4] A family of cliques of a hypercoherence X is said to be coherent if it is finite and non-empty, and if all its finite and non-empty sections belong to $\Gamma(X)$.

2 A motivating example

The goal of this section is to motivate the forthcoming definitions and constructions by a detailed analysis of the graphical structure of the hypercoherence of sequential functions from $\mathbf{Bool}^n$ to $\mathbf{Bool}$, showing in particular that the corresponding game can be retrieved from this graphical structure. It is also intended to be an illustration of the previous general definitions on hypercoherences.

Let $n \in \mathbf{N}$ be different from 0. Let $\mathbf{Bool}$ be the hypercoherence of booleans, defined by $|\mathbf{Bool}| = \{\mathbf{t}, \mathbf{f}\}$ and $\Gamma^*(\mathbf{Bool}) = \emptyset$. Let $X = (!\mathbf{Bool}^n)^\perp \parr \mathbf{Bool}$ be the hypercoherence of sequential functions from $\mathbf{Bool}^n = \overbrace{\mathbf{Bool} \,\&\, \mathbf{Bool} \,\&\, \ldots \,\&\, \mathbf{Bool}}^{n\times}$ (the n-ary cartesian product of $\mathbf{Bool}$) to $\mathbf{Bool}$. An element of $|X|$ is a pair (x, b) where $b \in |\mathbf{Bool}|$ and x is a (possibly empty) subset of $\{1, \ldots, n\} \times |\mathbf{Bool}|$ satisfying $(i, a_1), (i, a_2) \in x \Rightarrow a_1 = a_2$. We denote by x_i or $\pi_i(x)$ the set of all a such that $(i, a) \in x$, this set is either empty or is a singleton; it is the i-th projection of x.

A subset u of $|X|$ belongs to $\Gamma^*(X)$ iff $\pi_2(u) \in \Gamma^*(\mathbf{Bool})$ or $\pi_1(u) \notin \Gamma(!\mathbf{Bool}^n)$. But $\Gamma^*(\mathbf{Bool}) = \emptyset$, so $u \in \Gamma^*(X)$ iff $\pi_1(u) \notin \Gamma(!\mathbf{Bool}^n)$. This last condition in turn is equivalent to requiring that there exists $v \lhd \pi_1(u)$ such that $v \notin \Gamma(\mathbf{Bool}^n)$. But this holds iff there exists $i \in \{1, \ldots, n\}$ such that $\pi_i(\pi_1(u))$ is equal to $\{\{\mathbf{t}\}, \{\mathbf{f}\}\}$.

Let $(x, b) \in |X|$. There is a bijective correspondence between the $\subseteq$-maximal elements of $\Gamma^*(X)$ which contain (x, b) and the indexes i such that $x_i \neq \emptyset$. Indeed, let $i \in \{1, \ldots, n\}$ be such that $x_i \neq \emptyset$. Then the set $u_{\langle i \rangle} = \{(y, c) \in |X| \mid y_i \neq \emptyset\}$ belongs to $\Gamma^*(X)$ and contains (x, b). Moreover, this set contains some element $(z, d) \in |X|$ where z is such that $z_j = \emptyset$ for all $j \neq i$. From this, it results that $u_{\langle i \rangle}$ is maximal among the elements of $\Gamma(X)$. Observe also that for the same reason, the only element $j \in \{1, \ldots, n\}$ such that $y_j \neq \emptyset$ for all $(y, c) \in u_{\langle i \rangle}$ is i. Conversely, if u is an element of $\Gamma^*(X)$ such that $(x, b) \in u$, we have seen that there must exist some $i \in \{1, \ldots, n\}$ such that $y_i \neq \emptyset$ for all $(y, c) \in u$. In particular, $x_i \neq \emptyset$. For such an index i, we clearly have $u \subseteq u_{\langle i \rangle}$. So if u is maximal in $\Gamma(X)$, there exists a unique $i \in \{1, \ldots, n\}$ such that $u = u_{\langle i \rangle}$.

Let $i_1 \in \{1, \ldots, n\}$ be such that $x_{i_1} \neq \emptyset$. The set $u_{\langle i_1 \rangle}$ is the disjoint union of two non-empty subsets, namely

$$u_{\langle i_1 \mathbf{t} \rangle} = \{(y, c) \in u_{\langle i_1 \rangle} \mid y_{i_1} = \{\mathbf{t}\}\} \text{ and } u_{\langle i_1 \mathbf{f} \rangle} = \{(y, c) \in u_{\langle i_1 \rangle} \mid y_{i_1} = \{\mathbf{f}\}\} ,$$

and neither of these two sets belongs to $\Gamma(X)$, by the above characterization of $\Gamma^*(X)$. Observe also that $u_{\langle i_1 \mathbf{t} \rangle}$ and $u_{\langle i_1 \mathbf{f} \rangle}$ are the two maximal subsets of $u_{\langle i_1 \rangle}$ which do not belong to $\Gamma(X)$. Indeed, let $u \subseteq u_{\langle i_1 \rangle}$ be such that $u \not\subseteq u_{\langle i_1 \mathbf{t} \rangle}$ and $u \not\subseteq u_{\langle i_1 \mathbf{f} \rangle}$. Then, since $u \subseteq u_{\langle i_1 \rangle}$, one has $y_{i_1} \neq \emptyset$ for all $(y, c) \in u$. And since $u \not\subseteq u_{\langle i_1 \mathbf{t} \rangle}$, one has $y_{i_1} = \{\mathbf{f}\}$ for some $(y, c) \in u$. Similarly, $y'_{i_1} = \{\mathbf{t}\}$ for some $(y', c') \in u$. From this, it results that $\pi_{i_1}(\pi_1(u)) = \{\{\mathbf{t}\}, \{\mathbf{f}\}\}$, and hence $u \in \Gamma^*(X)$.

Of course, (x, b) belongs to exactly one of these two subsets of $u_{\langle i_1 \rangle}$. Let us say for instance that $(x, b) \in u_{\langle i_1 \mathbf{f} \rangle}$ (that is, $x_{i_1} = \{\mathbf{f}\}$).

Again, there is a bijective correspondence between the maximal subsets of $u_{\langle i_1 \mathbf{f} \rangle}$ which contain (x, b) and belong to $\Gamma(X)$ and the indexes $i \in \{1, \ldots, n\} \setminus \{i_1\}$ such that $x_i \neq \emptyset$. Let i_2 be such an index. The corresponding subset of $u_{\langle i_1 \mathbf{f} \rangle}$ is

$$u_{\langle i_1 \mathbf{f} i_2 \rangle} = \{(y, c) \in u_{\langle i_1 \mathbf{f} \rangle} \mid y_{i_2} \neq \emptyset\} .$$

This set again is the disjoint union of two non-empty subsets, namely

$$u_{\langle i_1 \mathbf{f} i_2 \mathbf{t} \rangle} = \{(y, c) \in u_{\langle i_1 \mathbf{f} i_2 \rangle} \mid y_{i_2} = \{\mathbf{t}\}\}$$

and

$$u_{\langle i_1 \mathbf{f} i_2 \mathbf{f}\rangle} = \{(y,c) \in u_{\langle i_1 \mathbf{f} i_2\rangle} \mid y_{i_2} = \{\mathbf{f}\}\} \ .$$

Neither of these sets belong to $\Gamma(X)$, and (x,b) belongs to exactly one of them.

This process can be iterated until the enumeration $i_1, i_2, \ldots, i_k$ we are choosing exhausts the set of all indexes i such that $x_i \neq \emptyset$, in other terms, until $k = \#x$. For each such enumeration, denoting by a_j the unique boolean such that $x_{i_j} = \{a_j\}$, we define a decreasing sequence of subsets of $|X|$, which alternatively belong to $\Gamma^*(X)$ and $\Gamma^*(X^\perp)$, namely $u_{\langle i_1\rangle} \supset u_{\langle i_1 a_1\rangle} \supset u_{\langle i_1 a_1 i_2\rangle} \supset \cdots \supset u_{\langle i_1 a_1 \ldots i_k a_k\rangle}$. The maximal subset of $u_{\langle i_1 a_1 \ldots i_k a_k\rangle}$ which belongs to $\Gamma(X)$ and contains (x,b) is the singleton $\{(x,b)\}$.

So we have a bijective correspondence between the sequences $\sigma = \langle i_1, \ldots, i_k\rangle$ which are enumerations without repetitions of the indexes i such that $x_i \neq \emptyset$ and the sequences $|X| = v_0 \supset v_1 \supset v_2 \supset \cdots \supset v_{2k}$ satisfying:

- For all $i \in \{0, \ldots, k-1\}$, v_{2i+1} is a maximal subset of v_{2i} that belongs to $\Gamma^*(X)$ and contains (x,b).
- For all $i \in \{0, \ldots, k-1\}$, v_{2i+2} is a maximal subset of v_{2i+1} that belongs to $\Gamma^*(X^\perp)$ and contains (x,b). Observe also that $v_0 = |X| \in \Gamma^*(X^\perp)$ and that $v_{2k} = \{(x,\mathbf{t}), (x,\mathbf{f})\}$.

With the notations above, we have $v_1 = u_{\langle i_1\rangle}$, $v_2 = u_{\langle i_1, a_1\rangle}$, and so on.

Let us call such a sequence $(v_j)_{j=0,\ldots,2k}$ a *tower* at (x,b). (We may have $k=0$, it corresponds to the case where $x = \emptyset$.)

Choosing a sequence σ of indexes which is an enumeration of the indexes i such that $x_i \neq \emptyset$ is just associating to (x,b) an evaluation order, that is, in terms of game theory, a *play*. We shall say that (x,b,σ) is a *play* at (x,b).

Indeed, in the theory of sequential algorithms as it is developed in [Cur94], the game corresponding to the type $\mathbf{Bool}^n \to \mathbf{Bool}$ can be presented as follows[5].

- A move by Player is either $(\mathbf{r}, b)$ where b is a boolean, and $\mathbf{r}$ means that this move is played in the right component of the $\to$ type constructor, or is $(\mathbf{l}, i)$ where $i \in \{1, \ldots, n\}$, and $\mathbf{l}$ means that this move is played in the left component of the $\to$ type constructor.
- A move by Opponent is either $(\mathbf{r}, *)$ where $*$ is the only initial move of the game **Bool**, or $(\mathbf{l}, a)$ where a is a boolean.
- A play is a sequence of moves $s = \langle m_1, \ldots, m_k\rangle$ where $m_1 = (\mathbf{r}, *)$ and such that, for all $q = 1, \ldots, k-1$, the moves m_q and m_{q+1} are not both played by Player or both played by Opponent. Moreover, the following conditions must be satisfied by s:
  - For $q = 2, \ldots, k$, if $m_q = (\mathbf{r}, \alpha)$, then necessarily α is a boolean, $q = k$, and for all $r = 2, \ldots, k-1$, the move m_r must be played in the left component of the $\to$ type constructor.
  - For q, r two distinct elements of $\{2, \ldots, k\}$, if $m_q = (\mathbf{l}, i)$ and $m_r = (\mathbf{l}, j)$ for some $i, j \in \{1, \ldots, n\}$, then $i \neq j$[6].

[5] This presentation is obtained by simply spelling out the general definitions of the interpretation of the $\multimap$ and ! connective in [Cur94]. The particular shape of the type under consideration leads to simplifications, especially concerning the moves in the left component of the $\to$ type constructor.

[6] This "no repetition" principle is characteristic of the interpretation of the ! connective in sequential algorithms. From the strongly stable viewpoint, it corresponds to the fact that, in the semantics we consider here, the web of

If we say that a play s is *complete* if its last move is of the shape $(\mathbf{r}, b)$, where b is a boolean, then it appears clearly that there is a bijective correspondence between the complete plays in the game associated to the type $\mathbf{Bool}^n \to \mathbf{Bool}$ in the theory of sequential algorithms, and the plays (x, b, σ) defined above.

When are two *different* plays (x, b, σ) and (y, c, τ) compatible, in the sense that they can both appear in a deterministic strategy, or sequential algorithm? Exactly when the longest common prefix $\langle i_1, \ldots, i_q \rangle$ of σ and τ is non-empty, and satisfies $x_{i_j} = y_{i_j}$ for all $j < q$, and $x_{i_q} \neq y_{i_q}$. And a sequential algorithm (or strategy) is essentially a set of plays which are pairwise compatible in this sense.

If (v_i) and (w_j) are the towers associated to (x, b, σ) and (y, c, τ), this compatibility condition translates to:

$$\text{there exists } i \text{ such that } v_i \neq w_i, \text{ and the least such } i \text{ is } \textit{even}. \tag{1}$$

So there is a way of retrieving from the hypercoherence X the structure of the coherence space E of sequential algorithms[7] from $\mathbf{Bool}^n$ to $\mathbf{Bool}$: the web of this space consists of the set of all possible $(x, b, (v_i))$ where $(x, b) \in |X|$ and (v_i) is a tower at (x, b) and its coherence relation is given by (1). Furthermore, there is an obvious forgetful map π from $|E|$ to $|X|$ defined by $\pi(x, b, (v_i)) = (x, b)$. One can check that this map is strongly stable (in the sense that its graph is a clique of $E \multimap X$), when E is considered as a hypercoherence as follows: $U \in \Gamma^*(E)$ if there exists i which is less than the length of all the towers of U and such that the v_i's are not all equal (for $(x, b, (v_j)) \in U$), and the least such i is even. This can be simply rephrased as follows: U is *connected* in E (considered as a graph).

Of course, the notion of tower is not very easy to handle, and it turns out fortunately that E can be defined in another, much more general way from X. Observe first that in a tower (v_i) at (x, b) the v_i's of even rank (those which belong to $\Gamma^*(X^\perp)$) are completely determined by (x, b) and by the previous v_i's of odd rank. The presence in general of several towers for a given $(x, b) \in |X|$ is essentially due to the fact that the union of two coherent subsets of $|X|$ which contain (x, b) is not necessarily coherent, and indeed, one can check that the towers at (x, b) are in bijective correspondence with the maximal subsets of $\Gamma(X)$ which are closed under finite unions and of which all elements contain (x, b).

This latter observation will serve as a definition when we build the *rigid parallel unfolding* of a hypercoherence.

3 Rigid objects

Before giving our general definitions and unfolding constructions on hypercoherences, we introduce a general categorical concept of *rigidity*, which is strictly weaker than the usual categorical notion of universality. The unfolding of hypercoherences will be characterized in terms of rigidity, and not in terms of universality.

Definition 3 Let A be an object of a category $\mathcal{C}$.

$!X$ is the set of all finite *cliques* (sets of points of the web of X) of X, and not of all finite *multi-cliques* (multisets of points of the web of X). In the games considered e.g. in the papers [AJM94, HO94, Nic94], repeated moves are allowed in the interpretation of !.

[7]These sequential algorithms are not really standard: they are sequential algorithms on sequential data structures (see [Cur94]) equipped with a notion of complete plays. This notion can be defined inductively on the construction of spaces, and the sequential algorithms we consider are strategies consisting only of complete plays.

- A is *rigid*[8] if $\mathrm{Hom}_{\mathcal{C}}(A, A) = \{\mathrm{Id}_A\}$.
- A is *weakly terminal* if $\mathrm{Hom}_{\mathcal{C}}(B, A) \neq \emptyset$ for all objects B of $\mathcal{C}$.

Lemma 4 *Let A and A' be isomorphic objects in a category $\mathcal{C}$. If A is rigid, then A' is rigid too.*

The proof is straightforward.

A terminal object is of course rigid. But a rigid weakly terminal object is not necessarily terminal, as we shall see. Being a rigid weakly terminal object is apparently not a universal property. However,

Proposition 5 *Let I and I' be two rigid weakly terminal objects in a category $\mathcal{C}$. Then* $\mathrm{Hom}_{\mathcal{C}}(I, I')$ *has exactly one element, and this unique morphism from I to I' is an isomorphism.*

The proof is straightforward.

We are interested in a particular situation. Let $\mathcal{C}$ be a category and let $\mathcal{P}$ be a class of objects of $\mathcal{C}$, which is closed under isomorphisms.

Definition 6 Let A be an object of $\mathcal{C}$. A *$\mathcal{P}$-unfolding* of A is a weakly terminal object of $\mathcal{P}/A$. A $\mathcal{P}$-unfolding of A is *rigid* if it is rigid as an object of $\mathcal{P}/A$.

