PROBABILISTIC CALL BY PUSH VALUE

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ABSTRACT. We introduce a probabilistic extension of Levy's Call-By-Push-Value. This extension consists simply in adding a "flipping coin" boolean closed atomic expression. This language can be understood as a major generalization of Scott's PCF encompassing both call-by-name and call-by-value and featuring recursive (possibly lazy) data types. We interpret the language in the previously introduced denotational model of probabilistic coherence spaces, a categorical model of full classical Linear Logic, interpreting data types as coalgebras for the resource comonad. We prove adequacy and full abstraction, generalizing earlier results to a much more realistic and powerful programming language.

1. INTRODUCTION

Call-by-Push-Value [22] is a class of functional languages generalizing the lambda-calculus in several directions. From the point of view of Linear Logic we understand it as a half-polarized system bearing some similarities with *e.g.* classical Parigot's lambda-mu-calculus, this is why we call it Λ_{HP}. The main idea of Laurent and Regnier interpretation of call-by-name lambda-mu in Linear Logic [20] (following actually [13]) is that all types of the minimal fragment of the propositional calculus (with $\Rightarrow$ as unique connective) are interpreted as *negative* types of Linear Logic which are therefore naturally equipped with structural morphisms: technically speaking, these types are algebras of the ?-monad of Linear Logic. This additional structure of negative types allows to perform logical structural rules on the *right side* of typing judgments even if these formulas are not necessarily of shape $?\sigma$, and this is the key towards giving a computational content to classical logical rules, generalizing the fundamental discovery of Griffin on typing call/cc with Peirce Law [15].

From our point of view, the basic idea of Λ_{HP} is quite similar, though somehow dual and used in a less systematic way: data types are interpreted as *positive* types of Linear Logic equipped therefore with structural morphisms (as linear duals of negative formulas, they are coalgebras of the !-comonad) and admit therefore structural rules on the *left side* of typing judgment even if they are not of shape $!\sigma$. This means that a function defined on a data type can have a *linear function type* even if it uses its argument in a non-linear way:

this non-linearity is automatically implemented by means of the structural morphisms the positive data type is equipped with.

The basic positive type in Linear Logic is $!\sigma$ (where σ is any type): it is the very idea of Girard's call-by-name translation of the lambda-calculus into Linear Logic to represent the implication type $\sigma \Rightarrow \tau$ by means of the decomposition $!\sigma \multimap \tau$. The new idea of Λ_{HP} is to generalize this use of the linear implication to any type construction of shape $\varphi \multimap \tau$ where φ is a positive type, without imposing any linearity restriction on the usage of the argument of type φ used by a function of type $\varphi \multimap \tau$. This non-symmetrical restriction in the use of the linear implication motivates our description of Λ_{HP} as a "half-polarized" system: in a fully polarized system like Laurent's *Polarized Linear Logic* LLP [20], one would also require the type σ to be negative in $\varphi \multimap \sigma$ (the last system presented in [5] implements this idea) and the resulting formalism would host classical computational primitives such as call/cc as well. The price to pay, as illustrated in [2], is a less direct access to data types: it is impossible to give a function from integers to integers the expected type $\iota \multimap \iota$ (where ι is the type of flat natural numbers satisfying $\iota = \mathbf{1} \oplus \iota$), the simplest type one can give to such a term is $\iota \multimap ?\iota$ which complicates its denotational interpretation[1].

Not being polarized on the right side of implications, Λ_{HP} remains "intuitionistic" just as standard functional programming languages whose paradigmatic example is PCF. So what is the benefit of this special status given to positive formulas considered as "data types"? There are several answers to this question.

- First, and most importantly, it gives a *call-by-value access* to data types: when translating PCF into Linear Logic, the simplest type for a function from integers to integers is $!\iota \multimap \iota$. This means that arguments of type ι are used in a call-by-name way: such arguments are evaluated again each time they are used. This can of course be quite inefficient. It is also simply *wrong* if we extend our language with a random integer generator since in that case each evaluation of such an argument can lead to a different value: in PCF there is no way to keep memory of the value obtained for one evaluation of such a parameter and probabilistic programming is therefore impossible. In Λ_{HP} data types such as ι can be accessed in call-by-value, meaning that they are evaluated once and that the resulting value is kept for further computation: this is typically the behavior of a function of type $\iota \multimap \iota$. This is not compulsory however and an explicit ! type constructor still allows to define functions of type $!\iota \multimap \iota$ in Λ_{HP}, with the usual PCF behavior.

- Positive types being closed under positive Linear Logic connectives (direct sums and tensor product) and under "least fixpoint" constructions, it is natural to allow corresponding constructions of positive types in Λ_{HP} as well, leading to a language with rich data type constructions (various kinds of trees, streams etc are freely available) and can be accessed in call-by-value as explained above for integers. From this data types point of view, the ! Linear Logic connective corresponds to the type of *suspensions* or *thunks* which are boxes (in the usual Linear Logic sense) containing unevaluated pieces of program.

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[1]One can also consider ? as the computational monad of linear continuations and use a translation from direct style into monadic style (which, for this monad, is just a version of the familiar CPS translation). This is just a matter of presentation and of syntactic sugar and does not change the denotational interpretation in the kind of concrete models of Linear Logic we have in mind such as the relational model, the coherence space model etc.

- As already mentioned, since the Linear Logic type constructors $\multimap$ and $!$ are available in Λ_{HP} (with the restriction explained above on the use of $\multimap$ that the left side type must be positive), one can represent in Λ_{HP} both Girard's translations from lambda-calculus into Linear Logic introduced in [12]: the usual one which is call-by-name and the "boring one" which is call-by-value. So our language Λ_{HP} is not intrinsically call-by-value and hosts very simply Girard's representation of call-by-name in Linear Logic as further explained in Section 2.4.

