MPRI 2-02: Call-By-Push-Value and Linear Logic

2015-2016

Syntax of CBPV

Types :

$$\varphi, \psi, \ldots \; := \; !\sigma \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \mathsf{Fix}\, \zeta \cdot \varphi$$
$$\sigma, \tau \ldots \; := \; \varphi \mid \varphi \multimap \sigma \mid \top$$

Terms :

$$\begin{aligned} M, N \ldots \; := \; & x \mid M^{!} \mid \langle M, N \rangle \mid \mathsf{in}_1 M \mid \mathsf{in}_2 M \\ & \mid \lambda x^{\varphi}\, M \mid \langle M \rangle N \mid \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) \\ & \mid \mathsf{pr}_1 M \mid \mathsf{pr}_2 M \mid \mathsf{der}(M) \mid \mathsf{fix}\, x^{!\sigma}\, M \end{aligned}$$

Typing rules

$$\frac{\mathcal{P} \vdash M : \sigma}{\mathcal{P} \vdash M^! : !\sigma} \qquad \frac{\mathcal{P} \vdash M_1 : \varphi_1 \qquad \mathcal{P} \vdash M_2 : \varphi_2}{\mathcal{P} \vdash \langle M_1, M_2 \rangle : \varphi_1 \otimes \varphi_2}$$

$$\frac{\mathcal{P} \vdash M : \varphi_i}{\mathcal{P} \vdash \mathrm{in}_i M : \varphi_1 \oplus \varphi_2} \qquad \frac{}{\mathcal{P}, x : \varphi \vdash x : \varphi}$$

$$\frac{\mathcal{P}, x : \varphi \vdash M : \sigma}{\mathcal{P} \vdash \lambda x^\varphi\, M : \varphi \multimap \sigma} \qquad \frac{\mathcal{P} \vdash M : \varphi \multimap \sigma \qquad \mathcal{P} \vdash N : \varphi}{\mathcal{P} \vdash \langle M \rangle N : \sigma}$$

$$\frac{\mathcal{P} \vdash M : !\sigma}{\mathcal{P} \vdash \mathrm{der}(M) : \sigma} \qquad \frac{\mathcal{P}, x : !\sigma \vdash M : \sigma}{\mathcal{P} \vdash \mathrm{fix}\, x^{!\sigma}\, M : \sigma}$$

$$\frac{\mathcal{P} \vdash M : \varphi_1 \oplus \varphi_2 \qquad \mathcal{P}, x_1 : \varphi_1 \vdash M_1 : \sigma \qquad \mathcal{P}, x_2 : \varphi_2 \vdash M_2 : \sigma}{\mathcal{P} \vdash \mathrm{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) : \sigma}$$

$$\frac{\mathcal{P} \vdash M : \varphi_1 \otimes \varphi_2}{\mathcal{P} \vdash \mathrm{pr}_i M : \varphi_i}$$

Values

Values are special terms of positive types :

- any variable x is a value
- for any term M, the term $M^!$ is a value
- if M is a value then $\mathrm{in}_i M$ is a value for $i = 1, 2$
- if M_1 and M_2 are values then $\langle M_1, M_2 \rangle$ is a value.

One uses letters V, W etc to denote values. If V is a value and $\mathcal{P} \vdash V : \sigma$ then σ is a positive type φ.

Reduction rules (weak reduction)

$$\frac{}{\mathrm{der}(M^!) \to_{\mathrm{w}} M} \qquad \frac{}{\langle \lambda x^{\varphi}\, M \rangle V \to_{\mathrm{w}} M\,[V/x]}$$

$$\frac{}{\mathrm{pr}_i \langle V_1, V_2 \rangle \to_{\mathrm{w}} V_i} \qquad \frac{}{\mathrm{fix}\, x^{!\sigma}\, M \to_{\mathrm{w}} M \left[(\mathrm{fix}\, x^{!\sigma}\, M)^! / x\right]}$$

$$\frac{M \to_{\mathrm{w}} M'}{\mathrm{der}(M) \to_{\mathrm{w}} \mathrm{der}(M')} \qquad \frac{M \to_{\mathrm{w}} M'}{\langle M \rangle N \to_{\mathrm{w}} \langle M' \rangle N}$$

$$\frac{N \to_{\mathrm{w}} N'}{\langle M \rangle N \to_{\mathrm{w}} \langle M \rangle N'} \qquad \frac{M \to_{\mathrm{w}} M'}{\mathrm{pr}_i M \to_{\mathrm{w}} \mathrm{pr}_i M'}$$