So, a $\mathcal{P}$-unfolding of A is an object P of $\mathcal{P}$ together with a morphism $p : P \to A$ such that for any $Q \in \mathcal{P}$ and any morphism $f : Q \to A$, there exists a (not necessarily unique) morphism $f' : Q \to P$ such that $p \circ f' = f$. We shall say that f' is a lifting of f along p. A very similar lifting condition played an essential role in [Ehr96].

Saying that (P, p) is a rigid $\mathcal{P}$-unfolding of A means furthermore that Id_P is the only morphism $g : P \to P$ such that $p \circ g = p$. By proposition 5, if (P', p') is another rigid $\mathcal{P}$-unfolding of A, there is exactly one morphism $f : P \to P'$ such that $p' \circ f = p$, and f is an isomorphism. And if $P' \in \mathcal{C}$ and $f : P' \to P$ is an isomorphism (so that actually $P' \in \mathcal{P}$), then $(P', p \circ f)$ is also a rigid $\mathcal{P}$-unfolding of A, by lemma 4.

Lemma 7 *Let A and A' be objects of $\mathcal{C}$, and let $\varphi : A \to A'$ be an isomorphism. If (P, p) is a rigid $\mathcal{P}$-unfolding of A, then $(P, \varphi \circ p)$ is a rigid $\mathcal{P}$-unfolding of A'.*

This is trivial.

When it exists, we denote by $(\widehat{A}, p_A)$ the rigid $\mathcal{P}$-unfolding of A. Observe that the operation $A \mapsto \widehat{A}$ has no reason to be functorial (by lack of universality).

We develop now a simple example of the abstract situation previously described. The interest of this example is that it is similar to the construction we shall introduce in section 6 for hypergraphs.

Let **Poset** be the category of locally finite posets (partially ordered sets where each element has a finite number of lower bounds) with a least element, and monotone functions.

Let **Tree** be the class of trees. A tree is a poset T having a least element and where, for all $t \in T$, the set

$$\downarrow t = \{s \in T \mid s \leq t\}$$

is finite and totally ordered by the order of T.

Let V be any object of **Poset**. We define a new poset $\mathcal{T}(V)$ as follows:

[8] Actually, one should rather use a term like "strongly rigid" as the word "rigid" is classically used for objects which have the identity as unique *automorphism* (and not endomorphism).

- An element of $\mathcal{T}(V)$ is a pair (v, I) where $v \in V$ and I is maximal among the subsets of $\downarrow v$ which are totally ordered (so $v \in I$).
- We endow $\mathcal{T}(V)$ with the following order: $(v, I) \leq (w, J)$ iff $I \subseteq J$ (which implies $v \leq w$).

As V is locally finite, for all $(v, I) \in \mathcal{T}(V)$, I is finite, and so $\mathcal{T}(V) \in$ **Tree**.

The map $\pi_V : \mathcal{T}(V) \to V$ which maps (v, I) to v is monotone.

Moreover, let $T \in$ **Tree** and let $f : T \to V$ be a monotone map. Let $(t_i)_{i \in A}$ be an enumeration without repetitions of T (assuming T to be denumerable for simplicity; A is either $\mathbf{N}$, the set of natural numbers, or an initial segment of it).

Assume furthermore this enumeration to be such that

$$t_i < t_j \Rightarrow i < j .$$

Such an enumeration exists by local finiteness of T as a poset.

We define a function $g : T \to \mathcal{T}(V)$ inductively: by induction on n, we define g on the set $\{t_1, \ldots, t_n\}$. So let $n \in \mathbf{N}$ and assume, as an inductive hypothesis, that, for each $i \leq n$ we have been able to define I_i, a maximal totally ordered subset of $\downarrow f(t_i)$ such that $f(\downarrow t_i)$, which is totally ordered, is a subset of I_i (the function g on $\{t_1, \ldots, t_n\}$ is given by $g(t_i) = (f(t_i), I_i)$). Our inductive hypothesis stipulates also that

$$\forall i, j \in \mathbf{N} \quad i, j \leq n \text{ and } t_i \leq t_j \Rightarrow I_i \subseteq I_j . \tag{2}$$

Our goal is to extend g to $\{t_1, \ldots, t_n, t_{n+1}\}$, that is, to define I_{n+1}, a maximal totally ordered subset of $\downarrow f(t_{n+1})$, in such a way that condition (2) still holds for $n + 1$.

Let t be the unique element of T which is maximal such that $t < t_{n+1}$ (the predecessor of t_{n+1}).

By our assumption on the enumeration (t_i), we know that $t = t_m$ for some $m \in \mathbf{N}$ such that $m \leq n$. Observe that $I_m \cup f(\downarrow t_{n+1}) = I_m \cup \{f(t_{n+1})\}$ is totally ordered. So define I_{n+1} as one of the totally ordered maximal subset of $\downarrow f(t_{n+1})$ containing $I_m \cup \{f(t_{n+1})\}$. It is clear that condition (2) still holds for $n + 1$.

The map $g : T \to \mathcal{T}(V)$ which to $t \in T$ associates $(f(t), I_n)$ (where n is the unique index such that $t_n = t$) is monotone. And so $(\mathcal{T}(V), \pi_V)$ is a **Tree**-unfolding of V.

Let us check that it is a rigid unfolding.

Let $f : \mathcal{T}(V) \to \mathcal{T}(V)$ be such that

$$\pi_V \circ f = \pi_V . \tag{3}$$

Let $(v, I) \in \mathcal{T}(V)$. By (3), one has $f(v, I) = (v, I')$ where I' is a maximal totally ordered subset of $\downarrow v$. Let $w \in I$, and let $J = I \cap \downarrow w$. One has $(w, J) \in \mathcal{T}(V)$ and $(w, J) \leq (v, I)$ in $\mathcal{T}(V)$. So $f(w, J) \leq (v, I')$, and hence $w \in I'$ since, by (3) again, $f(w, J)$ is equal to (w, J') for some $J' \subseteq \downarrow w$.

Thus $I \subseteq I'$, and since I is a maximal totally ordered subset of $\downarrow v$, one actually has $I = I'$. So f is the identity function.

Towards an application of this construction, observe that the category **Poset** is cartesian, the cartesian product of two posets being endowed with the product order. If T and T' are trees, $\mathcal{T}(T \times T')$ is a tree which is easily seen to be the "shuffle product" of the trees T and T'. Using the rigidity of this operation, one shows easily that it is associative. However, it is not a functorial operation.

We shall define a similar unfolding for hypercoherences.

4 Parallel and serial hypercoherences

We first introduce the class of parallel hypercoherences, and its dual class, the serial hypercoherences. The hypercoherence $(!\mathbf{Bool}^n)^\perp \parr \mathbf{Bool}$ considered in section 2 is a typical example of serial hypercoherence.

Definition 8 A hypercoherence X is *parallel* if for all $u, u' \in \Gamma(X)$, if $u \cap u' \neq \emptyset$, then $u \cup u' \in \Gamma(X)$.
A hypercoherence X is *serial* if its orthogonal $X^\perp$ is parallel.

Observe that any subspace of a parallel (resp. serial) hypercoherence is parallel (resp. serial).
Let X be a parallel hypercoherence, and let A be a non-empty subset of $|X|$. Then the binary relation $\sim_A$ defined on A by

$$a \sim_A a' \text{ iff there exists } u \in \Gamma(X) \text{ such that } a, a' \in u \subseteq A$$

is an equivalence relation. Furthermore, if A is finite then the two following properties are equivalent:

- $\sim_A$ has only one equivalence class
- $A \in \Gamma(X)$.

If A is a set, we denote by $\mathcal{P}^*_{\mathrm{fin}}(A)$ the set of all its finite and non-empty subsets.

Proposition 9 *Let X be a hypercoherence. The two following conditions are equivalent.*

i) X is serial.

ii) For all $u \in \Gamma^*(X)$, there exist $u_1, u_2 \in \mathcal{P}^*_{\mathrm{fin}}(|X|)$ such that $u_1 \cap u_2 = \emptyset$, $u_1 \cup u_2 = u$ and, for all $v \subseteq u$, if v intersects both u_1 and u_2, then $v \in \Gamma(X)$. We abbreviate this situation by writing simply $u = u_1 \,\&\, u_2$.

Proof: We first prove that (i) implies (ii). Let $u \in \Gamma^*(X)$. Then the relation $\sim_u$ (in $X^\perp$, which is parallel) is an equivalence relation which has more than one class. Let u_1 be one of these classes, and let $u_2 = u \setminus u_1$. Then $u_2 \neq \emptyset$. Let $v \subseteq u$ be such that $u_i \cap v \neq \emptyset$ for $i = 1, 2$. Let $a_i \in u_i \cap v$, for $i = 1, 2$. As $a_1 \not\sim_u a_2$ and as $a_1, a_2 \in v \subseteq u$, one has $v \notin \Gamma(X^\perp)$, that is $v \in \Gamma^*(X)$.

Conversely, assume that (ii) holds. We must prove that $X^\perp$ is parallel. Let $u, u' \in \Gamma(X^\perp)$ be such that $u \cap u' \neq \emptyset$. Assume that $u \cup u' \notin \Gamma(X^\perp)$, that is $u \cup u' \in \Gamma^*(X)$. Then we can find $u_1, u_2 \subseteq u \cup u'$, both non-empty, and such that

$$u \cup u' = u_1 \,\&\, u_2 \ .$$

Then u cannot intersect both u_1 and u_2, and similarly for u'. Without loss of generality, assume that $u \subseteq u_1$. As u' intersects u and hence intersects u_1, we must have $u' \subseteq u_1$. Hence $u \cup u' \subseteq u_1$, which is impossible since u_2 is not empty, and $u \cup u'$ is the disjoint union of u_1 and u_2. ■

Let us be more precise about this decomposition of the coherent subsets of the web of a serial hypercoherence.

Proposition 10 *Let X be a serial hypercoherence. Let $u \in \Gamma(X)$. Up to reindexing, there exists a unique family $u_1, \ldots, u_n$ of pairwise disjoint elements of $\Gamma(X^\perp)$ such that $u = u_1 \cup \cdots \cup u_n$, and such that, for all $v \subseteq u$, if $v \cap u_i \neq \emptyset$ for at least two distinct values of $i \in \{1, \ldots, n\}$, then $v \in \Gamma(X)$.*

If one considers u, $u_1,\ldots,u_n$ as subspaces of X, then the bijection from u to $(\{1\} \times u_1) \cup \cdots \cup (\{n\} \times u_n)$ which maps $a \in u$ to (i, a), where i is the unique index such that $a \in u_i$, is an isomorphisms between u and $u_1 \mathbin{\&} \cdots \mathbin{\&} u_n$.

Proof: The existence of this decomposition has essentially been established in the proof of proposition 9: for $(u_i)_{i=1,\ldots,n}$ we take an enumeration of the classes of the equivalence relation $\sim_u$ (in the parallel hypercoherence $X^\perp$). We just check that $u_i \in \Gamma(X^\perp)$. Since the elements of u_i are pairwise $\sim_u$-equivalent, there exists a subset v of u such that $u_i \subseteq v \in \Gamma(X^\perp)$. Now if $v \neq u_i$, then v meets u_j for some $j \neq i$, and hence $v \in \Gamma^*(X)$, contradiction. So $u_i = v \in \Gamma(X^\perp)$ as announced.

Now, we check uniqueness. Let $(v_j)_{j=1,\ldots,k}$ be another decomposition of u satisfying the same properties as $(u_i)_{i=1,\ldots,n}$. Without loss of generality, assume that $v_1 \neq u_i$ for each $i \in \{1,\ldots,n\}$. As $v_1 \in \Gamma(X^\perp)$, v_1 meets at most one of the u_i's, and since v_1 is not empty and is included in $u_1 \cup \cdots \cup u_n$, v_1 must meet one of the u_i's. Let i be the unique index such that v_1 meets u_i. We must have $v_1 \subseteq u_i$, and this inclusion is strict by our hypothesis on v_1. Since $u_i \subseteq v_1 \cup \cdots \cup v_k$, the set u_i must meet some set v_j with $j \neq 1$, and we have a contradiction with the fact that $u_i \in \Gamma(X^\perp)$. ∎

Next, we study the intersection of these two classes of hypercoherences.

Definition 11 Let E be a coherence space. One defines a hypercoherence E^{c} by setting $|E^{\mathrm{c}}| = |E|$, and by taking for $\Gamma(E^{\mathrm{c}})$ the set of all finite and non-empty connected subsets of $|E|$ (considering E as a graph). It is obvious that E^{c} is a parallel hypercoherence. If $u \subseteq |E|$ and if $a \in u$, we denote by $(a)_u$ the connected component of a in u (i.e. the set of all elements of u related to a by a path *contained in u*).

If X is a hypercoherence, one defines a coherence space X_{coh} by $|X_{\mathrm{coh}}| = |X|$ and $a \frown_{X_{\mathrm{coh}}} b$ iff $\{a, b\} \in \Gamma(X)$.

Definition 12 A coherence space E is *serial-parallel* if its web contains no tuple of four pairwise distinct elements (a_1, a_2, a_3, a_4) such that, for all $i, j \in \{1,2,3,4\}$ such that $i < j$, $a_i \frown_E a_j$ holds iff $j = i + 1$. (Such a tuple is called a "P_4" in graph theory, see figure 1.)

In a serial-parallel coherence space, connected sets have a very simple structure.

Lemma 13 *Let E be a serial-parallel coherence space. A subset u of $|E|$ is connected iff for all $a, b \in u$, there exists $c \in u$ such that $a \frown_E c$ and $c \frown_E b$.*

Proof: Consider a path between a and b in u, and if the length (number of edges) of this path is strictly greater than 2, apply iteratively the hypothesis that the graph E contains no P_4. ∎

The terminology previously introduced for hypercoherences is justified by the following result.

Theorem 14 *Let E be a serial-parallel coherence space. The hypercoherence E^{c} is both serial and parallel, and $E = {E^{\mathrm{c}}}_{\mathrm{coh}}$.*

Conversely, let X be a hypercoherence which is both serial and parallel. Then X_{coh} is a serial-parallel coherence space, and $X = {X_{\mathrm{coh}}}^{\mathrm{c}}$.

So that we can identify the notions of serial-parallel coherence space with the notion of serial and parallel hypercoherence.

Proof: Let us prove the first statement. We already know that E^{c} is parallel. We prove that this hypercoherence is serial (see figure 3). Let $u, v \in \Gamma(E^{\mathrm{c}\perp})$ be such that $u \cap v \neq \emptyset$. We show that $u \cup v \in \Gamma(E^{\mathrm{c}\perp})$. If one of the two sets u and v is a singleton, then we conclude immediately. So we assume that $u, v \in \Gamma^*(E^{\mathrm{c}\perp})$, that is, we assume that u and v are not connected. Assume moreover

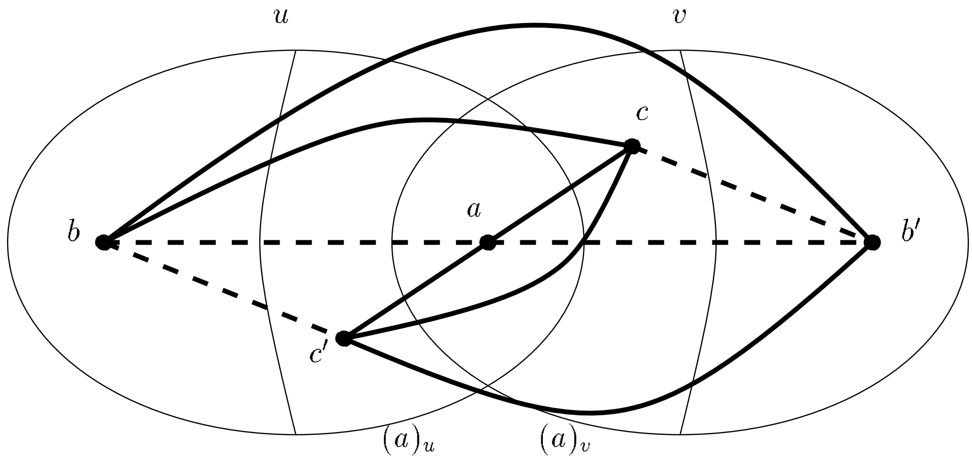

Figure 3: main step of the proof of theorem 14

that $u \cup v$ is connected, aiming at a contradiction. Let $a \in u \cap v$. Let $b \in u \setminus (a)_u$. By lemma 13, there exists $c \in u \cup v$ such that $a \frown_E c$ and $b \frown_E c$. Since $b \notin (a)_u$, we necessarily have that $c \in (a)_v \setminus u$. Similarly, let $b' \in v \setminus (a)_v$. We can find $c' \in (a)_u \setminus v$ such that $a \frown_E c'$ and $a \frown_E b'$. Since E is serial-parallel, (c', a, c, b) cannot be a P_4, and hence $c \frown_E c'$. Now, (b, c, c', b') cannot be a P_4, and hence $b \frown_E b'$. But now (a, c, b, b') is a P_4 in E, whence the contradiction.