Concretely, in Λ_{HP}, a term of positive type can be a *value*, and then it is discardable and duplicable and, accordingly, its denotational interpretation is a morphism of coalgebras: values are particular terms whose interpretation is easily checked to be such a morphism, which doesn't preclude other terms of positive type to have the same property of course, in particular terms reducing to values! Being a value is a property which can be decided in time at most the size of the term and values are freely duplicable and discardable. The "β-rules" of the calculus (the standard β-reduction as well as the similar reduction rules associated with tensor product and direct sum) are subject to restrictions on certain subterms of redexes to be values (because they must be duplicable and discardable) and these restrictions make sense thanks to this strong decidability property of being a value.

Probabilities in Λ_{HP}. Because of the possibility offered by Λ_{HP} of handling values in a call-by-value manner, this language is particularly suitable for probabilistic functional programming. Contrarily to the common monadic viewpoint on effects, we consider an extension of the language where probabilistic choice is a primitive $\mathsf{coin}(p)$ of type $\mathbf{1} \oplus \mathbf{1}$ (the type of booleans)[2] parameterized by $p \in [0,1] \cap \mathbb{Q}$ which is the probability of getting $\mathbf{t}$ (and $1 - p$ is the probability of getting $\mathbf{f}$). So our probabilistic extension $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ of Λ_{HP} is in direct style, but, more importantly, the denotational semantics we consider is itself in "direct style" and does not rely on any auxiliary computational monad of probability distributions [29, 17] (see [18] for the difficulties related with the monadic approach to probabilistic computations), random variables [3, 14, 25], or measures [31, 16].

On the contrary, we interpret our language in the model of *probabilistic coherence spaces* [4] that we already used for providing a fully abstract semantics for probabilistic PCF [9]. A probabilistic coherence space X is given by an at most countable set $|X|$ (the web of X) and a set $\mathsf{P}X$ of $|X|$-indexed families of non-negative real numbers, to be considered as some kind of "generalized probability distributions". This set of families of real numbers is subject to a closure property implementing a simple intuition of probabilistic observations. Probabilistic coherence spaces are a model of classical Linear Logic and can be seen as ω-continuous domains equipped with an operation of convex linear combination, and the linear morphisms of this model are exactly the Scott continuous functions commuting with these convex linear combinations.

Besides, probabilistic coherence spaces can be seen as particular *d-cones* [32] and even *Kegelspitzen* [19], that is, complete partial orders equipped with a Scott continuous "convex structure" allowing to compute probabilistic linear combinations of their elements. Kegelspitzen have been used recently by Rennela to define a denotational model of a probabilistic extension of FPC [28]. The main difference with respect to our approach seems to be the fact that non-linear morphisms (corresponding to morphisms of type $!\sigma \multimap \tau$ in

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[2]An not of type $T(\mathbf{1} \oplus \mathbf{1})$ where T would be a computational monad of probabilistic computations.

our setting) are general Scott continuous functions in Rennela's model[3], whereas they are analytic functions[4] in ours, which can be seen as very specific Scott continuous functions. See also [11] where these functions are seen to be *stable* in a generalized sense.

As shown in [4] probabilistic coherence spaces have all the required completeness properties for interpreting recursive type definitions (that we already used in [7] for interpreting the pure lambda-calculus) and so we are able to associate a probabilistic coherence space with all types of $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$.

In this model the type $\mathbf{1} \oplus \mathbf{1}$ is interpreted as the set of sub-probability distributions on $\{\mathbf{t}, \mathbf{f}\}$ so that we have a straightforward interpretation of $\mathsf{coin}(p)$. Similarly the type of flat integers ι is interpreted as a probabilistic coherence space N such that $|\mathsf{N}| = \mathbb{N}$ and PN is the set of all probability distributions on the natural numbers. Given probabilistic spaces X and Y, the space $X \multimap Y$ has $|X| \times |Y|$ as web and $\mathsf{P}(X \multimap Y)$ is the set of all $|X| \times |Y|$ matrices which, when applied to an element of $\mathsf{P}X$ gives an element of $\mathsf{P}Y$. The web of the space $!X$ is the set of all finite multisets of elements of $|X|$ so that an element of $!X \multimap Y$ can be considered as a power series on as many variables as there are elements in $|X|$ (the composition law associated with the Kleisli category of the !-comonad is compatible with this interpretation of morphisms as power series).

From a syntactic point of view, the only values of $\mathbf{1} \oplus \mathbf{1}$ are $\mathbf{t}$ and $\mathbf{f}$, so $\mathsf{coin}(p)$ is not a value. Therefore we cannot reduce $\langle \lambda x^{\mathbf{1} \oplus \mathbf{1}} M \rangle \mathsf{coin}(p)$ to $M[\mathsf{coin}(p)/x]$ and this is a good thing since then we would face the problem that the boolean values of the various occurrences of $\mathsf{coin}(p)$ might be different. We have first to reduce $\mathsf{coin}(p)$ to a value, and the reduction rules of our probabilistic Λ_{HP} stipulate that $\mathsf{coin}(p)$ reduces to $\mathbf{t}$ with probability p and to $\mathbf{f}$ with probability $1 - p$ (in accordance with the kind of operational semantics that we considered in our earlier work on this topic, starting with [4]). So $\langle \lambda x^{\mathbf{1} \oplus \mathbf{1}} M \rangle \mathsf{coin}(p)$ reduces to $M[\mathbf{t}/x]$ with probability p and to $M[\mathbf{f}/x]$ with probability $1 - p$, which is perfectly compatible with the intuition that in Λ_{HP} application is a linear operation (and that implication is linear: the type of $\lambda x^{\mathbf{1} \oplus \mathbf{1}} M$ is $(\mathbf{1} \oplus \mathbf{1}) \multimap \sigma$ for some type σ): in this operational semantics as well as in the denotational semantics outlined above, linearity corresponds to commutation with (probabilistic) convex linear combinations.

Contents. The results presented in this paper illustrate the tight connection between the syntactic and the denotational intuitions underpinning our understanding of this calculus.