$$\frac{M_1 \rightarrow_{\mathrm{w}} M_1'}{\langle M_1, M_2\rangle \rightarrow_{\mathrm{w}} \langle M_1', M_2\rangle} \qquad \frac{M_2 \rightarrow_{\mathrm{w}} M_2'}{\langle M_1, M_2\rangle \rightarrow_{\mathrm{w}} \langle M_1, M_2'\rangle}$$

$$\frac{}{\mathsf{case}(\mathsf{in}_i V, x_1 \cdot M_1, x_2 \cdot M_2) \rightarrow_{\mathrm{w}} M_i\,[V/x_i]} \qquad \frac{M \rightarrow_{\mathrm{w}} M'}{\mathsf{in}_i M \rightarrow_{\mathrm{w}} \mathsf{in}_i M'}$$

$$\frac{M \rightarrow_{\mathrm{w}} M'}{\mathsf{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) \rightarrow_{\mathrm{w}} \mathsf{case}(M', x_1 \cdot M_1, x_2 \cdot M_2)}$$

This is a weak reduction (not inside boxes, not under λ's).

Values and abstractions are normal for this reduction.

Relational semantics

$[\sigma]$ is a set, $[\varphi]^!$ is a coalgebra $([\varphi], \mathrm{h}_\varphi)$ where $\mathrm{h}_\varphi \in \mathbf{Rel}([\varphi], ![\varphi])$ satisfies two commutations (lecture notes). Concretely :

- $(a, [b]) \in \mathrm{h}_\varphi$ iff $a = b$
- and $(a, m_1 + \cdots + m_k) \in \mathrm{h}_\varphi$ iff there are $a_1, \ldots, a_k \in [\varphi]$ such that $(a, [a_1, \ldots, a_k]) \in \mathrm{h}_\varphi$ and $(a_i, m_i) \in \mathrm{h}_\varphi$ for $i = 1, \ldots, k$.

- $[\top] = \emptyset$, $[\varphi \multimap \sigma] = [\varphi] \multimap [\sigma] = [\varphi] \times [\sigma]$;
- $[!\sigma] = ![\sigma] = \mathcal{M}_{\mathrm{fin}}([\sigma])$ and $(m, [m_1, \ldots, m_k]) \in \mathrm{h}_{!\sigma}$ iff $m = m_1 + \cdots + m_k$.

- $[\varphi_1 \otimes \varphi_2] = [\varphi_1] \otimes [\varphi_2] = [\varphi_1] \times [\varphi_2]$ and $((a^1, a^2), [(a^1_1, a^2_1), \ldots, (a^1_k, a^2_k)]) \in \mathsf{h}_{\varphi_1 \otimes \varphi_2}$ iff $(a^j, [a^j_1, \ldots, a^j_k]) \in \mathsf{h}_{\varphi_j}$ for $j = 1, 2$.
- $[\varphi_1 \oplus \varphi_2] = [\varphi_1] \oplus [\varphi_2] = \{1\} \times [\varphi_1] \cup \{2\} \times [\varphi_2]$ and $((j, a), [(j_1, a_1), \ldots, (j_k, a_j)]) \in \mathsf{h}_{\varphi_1 \oplus \varphi_2}$ iff $j_1 = \cdots = j_k = j$ and $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_{\varphi_j}$.

For recursive types, the general definitions based on embedding/retraction pairs allows to get $[\mathsf{Fix}\, \zeta \cdot \varphi]^! = [\varphi\, [\mathsf{Fix}\, \zeta \cdot \varphi/\zeta]]^!$.

Examples

- $1 = !\top$, so $[1] = \{[]\}$ and $([], k[[]])$ for all $k \in \mathbb{N}$.
- $\iota = 1 \oplus \iota$ (that is $\iota = \mathsf{Fix}\,\zeta \cdot (1 \oplus \zeta)$, so that an element of $[\iota]$ has shape $(2, \dots, (2, (1, [])) \dots)$ (represents the integer n when there are n occurrences of "2"). We denote this element as $\overline{n}$. An element of h_ι is a pair $(\overline{n}, k[\overline{n}])$ for $k, n \in \mathbb{N}$.
- One can define a type of lists of integers by $\lambda = 1 \oplus (\iota \otimes \lambda)$ so that an element of $[\lambda]$ has shape $(2, (\overline{n_1}, (2, (\overline{n_2}, \dots, (2, (\overline{n_k}, (1, []))) \dots))))$ which represents the list $\vec{n} = \langle n_1, \dots, n_k \rangle$. An element of h_λ is a pair $(\vec{n}, p[\vec{n}])$ where $\vec{n}$ given list.