The equation $E = {E^{\mathrm{c}}}_{\mathrm{coh}}$ is obvious. We prove now the second statement, showing first that $X = {X_{\mathrm{coh}}}^{\mathrm{c}}$. The webs are clearly the same. Let $u \in \Gamma({X_{\mathrm{coh}}}^{\mathrm{c}})$. Since u is connected, one can find an enumeration $a_1, \ldots, a_n$ of u such that $a_i \frown_{X_{\mathrm{coh}}} a_{i+1}$ for all $i = 1, \ldots, n-1$ (of course, with possibly some repetitions), that is $\{a_i, a_{i+1}\} \in \Gamma(X)$. Using iteratively the fact that X is parallel, one concludes immediately that $u \in \Gamma(X)$. Conversely, let $u \in \Gamma^*(X)$ (if u is a singleton, there is nothing to prove). By proposition 9, we can find $u_1, u_2 \subseteq u$, both non-empty, such that $u = u_1 \,\&\, u_2$. Then for all $a_1 \in u_1$ and $a_2 \in u_2$ one has $a_1 \frown_{X_{\mathrm{coh}}} a_2$, hence u is connected in X_{coh}. Assume we have a P_4 (a, b, c, d) in X_{coh}. Then $u = \{a, b, c, d\}$ belongs to $\Gamma({X_{\mathrm{coh}}}^{\mathrm{c}}) = \Gamma(X)$, but $u = \{a, b, d\} \cup \{a, c, d\}$ and $\{a, b, d\}, \{a, c, d\} \in \Gamma(X^\perp)$ (both sets are non-connected), and $\{a, b, d\} \cap \{a, c, d\} \neq \emptyset$. This is contradictory because $X^\perp$ is parallel, and hence X_{coh} is serial-parallel. ■

We conclude this section by stating a few preservation properties of logical connectives with respect to parallel and serial hypercoherences.

Proposition 15 *Let X and Y be hypercoherences. If X and Y are parallel, then so are $X \,\&\, Y$, $X \oplus Y$, $X \otimes Y$ and $!X$, and $X^\perp$ is serial. If X and Y are serial, then so are $X \,\&\, Y$, $X \oplus Y$, $X \parr Y$ and $?X$, and $X^\perp$ is parallel.*

Proof: We just check the exponential case. Let $U, V \in \Gamma(!X)$ be such that $U \cap V \neq \emptyset$. Let $w \lhd U \cup V$. Let $u = w \cap \bigcup U$ and $v = w \cap \bigcup V$. Then $u \lhd U$ and $v \lhd V$, so $u, v \in \Gamma(X)$. Let $x \in U \cap V$ and let $a \in x$ be such that $a \in w$. Then $a \in u \cap v$, hence $u \cup v \in \Gamma(X)$ since X is parallel and we conclude since $u \cup v = w$. ■

5 Finite serial-parallel coherence spaces and games

There are various equivalent ways of presenting games. The most usual one consists in defining a game as a set of Opponent/Player-polarized *moves*, together with a prefix-closed set of *plays*, which are Opponent/Player-alternating sequences of moves. This set of plays constitutes a tree for the usual prefix ordering of sequences. We used this presentation in our informal discussion in section 2. But a game can also be presented directly as a tree of Opponent/Player-polarized *positions*, this choice has been done for example by Lamarche in [Lam92], and we prefer this presentation here. In this approach, a move is a transition from a position (starting position) to one of its immediate successors in the tree. A move is played by Player if the polarity of the starting position is Opponent, and by Opponent if the polarity of the starting position is Player.

We can apply proposition 10 and theorem 14 for establishing the connection we mentioned in the introduction between finite serial-parallel coherence spaces and finite games. In the non finite case, things are slightly more complicated, but, for instance, the notion of local finiteness introduced in section 8 can be used for extending this connection.

We start by an obvious observation on serial-parallel coherence spaces.

Lemma 16 *Let E and F be serial-parallel coherence spaces. Then $E \,\&\, F$ and $E \oplus F$ are serial-parallel, and one has $(E \,\&\, F)^{\mathrm{c}} = E^{\mathrm{c}} \,\&\, F^{\mathrm{c}}$ and $(E \oplus F)^{\mathrm{c}} = E^{\mathrm{c}} \oplus F^{\mathrm{c}}$.*

To any finite serial-parallel coherence space E, we want to associate an ordered set of positions P_E, which is a finite tree (see the definition of a tree in section 3), together with a labeling function $\lambda_E : \mathrm{P}_E \to \{\mathsf{O}, \mathsf{P}, \mathsf{N}\}$ which is alternating in the sense that, if $s, t \in \mathrm{P}_E$ and s is the predecessor of t, then $\lambda_E(s) \neq \lambda_E(t)$, and such that, moreover, $\lambda_E(s) = \mathsf{N}$ iff s is a maximal element of P_E (final positions are neutral). The elements of the poset P_E will be subsets of $|E|$, and the order relation of P_E will be the reversed inclusion on these subsets. We define now $(\mathrm{P}_E, \lambda_E)$ by induction on $\#|E|$.

For this purpose, we prefer to consider E as a serial and parallel hypercoherence (we identify E with E^{c}). Indeed, we know that the serial-parallel coherence spaces are in bijective correspondence with the serial and parallel hypercoherences by theorem 14, and, by lemma 16, that the additive connectives commute to this correspondence.

- If $|E| = \emptyset$, then $\mathrm{P}_E = \emptyset$ and there is nothing more to say.
- If $|E|$ is a singleton $\{a\}$, then $\mathrm{P}_E = \{\{a\}\}$, and $\lambda_E(\{a\}) = \mathsf{N}$.
- If $|E| \in \Gamma^*(E)$, then we know by proposition 10 that there exists a unique family of pairwise disjoint subspaces $E_1, \ldots, E_n$ (with $n \geq 2$ and $\#|E_i| \geq 1$ for $i = 1, \ldots, n$) of E such that $|E| = \bigcup_{i=1}^n |E_i|$, $|E_i| \in \Gamma(E_i{}^{\perp})$ and such that, up to the canonical bijection between $|E|$ and $|E_1 \,\&\, \cdots \,\&\, E_n|$, one has $E = E_1 \,\&\, \cdots \,\&\, E_n$. We set $\mathrm{P}_E = \{|E|\} \cup \bigcup_{i=1}^n \mathrm{P}_{E_i}$. Observe that this union is disjoint, as
  - if $s \in \mathrm{P}_{E_i}$, then s is a non-empty subset of $|E_i|$, and the sets $|E_i|$ are pairwise disjoint,
  - and as the inclusion $|E_i| \subseteq |E|$ is strict for each i.

  Last, we define λ_E by $\lambda_E(s) = \lambda_{E_i}(s)$ if $s \in \mathrm{P}_{E_i}$ and $\lambda_E(|E|) = \mathsf{P}$.
- Symmetrically, if $|E| \in \Gamma^*(E^{\perp})$, we find a unique family $E_1, \ldots, E_n$ (with $n \geq 2$) of pairwise disjoint non-empty subspaces of E such that $|E| = \bigcup_{i=1}^n |E_i|$, $|E_i| \in \Gamma(E_i{}^{\mathrm{c}})$ and $E = E_1 \oplus \cdots \oplus E_n$ (up to the canonical bijections between the web of these two spaces). Then we set as before $\mathrm{P}_E = \{|E|\} \cup \bigcup_{i=1}^n \mathrm{P}_{E_i}$ and we observe that this union is disjoint. Last, we define λ_E by $\lambda_E(s) = \lambda_{E_i}(s)$ if $s \in \mathrm{P}_{E_i}$ and $\lambda_E(|E|) = \mathsf{O}$.

Observe that, for $s \in \mathrm{P}_E$, $\lambda_E(s) = \mathsf{N}$ iff $\#s = 1$, $\lambda_E(s) = \mathsf{P}$ iff $s \in \Gamma^*(E)$, that is, iff $\#s \geq 2$ and s is connected in E (if one considers again E as a serial-parallel coherence space). And observe that $\lambda_E(s) = \mathsf{O}$ iff $s \in \Gamma^*(E^\perp)$, that is, iff $\#s \geq 2$ and s is connected in $E^\perp$ (again, considered as a serial-parallel coherence space), that is, iff s is not connected[9] in E. Observe also that, due to the uniqueness property stated by proposition 10, the game $(\mathrm{P}_E, \lambda_E)$ is uniquely determined by the serial-parallel space E.

Conversely, given a game (P, λ) where P is a finite tree and $\lambda : P \to \{\mathsf{O}, \mathsf{P}, \mathsf{N}\}$ is a function, we can define a hypercoherence $\mathrm{SP}_{(P,\lambda)}$ by $|\mathrm{SP}_{(P,\lambda)}| = \{s \in P \mid \lambda(s) = \mathsf{N}\}$ and, for $S \subseteq |\mathrm{SP}_{(P,\lambda)}|$, $S \in \Gamma^*(\mathrm{SP}_{(P,\lambda)})$ iff $\#S \geq 2$ and the glb of S in P (which exists, as P is a tree) is mapped to P by λ. Then it is easily checked that the hypercoherence $\mathrm{SP}_{(P,\lambda)}$ is always serial and parallel, and that, if the game (P, λ) we start from is given by $P = \mathrm{P}_E$ and $\lambda = \lambda_E$ for some finite serial and parallel hypercoherence E, then $\mathrm{SP}_{(P,\lambda)}$ is canonically isomorphic to E. It is in that sense that finite serial and parallel hypercoherences can be considered as games[10].

6 Parallel unfolding of a hypercoherence

We show in this section that any hypercoherence admits a rigid unfolding (in the sense of definition 6) with respect to the class of parallel hypercoherences, in the category of hypercoherences and strongly stable linear maps. This construction generalizes what has been done in a concrete case in section 2.

So for any hypercoherence X, we shall show that there exists a parallel hypercoherence Y, together with a linear strongly stable morphism $p : Y \multimap X$ satisfying the conditions prescribed in section 3. But it turns out that p will belong to a very particular class of morphisms, it will be a "web morphism".

Definition 17 Let X and Y be hypercoherences. A *web morphism* from X to Y is a morphism $f : X \multimap Y$ which is a function from $|X|$ to $|Y|$. (Remember that f, by definition of a morphism, is a subset of $|X| \times |Y|$; we just require this subset to be functional, in the usual set-theoretic sense.) Equivalently, a web morphism from X to Y is a function $f : |X| \to |Y|$ satisfying

$$\forall u \in \Gamma^*(X) \quad f(u) \in \Gamma^*(Y) \ .$$

When $f : X \multimap Y$ is a web morphism, we write $f : X \to Y$.

Let X and Y be hypercoherences, and let $p : Y \to X$ be a web morphism. Assume that Y is parallel, and that (Y, p) is a rigid unfolding of X with respect to parallel hypercoherences, in the category of hypercoherences and *web morphisms*. We show that (Y, p) is also a rigid unfolding of X with respect to parallel hypercoherences, in the category of hypercoherences and *arbitrary linear morphisms*.

Indeed, let Z be a parallel hypercoherence and let $f : Z \multimap X$ be a linear morphism. Let us define a hypercoherence T as follows: $|T| \subseteq |Z| \times |X|$ is the trace of f and a subset w of $|T|$ is in $\Gamma(T)$ iff it is finite, non-empty and satisfies $\pi_1(w) \in \Gamma(Z)$. Then it is clear that T is parallel and that

[9] This is another characterization of serial-parallel coherence spaces which derives from theorem 14: a coherence space E is serial-parallel iff for each finite subset u of the web of E such that $\#u \geq 2$, if u is connected in E, then u is not connected in $E^\perp$. The converse implication always holds, as easily checked (observe that u has at least two connected components in $E^\perp$.). Observe by the way that the P_4 is the smallest coherence space E which is connected both in E and in $E^\perp$.

[10] Observe however that if we start from a game (P, λ) and define $E = \mathrm{SP}_{(P,\lambda)}$, and then $P' = \mathrm{P}_E$ and $\lambda' = \lambda_E$, we arrive to a game (P', λ') which in general is not isomorphic to (P, λ).

π_2 is a web morphism $T \to X$, and so there exists a web morphism $g : T \to Y$ such that $p \circ g = \pi_2$. Observe then that there is a linear map $f' : Z \multimap T$, whose trace is $\{(c,(c,a)) \mid (c,a) \in \mathrm{tr}(f)\}$ such that $\pi_2 \circ f' = f$, so that $g' = g \circ f'$ is a linear map $Z \multimap Y$ such that $p \circ g' = f$. So arbitrary linear maps from a parallel hypercoherence to X can be lifted along p.

Last we show that any linear map $h : Y \multimap Y$ satisfying $p \circ h = p$ is actually a web morphism, and hence must be the identity morphism from Y to Y. Indeed, one has $p \circ h = \{(b, p(b')) \mid (b,b') \in h\}$ and so since $p \circ h = p$ and p is a web morphism, for all $b \in |Y|$, there exists $b' \in |Y|$ such that $(b,b') \in h$. Next, let $b, b'_1, b'_2 \in |Y|$ be such that $(b,b'_1),(b,b'_2) \in h$. Since $h \in \mathrm{qD}(Y \multimap Y)$, we must have $\{b'_1, b'_2\} \in \Gamma(Y)$. But as $p \circ h = p$, we have $p(b'_1) = p(b'_2) = p(b)$, and hence by the characterization above of web morphisms, $\{b'_1, b'_2\} \notin \Gamma^*(Y)$, so $b'_1 = b'_2$ and h is a web morphism.

Consequently, and without loss of generality, instead of constructing rigid parallel unfoldings in the category of hypercoherences and linear morphisms, we restrict our attention to the subcategory of hypercoherences and web morphisms.

Before proving that all hypercoherences admit a rigid parallel unfolding, let us introduce a few useful notations. Let X be a hypercoherence. If $u \in \Gamma(X)$, let us denote by $\Gamma_u(X)$ the set $\{v \in \Gamma(X) \mid u \subseteq v\}$ and by $\mathcal{F}_u(X)$ the set of all maximal subsets of $\Gamma_u(X)$ which are closed under finite unions.

Lemma 18 *Let α be a subset of $\Gamma_u(X)$. One has $\alpha \in \mathcal{F}_u(X)$ iff the two following conditions are satisfied:*

i) $\forall v, v' \in \alpha \quad v \cup v' \in \alpha$

ii) For all $v \in \Gamma_u(X)$, if $v \cup v' \in \Gamma(X)$ for all $v' \in \alpha$, then $v \in \alpha$.

Proof: Assume first that $\alpha \in \mathcal{F}_u(X)$ and let us prove property (ii). So let $v \in \Gamma_u(X)$ be such that

$$\forall v' \in \alpha \quad v \cup v' \in \Gamma(X) \;.$$

Let $\alpha' = \{v \cup v' \mid v' \in \alpha\}$ and $\beta = \alpha \cup \alpha'$. We have $\alpha \subseteq \beta \subseteq \Gamma_u(X)$. To conclude, it suffices to prove that β is closed under binary unions. So let $w, w' \in \beta$. Assume for instance that $w, w' \in \alpha'$, the other cases being simpler. Then $w = v \cup v'$ and $w' = v \cup v''$ for some $v', v'' \in \alpha$. But $v' \cup v'' \in \alpha$, and hence $w \cup w' = v \cup (v' \cup v'') \in \alpha'$.