We first introduce in Section 2 the syntax and operational semantics of $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$, an abstract programming language very close du Paul Levy's Call by Push Value (CBPV) [22]. It however differs from Levy's language mainly by the fact that CBPV computation types products and recursive type definitions have no couterparts in our language. This choice is mainly motivated by the wish of keeping the presentation reasonably short. It is argued in Sections 2.3 and 2.4 that $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ is expressive enough for containing well behaved lazy data types such as the type of streams, and for encoding call-by-name languages with products.

In Section 3, we present the Linear Logic model of probabilistic coherence spaces, introducing mainly the underlying linear category **Pcoh**, where $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ general types are interpreted, and the Eilenberg-Moore category $\mathbf{Pcoh}^!$, where the positive types are interpreted. In order to simplify the Adequacy and Full Abstraction proofs, we restrict actually our attention

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[3]More precisely, in his interpretation of FPC, Rennela uses strict Scott continuous functions, but, along the same lines, it seems clear that Kegelspitzen give rise to a model of PCF where morphisms are general Scott continuous functions.

[4]Meaning that it is definable by a power series.

to a well-pointed subcategory of $\mathbf{Pcoh}^!$ whose objects we call "dense coalgebras": this will allow to consider all morphisms as functions. As suggested by one of the referees, there are probably smaller well-pointed subcategories of $\mathbf{Pcoh}^!$ where we might interpret our positive types, and in particular the category of families introduced in [1] describing call-by-value games. This option will be explored in further work. We prefer here to work with the most general setting as it is also compatible with a probabilistic extension of the last system of [5], which features classical call/cc-like capabilities.

We prove then in Section 4 an Adequacy Theorem whose statement is extremely simple: given a closed term M of type $\mathbf{1}$ (which has exactly one value ()), the denotational semantics of M, which is an element of $[0, 1]$, coincides with its probability to reduce to () (such a term can only diverge or reduce to ()). In spite of its simple statement the proof of this result requires some efforts mainly because of the presence of unrestricted[5] recursive types in $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$. The method used in the proof relies on an idea of Pitts [26] and is described in the introduction of Section 4.

Last we prove Full Abstraction in Section 5 adapting the technique used in [7] to the present $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ setting. The basic idea consists in associating, with any element a of the web of the probabilistic coherence space $[\sigma]$ interpreting the type σ, a term a^- of type[6] $!\sigma \multimap !\iota \multimap \mathbf{1}$ such that, given two elements w and w' of $\mathsf{P}[\sigma]$ such that $w_a \neq w'_a$, the elements $[a^-] w^!$ and $[a^-] (w')^!$ of $\mathsf{P}(!\iota \multimap \mathbf{1})$ are different power series depending on a finite number n of parameters (this number n depends actually only on a) so that we can find a rational number valued sub-probability distribution for these parameters where these power series take different values in $[0, 1]$. Applying this to the case where w and w' are the interpretations of two closed terms M and M' of type σ, we obtain, by combining a^- with the rational sub-probability distribution which can be represented in the syntax using $\mathsf{coin}(p)$ for various values of p, a Λ_{HP} closed term C of type $!\sigma \multimap \mathbf{1}$ such that the probability of convergence of $\langle C \rangle M^!$ and $\langle C \rangle (M')^!$ are different (by adequacy). This proves that if two (closed) terms are operationally equivalent then they have the same semantics in probabilistic coherence spaces, that is, equational full abstraction.

Further developments. These results are similar to the ones reported in [10] but are actually different, and there is no clear logical connection between them, because the languages are quite different, and therefore, the observation contexts also. And this even in spite of the fact that PCF can be faithfully encoded in Λ_{HP}. This seems to show that the semantical framework for probabilistic functional programming offered by probabilistic coherence spaces is very robust and deserves further investigations. One major outcome of the present work is a natural extension of probabilistic computation to rich data-types, including types of potentially infinite values (streams etc).

Our full abstraction result cannot be extended to inequational full abstraction with respect to the natural order relation on the elements of probabilistic coherence spaces: a natural research direction will be to investigate other (pre)order relations and their possible interactive definitions. Also, it is quite tempting to replace the equality of probabilities in the definition of contextual equivalence by a distance; this clearly requires further developments.

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[5]Meaning that recursive type definitions are not restricted to covariant types.

[6]For technical reasons and readibility of the proof, the type we give to a^- in Section 5 slightly differs from this one.

2. Probabilistic Call By Push Value

We introduce the syntax of $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ of CBPV (where HP stands for "half polarized").

Types are given by the following BNF syntax. We define by mutual induction two kinds of types: *positive types* and *general types*, given type variables $\zeta, \xi \ldots$:

$$\text{positive}\quad \varphi, \psi, \ldots := \mathbf{1} \mid !\sigma \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \mathsf{Rec}\, \zeta \cdot \varphi \tag{2.1}$$

$$\text{general}\quad \sigma, \tau \ldots := \varphi \mid \varphi \multimap \sigma \tag{2.2}$$

The type $\mathbf{1}$ is the neutral element of $\otimes$ and it might seem natural to have also a type $\mathbf{0}$ as the neutral element of $\oplus$. We didn't do so because there is no associated constructor in the syntax of terms, and the only closed terms of type $\mathbf{0}$ that one can write are ever-looping terms.

Observe also that there are no restriction on the variance of types in the recursive type construction: for instance, in $\mathsf{Rec}\, \zeta \cdot \varphi$ is a well-formed positive type if $\varphi = !(\zeta \multimap \zeta)$, where ζ has a negative and a positive occurrence. Do well notice that our "positive types" are positive in the sense of logical polarities, and *not* of the variance of type variables!

