Integration in cones. . .

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. . . bis repetita placent

In 2018 I gave here a talk on a probabilistic model of LL using measurable cones, based on a joint work with Michele Pagani and Christine Tasson.

In 2022 we started discussing with Guillaume Geoffroy on the possible connections between his convex QBSs and cones.

We understood that convex QBSs are similar to cones where an operation of integration is available, and linear morphisms preserve integration.

Thanks to the measurability structure of cones, such an integral is very easy to axiomatize, following the idea of Pettis integral in topological vector spaces (weak integral, Gelfand-Pettis integral).

Selinger's cones

A cone is an $\mathbb{R}_{\geq 0}$ semi module P such that

- $x + y = 0 \Rightarrow x = 0$
- $x + y = x' + y \Rightarrow x = x'$

together with a norm $\|\_\| : P \to \mathbb{R}_{\geq 0}$

- $\|x\| = 0 \Rightarrow x = 0$
- $\|\lambda x\| = \lambda \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
- positivity: $\|x\| \leq \|x + y\|$

Cone order and completeness

Definition

$x \leq y$ if there is z such that $x + z = y$

Then z is unique, notation: $y - x = z$

Fact

This is an order relation.

A cone must also satisfy

Any monotone $(x_n)_{n\in\mathbb{N}}$ with $\forall n \ \ \|x_n\| \leq 1$ has a lub $x = \sup_{n\in\mathbb{N}} x_n$ such that $\|x\| \leq 1$.

Main example

X a measurable space.

Then FMeas(X), the set of all nonnegative finite measures on X, is a cone with, for all $U \in \sigma_X$:

- $(\mu + \nu)(U) = \mu(U) + \nu(U)$, $(\lambda\,\mu)(U) = \lambda\,\mu(U)$
- $\|\mu\| = \mu(U)$
- so $\mu \leq \nu$ means $\forall U \in \sigma_X\ \mu(U) \leq \nu(U)$
- $\|\mu\| = \mu(X)$ (total variation norm).

Any probabilistic coherence space can be seen as a cone.

Linear and continuous functions

Definition

$f : P \to Q$ is linear if $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$

$\mathcal{B}P = \{x \in P \mid \|x\| \leq 1\}$

Fact

If f is linear then

- *f is monotone*
- *$f(\mathcal{B}P)$ is bounded (for the norm)*

Definition

f is continuous if $f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n)$ for all monotone and bounded $(x_n \in P)_{n\in\mathbb{N}}$.

The category of cones

If $f : P \to Q$ linear,

$$\|f\| = \sup_{x \in \mathcal{B}P} \|f(x)\| \in \mathbb{R}_{\geq 0}$$

Definition

Cones is the category of cones and linear and continuous functions f such that $\|f\| \leq 1$.

Example of morphism

For us $\kappa : X \rightsquigarrow Y$ means that κ is a bounded kernel, that is:

- $\kappa : X \to \mathrm{FMeas}(Y)$
- $\{\|\kappa(r)\| = \kappa(r)(Y) \mid r \in X\}$ is bounded in $\mathbb{R}_{\geq 0}$
- for all $V \in \sigma_Y$ the function $r \mapsto \kappa(r)(V)$ is measurable.

Then κ induces a linear and continuous function

$$\begin{aligned} \widehat{\kappa} : \mathrm{FMeas}(X) &\to \mathrm{FMeas}(Y) \\ \mu &\mapsto \boldsymbol{\lambda} V \in \sigma_Y \cdot \int_{r \in X} \kappa(r)(V)\, \mu(dr) \end{aligned}$$

Problem

There are a lot of linear continuous $f : \mathrm{FMeas}(X) \to \mathrm{FMeas}(Y)$ which are not induced by kernels.

And even worse...

If $f : \mathrm{FMeas}(X) \to \mathrm{FMeas}(Y)$ is linear and continuous the function

$$\begin{aligned} \kappa' : X &\to \mathrm{FMeas}(Y) \\ r &\mapsto f(\boldsymbol{\delta}^X(r)) \end{aligned}$$

is not necessarily a kernel: for $V \in \sigma_Y$, the function $r \mapsto f(\boldsymbol{\delta}^X(r))(V)$ has no reason to be measurable.