The converse implication is straightforward. ∎

For $a \in |X|$, we abbreviate $\Gamma_{\{a\}}(X)$ by $\Gamma_a(X)$ and $\mathcal{F}_{\{a\}}(X)$ by $\mathcal{F}_a(X)$. Observe that, if $\alpha \in \mathcal{F}_u(X)$, then $u \in \alpha$ by maximality. For $\alpha \in \mathcal{F}_a(X)$, the only singleton belonging to α is $\{a\}$.

Observe that the three following conditions are equivalent:

- X is parallel.
- For all $a \in |X|$, the set $\mathcal{F}_a(X)$ is reduced to $\{\Gamma_a(X)\}$.
- For all $u \in \Gamma(X)$, the set $\mathcal{F}_u(X)$ is reduced to $\{\Gamma_u(X)\}$.

The cardinality of $\mathcal{F}_a(X)$ measures in some sense the lack of parallelism of X at a.

Definition 19 We define now a hypercoherence $\widehat{X}$ which is intended to be the rigid parallel unfolding of X.

Its web is given by

$$|\widehat{X}| = \bigcup_{a \in |X|} \mathcal{F}_a(X) \;.$$

Observe that this union is disjoint. Before giving $\Gamma(\widehat{X})$, we define a function $p_X : |\widehat{X}| \to |X|$ by: $p_X(\alpha)$ is the only $a \in |X|$ such that $\{a\} \in \alpha$. In other words, p_X is characterized by

$$\alpha \in \mathcal{F}_{p_X(\alpha)}(X) \ .$$

Let $U \subseteq |\widehat{X}|$ and let $u = p_X(U)$. We say that $U \in \Gamma(\widehat{X})$ iff U is finite and non-empty and satisfies

$$u \in \Gamma(X) \quad \text{and} \quad \bigcap U \in \mathcal{F}_u(X) \ .$$

This condition can be rephrased as follows. First, let $v \in \Gamma(X)$, let $\alpha \in \mathcal{F}_v(X)$ and let $w \in \alpha$. We denote by α_w the set $\alpha \cap \uparrow w = \{v' \in \alpha \mid w \subseteq v'\}$. Observe that $\alpha_w \in \mathcal{F}_w(X)$.

Lemma 20 *A subset U of $|\widehat{X}|$ belongs to $\Gamma(\widehat{X})$ iff U is finite and non-empty, and satisfies the following two conditions:*

i) For all $\alpha \in U$, the set $u = p_X(U)$ belongs to α.

ii) For all $\alpha, \alpha' \in U$, $\alpha_u = \alpha'_u$.

And if $U \in \Gamma(\widehat{X})$, one has $\alpha_u = \bigcap U$ for each $\alpha \in U$ (where $u = p_X(U)$).

Proof: First, assume that $U \in \Gamma(\widehat{X})$. We prove (i). We have $\bigcap U \in \mathcal{F}_u(X)$, so $u \in \bigcap U$. But $\bigcap U \subseteq \alpha$ and hence $u \in \alpha$ for all $\alpha \in U$. Next, let $\alpha \in U$. We have $\bigcap U \subseteq \alpha_u$. But $\alpha_u \in \mathcal{F}_u(X)$, and our hypothesis says that $\bigcap U \in \mathcal{F}_u(X)$, so $\bigcap U = \alpha_u$ and this proves (ii).

Conversely, let U be a finite and non-empty subset of $|\widehat{X}|$ satisfying (i) and (ii), and let $u = p_X(U)$. Since U is non-empty, $u \in \Gamma(X)$ by condition (i). The set $\bigcap U$ is closed under finite unions as an intersection of sets having that property. Let $v \in \bigcap U$. For all $\alpha \in U$ one has $v \in \alpha$, and hence $p_X(\alpha) \in v$. Hence $u \subseteq v$. Last, $\bigcap U$ belongs to $\mathcal{F}_u(X)$ since condition (ii) implies that $\alpha_u \subseteq \bigcap U$ and since $\alpha_u \in \mathcal{F}_u(X)$ (for each $\alpha \in U$). ∎

Theorem 21 *Let X be a hypercoherence.*

i) $(\widehat{X}, p_X)$ is a rigid parallel unfolding of X.

ii) Furthermore, let Y be a parallel hypercoherence, let $f : |Y| \to |X|$ be a web morphism, let $b \in |Y|$ and let $\alpha \in \mathcal{F}_{f(b)}(x)$ be such that $f(\Gamma_b(Y)) \subseteq \alpha$. Then there exists a lifting g of f along p_X such that $g(b) = \alpha$.

iii) Specifically, for all $\alpha \in |\widehat{X}|$, for all $u \in \alpha$, there exists $U \in \Gamma_\alpha(\widehat{X})$ such that $p_X(U) = u$.

Proof: Let us first check that $\widehat{X}$ is parallel. Let $U, V \in \Gamma(\widehat{X})$ be such that $U \cap V \neq \emptyset$, and let α be an element of this intersection. Let $u = p_X(U)$, $v = p_X(V)$. Since $U \in \Gamma(\widehat{X})$ we have $u \in \alpha$. Similarly $v \in \alpha$. But α is closed under unions so $u \cup v \in \alpha$. Now let $\beta, \gamma \in U \cup V$ and let $w \subseteq |X|$ be finite and such that $u \cup v \subseteq w$. If $w \in \beta$, since $\beta, \alpha \in U$ or $\beta, \alpha \in V$, and since $U, V \in \Gamma(\widehat{X})$, we have $w \in \alpha$. Then since $\gamma, \alpha \in U$ or $\gamma, \alpha \in V$, we have $w \in \gamma$. By lemma 20, we conclude that $U \cup V \in \Gamma(\widehat{X})$.

Let $U \in \Gamma(\widehat{X})$. By definition, $p_X(U) \in \Gamma(X)$. If this set is a singleton $\{a\}$, then each element of U is in $\mathcal{F}_a(X)$ and for two such α and α' we must have $\alpha_{\{a\}} = \alpha'_{\{a\}}$, that is $\alpha = \alpha'$. So p_X is a web morphism.

Let Y be a parallel hypercoherence, and let $f : |Y| \to |X|$ be a web morphism. We want to build a web morphism $g : |Y| \to |\widehat{X}|$ such that $p_X \circ g = f$.

Let $B \subseteq |Y|$. Assume that, for each $b \in B$, we have found $g(b) \in \mathcal{F}_{f(b)}(X)$ in such a way that the two following conditions are satisfied.

(a) $\forall b \in B \quad f(\Gamma_b(Y)) \subseteq g(b)$.

(b) $\forall v \in \Gamma(Y)\, \forall b, b' \in B \cap v \quad g(b)_{f(v)} = g(b')_{f(v)}$.

These conditions are very natural. Indeed, let $v \in \Gamma_b(Y)$. First, $g(b) \in g(v)$, so $p_X(g(v)) \in g(b)$ (by lemma 20, (i)). Since we want to have $p_X \circ g = f$, this implies that $f(v) \in g(b)$ so condition (a) must hold. Condition (b) comes from the fact that $g(v) \in \Gamma(\widehat{X})$, and from lemma 20, (ii).

Let $c \in |Y|$. We prove that we can extend g to $B \cup \{c\}$ in such a way that these two properties still hold for this extension. For $v \in \Gamma_c(Y)$ such that $v \cap B \neq \emptyset$, let us denote by F_v the common value of all the $g(b)_{f(v)}$'s for $b \in B \cap v$. Let

$$F = \bigcup \{F_v \mid v \in \Gamma_c(Y) \text{ and } v \cap B \neq \emptyset\} .$$

We first prove that F is closed under unions. For $i = 1, 2$, let $u_i \in F$. Let $v_i \in \Gamma_c(Y)$ be such that $v_i \cap B \neq \emptyset$ and $u_i \in F_{v_i}$. Let $b \in v_1 \cap B$. We have $u_1 \in g(b)$. Since Y is parallel and since $c \in v_1 \cap v_2$, we have $v_1 \cup v_2 \in \Gamma(Y)$, and so $v_1 \cup v_2 \in \Gamma_b(Y)$ since $b \in v_1$. Hence $u_1 \cup f(v_1 \cup v_2) \in g(b)$ as $g(b)$ is closed under unions and contains $f(\Gamma_b(Y))$ as a subset. Hence

$$u_1 \cup f(v_1 \cup v_2) \in g(b)_{f(v_1 \cup v_2)} = F_{v_1 \cup v_2} .$$

Symmetrically one proves that $u_2 \cup f(v_1 \cup v_2) \in F_{v_1 \cup v_2}$ and hence $u_1 \cup u_2 \cup f(v_1 \cup v_2) \in F_{v_1 \cup v_2}$, that is

$$u_1 \cup u_2 \in F_{v_1 \cup v_2}$$

since $f(v_i) \subseteq u_i$.

Next, we prove that $F \cup f(\Gamma_c(Y))$ is closed under unions. Since this property holds for F and for $f(\Gamma_c(Y))$, we have just one case to check. Let $u \in F$ and let $v' \in \Gamma_c(Y)$. Let $v \in \Gamma_c(Y)$ be such that $v \cap B \neq \emptyset$ and $u \in F_v$. Again we choose $b \in v \cap B$. We have

$$u \in g(b) \quad \text{and} \quad f(v \cup v') \in g(b) ,$$

hence

$$u \cup f(v \cup v') \in g(b)_{f(v \cup v')} = F_{v \cup v'}$$

and we conclude since $u \cup f(v \cup v') = u \cup f(v')$.

Let us choose for $g(c)$ any element of $\mathcal{F}_{f(c)}(X)$ such that $F \cup f(\Gamma_c(Y)) \subseteq g(c)$. Indeed, we may apply Zorn's lemma, since, denoting by $\mathcal{C}$ the set of all subsets of $\Gamma_{f(c)}(X)$ which are closed under finite unions, each totally ordered subset $\mathcal{T}$ of $\mathcal{C}$ is upper-bounded by $\bigcup \mathcal{T}$ which belongs to $\mathcal{C}$, and we have proved that $F \cup f(\Gamma_c(Y)) \in \mathcal{C}$. Property (a) obviously holds for this extension of g; let us check property (b). The only non-trivial case is when $b \in v \cap B$ and $b' = c$ (and hence $v \in \Gamma_c(Y)$). But we have $g(b)_{f(v)} = F_v$ by definition of F_v, and by definition of $g(c)$, we have that

$$F_v \subseteq g(c)_{f(v)} .$$

This inclusion is actually an equality by maximality of F_v and because $g(c)_{f(v)}$ is closed under unions.

To build the required function g on $|Y|$, one chooses an ordinal enumeration of $|Y|$ and one uses the property above in a trivial transfinite induction. As a result, we get a function $g : |Y| \to |\widehat{X}|$ satisfying (a) and (b) for $B = |Y|$. These two properties, together with lemma 20, imply that $g(v) \in \Gamma(\widehat{X})$ for all $v \in \Gamma(Y)$. It is also clear that, by construction of g, for all $b \in |Y|$, one has $p_X(g(b)) = f(b)$. For showing that g is a web morphism from Y to $\widehat{X}$, it remains to check that if

$v \in \Gamma^*(Y)$, one has $\#g(v) \geq 2$. As f is a web morphism, one has $\#f(v) \geq 2$, that is $\#p_X(g(v)) \geq 2$ and hence $g(v)$ cannot be a singleton.

Item (ii) of the theorem is an obvious consequence of this construction as we can choose the enumeration of $|Y|$ in such a way that $b_1 = b$, and for $g(b_1)$, we can choose $g(b_1)$ freely among all the $\alpha \in \mathcal{F}_{f(b_1)}(X)$ such that $f(\Gamma_{b_1}(Y)) \subseteq \alpha$. Item (iii) is a special case of (ii). Indeed, let Y be the parallel hypercoherence defined by $|Y| = u$ and $\Gamma^*(Y) = \{u\}$ (if u is not a singleton; otherwise, there is nothing to prove). Take for f the inclusion of $|Y|$ into $|X|$ which is obviously a web morphism. Let g be a lifting of f along p_X, and set $U = g(u)$.

To conclude, let $h : \widehat{X} \to \widehat{X}$ be a web morphism such that $p_X \circ h = p_X$, and assume that $h \neq \mathrm{Id}$. Let $\alpha \in |\widehat{X}|$ be such that $\beta = h(\alpha) \neq \alpha$. Let $a = p_X(\alpha) = p_X(\beta)$. Then by maximality of α and β, there exist $u \in \alpha$ and $v \in \beta$ such that $u \cup v \notin \Gamma(X)$. By (iii), we can find $U \in \Gamma_\alpha(\widehat{X})$ and $V \in \Gamma_\beta(\widehat{X})$ such that $p_X(U) = u$ and $p_X(V) = v$. Since h is a web morphism, we must have $h(U) \in \Gamma_\beta(\widehat{X})$, and since $\widehat{X}$ is parallel, we have $h(U) \cup V \in \Gamma_\beta(\widehat{X})$, hence $p_X(h(U) \cup V) \in \Gamma_a(X)$ since p_X is a web morphism. But $p_X(h(U) \cup V) = u \cup v$, and we have a contradiction. ∎

Remark: Another important consequence of the lifting property is that, whenever x is a clique of X, there exists a clique A of $\widehat{X}$ such that $p_X(A) = x$. Indeed, x may be considered as a (trivially) parallel subspace of X. Usually, there are many cliques A in $\widehat{X}$ such that $p_X(A) = x$. But if x is sufficiently "large", the clique A is unique. It can be checked for instance that if Z is a hypercoherence, if $X = Z^\perp \parr Z$ and if x is the identity clique of X (that is, $x = \{(c, c) \mid c \in |Z|\}$), then there is exactly one clique A of $\widehat{X}$ such that $p_X(A) = x$: there is only one way of unfolding the identity. This possibility of lifting all cliques along p_X presents some similarity with part (iii) of theorem 21 above. It is in some sense much stronger in that it deals with non necessarily finite subsets of the web of X, and moreover, when x is finite, it says not only that x can be lifted in a coherent subset of $|\widehat{X}|$, but moreover that all non-empty subsets of x can be *simultaneously* lifted as coherent subsets of $|\widehat{X}|$.

Remark: As observed by one of the referees of this paper, there is another (and simpler) way of associating to a hypercoherence X a parallel hypercoherence Y: for $|Y|$, take the same definition as for $|\widehat{X}|$, but remove the maximality requirement (that is, an element of $|Y|$ is a pair (a, A) where $a \in |X|$ and $A \subseteq \Gamma_a(X)$ is closed under finite unions, but not necessarily maximal such), and for $\Gamma(Y)$, take lemma 20 as a definition. Then one can also define a projection web morphism $p : Y \to X$ by $p(a, A) = a$, and it is straightforward that each web morphism from some parallel hypercoherence to X can be lifted along p. Moreover, this construction can be characterized by a universal property of initiality, and is clearly functorial. However, this very natural construction is too "generous" in the sense that when X is already parallel, the hypercoherence associated to X is not isomorphic to X itself. Moreover, this construction does not satisfy theorem 23 that we consider as essential. A similar construction is also possible in the poset example of section 3 (replace "maximal totally ordered subsets" by "totally ordered subsets").

The next proposition provides a characterization of coherence and incoherence in $\widehat{X}$ which is very simple and will be useful in the proof of the next theorem.

Proposition 22 *Let U be a non-empty and finite subset of $|\widehat{X}|$.*

i) U belongs to $\Gamma(\widehat{X})$ iff, for all $(u_\alpha)_{\alpha \in U}$ such that $u_\alpha \in \alpha$ for each $\alpha \in U$, one has $\bigcup_{\alpha \in U} u_\alpha \in \Gamma(X)$.

ii) U belongs to $\Gamma(\widehat{X}^\perp)$ iff there exists $(u_\alpha)_{\alpha \in U}$ such that $u_\alpha \in \alpha$ for each $\alpha \in U$, and $\bigcup_{\alpha \in U} u_\alpha \in \Gamma(X^\perp)$.