Terms are given by the following BNF syntax, given variables $x, y, \ldots$:

$$\begin{aligned}
M, N \ldots := \ &x \mid () \mid M^! \mid (M, N) \mid \mathsf{in}_\ell M \mid \mathsf{in}_r M \\
&\mid \lambda x^\varphi\, M \mid \langle M \rangle N \mid \mathsf{case}(M, x_\ell \cdot N_\ell, x_r \cdot N_r) \\
&\mid \mathsf{pr}_\ell M \mid \mathsf{pr}_r M \mid \mathsf{der}\, M \mid \mathsf{fix}\, x^{!\sigma}\, M \\
&\mid \mathsf{fold}(M) \mid \mathsf{unfold}(M) \\
&\mid \mathsf{coin}(p),\ p \in [0, 1] \cap \mathbb{Q}
\end{aligned}$$

Remark. This functional language $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$, or rather the sublanguage Λ_{HP} which is $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ stripped from the $\mathsf{coin}(p)$ construct, is formally very close to Levy's CBPV. As explained in the Introduction, our intuition however are more related to Linear Logic than to CBPV and its general adjunction-based models. This explains why our syntax slightly departs from Levy's original syntax as described *e.g.* in [21] and is much closer to the SFPL language of [23]: Levy's type constructor F is kept implicit and U is "!". We use LL inspired notations: $M^!$ corresponds to $\mathsf{thunk}(M)$ and $\mathsf{der}\, M$ to $\mathsf{force}(M)$. Our syntax is also slightly simpler than that of SFPL in that our general types do not feature products and recursive types definitions, we will explain in Section 2.4 that this is not a serious limitation in terms of expressiveness.

Figure 1 provides the typing rules for these terms. A typing context is an expression $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$ where all types are positive and the x_is are pairwise distinct variables.

2.1. Reduction rules.

Values are particular $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ terms (they are not a new syntactic category) defined by the following BNF syntax:

$$V, W \ldots := x \mid () \mid M^! \mid (V, W) \mid \mathsf{in}_\ell V \mid \mathsf{in}_r V \mid \mathsf{fold}(V)\,.$$

Figure 2 defines a deterministic *weak* reduction relation $\to_{\mathsf{w}}$ and a probabilistic reduction $\overset{p}{\to}$ relation. This reduction is weak in the sense that we never reduce within a "box" $M^!$ or under a λ.

The distinguishing feature of this reduction system is the role played by values in the definition of $\to_{\mathsf{w}}$. Consider for instance the case of the term $\mathsf{pr}_\ell (M_\ell, M_r)$; one might expect

$$\frac{}{\mathcal{P}, x : \varphi \vdash x : \varphi} \qquad \frac{\mathcal{P}, x : \varphi \vdash M : \sigma}{\mathcal{P} \vdash \lambda x^\varphi \, M : \varphi \multimap \sigma} \qquad \frac{\mathcal{P} \vdash M : \varphi \multimap \sigma \qquad \mathcal{P} \vdash N : \varphi}{\mathcal{P} \vdash \langle M \rangle N : \sigma}$$

$$\frac{\mathcal{P} \vdash M : \sigma}{\mathcal{P} \vdash M^! : !\sigma} \qquad \frac{}{\mathcal{P} \vdash () : \mathbf{1}} \qquad \frac{\mathcal{P} \vdash M_\ell : \varphi_\ell \qquad \mathcal{P} \vdash M_r : \varphi_r}{\mathcal{P} \vdash (M_\ell, M_r) : \varphi_\ell \otimes \varphi_r} \qquad \frac{\mathcal{P} \vdash M : \varphi_i \quad i \in \{\ell, r\}}{\mathcal{P} \vdash \mathsf{in}_i M : \varphi_\ell \oplus \varphi_r}$$

$$\frac{\mathcal{P} \vdash M : !\sigma}{\mathcal{P} \vdash \mathsf{der}\, M : \sigma} \qquad \frac{\mathcal{P} \vdash M : \varphi_\ell \otimes \varphi_r \quad i \in \{\ell, r\}}{\mathcal{P} \vdash \mathsf{pr}_i M : \varphi_i} \qquad \frac{\mathcal{P}, x : !\sigma \vdash M : \sigma}{\mathcal{P} \vdash \mathsf{fix}\, x^{!\sigma} \, M : \sigma}$$

$$\frac{\mathcal{P} \vdash M : \varphi_\ell \oplus \varphi_r \qquad \mathcal{P}, x_\ell : \varphi_\ell \vdash M_\ell : \sigma \qquad \mathcal{P}, x_r : \varphi_r \vdash M_r : \sigma}{\mathcal{P} \vdash \mathsf{case}(M, x_\ell \cdot M_\ell, x_r \cdot M_r) : \sigma}$$

$$\frac{}{\mathcal{P} \vdash \mathsf{coin}(p) : \mathbf{1} \oplus \mathbf{1}}$$

$$\frac{\mathcal{P} \vdash M : \psi \,[\mathsf{Rec}\, \zeta \cdot \psi / \zeta]}{\mathcal{P} \vdash \mathsf{fold}(M) : \mathsf{Rec}\, \zeta \cdot \psi} \qquad \frac{\mathcal{P} \vdash M : \mathsf{Rec}\, \zeta \cdot \psi}{\mathcal{P} \vdash \mathsf{unfold}(M) : \psi \,[\mathsf{Rec}\, \zeta \cdot \psi / \zeta]}$$

Figure 1: Typing system for $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$

this term to reduce directly to M_ℓ but this is not the case. One needs first to reduce M_ℓ *and* M_r to values before extracting the first component of the pair (the terms $\mathsf{pr}_\ell(M_\ell, M_r)$ and M_ℓ have not the same denotational interpretation in general). Of course replacing M_i with $M_i^!$ allows a lazy behavior. Similarly, in the $\to_{\mathsf{w}}$ rule for case, the term on which the test is made must be reduced to a value (necessarily of shape $\mathsf{in}_\ell V$ or $\mathsf{in}_r V$ if the expression is well typed) before the reduction is performed. As explained in the Introduction this allows to "memoize" the value V for futher usage: the value is passed to the relevant branch of the case through the variable x_i.