Of course if $f = \widehat{\kappa}$ for a kernel κ then $\kappa' = \kappa$.

So we need...

... an additional structure on a cone P to speak about measurability.

Measurability structure on a cone

Ar a full subcategory of the category of measurable spaces and measurable functions which is closed by cartesian product. Terminal object 0.

A *measurability structure* on the cone P is a family $\mathcal{M} = (\mathcal{M}_X)_{X \in \mathbf{Ar}}$ where

$$\mathcal{M}_X \subseteq (\mathbb{R}_{\geq 0})^{X \times P}$$

is a collection of *measurability tests* on P.

- If $m \in \mathcal{M}_X$ and $r \in X$, $m(r, \_) \in \mathbf{Cones}(P, \mathbb{R}_{\geq 0})$;
- If $m \in \mathcal{M}_X$ and $x \in P$, $m(\_, x) : X \to \mathbb{R}_{\geq 0}$ is measurable;
- If $m \in \mathcal{M}_X$ and $\varphi \in \mathbf{Ar}(Y, X)$, then $m(\varphi(\_), \_) \in \mathcal{M}_Y$;
- If $x \neq y \in P$, there is $m \in \mathcal{M}_0$ such that $m(x) \neq m(y)$;
- $\|x\| = \sup \left\{ \frac{m(x)}{\|m\|} \mid m \in \mathcal{M}_0 \setminus \{0\} \right\}$

Example

$X \in \mathbf{Ar}$.

Given $Y \in \mathbf{Ar}$ and $U \in \sigma_X$, define

$$\begin{aligned} \widetilde{U} : Y \times \mathrm{FMeas}(X) &\to \mathbb{R}_{\geq 0} \\ (s, \mu) &\mapsto \mu(U) \end{aligned}$$

and $\mathcal{M}_Y = \{\widetilde{U} \mid U \in \sigma_Y\}$.

$\mathcal{M}$ is a measurability structure on $\mathrm{FMeas}(X)$.

Measurable paths

A measurable cone is a pair $B = (\underline{B}, \mathcal{M}^B)$ where

- $\underline{B}$ is a cone
- $\mathcal{M}^B$ is a measurability structure on $\underline{B}$.

Definition

A measurable path from $X \in \mathbf{Ar}$ to B is a function $\beta : X \to \underline{B}$ which is bounded ($\beta(X)$ bounded in $\underline{B}$) and such that:

for all $Y \in \mathbf{Ar}$ and all $m \in \mathcal{M}^B_Y$, the function

$$\begin{aligned} Y \times X &\to \mathbb{R}_{\geq 0} \\ (s, r) &\mapsto m(s, \beta(r)) \end{aligned}$$

is measurable.

Remark: $\underline{B}$, equipped with its measurable paths, is a QBS.

Lin., cont. and measurable functions

Definition

Let B and C be measurable cones.

A linear and continuous function $f : \underline{B} \to \underline{C}$ is measurable if, for any B-measurable path $\beta : X \to \underline{B}$, the function $f \circ \beta$ is a C-measurable path $X \to \underline{C}$.

Example

We consider $\mathrm{FMeas}(X)$ as a measurable cone.

If $\kappa : X \rightsquigarrow Y$ then $\widehat{\kappa} : \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$ is a linear, continuous and measurable functions.

But there are still linear, continuous and measurable $\widehat{\kappa} : \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$ which are not of shape $\widehat{\kappa}\ldots$

Integral of a measurable path

Let $\beta : X \to \underline{B}$ be a measurable path and $\mu \in \underline{\mathrm{FMeas}(X)}$.

Definition

An integral of β wrt. μ is an $x \in \underline{B}$ such that

$$\forall m \in \mathcal{M}_0^B \quad m(x) = \int m(\beta(r))\, \mu(dr)$$

This Lebesgue integral $\in \mathbb{R}_{\geq 0}$ because $m \circ \beta : X \to \mathbb{R}_{\geq 0}$ is a bounded measurable function and μ is a finite measure.