Examples

- The type of streams of elements of positive type φ can be defined by $\rho_\varphi = \varphi \otimes !\rho_\varphi$, so an element of $[\rho_\varphi]$ is a pair $(a, [s_1, \ldots, s_k])$ where $s_1, \ldots, s_k$ are elements of $[\rho_\varphi]$. For instance $(\overline{3}, [(\overline{0}, [(\overline{7}, [])]), (\overline{2}, [])])$ is an element of $[\rho_\varphi]$. An element of $\mathsf{h}_{\rho_\varphi}$ is a pair $((a, m_1 + \cdots + m_k), [(a_1, m_1), \ldots, (a_k, m_k)])$ such that $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_\varphi$.

A term M such that $\mathcal{P} \vdash M : \sigma$ where $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$ is interpreted as a morphism $[M]_\mathcal{P} \in \mathbf{Rel}([\varphi_1] \otimes \cdots \otimes [\varphi_k], [\sigma])$, that is

$$[M]_\mathcal{P} \subseteq [\varphi_1] \times \cdots \times [\varphi_k] \times [\sigma]$$

When M is a value V and hence $\mathcal{P} \vdash V : \varphi$ for some positive φ we have $[V]_\mathcal{P} \in \mathbf{Rel}^!([\varphi_1]^! \otimes \cdots \otimes [\varphi_k]^!, [\varphi]^!)$.

When $a_i \in [\varphi_i]$ for $i = 1, \ldots, k$ and $b \in [\sigma]$ satisfy $(a_1, \ldots, a_k, b) \in [M]_{\mathcal{P}}$ we say that the following *semantic judgment* holds :

$$\Phi \vdash M : b : \sigma$$

where $\Phi = (x_1 : a_1 : \varphi_1, \ldots, x_k : a_k : \varphi_k)$ is a *semantic context*.

If $\Phi = (x_1 : a_1 : \varphi_1, \ldots, x_k : a_k : \varphi_k)$ is a semantic context, we use $\underline{\Phi}$ for the ordinary context $\Phi = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$ and $\widehat{\Phi}$ for the sequence $(a_I, \ldots, a_k)$.

There is a "typing derivation system" for these semantic judgments such that $x_1 : a_1 : \varphi_1, \ldots, x_k : a_k : \varphi_k \vdash M : b : \sigma$ is derivable iff $(a_1, \ldots, a_k, b) \in [M]_{\mathcal{P}}$.

We give now the deduction rules for this system.

$$\frac{(\widehat{\Phi}, []) \in \mathsf{h}_{\underline{\Phi}}}{\Phi, x : a : \varphi \vdash x : a : \varphi}$$

The premise of this rule means that the points a_i mentioned in Φ are "concealable".

$$\frac{\Phi_i \vdash M : a_i : \sigma \text{ for } i = 1, \ldots, k \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \ldots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash M^! : [a_1, \ldots, a_k] : !\sigma}$$

where we also assume that $\underline{\Phi} = \underline{\Phi_i}$ for each i (similar assumptions in the next rules). The last premise means that the a_i in Φ are k-duplicable.

$$\frac{\Phi_1 \vdash M_1 : a_1 : \varphi_1 \qquad \Phi_2 \vdash M_2 : a_2 : \varphi_2 \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \langle M_1, M_2 \rangle : (a_1, a_2) : \varphi_1 \otimes \varphi_2}$$

$$\frac{\Phi \vdash M : a : \varphi_i}{\Phi \vdash \mathsf{in}_i M : (i, a) : \varphi_1 \oplus \varphi_2} \qquad \frac{\Phi, x : a : \varphi \vdash M : b : \sigma}{\Phi \vdash \lambda x^{\varphi}\, M : (a, b) : \varphi \multimap \sigma}$$

$$\frac{\Phi_1 \vdash M : (a, b) : \varphi \multimap \sigma \qquad \Phi_2 \vdash N : a : \varphi \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \langle M \rangle N : b : \sigma}$$

$$\frac{\Phi \vdash M : [a] : !\sigma}{\Phi \vdash \mathsf{der}(M) : a : \sigma}$$

$$\frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2 \qquad (a_2, []) \in \mathsf{h}_{\varphi_2}}{\Phi \vdash \mathsf{pr}_1 M : a_1 : \varphi_1}$$

$$\frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2 \qquad (a_1, []) \in \mathsf{h}_{\varphi_1}}{\Phi \vdash \mathsf{pr}_2 M : a_2 : \varphi_2}$$

$$\frac{\Phi_0 \vdash M : (1, a_1) : \varphi_1 \oplus \varphi_2 \qquad \Phi_1, x : a_1; \varphi_1 \vdash N_1 : b : \sigma}{\Phi \vdash \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) : b : \sigma}$$