Given two terms M, M' and a real number $p \in [0, 1]$, $M \xrightarrow{p} M'$ means that M reduces in one step to M' with probability p.

We say that M is *weak normal* if there is no reduction $M \xrightarrow{p} M'$. It is clear that any value is weak normal. When M is closed, M is weak normal iff it is a value or an abstraction.

In order to simplify the presentation we *choose* in Figure 2 a reduction strategy. For instance we decide that, for reducing (M_ℓ, M_r) to a value, one needs first to reduce M_ℓ to a value, and then M_r; this choice is of course completely arbitrary. A similar choice is made for reducing terms of shape $\langle M \rangle N$, where we require the argument to be reduced first. This choice is less arbitrary as it will simplify a little bit the proof of adequacy in Section 4 (see for instance the proof of Lemma 22).

We could perfectly define a more general weak reduction relation as in [5] for which we could prove a "diamond" confluence property but we would then need to deal with a reduction transition system where, at each node (term), several probability distributions of reduced terms are available and so we would not be able to describe reduction as a simple (infinite dimensional) stochastic matrix. We could certainly also define more general reduction rules allowing to reduce redexes anywhere in terms (apart for $\mathsf{coin}(p)$ which can be reduced only when in linear position) but this would require the introduction of additional σ-rules as in [6]. As in that paper, confluence can probably be proven, using ideas coming from [8, 33] for dealing with reduction in an algebraic lambda-calculus setting.

2.2. **Observational equivalence.** In order to define observational equivalence, we need to represent the probability of convergence of a term to a normal form. As in [4], we consider the reduction as a discrete time Markov chain whose states are terms and stationary states

$$\frac{}{\mathsf{der}\,M^! \to_\mathsf{w} M} \qquad \frac{}{\langle\lambda x^\varphi\,M\rangle V \to_\mathsf{w} M\,[V/x]} \qquad \frac{}{\mathsf{pr}_i\,(V_\ell, V_r) \to_\mathsf{w} V_i}\; i \in \{\ell, r\}$$

$$\frac{}{\mathsf{fix}\,x^{!\sigma}\,M \to_\mathsf{w} M\,\big[(\mathsf{fix}\,x^{!\sigma}\,M)^!/x\big]} \qquad \frac{}{\mathsf{case}(\mathsf{in}_i V, x_\ell \cdot M_\ell, x_r \cdot M_r) \to_\mathsf{w} M_i\,[V/x_i]}\; i \in \{\ell, r\}$$

$$\frac{}{\mathsf{unfold}(\mathsf{fold}(V)) \to_\mathsf{w} V}$$

$$\frac{M \to_\mathsf{w} M'}{M \xrightarrow{1} M'} \qquad \frac{}{\mathsf{coin}(p) \xrightarrow{p} \mathsf{in}_\ell()} \qquad \frac{}{\mathsf{coin}(p) \xrightarrow{1-p} \mathsf{in}_r()}$$

$$\frac{M \xrightarrow{p} M'}{\mathsf{der}\,M \xrightarrow{p} \mathsf{der}\,M'} \qquad \frac{M \xrightarrow{p} M'}{\langle M\rangle V \xrightarrow{p} \langle M'\rangle V} \qquad \frac{N \xrightarrow{p} N'}{\langle M\rangle N \xrightarrow{p} \langle M\rangle N'} \qquad \frac{M \xrightarrow{p} M'}{\mathsf{pr}_i M \xrightarrow{p} \mathsf{pr}_i M'}\; i \in \{\ell, r\}$$

$$\frac{M_\ell \xrightarrow{p} M'_\ell}{(M_\ell, M_r) \xrightarrow{p} (M'_\ell, M_r)} \qquad \frac{M_r \xrightarrow{p} M'_r}{(V, M_r) \xrightarrow{p} (V, M'_r)} \qquad \frac{M \xrightarrow{p} M'}{\mathsf{in}_i M \xrightarrow{p} \mathsf{in}_i M'}\; i \in \{\ell, r\}$$

$$\frac{M \xrightarrow{p} M'}{\mathsf{case}(M, x_\ell \cdot M_\ell, x_r \cdot M_r) \xrightarrow{p} \mathsf{case}(M', x_\ell \cdot M_\ell, x_r \cdot M_r)}$$

$$\frac{M \xrightarrow{p} M'}{\mathsf{fold}(M) \xrightarrow{p} \mathsf{fold}(M')} \qquad \frac{M \xrightarrow{p} M'}{\mathsf{unfold}(M) \xrightarrow{p} \mathsf{unfold}(M')}$$

Figure 2: Weak and Probabilistic reduction axioms and rules for $\Lambda^\mathsf{p}_\mathsf{HP}$

are weak normal terms. We then define a stochastic matrix $\mathsf{Red} \in [0,1]^{\Lambda^\mathsf{p}_\mathsf{HP} \times \Lambda^\mathsf{p}_\mathsf{HP}}$ (indexed by terms) as

$$\mathsf{Red}_{M,M'} = \begin{cases} p & \text{if } M \xrightarrow{p} M' \\ 1 & \text{if } M \text{ is weak-normal and } M' = M \\ 0 & \text{otherwise.} \end{cases}$$

Saying that Red is stochastic means that the coefficients of Red belong to $[0,1]$ and that, for any given term M, one has $\sum_{M'} \mathsf{Red}_{M,M'} = 1$ (actually there are at most two terms M' such that $\mathsf{Red}_{M,M'} \neq 0$).

For all $M, M' \in \Lambda^\mathsf{p}_\mathsf{HP}$, if M' is weak-normal then the sequence $(\mathsf{Red}^n_{M,M'})^\infty_{n=1}$ is monotone and included in $[0,1]$, and therefore has a lub that we denote as $\mathsf{Red}^\infty_{M,M'}$ which defines a sub-stochastic matrix (taking $\mathsf{Red}^\infty_{M,M'} = 0$ when M' is not weak-normal).