If x exists, it is unique because $\mathcal{M}_0^B$ separates $\underline{B}$, notation:

$$x = \int \beta(r)\, \mu(dr)$$

Similar to the Pettis integral in top. vect. spaces (1938).

Integrable cone

Definition

A measurable cone B is integrable if, for any $X \in \mathbf{Ar}$ and any $\mu \in \underline{\mathsf{FMeas}(X)}$, all the measurable paths $X \to \underline{B}$ have an integral wrt. μ.

This is *property* of B, not a structure.

Definition

Let B and C be integrable cones. A linear and continuous $f : \underline{B} \to \underline{C}$ is integrable if it is measurable and, for any $\mu \in \underline{\mathsf{FMeas}(X)}$ and any measurable path $\beta : X \to \underline{B}$, one has

$$f\Big(\int \beta(r)\,\mu(dr)\Big) = \int f(\beta(r))\,\mu(dr)$$

Integration of measurable paths in integrable cones has all the good properties:

- Fubini theorem
- change of variable
- integrals with parameters
- etc

inherited from the standard Lebesgue integrals wrt. finite measures.

In a cone, we perform infinite sums which cannot be represented by integrals wrt. *finite* measures.

For instance if $(x_n \in \underline{B})_{n\in\mathbb{N}}$ such that

$$\forall N \in \mathbb{N} \quad \left\| \sum_{n=0}^{N} x_n \right\| \leq 1$$

then

$$\sum_{n=0}^{\infty} x_n = \sup_{N\in\mathbb{N}} \sum_{n=0}^{N} x_n$$

exists and has norm ≤ 1.

Such sums are taken into account by the ω-completeness of cones.

ICones the category whose objects are the integrable cones and

$$\mathbf{ICones}(B, C) = \{f \in \mathbf{Cones}(\underline{B}, \underline{C}) \mid f \text{ integrable } \}$$

Example

A measurable path $Y \to \underline{\mathrm{FMeas}(X)}$ is a finite kernel $\kappa : Y \rightsquigarrow X$.

If $\nu \in \underline{\mathrm{FMeas}(Y)}$, κ has an integral wrt. ν, namely

$$\widehat{\kappa}(\nu) = \boldsymbol{\lambda} U \in \sigma_X \cdot \int \kappa(s)(U)\, \nu(ds) \in \underline{\mathrm{FMeas}(X)}$$

So $\mathrm{FMeas}(X)$ is an integrable cone.

Nice consequences of integration

Let $f, g : \underline{\mathrm{FMeas}(X)} \to \underline{B}$ be linear continuous and integrable.

If $f(\boldsymbol{\delta}^X(r)) = g(\boldsymbol{\delta}^X(r))$ for all $r \in X$, then $f = g$:

if $\mu \in \underline{\mathrm{FMeas}(X)}$, we have

$$\begin{aligned} f(\mu) &= f\Big(\int \boldsymbol{\delta}^X(r)\,\mu(dr)\Big) \\ &= \int f(\boldsymbol{\delta}^X(r))\,\mu(dr) = \int g(\boldsymbol{\delta}^X(r))\,\mu(dr) = g(\mu) \end{aligned}$$

Let $f : \underline{\mathrm{FMeas}(X)} \to \underline{\mathrm{FMeas}(Y)}$ be linear continuous and integrable.

$\boldsymbol{\delta}^X : X \to \underline{\mathrm{FMeas}(X)}$ is a finite kernel (the identity kernel actually).

Since f is measurable, $f \circ \boldsymbol{\delta}^X : X \to \underline{\mathrm{FMeas}(Y)}$ is a measurable path, *ie.* a finte kernel $X \rightsquigarrow Y$.

We have $\widehat{f \circ \boldsymbol{\delta}^X}(\boldsymbol{\delta}^X(r)) = \int f(\boldsymbol{\delta}^X(r'))\, \boldsymbol{\delta}^X(r)(dr') = f(\boldsymbol{\delta}^X(r))$.

Hence $f = \widehat{f \circ \boldsymbol{\delta}^X}$: f is induced by a finite kernel.

Existence of left adjoints

ICones is locally small and complete, limits are computed as in **Set**.