Proof: We prove (i). Let $U \in \Gamma(\widehat{X})$, and let $(u_\alpha)_{\alpha\in U}$ be such that $u_\alpha \in \alpha$ for each $\alpha \in U$. Let $u = p_X(U)$. We know that for each $\alpha \in U$, $u \in \alpha$, so that $u \cup u_\alpha \in \alpha$ and hence $u \cup u_\alpha \in \alpha_u$. Now since $U \in \Gamma(\widehat{X})$, one has $\alpha_u = \bigcap U$ by lemma 20, and hence $u \cup u_\alpha \in \cap U$. As this holds for each $\alpha \in U$, one has in particular $\bigcup_{\alpha\in U}(u \cup u_\alpha) \in \Gamma(X)$ but this last set is equal to $\bigcup_{\alpha\in U} u_\alpha$ as, for each $\alpha \in U$, $p_X(\alpha) \in u_\alpha$.

Conversely, assume that $\bigcup_{\alpha\in U} u_\alpha \in \Gamma(X)$ whenever $u_\alpha \in \alpha$ for each $\alpha \in U$. Let $u = p_X(U)$. As $u = \bigcup_{\alpha\in U} u_\alpha$ where $u_\alpha = \{p_X(\alpha)\} \in \alpha$ for each $\alpha \in U$, we have $u \in \Gamma(X)$. Now let $\alpha \in U$ and let us prove that $u \in \alpha$. If this were not the case, there would exist some $v \in \alpha$ such that $u \cup v \notin \Gamma(X)$. Now set $u_\beta = \{p_X(\beta)\}$ if $\beta \neq \alpha$ and $u_\alpha = v$. We have $\bigcup_{\beta\in U} u_\beta = u \cup v \notin \Gamma(X)$, and this is a contradiction. Last, let $\alpha, \beta \in U$, and let $v \in \alpha$ be such that $u \subseteq v$, and assume that $v \notin \beta$. Then, there exists $w \in \beta$ such that $v \cup w \notin \Gamma(X)$. As previously, one derives a contradiction, defining a family $(u_\gamma)_{\gamma\in U}$ as follows:

$$u_\gamma = \begin{cases} v & \text{if } \gamma = \alpha \\ w & \text{if } \gamma = \beta \\ \{p_X(\gamma)\} & \text{otherwise.} \end{cases}$$

The union of that family is $v \cup w$, as $u \subseteq v$.

Now we prove (ii). Assume first that $U \in \Gamma(\widehat{X}^\perp)$. If U is a singleton $\{\alpha\}$, we can take $u_\alpha = \{p_X(\alpha)\} \in \Gamma(X^\perp)$. Otherwise, $U \notin \Gamma(\widehat{X})$ and we apply (i). Conversely, let $(u_\alpha)_{\alpha\in U}$ be such that $u_\alpha \in \alpha$ for each $\alpha \in U$, and $v = \bigcup_{\alpha\in U} u_\alpha \in \Gamma(X^\perp)$. If v is not a singleton, we conclude directly, applying (i). Otherwise, $v = \{a\}$ with $p_X(\alpha) = \{a\}$ for each $\alpha \in U$. Then $U \in \Gamma(\widehat{X}^\perp)$ because p_X is a web morphism. ∎

Theorem 23 *Let X be a serial hypercoherence. Then $\widehat{X}$ is serial too (and hence is serial and parallel).*

Proof: Let $U, V \in \Gamma(\widehat{X}^\perp)$ having a non-empty intersection, and let $\alpha \in U \cap V$. By proposition 22 we can find a family $(v_\beta)_{\beta\in U}$ such that $v_\beta \in \beta$ for each $\beta \in U$ and a family $(w_\gamma)_{\gamma\in V}$ such that $w_\gamma \in \gamma$ for each $\gamma \in V$, such that moreover

$$v = \bigcup_{\beta\in U} v_\beta \in \Gamma(X^\perp) \quad \text{and} \quad w = \bigcup_{\gamma\in V} w_\gamma \in \Gamma(X^\perp)\ .$$

We define a family $(u_\delta)_{\delta\in U\cup V}$ as follows:

$$u_\delta = \begin{cases} v_\delta & \text{if } \delta \in U \setminus V \\ w_\delta & \text{if } \delta \in V \setminus U \\ v_\delta \cup w_\delta & \text{if } \delta \in U \cap V\ , \end{cases}$$

then $u_\delta \in \delta$ for each $\delta \in U \cup V$. Since X is serial, and since clearly $p_X(\alpha) \in v \cap w$, we have $v \cup w \in \Gamma(X^\perp)$. But

$$\bigcup_{\delta\in U\cup V} u_\delta = v \cup w$$

and we conclude, by proposition 22. ∎

7 Unfolding a finite serial hypercoherence

We present now another, and maybe more intuitive, way of constructing $\widehat{X}$ in the special case where X is a finite and serial hypercoherence. For all such X, let us define a hypercoherence $\widetilde{X}$ together with a web morphism $q_X : \widetilde{X} \to X$ by induction on $\#|X|$ as follows:

i) If $\#|X| = 1$, then $\widetilde{X} = X$ and $q_X = \text{Id}$.

ii) If $|X| \in \Gamma^*(X)$, then by proposition 10, as X is serial, it can be written in a unique way (up to permutations of indexes) as $X = X_1 \,\&\, \cdots \,\&\, X_n$ where the X_i's are pairwise disjoint non-empty subspaces of X verifying $|X_i| \in \Gamma(X^\perp)$. So the sets $|X_i|$ are the maximal subsets of X which belong to $\Gamma(X^\perp)$. Then we set

$$\widetilde{X} = \widetilde{X_1} \,\&\, \cdots \,\&\, \widetilde{X_n} \quad \text{and} \quad q_X = q_{X_1} \,\&\, \cdots \,\&\, q_{X_n} \ .$$

iii) If $|X| \notin \Gamma(X)$, then let $X_1, \ldots, X_n$ be the maximal subspaces of X whose web belongs to $\Gamma(X)$. Observe that these subspaces are not necessarily disjoint (because X may not be parallel). Then we set

$$\widetilde{X} = \widetilde{X_1} \oplus \cdots \oplus \widetilde{X_n} \ .$$

We define q_X as

$$q_X = q \circ (q_{X_1} \oplus \cdots \oplus q_{X_n}) \ ,$$

where

$$\begin{aligned} q : \bigcup_{i=1}^{n} (\{i\} \times |X_i|) \quad &\to \quad |X| \\ (i, a) \quad &\mapsto \quad a \end{aligned}$$

Indeed, q is a web morphism from $X_1 \oplus \cdots \oplus X_n \to X$ as easily checked.

The hypothesis that X is serial is heavily used for proving that q_X is a web morphism. Indeed, otherwise, in the case where $|X| \in \Gamma^*(X)$, the X_i's (maximal subspaces of X such that $|X_i| \notin \Gamma^*(X)$) would not define a partition of X and then, setting $\widetilde{X} = \widetilde{X_1} \,\&\, \cdots \,\&\, \widetilde{X_n}$ and $q_X = r \circ (q_{X_1} \,\&\, \cdots \,\&\, q_{X_n})$ (where $r : |X_1 \,\&\, \cdots \,\&\, X_n| \to |X|$ is defined as the function q above) would not give rise to a web morphism in general. It turns out that when X is serial, r is an isomorphism, and this makes this construction possible.

The following property immediately results from this construction.

Lemma 24 *Let Z be a finite serial hypercoherence. If $|Z| \in \Gamma(Z)$, then $|\widetilde{Z}| \in \Gamma(\widetilde{Z})$.*

We shall use the following general lemma.

Lemma 25 *Let $(S, \leq)$ be a poset, let $A \subseteq S$ be directed and $B \subseteq S$ be finite. Then*

$$(\forall s \in A \, \exists t \in B \ s \leq t) \ \Rightarrow \ (\exists t \in B \, \forall s \in A \ s \leq t)$$

Proposition 26 *Let X be finite and serial. Then $(\widetilde{X}, q_X)$ is a rigid parallel unfolding of X. Consequently, there is a unique morphism $\varphi : \widetilde{X} \to \widehat{X}$ such that $p_X \circ \varphi = q_X$, and φ is an isomorphism.*

Proof: We prove the result by induction on $\#|X|$. Let Y be a parallel hypercoherence and let $f : Y \to X$ be a web morphism.

- For $\#|X| \leq 1$, the result is obvious.

- Assume that $|X| \in \Gamma^*(X)$. Let $X = X_1 \mathbin{\&} \cdots \mathbin{\&} X_n$ be the decomposition of X in maximal subspaces X_i such that $|X_i| \in \Gamma(X^\perp)$ given by proposition 10. For $i = 1, \dots, n$, let Y_i be the subspace of Y whose web is $f^{-1}(|X_i|)$, and let f_i be the restriction of f to this subspace. By inductive hypothesis, we can find $g_i : Y_i \to \widetilde{X_i}$ such that $q_{X_i} \circ g_i = f_i$. We set $g = (g_1 \mathbin{\&} \cdots \mathbin{\&} g_n) \circ j$ where $j : |Y| \to |Y_1 \mathbin{\&} \cdots \mathbin{\&} Y_n|$ maps each $b \in |Y|$ to (i, b) where i is the unique index such that $b \in |Y_i|$. As j is clearly a web morphism from Y to $Y_1 \mathbin{\&} \cdots \mathbin{\&} Y_n$, the function g is a web morphism from Y to $\widetilde{X}$, and we have $q_X \circ g = f$. Now, let $h : \widetilde{X} \to \widetilde{X}$ be such that $q_X \circ h = q_X$, and let h_i be its restriction to $\widetilde{X_i}$. It is easily checked that h_i is a web morphism $\widetilde{X_i} \to \widetilde{X_i}$ such that $q_{X_i} \circ h_i = q_{X_i}$ and hence by inductive hypothesis, $h_i = \mathrm{Id}$, so that $h = \mathrm{Id}$.
- Assume last that $|X| \notin \Gamma(X)$, and let $X_1, \dots, X_n$ be its maximal subspaces such that $|X_i| \in \Gamma(X)$. Since Y is parallel, it can be written as $Y = \bigoplus_{j \in J} Y_j$ where the family $(|Y_j|)_{j \in J}$ is an enumeration (without repetitions) of $|Y|/\sim_{|Y|}$, the equivalence relation $\sim_{|Y|}$ on $|Y|$ having been defined at the beginning of section 4 [11]. For each $j \in J$, $\Gamma(Y_j)$ is a directed set. Indeed, as $|Y_j|$ is an equivalence class of the relation $\sim_{|Y|}$, each finite subset of $|Y_j|$ is upper bounded by an element v of $\Gamma(Y)$, and v is necessarily a subset of $|Y_j|$, as two elements of v are always $\sim_{|Y|}$-equivalent. Hence by lemma 25 (with $A = f(\Gamma(Y_j))$, $B = \{|X_1|, \dots, |X_n|\}$, the order being of course the inclusion) there exists a function $l : J \to \{1, \dots, n\}$ such that the restriction f_j of f to $|Y_j|$ is a web morphism $f_j : Y_j \to X_{l(j)}$. By inductive hypothesis, we can lift f_j along $q_{X_{l(j)}}$ by a web morphism $g_j : Y_j \to \widetilde{X_{l(j)}}$. Using the fact that $Y = \bigoplus_{j \in J} Y_j$, we obtain in that way a web morphism $g : Y \to \bigoplus_{i=1}^n \widetilde{X_i} = \widetilde{X}$ which satisfies $q_X \circ g = f$. Now let $h : \widetilde{X} \to \widetilde{X}$ be a web morphism such that $q_X \circ h = q_X$. Let $i \in \{1, \dots, n\}$. By lemma 24, there exists $j \in \{1, \dots, n\}$ such that $h(|\widetilde{X_i}|) \subseteq |\widetilde{X_j}|$. By applying q_X to both members of this inclusion, we get $|X_i| \subseteq |X_j|$ so that $i = j$ by maximality of the X_k's, and we conclude by inductive hypothesis.

■

Let us give yet another way of presenting this construction, establishing a direct link with section 2.

Definition 27 Let X be a finite hypercoherence. A *tower* of X is a sequence $s = \langle u_0, \dots, u_n \rangle$ of subsets of $|X|$ such that

- $u_0 = |X|$,
- $\# u_n = 1$,
- if $0 \leq i < n$, then u_i is not a singleton, and if $u_i \in \Gamma^*(X)$, then u_{i+1} is a maximal subset of u_i which belongs to $\Gamma(X^\perp)$, and if $u_i \in \Gamma^*(X^\perp)$, then u_{i+1} is a maximal subset of u_i which belongs to $\Gamma(X)$.

If a is the element of $|X|$ such that $u_n = \{a\}$, one says that s is a tower at a. One writes $a = q_X^{\mathrm{T}}(s)$ as a is uniquely determined by s.

[11] The proof that Y is the sum of its subspaces Y_j proceeds like the proof of proposition 9; by the way, one might derive this decomposition of Y from proposition 10 applied to $Y^\perp$ if $|Y|$ were assumed to be finite.

Observe that if two towers of X are comparable for the prefix ordering of sequences, they must be equal. Observe also that the first element of any tower of X must be $|X|$, so that two towers have always a non-empty common prefix.

The set $|\mathrm{T}(X)|$ of all towers of X can naturally be considered as the web of a coherence space: say that $s, s' \in |\mathrm{T}(X)|$ are strictly coherent if they are different and the last element u of their longest common prefix belongs to $\Gamma^*(X)$ (observe that as $s \neq s'$, the set u cannot be a singleton). We denote by $\mathrm{T}(X)$ this coherence space, which is serial-parallel.

Proposition 28 *If X is a serial and finite hypercoherence, then there is a bijection $\varphi : |\mathrm{T}(X)| \to |\widetilde{X}|$ which is an isomorphism of hypercoherences from $\mathrm{T}(X)^{\mathrm{c}}$ to $\widetilde{X}$ and which moreover satisfies $q_X \circ \varphi = q_X^{\mathrm{T}}$.*

Hence q_X^{T} is a web morphism from $\mathrm{T}(X)^{\mathrm{c}}$ to X and $(\mathrm{T}(X)^{\mathrm{c}}, q_X^{\mathrm{T}})$ is a rigid parallel unfolding of X, by proposition 26.

Proof: Straightforward induction based on the observation that in the definition of $\widetilde{X}$, the $|X_i|$'s are the maximal subsets of $|X|$ such that $|X_i| \in \Gamma(X^\perp)$ when $|X| \in \Gamma^*(X)$ (case (ii) of the construction), and the maximal subsets of $|X|$ such that $|X_i| \in \Gamma(X)$ when $|X| \in \Gamma^*(X^\perp)$ (case (iii) of the construction). ∎

If X is a serial and finite hypercoherence, we have established an isomorphism between $\mathrm{T}(X)^{\mathrm{c}}$ and $\widehat{X}$, in a rather indirect way. This correspondence can be made more explicit as follows. Given $a \in |X|$ and $s = \langle u_0, \ldots, u_n \rangle \in |\mathrm{T}(X)|$ a tower at a, consider the set $S = \{u_i \mid i \in \{0, \ldots, n\}$ and $u_i \in \Gamma(X)\}$. This is a subset of $\Gamma_a(X)$ which is obviously closed under unions (indeed, it is totally ordered by the inclusion relation). It can be proved that there is exactly one element $\alpha(s)$ of $\mathcal{F}_a(X)$ such that $S \subseteq \alpha(s)$, and that the map associating to s this unique element $\alpha(s)$ of $|\widehat{X}|$ is an isomorphism from $\mathrm{T}(X)^{\mathrm{c}}$ to $\widehat{X}$.

The serial-parallel coherence space associated to the serial and finite hypercoherence $X = (!\mathbf{Bool}^n)^\perp \parr \mathbf{Bool}$ in section 2 was $\mathrm{T}(X)$. So the coherence space of all complete plays of the game associated to the type $\mathbf{Bool}^n \to \mathbf{Bool}$ in the theory of sequential algorithms is canonically isomorphic to the rigid parallel unfolding of the hypercoherence interpreting this type in the hypercoherent semantics.

8 A cardinality issue

The web of the rigid parallel unfolding of a hypercoherence X has a cardinality which generally is strictly greater than the cardinality of $|X|$. Consider for instance the hypercoherence X whose web is the set of integers, and where the only elements of $\Gamma^*(X)$ are the sets of the shape

$$\{-n, \ldots, n, n+1\} \quad \text{and} \quad \{-n-1, -n, \ldots, n\}$$

for all $n \in \mathbf{N}$. It is easily checked that there is a bijective correspondence between $\mathcal{F}_0(X)$ and the set of all subsets of $\mathbf{N}$, so that $\#|\widehat{X}| = 2^{\#|X|}$.