When M' is weak-normal, the number $p = \mathsf{Red}^\infty_{M,M'}$ is the probability that M reduces to M' after a finite number of steps.

Let us say when two closed terms M_1, M_2 of type σ are *observationally equivalent*:

$M_1 \sim M_2$, if for all closed term C of type $!\sigma \multimap \mathbf{1}$, $\mathsf{Red}^\infty_{\langle C\rangle M^!_1,()} = \mathsf{Red}^\infty_{\langle C\rangle M^!_1,()}$.

For simplicity we consider only closed terms M_1 and M_2. We could also define an observational equivalence on non closed terms, replacing the term C with a context $C[\,]$ which could bind free variables of the M_i's, this would not change the results of the paper.

2.3. **Examples.** For the sake of readability, we drop the fold/unfold constructs associated with recursive types definitions; they can easily be inserted at the right places.

Ever-looping program. Given any type σ, we define $\Omega^\sigma = \mathsf{fix}\, x^{!\sigma}\, \mathsf{der}\, x$ which satisfies $\vdash \Omega^\sigma : \sigma$. It is clear that $\Omega^\sigma \to_{\mathsf{w}} \mathsf{der}\, (\Omega^\sigma)^! \to_{\mathsf{w}} \Omega^\sigma$ so that we can consider Ω^σ as the ever-looping program of type σ.

Booleans. We define the type $o = \mathbf{1} \oplus \mathbf{1}$, so that $\mathcal{P} \vdash \mathsf{coin}(p) : o$. We define the "true" constant as $\mathbf{t} = \mathsf{in}_\ell()$ and the "false" constant as $\mathbf{f} = \mathsf{in}_r()$. The corrresponding eliminator is defined as follows. Given terms M, N_ℓ and N_r we set $\mathsf{if}(M, N_\ell, N_r) = \mathsf{case}(M, x_\ell \cdot N_\ell, x_r \cdot N_r)$ where x_i is not free in N_i for $i \in \{\ell, r\}$, so that

$$\frac{\mathcal{P} \vdash M : o \qquad \mathcal{P} \vdash N_\ell : \sigma \qquad \mathcal{P} \vdash N_r : \sigma}{\mathcal{P} \vdash \mathsf{if}(M, N_\ell, N_r) : \sigma}$$

We have the following weak and probabilistic reduction rules, derived from Figure 2:

$$\frac{}{\mathsf{if}(\mathbf{t}, N_\ell, N_r) \to_{\mathsf{w}} N_\ell} \qquad \frac{}{\mathsf{if}(\mathbf{f}, N_\ell, N_r) \to_{\mathsf{w}} N_r} \qquad \frac{M \xrightarrow{p} M'}{\mathsf{if}(M, N_\ell, N_r) \xrightarrow{p} \mathsf{if}(M', N_\ell, N_r)}$$

Natural numbers. We define the type ι of unary natural numbers by $\iota = \mathbf{1} \oplus \iota$ (by this we mean that $\iota = \mathsf{Rec}\, \zeta \cdot (\mathbf{1} \oplus \zeta)$). We define $\underline{0} = \mathsf{in}_\ell()$ and $\underline{n+1} = \mathsf{in}_r\underline{n}$ so that we have $\mathcal{P} \vdash \underline{n} : \iota$ for each $n \in \mathbb{N}$.

Then, given a term M, we define the term $\mathsf{suc}(M) = \mathsf{in}_r M$, so that we have

$$\frac{\mathcal{P} \vdash M : \iota}{\mathcal{P} \vdash \mathsf{suc}(M) : \iota}$$

Last, given terms M, N_ℓ and N_r and a variable x, we define an "ifz" conditional by $\mathsf{ifz}(M, N_\ell, x \cdot N_r) = \mathsf{case}(M, z \cdot N_\ell, x \cdot N_r)$ where z is not free in N_ℓ, so that

$$\frac{\mathcal{P} \vdash M : \iota \qquad \mathcal{P} \vdash N_\ell : \sigma \qquad \mathcal{P}, x : \iota \vdash N_r : \sigma}{\mathcal{P} \vdash \mathsf{ifz}(M, N_\ell, x \cdot N_r) : \sigma}$$

We have the following weak and probabilistic reduction rules, derived from Figure 2:

$$\frac{}{\mathsf{ifz}(\mathsf{in}_i V, M_\ell, x \cdot M_r) \to_{\mathsf{w}} M_i\,[V/x]}\; i \in \{\ell, r\} \qquad \frac{M \xrightarrow{p} M'}{\mathsf{ifz}(M, N_\ell, x \cdot N_r) \xrightarrow{p} \mathsf{ifz}(M', N_\ell, x \cdot N_r)}$$

These conditionals will be used in the examples below.

Streams. Let φ be a positive type and S_φ be the positive type defined by $\mathsf{S}_\varphi = !(\varphi \otimes \mathsf{S}_\varphi)$, that is $\mathsf{S}_\varphi = \mathsf{Rec}\, \zeta \cdot !(\varphi \otimes \zeta)$. We can define a term M such that $\vdash M : \mathsf{S}_\varphi \multimap \iota \multimap \varphi$ which computes the nth element of a stream:

$$M = \mathsf{fix}\, f^{!(\mathsf{S}_\varphi \multimap \iota \multimap \varphi)}\, \lambda x^{\mathsf{S}_\varphi}\, \lambda y^\iota\, \mathsf{ifz}(y, \mathsf{pr}_\ell(\mathsf{der}\, x), z \cdot \langle \mathsf{der}\, f \rangle \mathsf{pr}_r(\mathsf{der}\, x)\, z)$$

Let $O = (\Omega^{\varphi \otimes \mathsf{S}_\varphi})^!$, a term which represents the "undefined stream" (more precisely, it is a stream which is a value, but contains nothing, not to be confused with $\Omega^{\mathsf{S}_\varphi}$ which has the same type but is not a value). We have $\vdash O : \mathsf{S}_\varphi$, and observe that the reduction of $\langle M \rangle O$ converges (to an abstraction) and that $\langle M \rangle O\, \underline{0}$ diverges.