$\mathbb{R}_{\geq 0}$ is cogenerating: if $f \neq g \in \mathbf{ICones}(B, C)$ there is $h \in \mathbf{ICones}(C, \mathbb{R}_{\geq 0})$ with $h\,f \neq h\,g$. Because $\mathcal{M}_0^B$ separates $\underline{B}$.

ICones is well-powered: for any integrable cone B there is a *set* $\mathcal{S}$ of subobjects $(B_0, h_0 \in \mathbf{ICones}(B_0, B))$ such that, for any subobject $(C, h \in \mathbf{ICones}(C, B))$, there is a $(B_0, h_0) \in \mathcal{S}$ and an iso $f \in \mathbf{Cones}(B_0, C)$ such that $h\,f = h_0$. Because **Ar** is small.

Consequence

For any locally small $\mathcal{C}$, any $F : \mathbf{ICones} \to \mathcal{C}$ preserving all limits has a left adjoint. By the special adjoint functor theorem (Freyd).

Internal hom

If B and C are integrable cones, the set P of all linear, continuous and integrable functions $\underline{B} \to \underline{C}$ is a cone.

It has a measurability structure $\mathcal{M} = (\mathcal{M}_X)_{X \in \mathbf{Ar}}$ where

$$\mathcal{M}_X = \{\beta \triangleright p \mid \beta \in \underline{\mathrm{Path}(X, B)} \text{ and } p \in \mathcal{M}^C_X\}$$

where

$$\begin{aligned} \beta \triangleright p : X \times P &\to \mathbb{R}_{\geq 0} \\ (r, f) &\mapsto p(r, f(\beta(r))) \end{aligned}$$

Fact

In that way one defines a measurable cone $B \multimap C$.

This measurable cone is integrable. The proof uses the monotone convergence theorem.

Integrals are defined pointwise: if $\theta \in \underline{\mathrm{Path}(X, B \multimap C)}$ and $\mu \in \underline{\mathrm{FMeas}(X)}$, then

$$f = \int^{C \multimap D} \theta(r)\, \mu(dr) \in \underline{C \multimap D}$$

is given by

$$f(x) = \int^{C} \theta(r)(x)\, \mu(dr)$$

Fact

$\_ \multimap \_$ is a functor $\mathbf{ICones}^{\mathrm{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ defined on morphisms by pre- and post-composition.

The tensor product

Fact

For each integrable cone, the functor $B \Rightarrow \_ : \mathbf{ICones} \to \mathbf{ICones}$ *preserves all limits.*

So this functor has a left adjoint $\_ \otimes B : \mathbf{ICones} \to \mathbf{ICones}$.

Actually $\otimes : \mathbf{ICones} \times \mathbf{ICones} \to \mathbf{ICones}$ *is a functor (by functoriality of* $\_ \multimap \_$*).*

We have no explicit description of $A \otimes B$!

The cone of paths

If $X \in \mathbf{Ar}$, the set of measurable paths $X \to \underline{B}$ has a structure of cone: operations defined pointwise and

$$\|\beta\| = \sup_{r \in X} \|\beta(r)\| \ .$$

Can be equipped with a measurability structure defined as in $\multimap$.

This is an integrable cone $\mathsf{Path}(X, B)$, integrals defined pointwise.

Thanks to integration again, $\mathsf{Path}(X, B) \simeq (\mathsf{FMeas}(X) \multimap B)$, by

$$\beta \mapsto \boldsymbol{\lambda}\mu \in \underline{\mathsf{FMeas}(X)} \cdot \int \beta(r)\, \mu(dr)$$

ICones as a monoidal category

By adjunction we have natural bijection

$$\Phi_{A,B,C} : \mathbf{ICones}(A \otimes B, C) \to \mathbf{ICones}(A, B \multimap C)$$

Fact

We have

$$\Phi_{A,B,C} \in \mathbf{ICones}(A \otimes B \multimap C, A \multimap (B \multimap C))$$

This gives us the monoidality isomorphisms!