But in denotational semantics, one tends to consider that the spaces used for interpreting formulae or types should have a denumerable number of generators: this corresponds to the standard requirement of ω-algebraicity in Scott semantics for instance. When one deals with coherence spaces, qualitative domains or hypercoherences, the corresponding condition is the countability of the webs.

We present a condition on hypercoherences that allows to control the cardinality of webs through the general parallel unfolding construction of section 6, and which is preserved by all the standard constructions of linear logic.

For a hypercoherence X satisfying this condition, we shall have, for all $a \in |X|$,

$$\#p_X^{-1}(a) < \infty \ .$$

The *degree* of $a \in |X|$ is classically the number of hyperedges of X which contain a (that is, $\#\Gamma_a(X)$). Requiring the degree of a to be finite guarantees of course that $\#p_X^{-1}(a) < \infty$. Unfortunately, this condition is not preserved under the constructions of linear logic. For instance, if $\Gamma^*(X) = \emptyset$, the degree of a in X is 1, whereas its degree in $X^\perp$ is $\#|X|$ (when this cardinal is infinite). So we shall define a notion of *reduced degree* which will be better behaved.

If A is a set and $a \in A$, we denote by $\mathcal{P}^a_{\text{fin}}(A)$ the set of all finite subsets of A which contain a.

Let X be a hypercoherence and let $a \in |X|$. We define on $\mathcal{P}^a_{\text{fin}}(|X|)$ an equivalence relation as follows:

$$u \approx_X u' \quad \text{iff} \quad \forall v \in \mathcal{P}^a_{\text{fin}}(|X|) \quad u \cup v \in \Gamma(X) \Leftrightarrow u' \cup v \in \Gamma(X) \ .$$

Actually, this equivalence relation can be more globally defined on $\mathcal{P}^*_{\text{fin}}(|X|)$, but we consider here only the local version.

Definition 29 Let X be a hypercoherence and let $a \in |X|$. The *reduced degree* of a in X is

$$\mathrm{d}_X(a) = \#\mathcal{P}^a_{\text{fin}}(|X|)/{\approx_X} \ .$$

One says that X is *locally finite* if all the elements of $|X|$ are of finite reduced degree.

Before studying these notions, we state a few trivial lemmas on equivalence relations.

Lemma 30 *Let E and F be sets and let R and S be two equivalence relations on E and F respectively. Let $R \times S$ be the product of these two relations (so that $(a,b)\ R \times S\ (a',b')$ iff $a\ R\ a'$ and $b\ S\ b'$), which is an equivalence relation. Then*

$$\#(E \times F)/(R \times S) = (\#E/R)(\#F/S) \ .$$

This is obvious.

Lemma 31 *Let E and F be two sets. Let R and S be two equivalence relations on E and F respectively. If there is a function $f : E \to F$ such that*

$$f(a)\ S\ f(a') \Rightarrow a\ R\ a' \ ,$$

then

$$\#E/R \leq \#F/S \ .$$

This is obvious.

Lemma 32 *Let E be a set and R be an equivalence relation on E. Let R^* be the equivalence relation on $\mathcal{P}(E)$ defined as follows: $x\ R^*\ y$ iff*

$$\forall a \in x\ \exists b \in y \quad a\ R\ b \quad \text{and} \quad \forall b \in y\ \exists a \in x \quad a\ R\ b \ .$$

Then

$$\#\mathcal{P}(E)/R^* = 2^{\#E/R} \ .$$

Proof: Observe that any element of $\mathcal{P}(E)/R^*$ is the class of a subset x of E satisfying:

$$\forall a, a' \in x \quad a\, R\, a' \Rightarrow a = a'$$

and that two such sets x and x' are equivalent (for R^*) iff there is a bijection $\varphi : x \to x'$ such that $a\, R\, \varphi(a)$ for all $a \in x$, so that there is a bijective correspondence between $\mathcal{P}(E)/R^*$ and $\mathcal{P}(E/R)$. ∎

Lemma 33 *Let X be a hypercoherence and $a \in |X|$. If a is of finite reduced degree in X, it is also of finite reduced degree in $X^\perp$ and more precisely*

$$|\mathrm{d}_{X^\perp}(a) - \mathrm{d}_X(a)| \leq 1$$

Proof: Let P be the set of elements of $\mathcal{P}^a_{\mathrm{fin}}(|X|)$ which are not singletons. It is clear that $P/{\approx_X} = P/{\approx_{X^\perp}}$, but $\mathrm{d}_X(a) \in \{\#P/{\approx_X}, \#P/{\approx_X} + 1\}$, whence the result. ∎

Lemma 34 *Let X and Y be hypercoherences and let $a \in |X|$ be of finite reduced degree in X. Then $(1, a)$ is of finite reduced degree in $X \,\&\, Y$, and more precisely*

$$\mathrm{d}_{X \,\&\, Y}(1, a) \in \{\mathrm{d}_X(a), \mathrm{d}_X(a) + 1\} \ .$$

The proof is straightforward.

Lemma 35 *Let X and Y be two hypercoherences. Let $a \in |X|$ and $b \in |Y|$ be of finite reduced degrees in X and Y respectively. Then (a, b) is of finite reduced degree in $X \otimes Y$, and more precisely*

$$\mathrm{d}_{X \otimes Y}(a, b) \leq \mathrm{d}_X(a)\, \mathrm{d}_Y(b) \ .$$

Proof: Consider the map

$$\begin{array}{rcl} \pi : \mathcal{P}^{(a,b)}_{\mathrm{fin}}(|X \otimes Y|) & \to & \mathcal{P}^a_{\mathrm{fin}}(|X|) \times \mathcal{P}^b_{\mathrm{fin}}(|Y|) \\ w & \mapsto & (\pi_1(w), \pi_2(w)) \end{array}$$

If $\pi(w)$ is equivalent to $\pi(w')$ for the product of the equivalence relations $\approx_X$ and $\approx_Y$, then $w \approx_{X \otimes Y} w'$.

Applying lemmas 30 and 31, we get the required inequation. ∎

Lemma 36 *Let X be a hypercoherence and let $x \in |!X|$ be such that all the elements of x have finite reduced degree in X. Then x has finite reduced degree in $!X$, and more precisely*

$$\mathrm{d}_{!X}(x) \leq 1 + \prod_{a \in x} 2^{\mathrm{d}_X(a)} \ .$$

Proof: Let $\mathrm{S}(X)$ be the subspace of $!X$ defined by

$$|\mathrm{S}(X)| = |!X| \setminus \{\emptyset\} \ .$$

We prove that

$$\mathrm{d}_{\mathrm{S}(X)}(x) \leq \prod_{a \in x} 2^{\mathrm{d}_X(a)}$$

and the result will follow from lemma 34, as clearly $!X \simeq 1 \,\&\, \mathrm{S}(X)$, where 1 stands here for the hypercoherence whose web is a singleton.

So let $x \in |\mathrm{S}(X)|$. For each $a \in x$, let us define a function as follows

$$\begin{array}{rcl} \mathrm{Sec}_a : \mathcal{P}^x_{\mathrm{fin}}(|\mathrm{S}(X)|) & \to & \mathcal{P}(\mathcal{P}^a_{\mathrm{fin}}(|X|)) \\ U & \mapsto & \{u \in \mathcal{P}^a_{\mathrm{fin}}(|X|) \mid u \lhd U\} \end{array}$$

and let R_a be the equivalence relation defined on $\mathcal{P}(\mathcal{P}^a_{\mathrm{fin}}(|X|))$ by

$$\mathcal{U} \; R_a \, \mathcal{U}' \quad \text{iff} \quad \left\{ \begin{array}{l} \forall u \in \mathcal{U} \; \exists u' \in \mathcal{U}' \; u \approx_X u' \\ \forall u' \in \mathcal{U}' \; \exists u \in \mathcal{U} \; u \approx_X u' \end{array} \right. .$$

Let $U, U' \in \mathcal{P}^x_{\mathrm{fin}}(|\mathrm{S}(X)|)$. Assume that $\mathrm{Sec}_a(U) \; R_a \; \mathrm{Sec}_a(U')$ for all $a \in x$. We claim that $U \approx_{\mathrm{S}(X)} U'$.

Indeed, let $V \in \mathcal{P}^x_{\mathrm{fin}}(|\mathrm{S}(X)|)$ and assume that $U \cup V \in \Gamma_x(\mathrm{S}(X))$. Let $w \lhd U' \cup V$, let $u' = w \cap \bigcup U'$ and $v = w \cap \bigcup V$. As $x \neq \emptyset$, we have $w \cap x \neq \emptyset$, so let $a \in w \cap x$. We have $a \in u' \cap v$, $u' \lhd U'$ and $v \lhd V$. Since $\mathrm{Sec}_a(U) \; R_a \; \mathrm{Sec}_a(U')$, there exists u such that $a \in u$, $u \lhd U$ and $u \approx_X u'$. But $u \cup v \lhd U \cup V$ and $U \cup V \in \Gamma_x(\mathrm{S}(X))$ by assumption, so $u \cup v \in \Gamma_a(X)$, and hence $u' \cup v \in \Gamma_a(X)$. Since clearly $w = u' \cup v$, we have proven that $U' \cup V \in \Gamma_x(\mathrm{S}(X))$ as required.

To conclude, consider the map

$$\begin{array}{rcl} \mathcal{P}^x_{\mathrm{fin}}(|\mathrm{S}(X)|) & \to & \prod_{a \in x} \mathcal{P}(\mathcal{P}^a_{\mathrm{fin}}(|X|)) \\ U & \mapsto & (\mathrm{Sec}_a(U))_{a \in x} \end{array}$$

and apply lemmas 30, 31 and 32. ∎

Theorem 37 *If X and Y are locally finite hypercoherences, then so are $X^\perp$, $X \,\&\, Y$, $X \oplus Y$, $X \otimes Y$, $X \parr Y$, $!X$ and $?X$.*

It is an immediate consequence of the previous lemmas.

Theorem 38 *Let X be a hypercoherence and let $a \in |X|$ be of finite reduced degree. Then $p_X^{-1}(a)$ is a finite set. More precisely,*

$$\# p_X^{-1}(a) \leq 2^{\mathrm{d}_X(a)} .$$

So if X is locally finite and if the cardinality of $|X|$ is infinite, then

$$\# |\widehat{X}| = \# |X| .$$

Proof: Observe that the elements α of $p_X^{-1}(a)$ are closed under the equivalence relation $\approx_X$. By this, we mean that they satisfy

$$\forall u, u' \in \mathcal{P}^a_{\mathrm{fin}}(|X|) \quad (u \in \alpha \text{ and } u \approx_X u') \Rightarrow u' \in \alpha .$$

∎

The next technical lemma will be useful in the proof of the last theorem of this section.

Lemma 39 *Let X be a hypercoherence, and let $W \subseteq |\widehat{X}|$ be finite and non-empty. Let $w = p_X(W)$.*

i) For all $u \in \bigcap W$, one has $w \subseteq u$.

ii) Let $u \in \bigcap W$ and let u' be a finite subset of $|X|$ such that $w \subseteq u'$. If $u' \approx_X u$, then $u' \in \bigcap W$.

The proof is straightforward.

Theorem 40 *Let X be a hypercoherence and let $\alpha \in |\widehat{X}|$. If $a = p_X(\alpha)$ is of finite reduced degree in X, then α is of finite reduced degree in $\widehat{X}$. More precisely,*

$$\mathrm{d}_{\widehat{X}}(\alpha) \leq \mathrm{d}_X(a) 2^{\mathrm{d}_X(a)} .$$

So, if X is locally finite, $\widehat{X}$ is locally finite.

Proof: We denote by $\approx^*_X$ the equivalence relation defined on $\mathcal{P}(\mathcal{P}^a_{\mathrm{fin}}(|X|))$ by $\mathcal{U} \approx^*_X \mathcal{V}$ iff

$$\forall u \in \mathcal{U}\ \exists v \in \mathcal{V}\ u \approx_X v \quad \text{and} \quad \forall v \in \mathcal{V}\ \exists u \in \mathcal{U}\ u \approx_X v .$$

By lemma 32, this equivalence relation has $2^{\mathrm{d}_X(a)}$ classes. Let $U, U' \in \mathcal{P}^\alpha_{\mathrm{fin}}(|\widehat{X}|)$, and set $u = p_X(U)$, $u' = p_X(U')$. Assume that

$$u \approx_X u' \quad \text{and} \quad \bigcap U \approx^*_X \bigcap U' .$$

We claim that $U \approx_{\widehat{X}} U'$, and the theorem will follow, by lemmas 30 and 31.

We prove now this claim. Let $V \in \mathcal{P}^\alpha_{\mathrm{fin}}(|\widehat{X}|)$ and let $v = p_X(V)$. Assume that $U \cup V \in \Gamma(\widehat{X})$. This means that $u \cup v \in \Gamma(X)$ and that $\bigcap U \cap \bigcap V \in \mathcal{F}_{u \cup v}(X)$. As $u \cup v \in \Gamma(X)$ and as $u \approx_X u'$, we have $u' \cup v \in \Gamma(X)$. It remains to prove that $\bigcap U' \cap \bigcap V \in \mathcal{F}_{u' \cup v}(X)$.

So let w' be a finite subset of $|X|$ such that $u' \cup v \subseteq w'$, and assume that

$$\forall t' \in \bigcap U' \cap \bigcap V \quad w' \cup t' \in \Gamma(X) . \tag{4}$$

We have to prove that $w' \in \bigcap U' \cap \bigcap V$ (the set $\bigcap U' \cap \bigcap V$ is obviously closed under finite unions, as an intersection of sets having that property).

As $u \approx_X u'$, one has $u \cup w' \approx_X u' \cup w'$ (indeed, $\approx_X$ is a congruence with respect to $\cup$). That is $w' \approx_X w' \cup u$. Let $t \in \bigcap U \cap \bigcap V$. As $t \in \bigcap U$ and as $\bigcap U \approx^*_X \bigcap U'$, there exists $t' \in \bigcap U'$ such that $t' \approx_X t$. We have then $t \cup v \approx_X t' \cup v$, that is (since $t \in \bigcap V$ and hence $v \subseteq t$ by lemma 39 (i)) $t \approx_X t' \cup v$. But we have $t \in \bigcap V$, and so, by lemma 39 (ii), we have $t' \cup v \in \bigcap V$ and also, since $t' \cup v \approx_X t \approx_X t' \in \bigcap U'$, by lemma 39 (ii) again, we have $t' \cup v \in \bigcap U'$ and so $t' \cup v \in \bigcap U' \cap \bigcap V$. But w' satisfies the property (4) above, hence we have $w' \cup t' \cup v \in \Gamma(X)$, that is $w' \cup t' \in \Gamma(X)$. But $t \approx_X t'$, hence $w' \cup t \in \Gamma(X)$, that is $(w' \cup u) \cup t \in \Gamma(X)$ (as $u \subseteq t$). This holds for all $t \in \bigcap U \cap \bigcap V$, and we have $u \cup v \subseteq w' \cup u$, hence

$$w' \cup u \in \bigcap U \cap \bigcap V$$

since $\bigcap U \cap \bigcap V \in \mathcal{F}_{u \cup v}(X)$. Remember now that $w' \cup u \approx_X w'$. So, since $v \subseteq w'$ we have $w' \in \bigcap V$ by lemma 39 (ii).

On the other hand, since $\bigcap U \approx^*_X \bigcap U'$, and since we have proved above that $w' \cup u \in \bigcap U$, there exists $s' \in \bigcap U'$ such that $w' \cup u \approx_X s'$. Then we have $u' \subseteq w'$ and $w' \approx_X w' \cup u \approx_X s' \in \bigcap U'$, and hence, by lemma 39 (ii) again, we get $w' \in \bigcap U'$, that is $w' \in \bigcap U' \cap \bigcap V$ and this concludes the proof of the claim, and of the theorem. ∎

9 Some remarkable isomorphisms

This section presents some isomorphisms satisfied by the rigid parallel unfolding of hypercoherences. As this operation gives rise to parallel hypercoherences and as the operations "&", "⊕", "⊗" and "!" preserve parallelism of hypercoherences, it is not very surprising that the rigid parallel unfolding commutes with these operations. This is the object of the four next statements.