To be precise one has also to assume that $\underline{\Phi}, x_2 : \varphi_2 \vdash N_2 : \varphi_2$, and of course that $(\widehat{\Phi}, [\widehat{\Phi_0}, \widehat{\Phi_1}]) \in \mathsf{h}_{\underline{\Phi}}$. Similarly :

$$\frac{\Phi_0 \vdash M : (2, a_2) : \varphi_1 \oplus \varphi_2 \qquad \Phi_2, x : a_2; \varphi_2 \vdash N_2 : b : \sigma}{\Phi \vdash \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) : b : \sigma}$$

$$\frac{\Phi_0, x : [a_1, \dots, a_k] : !\sigma \vdash M : a : \sigma \qquad \forall i \ \Phi_i \vdash \mathsf{fix}\, x^{!\sigma} \, M : a_i : \sigma}{\Phi \vdash \mathsf{fix}\, x^{!\sigma} \, M : a : \sigma}$$

with the additional assumption that $(\widehat{\Phi}, [\widehat{\Phi_0}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}$.

The main feature of values is that, if $\mathcal{P} \vdash V : \varphi$ then $[V]_\mathcal{P} \in \mathbf{Rel}^!([\mathcal{P}]^!, [\varphi]^!)$, that is :

if $\Phi \vdash V : a : \varphi$ and $a_1, \ldots, a_k \in [\varphi]$, one has $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_\varphi$ if and only if there are $\Phi_1, \ldots, \Phi_k$ such that :

- $\underline{\Phi_i} = \underline{\Phi}$ for each i
- $\Phi_i \vdash V : a_i : \varphi$ for each i
- and $(\widehat{\Phi}, [\widehat{\Phi_1}, \ldots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}$.

Examples of term interpretations

- $[\lambda x^{\varphi}\, x] = \{(a, a) \mid a \in [\varphi]\}$.
- $\Omega^{\sigma} = \mathsf{fix}\, x^{!\sigma}\, x$ satisfies $\vdash \Omega^{\sigma} : \sigma$. Then $[\Omega^{\sigma}] = \emptyset$.
- $() = (\Omega^{\top})^{!}$, then $\vdash () : 1$ and $[()] = \{[]\}$.
- If $n \in \mathbb{N}$ one defines $\underline{n}$ such that $\vdash \underline{n} : \iota$ by $\underline{0} = \mathsf{in}_1()$ and $\underline{n+1} = \mathsf{in}_2\underline{n}$. Then $[\underline{n}] = \{\overline{n}\}$.
- $\mathsf{succ} = \lambda x^{\iota}\, \mathsf{in}_2(x)$, then $\vdash \mathsf{succ} : \iota \multimap \iota$ and $\mathsf{succ} = \{(\overline{n}, \overline{n+1}) \mid n \in \mathbb{N}\}$.
- $\mathsf{add} = \lambda x^{\iota}\, \mathsf{fix}\, f^{!(\iota \multimap \iota)}\, \lambda y^{\iota}\, \mathsf{case}(y, d \cdot \underline{0}, z \cdot \langle \mathsf{succ} \rangle \langle \mathsf{der}(f) \rangle z)$ then $\vdash \mathsf{add} : \iota \multimap \iota \multimap \iota$ and one has $[\mathsf{add}] = \{(n_1, n_2, n_1 + n_2) \mid n_1, n_2 \in \mathbb{N}\}$.

- maps $=$
$\lambda f^{!(\varphi \multimap \psi)}\,\mathsf{fix}\,h^{!(\rho_\varphi \multimap \rho_\psi)}\,\lambda y^{\rho_\varphi}\,\langle\langle \mathsf{der}(f)\rangle \mathsf{pr}_1 y, (\langle \mathsf{der}(h)\rangle \mathsf{pr}_2 y)^!\rangle$.
Then $\vdash \mathsf{maps} : !(\varphi \multimap \psi) \multimap \rho_\varphi \multimap \rho_\psi$ is a map functional for streams. Then $[\mathsf{maps}]$ is the least set of tuples
$(([(a,b)] + m_1 + \cdots + m_k), (a, [s_1, \ldots, s_k]), (b, [t_1, \ldots, t_k]))$
such that $(m_i, (s_i, t_i)) \in [\mathsf{maps}]$ for each $i \in \{1, \ldots, k\}$.
- Using this we can define for instance
$M = \lambda f^{!(\varphi \multimap \varphi)}\,\lambda x^\varphi\,\mathsf{fix}\,y^{!\rho_\varphi}\,\langle x, (\langle \mathsf{maps}\rangle f\,\mathsf{der}(y))^!\rangle$ such that $\vdash M : !(\varphi \multimap \varphi) \multimap \varphi \multimap \rho_\varphi$. What does this function compute ? What is its relational interpretation ? Execute a few step of $\to_{\mathsf{w}}$-reduction on $S = \langle M\rangle \mathsf{succ}^!\,\underline{0}$ and give the relational interpretation of S (observe that $\vdash S : \rho_\iota$).