Conversely, we can define a term N such that $\vdash N : !(\iota \multimap \varphi) \multimap \mathsf{S}_\varphi$ which turns a function into the stream of its successive applications to an integer.

$$N = \mathsf{fix}\, F^{!(!(\iota \multimap \varphi) \multimap \mathsf{S}_\varphi)}\, \lambda f^{!(\iota \multimap \varphi)}\, \left(\langle \mathsf{der}\, f \rangle \underline{0}, \langle \mathsf{der}\, F \rangle (\lambda x^\iota\, \langle \mathsf{der}\, f \rangle \mathsf{suc}(x))^! \right)^!$$

Observe that the recursive call of F is encapsulated into a box, which makes the construction lazy. As a last example, consider the following term P such that $\vdash P :$ $(\mathsf{S}_\varphi \otimes \mathsf{S}_\varphi) \multimap (\iota \oplus \iota) \multimap \varphi$ given by

$$P = \lambda y^{\mathsf{S}_\varphi \otimes \mathsf{S}_\varphi} \, \lambda c^{\iota \oplus \iota} \, \mathsf{case}(c, x \cdot \langle M \rangle x \, \mathsf{pr}_\ell y, x \cdot \langle M \rangle x \, \mathsf{pr}_r y)$$

Take $\varphi = \mathbf{1}$ and consider the term $Q = \left(((), O)^!, O \right)$, then we have $\vdash Q : \mathsf{S}_1 \otimes \mathsf{S}_1$, and observe that $\langle P \rangle Q \, \mathsf{in}_\ell \underline{0}$ converges to () whereas $\langle P \rangle Q \, \mathsf{in}_r \underline{0}$ diverges.

These examples suggest that S_φ behaves as should behave a type of streams of elements of type φ.

Lists. There are various possibilities for defining a type of lists of elements of a positive type φ. The simplest definition is $\lambda_0 = \mathbf{1} \oplus (\varphi \otimes \lambda_0)$. This corresponds to the ordinary ML type of lists. But we can also define $\lambda_1 = \mathbf{1} \oplus {!}(\varphi \otimes \lambda_1)$ and then we have a type of lazy lists (or terminable streams) where the tail of the list is computed only when required. Here is an example of a term L such that $\vdash L : \lambda_1$, with $\varphi = o = \mathbf{1} \oplus \mathbf{1}$ which is a list of random length containing random booleans:

$$L = \mathsf{fix}\, x^{!\lambda_1}\, \mathsf{if}(\mathsf{coin}(1/4), \mathsf{in}_\ell(), \mathsf{in}_r(\mathsf{coin}(1/2) \otimes \mathsf{der}\, x)^!)$$

Then L will reduce with probability $\frac{1}{4}$ to the empty list $\mathsf{in}_\ell()$, and with probability $\frac{3}{4}$ to the list $L' = \mathsf{in}_r(\mathsf{coin}(1/2) \otimes L)^!$ (up to the equivalence of $\mathsf{der}\, L^!$ with L) which is a value. One can access this value by evaluating the term $\mathsf{case}(L', z \cdot \_, y \cdot \mathsf{der}\, y)$ where $\_$ is any term of type $o \otimes \lambda_1$ (we know that this branch will not be evaluated) and this term reduces to $(\mathbf{t}, L)$ or $(\mathbf{f}, L)$ with probability $\frac{1}{2}$. In turn each of these terms (b, L) will reduce to $(b, \mathsf{in}_\ell())$ with probability $\frac{1}{4}$ and to (b, L') with probability $\frac{3}{4}$.

We can iterate this process, defining a term R of type $\lambda_1 \multimap \lambda_0$ which evaluates completely a terminable stream to a list:

$$R = \mathsf{fix}\, f^{!(\lambda_1 \multimap \lambda_0)} \, \lambda x^{\lambda_1} \, \mathsf{case}(x, z \cdot \mathsf{in}_\ell(), z \cdot \langle \lambda y^{\mathbf{Bool} \otimes \lambda_1} \, (\mathsf{pr}_\ell y, \langle \mathsf{der}\, f \rangle \mathsf{pr}_r y) \rangle \mathsf{der}\, z)\,.$$

Then $\langle R \rangle L$, which is a closed term of type λ_0, terminates with probability 1. The expectation of the length of this "random list" is $\sum_{n=0}^\infty n(\frac{3}{4})^n = 12$.

We could also consider $\lambda_2 = \mathbf{1} \oplus ({!}\sigma \otimes \lambda_2)$ which allows to manipulate lists of objects of type σ (which can be a general type) without accessing their elements.

Probabilistic tests. If $\mathcal{P} \vdash M_i : \sigma$ for $i = 1, 2$, we set $\mathsf{dice}_p(M_1, M_2) = \mathsf{if}(\mathsf{coin}(p), M_1, M_2)$ and this term satisfies $\mathcal{P} \vdash \mathsf{dice}_p(M_1, M_2) : \sigma$. If M_i reduces to a value V_i with probability q_i, then $\mathsf{dice}_p(M_1, M_2)$ reduces to V_1 with probability $p\, q_1$ and to V_2 with probability $(1 - p)q_2$.