Proposition 41 *Let X and Y be two hypercoherences. Then $(\widehat{X} \mathbin{\&} \widehat{Y}, p_X \mathbin{\&} p_Y)$ is a rigid parallel unfolding of $X \mathbin{\&} Y$.*

Proof: Let Z be a parallel hypercoherence and let $f : Z \to X \mathbin{\&} Y$ be a web morphism. Consider the subspaces Z_X and Z_Y of Z defined by

$$|Z_X| = f^{-1}(|X|) \quad \text{and} \quad |Z_Y| = f^{-1}(|Y|) \ .$$

Then $|Z|$ is the disjoint union of $|Z_X|$ and $|Z_Y|$. Let f_X and f_Y be the restrictions of f to $|Z_X|$ and $|Z_Y|$. We can lift f_X and f_Y along p_X and p_Y respectively, getting $g_X : Z_X \to \widehat{X}$ and $g_Y : Z_Y \to \widehat{Y}$. On the other hand, the canonical bijection $|Z| \to |Z_X \mathbin{\&} Z_Y|$ is obviously a web morphism $j : Z \to Z_X \mathbin{\&} Z_Y$. Now $(g_X \mathbin{\&} g_Y) \circ j$ is a lifting of f along $p_X \mathbin{\&} p_Y$.

We conclude by the observation that any web morphism $h : \widehat{X} \mathbin{\&} \widehat{Y} \to \widehat{X} \mathbin{\&} \widehat{Y}$ such that $(p_X \mathbin{\&} p_Y) \circ h = p_X \mathbin{\&} p_Y$ is of the shape $h = h_X \mathbin{\&} h_Y$ where $h_X : \widehat{X} \to \widehat{X}$ satisfies $p_X \circ h_X = p_X$, and similarly for h_Y. ∎

We can easily describe this isomorphism explicitly. The map $|\widehat{X} \mathbin{\&} \widehat{Y}| \to |\widehat{X \mathbin{\&} Y}|$ associates to $(1, \alpha)$ (where $\alpha \in \mathcal{F}_a(X)$) the element

$$\{\{1\} \times u \mid u \in \alpha\} \cup \{w \in \mathcal{P}^*_{\mathrm{fin}}(|X \mathbin{\&} Y|) \mid (1, a) \in w \text{ and } w \cap (\{2\} \times |Y|) \neq \emptyset\}$$

of $\mathcal{F}_{(1,a)}(X \mathbin{\&} Y)$. Its inverse associates to $\gamma \in \mathcal{F}_{(1,a)}(X \mathbin{\&} Y)$ the element

$$(1, \{\pi_2(w) \mid w \in \gamma \text{ and } \pi_1(w) = \{1\}\}) \ .$$

Proposition 42 *Let X and Y be two hypercoherences. Then $(\widehat{X} \oplus \widehat{Y}, p_X \oplus p_Y)$ is a rigid parallel unfolding of $X \oplus Y$.*

The proof is straightforward.

Proposition 43 *Let X and Y be two hypercoherences. Then $(\widehat{X} \otimes \widehat{Y}, p_X \otimes p_Y)$ is a rigid parallel unfolding of $X \otimes Y$.*

In other words, there is a unique isomorphism $\varphi : \widehat{X \otimes Y} \to \widehat{X} \otimes \widehat{Y}$ such that

$$(p_X \otimes p_Y) \circ \varphi = p_{X \otimes Y} \ .$$

Proof: We construct directly the map φ, by setting

$$\varphi(\gamma) = (\pi_1\gamma, \pi_2\gamma) = (\{\pi_1(w) \mid w \in \gamma\}, \{\pi_2(w) \mid w \in \gamma\})$$

for all $\gamma \in |\widehat{X \otimes Y}|$.

Let $(a, b) = p_{X \otimes Y}(\gamma)$. It is clear that $\pi_1\gamma \subseteq \Gamma_a(X)$ and that $\pi_1\gamma$ is closed under binary unions, and similarly for $\pi_2\gamma$.

Let us check that $\pi_1\gamma$ is maximal. So let $u \in \Gamma_a(X)$ be such that $u \cup \pi_1(w) \in \Gamma(X)$ for all $w \in \gamma$. We have $(u \times \{b\}) \cup w \in \Gamma(X \otimes Y)$ for all $w \in \gamma$, and hence, by maximality of γ, we have $u \times \{b\} \in \gamma$, hence $u \in \pi_1\gamma$. And similarly for $\pi_2\gamma$, hence φ is a well defined function from $|\widehat{X \otimes Y}|$ to $|\widehat{X} \otimes \widehat{Y}|$.

We check now that φ is a web morphism. Let $W \in \Gamma^*(\widehat{X \otimes Y})$ and let $w = p_{X\otimes Y}(W)$, which belongs to $\Gamma^*(X \otimes Y)$. As w is not a singleton, $\varphi(W)$ cannot be a singleton, so we just have to check that $\varphi(W) \in \Gamma(\widehat{X} \otimes \widehat{Y})$. Let us check that $\pi_1(\varphi(W)) \in \Gamma(\widehat{X})$. Let $\gamma, \gamma' \in W$ and let $u \in \pi_1\gamma$ be such that $\pi_1(w) \subseteq u$. We must show that $u \in \pi_1\gamma'$. Let $w' \in \gamma$ be such that $\pi_1(w') = u$. As $W \in \Gamma(\widehat{X \otimes Y})$, we have $w \in \gamma$ and so, as γ is closed under binary unions, we have $w' \cup w \in \gamma$. But $\pi_1(w' \cup w) = u$, so we can assume that $w \subseteq w'$ (otherwise use $w \cup w'$ instead of w'). Consequently, $w' \in \gamma'$ and hence $u \in \pi_1\gamma'$ as required, so φ is a web morphism, and we have

$$(p_X \otimes p_Y) \circ \varphi = p_{X\otimes Y}$$

by definition of φ. Consequently, for any parallel hypercoherence Z and any web morphism $f : Z \to X \otimes Y$, there exists a web morphism $f' : Z \to \widehat{X} \otimes \widehat{Y}$ such that $(p_X \otimes p_Y) \circ f' = f$: take a morphism $g : Z \to \widehat{X \otimes Y}$ such that $p_{X\otimes Y} \circ g = f$ and set $f' = \varphi \circ g$.

As to rigidity, consider a web morphism $h : \widehat{X} \otimes \widehat{Y} \to \widehat{X} \otimes \widehat{Y}$ such that

$$p_X \otimes p_Y = (p_X \otimes p_Y) \circ h \ ,$$

and let us show that $h = \mathrm{Id}$. Assume it is not the case, and let $(\alpha, \beta) \in |\widehat{X} \otimes \widehat{Y}|$ be such that $(\alpha', \beta') = h(\alpha, \beta) \neq (\alpha, \beta)$. Without loss of generality, assume that $\alpha' \neq \alpha$. So let $u \in \alpha$ and $u' \in \alpha'$ be such that $u \cup u' \notin \Gamma(X)$. Let $a = p_X(\alpha) = p_X(\alpha')$. By theorem 21 (iii), there exists $U \in \Gamma_\alpha(\widehat{X})$ such that $p_X(U) = u$. We have $U \times \{\beta\} \in \Gamma_{(\alpha,\beta)}(\widehat{X} \otimes \widehat{Y})$ and so $h(U \times \{\beta\}) \in \Gamma_{(\alpha',\beta')}(\widehat{X} \otimes \widehat{Y})$. Similarly, there exists $U' \in \Gamma_{\alpha'}(\widehat{X})$ such that $p_X(U') = u'$. We have $U' \times \{\beta'\} \in \Gamma_{(\alpha',\beta')}(\widehat{X} \otimes \widehat{Y})$, and so

$$h(U \times \{\beta\}) \cup (U' \times \{\beta'\}) \in \Gamma_{(\alpha',\beta')}(\widehat{X} \otimes \widehat{Y})$$

as $\widehat{X} \otimes \widehat{Y}$ is a parallel hypercoherence. But then we must have

$$(p_X \otimes p_Y)(h(U \times \{\beta\}) \cup (U' \times \{\beta'\})) \in \Gamma(X \otimes Y) \ ,$$

that is $(u \times \{b\}) \cup (u' \times \{b\}) \in \Gamma(X \otimes Y)$, which is not the case since $u \cup u' \notin \Gamma(X)$. ∎

Proposition 44 *Let X be a hypercoherence. $(!\widehat{X}, !p_X)$ is a rigid parallel unfolding of $!X$.*

In other words, there is a unique isomorphism $\varphi : \widehat{!X} \to !\widehat{X}$ such that $!p_X \circ \varphi = p_{!X}$.

Proof: Let $x \in |!X|$ and let $\Theta \in \mathcal{F}_x(!X)$. For all $a \in x$, we define

$$\varphi_a(\Theta) = \{u \in \mathcal{P}^*_{\mathrm{fin}}(|X|) \mid a \in u \text{ and } \exists U \in \Theta \ u \lhd U\} \ .$$

Let us prove that $\alpha = \varphi_a(\Theta)$ belongs to $\mathcal{F}_a(X)$.

First, α is closed under binary unions. Indeed, if $u, u' \in \alpha$, let $U, U' \in \Theta$ be such that $u \lhd U$ and $u' \lhd U'$. As clearly $u \cup u' \lhd U \cup U'$ and as $U \cup U' \in \Theta$, we have $u \cup u' \in \alpha$.

As to the maximality of α, let $v \in \Gamma_a(X)$ be such that $v \cup u \in \Gamma(X)$ for all $u \in \alpha$. Let

$$V = \{x\} \cup \{\{c\} \mid c \in v\} \ .$$

Let $U \in \Theta$. We want to prove that $U \cup V \in \Gamma(!X)$. So let $w \lhd U \cup V$. Let $u = w \cap \bigcup U$. We have $u \lhd U$ and $a \in u$ (since $\{a\} \in V \subseteq U \cup V$ and hence $a \in w$, and $a \in x \in U$, so $a \in \bigcup U$), hence $u \in \alpha$. Furthermore, $w = u \cup v$. Indeed, if $b \in v$, we have $\{b\} \in U \cup V$, so $b \in w$, hence $v \subseteq w$, which implies $u \cup v \subseteq w$. Conversely, let $b \in w$. If $b \in \bigcup U$, then $b \in u$ and we are done. Otherwise, let $y \in U \cup V$ be such that $b \in y$, we know that $y \notin U$, so $y \in V$ and $y \neq x$, so $y = \{c\}$ for some $c \in v$ and we are done. So $w \in \Gamma(X)$ and hence $U \cup V \in \Gamma(!X)$.

As this holds for all $U \in \Theta$ we must have $V \in \Theta$, but $v \lhd V$ and $a \in v$, hence $v \in \alpha$. Hence

$$\varphi_a(\Theta) \in \mathcal{F}_a(X) \ .$$

Set

$$\varphi(\Theta) = \{\varphi_a(\Theta) \mid a \in x\} \ .$$

Let $u \subseteq x$ be non empty. We prove that $U = \{\varphi_a(\Theta) \mid a \in u\}$ belongs to $\Gamma(\widehat{X})$. First we have $p_X(U) = u \in \Gamma(X)$ as $x \in \mathrm{qD}(X)$. Next, let $a, a' \in u$ and let $v \in \varphi_a(\Theta)$ be such that $u \subseteq v$. We have $a' \in v$, hence also $v \in \varphi_{a'}(\Theta)$. So

$$\varphi(\Theta) \in \mathrm{qD}(\widehat{X}) \ .$$

Hence φ is a well defined map from $|\widehat{!X}|$ to $|!\widehat{X}|$ and it is clear that

$$!p_X \circ \varphi = p_{!X} \ .$$

We check now that φ is a web morphism. Let $\mathcal{U} \in \Gamma^*(\widehat{!X})$. We just have to prove that $\varphi(\mathcal{U}) \in \Gamma(!\widehat{X})$. So let $C \lhd \varphi(\mathcal{U})$ and let $u = p_X(C)$.

Let us first check that $u \lhd p_{!X}(\mathcal{U})$ which belongs to $\Gamma(!X)$, as $p_{!X}$ is a web morphism. From this, we shall deduce that $u \in \Gamma(X)$. So let $a \in u$. Let $\alpha \in C$ be such that $a = p_X(\alpha)$. Let $\Theta \in \mathcal{U}$ be such that $\alpha \in \varphi(\Theta)$, that is $\alpha = \varphi_b(\Theta)$ for some $b \in p_{!X}(\Theta)$. We have

$$b = p_X(\varphi_b(\Theta)) = p_X(\alpha) = a \ ,$$

hence $a \in p_{!X}(\Theta)$. Conversely, let $\Theta \in \mathcal{U}$ and let $x = p_{!X}(\Theta)$. Let $\alpha \in C$ be such that $\alpha \in \varphi(\Theta)$, that is $\alpha = \varphi_a(\Theta)$ for some $a \in x$. So we have

$$a = p_X(\alpha) \in p_X(C) = u \ .$$

We want now to prove that $C \in \Gamma(\widehat{X})$. We already know that $u = p_X(C) \in \Gamma(X)$. So let $\alpha, \alpha' \in C$ and let $v \in \alpha$ be such that $u \subseteq v$. We have to prove that $v \in \alpha'$. As $C \lhd \varphi(\mathcal{U})$, there exist $\Theta, \Theta' \in \mathcal{U}$ such that $\alpha \in \varphi(\Theta)$ and $\alpha' \in \varphi(\Theta')$, that is $\alpha = \varphi_a(\Theta)$ and $\alpha' = \varphi_{a'}(\Theta')$ where $a = p_X(\alpha)$ and $a' = p_X(\alpha')$ (and hence $a, a' \in u$). Since $v \in \alpha = \varphi_a(\Theta)$, there exists $V \in \Theta$ such that $v \lhd V$ (see the definition of $\varphi_a(\Theta)$ at the beginning of the proof). As $u \subseteq v$ and $u \lhd p_{!X}(\mathcal{U})$, we also have

$$v \lhd V \cup p_{!X}(\mathcal{U}) \ .$$

As $\mathcal{U} \in \Gamma(\widehat{!X})$ and $V \cup p_{!X}(\mathcal{U}) \in \Theta$, we have also

$$V \cup p_{!X}(\mathcal{U}) \in \Theta' \ ,$$

and since $a' \in u \subseteq v$, we conclude that $v \in \alpha'$ and we are done.

So φ is a web morphism. From this, it results that $!p_X$ has the lifting property.

We want now to prove rigidity.

Observe first that, for all $A \in |!\widehat{X}|$ and $a \in !p_X(A)$, there is exactly one $\alpha \in A$ such that $p_X(\alpha) = a$, since p_X is a web morphism.

Let $h : !\widehat{X} \to !\widehat{X}$ be a web morphism such that

$$!p_X \circ h = !p_X \ .$$

We must prove that $h = \mathrm{Id}$. Assume it is not the case, so let $A \in |!\widehat{X}|$ be such that $h(A) \neq A$ and set $x = !p_X(A)$. As $!p_X(A) = !p_X(h(A))$, we can find $\alpha \in A$ and $\beta \in h(A)$ such that $p_X(\alpha) = p_X(\beta)$ but $\alpha \neq \beta$. Let $u \in \alpha$ and $v \in \beta$ be such that $u \cup v \notin \Gamma(X)$. By theorem 21 (iii), there exists $C \in \Gamma(\widehat{X})$ such that $\alpha \in C$ and $p_X(C) = u$. Let

$$\mathcal{A} = \{A\} \cup \{\{\gamma\} \mid \gamma \in C\} \ .$$

Each section D of $\mathcal{A}$ satisfies $D = C \cup (D \cap A)$, but $\alpha \in C \cap (D \cap A)$ (since $\alpha \in C \subseteq D$ and $\alpha \in A$) and $C, D \cap A \in \Gamma(\widehat{X})$, so $D \in \Gamma(\widehat{X})$ since $\widehat{X}$ is a parallel hypercoherence. So

$$\mathcal{A} \in \Gamma(!\widehat{X}) \ ,$$

and we have

$$!p_X(\mathcal{A}) = \{x\} \cup \{\{c\} \mid c \in u\} \ .$$

In a similar way, we can find $\mathcal{B} \in \Gamma(!\widehat{X})$ such that $h(A) \in \mathcal{B}$ and $!p_X(\mathcal{B}) = \{x\} \cup \{\{c\} \mid c \in v\}$. As $!\widehat{X}$ is a parallel hypercoherence and as h is a web morphism, we have $h(\mathcal{A}) \cup \mathcal{B} \in \Gamma(!\widehat{X})$ (since $h(A) \in h(\mathcal{A}) \cap \mathcal{B}$), and hence $!p_X(h(\mathcal{A}) \cup \mathcal{B}) \in \Gamma(!X)$. But

$$!p_X(h(\mathcal{A}) \cup \mathcal{B}) = \{x\} \cup \{\{c\} \mid c \in u \cup v\} \ ,$$

hence

$$u \cup v \lhd !p_X(h(\mathcal{A}) \cup \mathcal{B})$$

whence a contradiction, since $u \cup v \notin \Gamma(X)$. ■

The "$\parr$" connective transforms parallel hypercoherences in non parallel ones, so we cannot hope that the rigid parallel unfolding commute with it. We can however prove a result which states that, when unfolding $X \parr Y$, one can indifferently unfold X and Y before. In our proof, we need the assumption that both X and Y are serial. We do not know if the result can be extended to more general situations.