For instance let $C = B_1 \otimes (B_2 \otimes B_3)$, we have

$$\begin{aligned}
\mathrm{Id} \in \mathbf{ICones}(B_1 \otimes (B_2 \otimes B_3), C) & \\
\simeq \mathbf{ICones}(B_1, B_2 \otimes B_3 \multimap C) & \\
\simeq \mathbf{ICones}(B_1, B_2 \multimap (B_3 \multimap C)) & \\
\text{by the iso of the previous slide} & \\
\simeq \mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, C) &
\end{aligned}$$

and this gives us $\alpha \in \mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, B_1 \otimes (B_2 \otimes B_3))$, the associator.

Then it is easy to prove that α is a natural iso, that the MacLane coherence conditions hold etc.

This is essentially the idea of *closed categories* (Eilenberg & Kelly, 1965). MLL without tensor product?

Theorem

Equipped with $\otimes$ *and* $1 = \mathbb{R}_{\geq 0}$*, the category* **ICones** *is an SMCC.*

We already know that **ICones** is cartesian (it is complete).

Fact

$$\underline{\underset{i \in I}{\&} B_i} = \left\{ \overrightarrow{x} \in \prod_{i \in I} \underline{B_i} \mid (\|x_i\|_{B_i})_{i \in I} \text{ is bounded} \right\}$$

$$\left\| \overrightarrow{x} \right\| = \sup_{i \in I} \|x_i\|$$

Fact

ICones *is also cocomplete: this is another consequence of the SAFT. So we have cokernels etc., no clue about how to describe them concretely!*

A function $f : \underline{B}^n \to \underline{C}$ is n-linear, continuous and integrable if it is so, separately in each argument.

A function $h : \underline{B} \to \underline{C}$ is n-homogeneous polynomial is there is an $f : \underline{B}^n \to \underline{C}$ which is n-linear, continuous and integrable such that

$$h(x) = f(x, \ldots, x)$$

Fact

There is only one such n-linear, continuous, integrable and symmetric f.

It is obtained from h by polarization.

A function $f : \mathcal{B}\underline{B} \to \underline{C}$ is analytic if it is bounded and there is a family $(f_n)_{n\in\mathbb{N}}$ such that $f_n : \underline{B} \to \underline{C}$ is n-homogeneous and

$$\forall x \in \mathcal{B}B \quad f(x) = \sum_{n\in\mathbb{N}} f_n(x) = \sup_{N\in\mathbb{N}} \sum_{n=0}^{N} f_n(x)$$

Example

$f : [0,1] = \mathcal{B}\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $f(x) = 1 - \sqrt{1-x}$.

Cannot be extended beyond $[0,1]$.

The $(f_n)_{n\in\mathbb{N}}$ is unique: for all $m \in \mathcal{M}^B$,

$$m(f_n(x)) = \frac{1}{n!}\frac{d^n}{dx^n}\, m(f(tx))\,|_{t=0}$$

Taylor expansion

So when $f : \mathcal{B}\underline{B} \to \underline{C}$, there are uniquely determined symmetric multilinear continuous and integrable functions

$$D^n f(0) : \underline{B}^n \to \underline{C}$$

(the derivatives of f at 0) such that

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) \cdot (x, \ldots, x)$$

Total monotonicity

Analytic functions are $\overbrace{\text{very, very,}\ldots}^{\infty\times}$ monotonic. When this makes sense:

$$f(x) \le f(x+u)$$

$$f(x+u_2) - f(x) \le f(x+u_1+u_2) - f(x+u_1)$$

$$\begin{aligned} f(x+u_2+u_3) - f(x+u_2) - (f(x+u_3) - f(x)) \\ \le f(x+u_1+u_2+u_3) - f(x+u_1+u_2) \\ - (f(x+u_1+u_3) - f(x+u_1)) \\ \vdots \end{aligned}$$

In a more civilized way:

$$f(x) \leq f(x+u)$$

$$f(x+u_1) + f(x+u_2) \leq f(x+u_1+u_2) + f(x)$$

$$\begin{aligned} f(x+u_1+u_2) + f(x+u_2+u_3) + f(x+u_1+u_3) \\ \leq f(x+u_1+u_2+u_3) + f(x+u_1) + f(x+u_2) + f(x+u_3) \\ \vdots \end{aligned}$$

Any analytic function $f : \mathcal{B}\underline{B} \to \underline{C}$ is also Scott continuous: if $(x_n \in \mathcal{B}B)_{n\in\mathbb{N}}$ is monotone then

$$f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n)$$

The analytic functions $f : \mathcal{B}\underline{B} \to \underline{C}$ form an integrable cone (norm, measurability structure and integration as in $B \multimap C$):

$$B \Rightarrow_{\mathrm{a}} C$$

The category of analytic functions

$\mathbf{ACones}(B, C) = \mathcal{B}\underline{B \Rightarrow_{\mathrm{a}} C}$, composition as in **Set**.