Theorem 45 *Let X and Y be serial hypercoherences. Then there is exactly one morphism φ : $\widehat{X \parr Y} \to \widehat{\widehat{X} \parr \widehat{Y}}$ such that*

$$(p_X \parr p_Y) \circ p_{\widehat{X} \parr \widehat{Y}} \circ \varphi = p_{X \parr Y} \ ,$$

and φ is an isomorphism.

Proof: It is sufficient to prove that $(\widehat{\widehat{X} \parr \widehat{Y}}, (p_X \parr p_Y) \circ p_{\widehat{X} \parr \widehat{Y}})$ is a rigid parallel unfolding of $X \parr Y$.

Let us first prove the lifting property. So let Z be a parallel hypercoherence and let $f : Z \to X \parr Y$ be a web morphism. Let $f_X : |Z| \to |X|$ and $f_Y : |Z| \to |Y|$ be obtained by composing f with the two projections (these functions have no reason to be web morphisms).

We define a hypercoherence Z' by setting $|Z'| = |Z|$ and

$$\Gamma(Z') = \{w \in \Gamma(Z) \mid f_Y(w) \in \Gamma(Y^{\perp})\} \ .$$

This hypercoherence is parallel because Z and $Y^{\perp}$ are. Furthermore, f_X is a web morphism from Z' to X. So let $g_X : Z' \to \widehat{X}$ be a lifting of f_X along p_X. Let

$$f' : |Z| \to |\widehat{X}| \times |Y|$$

be defined by

$$f'(c) = (g_X(c), f_Y(c)) \ .$$

Then f' is a web morphism from Z to $\widehat{X} \parr Y$ such that $(p_X \parr Y) \circ f' = f$. As $\widehat{X}$ is still serial by theorem 23, we can perform the same operation on the other side, and we get a web morphism

$$f'' : Z \to \widehat{X} \parr \widehat{Y}$$

such that

$$(p_X \parr p_Y) \circ f'' = f \ ,$$

and we conclude by lifting f'' along $p_{\widehat{X} \parr \widehat{Y}}$.

Now let

$$h : \widehat{\widehat{X} \parr \widehat{Y}} \to \widehat{\widehat{X} \parr \widehat{Y}}$$

be a web morphism such that

$$(p_X \parr p_Y) \circ p_{\widehat{X} \parr \widehat{Y}} \circ h = (p_X \parr p_Y) \circ p_{\widehat{X} \parr \widehat{Y}} \ .$$

Assume that $p_{\widehat{X} \parr \widehat{Y}} \circ h \neq p_{\widehat{X} \parr \widehat{Y}}$, otherwise we immediately conclude that $h = \mathrm{Id}$, since $(\widehat{\widehat{X} \parr \widehat{Y}}, p_{\widehat{X} \parr \widehat{Y}})$ is rigid.

Let $\gamma \in |\widehat{\widehat{X} \parr \widehat{Y}}|$ be such that

$$p_{\widehat{X} \parr \widehat{Y}}(h(\gamma)) \neq p_{\widehat{X} \parr \widehat{Y}}(\gamma) \ .$$

Set

$$(\alpha', \beta') = p_{\widehat{X} \parr \widehat{Y}}(h(\gamma)) \quad \text{and} \quad (\alpha, \beta) = p_{\widehat{X} \parr \widehat{Y}}(\gamma) \ .$$

Let $a = p_X(\alpha) = p_X(\alpha')$ and $b = p_Y(\beta) = p_Y(\beta')$. Assume for instance that $\alpha \neq \alpha'$ and $\beta \neq \beta'$ (the other cases are similar).

Let $u \in \alpha, u' \in \alpha'$ and $v \in \beta, v' \in \beta'$ be such that $u \cup u' \notin \Gamma(X)$ and $v \cup v' \notin \Gamma(Y)$.

By theorem 21 (iii), there exists $U \in \Gamma_\alpha(\widehat{X})$, $U' \in \Gamma_{\alpha'}(\widehat{X})$, $V \in \Gamma_\beta(\widehat{Y})$ and $V' \in \Gamma_{\beta'}(\widehat{Y})$ such that $p_X(U) = u$, $p_X(U') = u'$, $p_Y(V) = v$ and $p_Y(V') = v'$.

We have $U \times V \in \gamma$. Indeed, observe first that $U \times V \in \Gamma_{(\alpha,\beta)}(\widehat{X} \parr \widehat{Y})$. Let W be an element of γ, and let us check that $(U \times V) \cup W \in \Gamma(\widehat{X} \parr \widehat{Y})$. We can assume that W is not a singleton and hence

$$\pi_1(W) \in \Gamma^*(\widehat{X}) \quad \text{or} \quad \pi_2(W) \in \Gamma^*(\widehat{Y}) \ .$$

If we are in the first case, then $\pi_1(W \cup (U \times V)) = \pi_1(W) \cup U \in \Gamma^*(\widehat{X})$, since $\widehat{X}$ is parallel and we are done, and similarly in the other case.

Since $U \times V \in \gamma$, there exists $\mathcal{U} \in \Gamma_\gamma(\widehat{\widehat{X} \parr \widehat{Y}})$ such that $p_{\widehat{X} \parr \widehat{Y}}(\mathcal{U}) = U \times V$ by theorem 21.

Consequently,

$$W' = p_{\widehat{X} \parr \widehat{Y}}(h(\mathcal{U})) \in \Gamma_{(\alpha',\beta')}(\widehat{X} \parr \widehat{Y})$$

and, because both U' and V' are coherent, we get

$$W' \cup (U' \times V') \in \Gamma(\widehat{X} \parr \widehat{Y}) \ ,$$

hence

$$(p_X \parr p_Y)(W' \cup (U' \times V')) \in \Gamma(X \parr Y) \ .$$

Since $(p_X \parr p_Y) \circ p_{\widehat{X} \parr \widehat{Y}} \circ h = (p_X \parr p_Y) \circ p_{\widehat{X} \parr \widehat{Y}}$, we have

$$(p_X \parr p_Y)(W') = (p_X \parr p_Y)(U \times V) = u \times v \ ,$$

so that we have $(u \times v) \cup (u' \times v') \in \Gamma(X \parr Y)$ which is impossible, since the first projection of that set is $u \cup u'$ and the second is $v \cup v'$, both strictly incoherent sets. ∎

10 Interpretation of formulae

We define an interpretation of formulae of propositional linear logic as serial and parallel hypercoherences (or, equivalently, serial-parallel coherence spaces). For this purpose, we define the connectives $\hat{\parr}$ and $\hat{?}$ which, applied to serial and parallel hypercoherences will give rise to serial and parallel hypercoherences. The constants and the additive connectives will be left unchanged, as well as linear negation. The other connectives will be defined using the De Morgan laws for linear logic.

A very natural question arises here: since these connectives act on coherence spaces, why this roundabout through hypercoherences for defining them? Of course, a direct definition is possible (it is just a matter of translation), but does not enlighten at all the situation. The point is that, even when defining for instance the web of $E \hat{\parr} F$ (for E and F serial-parallel coherence spaces), we are really using the whole structure of the hypercoherence $E^{\mathrm{c}} \parr F^{\mathrm{c}}$, which seems non-trivial in general; in particular, we do not see any way of extracting the structure of this hypercoherence from the mere coherence space $E \parr F$ (here, the $\parr$ is performed in the category of coherence spaces, according to the definitions given in [Gir95]), for instance. This means that the coherence space structure of E and F is not really relevant, although it completely defines the objects E and F.

We are *not* giving a denotational semantics of linear logic in serial and parallel hypercoherences, as we are not (yet) able to interpret proofs as cliques of the spaces we define in what follows. We shall just show, using some of the results proven until now, that these constructions satisfy some of the main isomorphisms of linear logic.

Definition 46 Let E and F be serial and parallel hypercoherences. One sets $E \hat{\parr} F = \widehat{E \parr F}$ and $\hat{?}E = \widehat{?E}$.

By propositions 15 and by theorem 23, the hypercoherences defined in this way are serial and parallel.

Let us give some more concrete hints on the structure of $E \hat{\parr} F$, just for the purpose of convincing ourselves that it has to do with games. Let E and F be two serial and parallel hypercoherences, that we assume to be finite for simplicity.

i) Assume first that $|E| \in \Gamma^*(E^{\perp})$ and that $|F| \in \Gamma^*(F^{\perp})$ (and then $|E \hat{\parr} F| \in \Gamma^*((E \parr F)^{\perp})$). Then, according to what has been said in section 5 about the connection between serial-parallel finite coherent spaces and games, Player plays first in the game associated to E and in the game associated to F. We have, up to isomorphism,

$$E = E_1 \oplus \cdots \oplus E_n \quad \text{and} \quad F = F_1 \oplus \cdots \oplus F_m$$

where $|E_1|, \ldots, |E_n|$ are the maximal elements of $\Gamma(E)$ and $|F_1|, \ldots, |F_m|$ are the maximal elements of $\Gamma(F)$ (by proposition 10 applied to $E^{\perp}$). For the sake of simplicity again, assume that all these sets are *strictly* coherent (that is, are not singletons). These subspaces should be considered as representing the various first possible moves for Player in the games associated to E and F respectively (again, see section 5). It is clear that, for $i = 1, \ldots, n$, $|E_i| \times |F| \in \Gamma^*(E \parr F)$ and that, for $j = 1, \ldots, m$, $|E| \times |F_j| \in \Gamma^*(E \parr F)$. Moreover, the sets $|E_i| \times |F|$ and $|E| \times |F_j|$ are the maximal subsets of $|E \parr F|$ which belong to $\Gamma(E \parr F)$, due to the maximality properties of the sets $|E_i|$ and $|F_j|$. Then the construction presented in section 7 shows that, up to a canonical isomorphism,

$$E \hat{\parr} F = \bigoplus_{i=1}^{n} (E_i \hat{\parr} F) \oplus \bigoplus_{j=1}^{m} (E \hat{\parr} F_j)$$

which means that, in $E \hat{\parr} F$, Player plays first, choosing one component of the $\parr$ and playing in that component according to the corresponding game.

ii) The other cases, when one at least of the spaces is strictly coherent as a whole, are simpler. Assume for instance that $|E| \in \Gamma^*(E)$ and that $|F| \in \Gamma^*(F^\perp)$ (and then $|E \parr F| \in \Gamma^*(E \parr F)$). Then by proposition 10, and up to a canonical isomorphism, $E = E_1 \,\&\, \cdots \,\&\, E_n$ where the spaces E_i are the maximal subspaces of E whose web belongs to $\Gamma(E^\perp)$. Then the sets $|E_i| \times |F|$ are the maximal subsets of $|E \parr F|$ which belong to $\Gamma((E \parr F)^\perp)$ and one has, according to the construction presented in section 7,

$$E \hat{\parr} F = (E_1 \hat{\parr} F) \,\&\, \cdots \,\&\, (E_n \hat{\parr} F) \ ,$$

up to a canonical isomorphism (we shall see by the way that $\hat{\parr}$ is distributive over $\&$, as suggested by this isomorphism). This corresponds to the fact that, in the game-theoretic $\parr$, Opponent cannot switch between the two components of the $\parr$.

- When both spaces are strictly coherent as a whole, Opponent must play simultaneously in both components.

Observations (i) and (ii) above express the well-known switching condition of the $\parr$ connective in its game-theoretic interpretations.

Proposition 47 *The operation $\hat{\parr}$ is associative. More precisely, there is exactly one isomorphism φ making the following diagram commutative:*

$$\begin{array}{ccc}
(E \hat{\parr} F) \hat{\parr} G & \xrightarrow{\ \varphi\ } & E \hat{\parr} (F \hat{\parr} G) \\
\Big\downarrow {\scriptstyle p_{(E \hat{\parr} F) \parr G}} & & \Big\downarrow {\scriptstyle p_{E \parr (F \hat{\parr} G)}} \\
(E \hat{\parr} F) \parr G & & E \parr (F \hat{\parr} G) \\
\Big\downarrow {\scriptstyle p_{E \parr F} \parr G} & & \Big\downarrow {\scriptstyle E \parr p_{F \parr G}} \\
(E \parr F) \parr G & \xrightarrow{\ \varphi_0\ } & E \parr (F \parr G)
\end{array}$$

where φ_0 is the usual isomorphism.

Proof: By theorem 45 and lemma 7, we know that composing the maps

$$(E \hat{\parr} F) \hat{\parr} G \xrightarrow{p_{(E \hat{\parr} F) \parr G}} (E \hat{\parr} F) \parr G \xrightarrow{p_{E \parr F} \parr G} (E \parr F) \parr G \xrightarrow{\varphi_0} E \parr (F \parr G)$$

we get a rigid parallel unfolding of $E \parr (F \parr G)$ and we conclude by proposition 5. When applying theorem 45, one uses the fact that $G = \widehat{G}$, up to a canonical isomorphism, since G is parallel. ∎

Proposition 48 *Let E, F and G be serial and parallel hypercoherences. There is a unique isomorphism φ making the following diagram commutative.*

$$\begin{array}{ccc}
E \hat{\parr} (F \,\&\, G) & \xrightarrow{\varphi} & (E \hat{\parr} F) \,\&\, (E \hat{\parr} G) \\
\Big\downarrow{\scriptstyle p_{E \parr (F \& G)}} & & \Big\downarrow{\scriptstyle p_{E \parr F} \,\&\, p_{E \parr G}} \\
E \parr (F \,\&\, G) & \xrightarrow{\varphi_0} & (E \parr F) \,\&\, (E \parr G)
\end{array}$$

where φ_0 is the usual isomorphism.

It is a consequence of proposition 41, proposition 5 and lemma 7.

Proposition 49 *Let E and F be serial and parallel hypercoherences. There is a unique isomorphism φ making the following diagram commutative.*

$$\begin{array}{ccc}
\hat{?}(E \oplus F) & \xrightarrow{\varphi} & (\hat{?}E) \hat{\parr} (\hat{?}F) \\
 & & \Big\downarrow{\scriptstyle p_{(\hat{?}E) \parr (\hat{?}F)}} \\
\Big\downarrow{\scriptstyle p_{?(E \oplus F)}} & & (\hat{?}E) \parr (\hat{?}F) \\
 & & \Big\downarrow{\scriptstyle p_{?E} \parr p_{?F}} \\
?(E \oplus F) & \xrightarrow{\varphi_0} & (?E) \parr (?F)
\end{array}$$

where φ_0 is the usual isomorphism.

Proof: By theorem 45, composing the maps

$$(\hat{?}E) \hat{\parr} (\hat{?}F) \xrightarrow{p_{(\hat{?}E) \parr (\hat{?}F)}} (\hat{?}E) \parr (\hat{?}F) \xrightarrow{p_{?E} \parr p_{?F}} (?E) \parr (?F)$$

we get a rigid parallel unfolding of $(?E) \parr (?F)$, and we conclude by proposition 5 and lemma 7. ∎

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