Theorem

ACones *is a CCC.*

For any integrable cone, there is an analytic least fixpoint operator $\mathcal{Y} \in \mathbf{ACones}(B \Rightarrow_{\mathrm{a}} B, B)$ *such that*

$$\forall f \in \mathcal{B}\underline{B \Rightarrow_{\mathrm{a}} B} \quad \mathcal{Y}(f) = \sup_{n \in \mathbb{N}} f^n(0)$$

so that $\mathcal{Y}(f)$ *is the least fixpoint of* f*.*

The analytic exponential

There is a functor $\mathsf{Der}^{\mathsf{a}} : \mathbf{ICones} \to \mathbf{ACones}$ such that

$$\mathsf{Der}^{\mathsf{a}}(B) = B \quad \text{and} \quad \mathsf{Der}^{\mathsf{a}}(f) = f$$

since $f \in \mathbf{ICones}(B, C) \Rightarrow f \in \mathbf{ACones}(B, C)$ (more precisely, the restriction of f to $\mathcal{B}\underline{B}$).

The functor $\mathsf{Der}^{\mathsf{a}}$ preserves all limits.

So it has a left adjoint $\mathsf{E}^{\mathsf{a}} : \mathbf{ACones} \to \mathbf{ICones}$.

$$\Psi_{B,C} : \mathbf{ICones}(\mathsf{E}^{\mathsf{a}}(B), C) \simeq \mathbf{ACones}(B, C)$$

which induces a comonad $!^{\mathsf{a}} = \mathsf{E}^{\mathsf{a}} \circ \mathsf{Der}^{\mathsf{a}} : \mathbf{ICones} \to \mathbf{ICones}$ with counit der and comultiplication dig, and

$$\mathbf{ICones}_{!} \simeq \mathbf{ACones}$$

Promotion

We have

$$\mathsf{an} = \Psi_{B,\mathsf{E}^{\mathsf{a}}(B)}(Id) \in \mathbf{ACones}(B, !^{\mathsf{a}} B)$$

the universal analytic function: for any $f \in \mathbf{ACones}(B, C)$, there is exactly one $g \in \mathbf{ICones}(!^{\mathsf{a}} B, C)$ such that

$$f = g \circ \mathsf{an}$$

namely $g = \Psi^{-1}_{B,C}(f)$.

For $x \in \mathcal{B}\underline{B}$, we set $x^{!_{\mathsf{a}}} = \mathsf{an}(x) \in \mathcal{B}\underline{!^{\mathsf{a}} B}$.

Remark

If $f, g \in \mathbf{ICones}(!^{\mathsf{a}} B, C)$ satisfy $f(x^{!_{\mathsf{a}}}) = g(x^{!_{\mathsf{a}}})$ for all $x \in \mathcal{B}\underline{B}$, then $f = g$.

Cones of finite measures as data-types

Let $X \in \mathbf{Ar}$.

We define $h_X \in \mathbf{ICones}(\mathsf{FMeas}(X), !^{\mathsf{a}}\mathsf{FMeas}(X))$ by

$$h_X(\mu) = \int (\boldsymbol{\delta}^X(r))^{!_{\mathsf{a}}} \, \mu(dr)$$

Fact

$(\mathsf{FMeas}(X), h_X)$ *is a* $!^{\mathsf{a}}$*-coalgebra.*

We have used the fact that integration is possible in $!^{\mathsf{a}}\mathsf{FMeas}(X)$, a major outcome of this approach!

We must prove

$$
\begin{tikzcd}
\underline{\mathsf{FMeas}(X)} \arrow[r, "h_X"] \arrow[d, "h_X"'] & \underline{!^{\mathsf{a}}\mathsf{FMeas}(X)} \arrow[d, "!^{\mathsf{a}} h_X"] \\
\underline{!^{\mathsf{a}}\mathsf{FMeas}(X)} \arrow[r, "\mathsf{dig}_X"] & \underline{!^{\mathsf{a}}!^{\mathsf{a}}\mathsf{FMeas}(X)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\underline{\mathsf{FMeas}(X)} \arrow[d, "h_X"'] \arrow[dr, "h_X"] & \\
\underline{!^{\mathsf{a}}\mathsf{FMeas}(X)} \arrow[r, "\mathsf{der}_X"] & \underline{\mathsf{FMeas}(X)}
\end{tikzcd}
$$

By integrability it suffices to prove the commutations on the $\mu \in \underline{\mathsf{FMeas}(X)}$ of shape $\mu = \boldsymbol{\delta}^X(r)$ and this is trivial.

Moreover for $\varphi \in \mathbf{Ar}(X, Y)$, we have the push-foward

$$\begin{aligned} \varphi_* : \underline{\mathrm{FMeas}(X)} &\to \underline{\mathrm{FMeas}(Y)} \\ \mu &\mapsto \boldsymbol{\lambda} V \in \sigma_Y \cdot \mu(\varphi^{-1}(V)) \end{aligned}$$

Fact

φ_* is a coalgebra morphism.
$\varphi_* \in \mathbf{ICones}^!((\mathrm{FMeas}(X), h_X), (\mathrm{FMeas}(Y), h_Y)))$

So we have a functor $\mathbf{Ar} \to \mathbf{ICones}^!$ which is easily seen to be faithful.

*If all the objects or **Ar** are standard Borel spaces (that is, they are Polish spaces equipped with their Borel σ-algebra), then this functor is also full.*

This is not a serious restriction (discrete $\mathbb{N}$, $\mathbb{R}$, Cantor space etc. are such).

How to interpret sampling

Imagine we have a programming language which has at the types

$$\sigma, \tau, \cdots := \rho \mid \sigma \Rightarrow \tau \mid \cdots$$

where ρ is the type of real numbers. We choose **Ar** with $\mathbb{R} \in \mathbf{Ar}$.

$[\![\sigma]\!]$ is a measurable cone, $[\![\rho]\!] = \mathrm{FMeas}(\mathbb{R})$,
$[\![\sigma \Rightarrow \tau]\!] = [\![\sigma]\!] \Rightarrow_{\mathrm{a}} [\![\tau]\!]$

If $\vdash M : \rho$ then $[\![M]\!] \in \mathcal{B}\underline{\mathrm{FMeas}(\mathbb{R})}$.

If $x : \rho \vdash N : \sigma$ then $[\![N]\!] \in \mathbf{ACones}([\![\rho]\!], [\![\sigma]\!])$.

Then we can sample a real number according to the subdistribution M in N

$$\vdash \mathtt{sample}(x, M, N) : \sigma$$

Considering $[\![N]\!] \in \mathbf{ICones}(!^{\mathsf{a}}\mathsf{FMeas}(\mathbb{R}), [\![\sigma]\!])$ then we have

$$[\![N]\!]\, h_{\mathsf{FMeas}(\mathbb{R})} \in \mathbf{ICones}(\mathsf{FMeas}(\mathbb{R}), [\![\sigma]\!])$$

and we take

$$[\![\mathtt{sample}(x, M, N)]\!] = [\![N]\!](h_{\mathsf{FMeas}(\mathbb{R})}([\![M]\!])) \tag{1}$$

that is, considering $[\![N]\!]$ as an analytic function $\mathcal{B}\underline{\mathsf{FMeas}(\mathbb{R})} \to \underline{[\![\sigma]\!]}$,

$$[\![\mathtt{sample}(x, M, N)]\!] = \int [\![N]\!](\boldsymbol{\delta}^{\mathbb{R}}(r))\, [\![M]\!](dr)$$

(1) means that sampling is just a `let` construct which allows to use the type ρ in call-by-value.