Let $n \in \mathbb{N}$ and let $\vec{p} = (p_0, \dots, p_n)$ be such that $p_i \in [0, 1] \cap \mathbb{Q}$ and $p_0 + \cdots + p_n \leq 1$. Then one defines a closed term $\mathsf{ran}(\vec{p})$, such that $\vdash \mathsf{ran}(\vec{p}) : \iota$, which reduces to $\underline{i}$ with probability p_i for each $i \in \{0, \dots, n\}$. The definition is by induction on n.

$$\mathsf{ran}(\vec{p}) = \begin{cases} \underline{0} & \text{if } p_0 = 1 \text{ whatever be the value of } n \\ \mathsf{if}(\mathsf{coin}(p_0), \underline{0}, \Omega^\iota) & \text{if } n = 0 \\ \mathsf{if}(\mathsf{coin}(p_0), \underline{0}, \mathsf{suc}(\mathsf{ran}(\frac{p_1}{1-p_0}, \dots, \frac{p_n}{1-p_0}))) & \text{otherwise} \end{cases}$$

As an example of use of the test to zero conditional, we define, by induction on k, a family of terms eq_k such that $\vdash \mathsf{eq}_k : \iota \multimap \mathbf{1}$ and that tests the equality to k:

$$\mathsf{eq}_0 = \lambda x^\iota \, \mathsf{ifz}(x, (), z \cdot \Omega^{\mathbf{1}}) \qquad \mathsf{eq}_{k+1} = \lambda x^\iota \, \mathsf{ifz}(x, \Omega^{\mathbf{1}}, z \cdot \langle \mathsf{eq}_k \rangle z)$$

For M such that $\vdash M : \iota$, the term $\langle \mathsf{eq}_k \rangle M$ reduces to () with a probability which is equal to the probability of M to reduce to $\underline{k}$.

2.3.1. *Notation.* Now, we introduce terms that will be used in the definition of testing terms in the proof of Full Abstraction in Section 5.

First, we define prod_k such that $\vdash \mathsf{prod}_k : \mathbf{1} \multimap \cdots \multimap \mathbf{1} \multimap \varphi \multimap \varphi$ (with k occurrences of $\mathbf{1}$):

$$\mathsf{prod}_0 = \lambda y^\varphi \, y \qquad \mathsf{prod}_{k+1} = \lambda x^{\mathbf{1}} \, \mathsf{prod}_k \,.$$

Given for each $i \in \{0, \dots, k\}$, M_i such that $\mathcal{P} \vdash M_i : \mathbf{1}$ and $\mathcal{P} \vdash N : \varphi$, the term $\langle \langle \langle \mathsf{prod}_{k+1} \rangle M_0 \cdots \rangle M_k \rangle N$ reduces to a value V with probability $p_0 \cdots p_k \, q$ where p_i is the probability of M_i to reduce to () and q is the probability of N to reduce to V. We use the notations:

$$M_0 \cdot N = \langle \langle \mathsf{prod}_1 \rangle M_0 \rangle N \qquad M_0 \wedge \cdots \wedge M_{k-1} = \left\{ \begin{array}{l} () \text{ if } k = 0 \\ \langle \langle \mathsf{prod}_k \rangle M_0 \cdots \rangle M_{k-1} \text{ otherwise,} \end{array} \right.$$

so that $\mathcal{P} \vdash M_0 \cdot N : \varphi$ and the probability that $M_0 \cdot N$ reduces to V is $p_0 \, q$ and $\mathcal{P} \vdash M_0 \wedge \cdots \wedge M_{k-1} : \mathbf{1}$ and $M_0 \wedge \cdots \wedge M_{k-1}$ reduces to () with probability $p_0 \cdots p_{k-1}$.

Given a general type σ and terms $M_0, \dots, M_{k-1}$ such that, for any $i \in \{0, \dots, k-1\}$, $\vdash M_i : \sigma$, we define close terms $\mathsf{choose}_i^\sigma(M_0, \dots, M_{k-1})$ for $i \in \{0, \dots, n-1\}$ such that $\vdash \mathsf{choose}_i^\sigma(M_0, \dots, M_{k-1}) : \iota \multimap \sigma$

$$\mathsf{choose}_0^\sigma(M_0, \dots, M_{k-1}) = \lambda z^\iota \, \Omega^\sigma$$

$$\mathsf{choose}_{i+1}^\sigma(M_0, \dots, M_{k-1}) = \lambda z^\iota \, \mathsf{ifz}(z, M_0, y \cdot \langle \mathsf{choose}_i^\sigma(M_1, \dots, M_{k-1}) \rangle y) \text{ if } i \leq k-1$$

Given a term P such that $\mathcal{P} \vdash P : \iota$ and p_i the probability of P to reduce to $\underline{i}$ for any i, the first steps of the reduction are probabilistic:

$$\forall i \in \{0, \dots, k\}, \ \langle \mathsf{choose}_{k+1}^\sigma(M_0, \dots, M_k) \rangle P \overset{p_i}{\to}{}^* \langle \mathsf{choose}_{k+1}^\varphi(M_0, \dots, M_k) \rangle \underline{i}$$

the next steps of the reduction are deterministic:

$$\langle \mathsf{choose}_{k+1}^\varphi(M_0, \dots, M_k) \rangle \underline{i} \to_{\mathsf{w}}^* M_i$$

As we will see more precisely in Paragraph 3.2.12, a term of type ι can be seen as a sub-probability distribution over $\mathbb{N}$. Given integers $0 \leq l \leq r$, we define by induction the term $\mathsf{ext}\,(l, r)$ of type $\iota \multimap \iota$:

$$\begin{array}{rcl} \mathsf{ext}\,(0, 0) & = & \lambda z^\iota \, \mathsf{ifz}(z, \underline{0}, x \cdot \Omega^\iota) \\ \forall r > 0, \ \mathsf{ext}\,(0, r) & = & \mathsf{choose}_{r+1}^\iota(\underline{0}, \dots, \underline{r}) \\ \mathsf{ext}\,(l+1, r+1) & = & \lambda z^\iota \, \mathsf{ifz}(z, \Omega^\iota, x \cdot \mathsf{suc}(\langle \mathsf{ext}\,(l, r) \rangle x)) \end{array}$$

such that if $\vdash P : \iota$, then $\langle \mathsf{ext}\,(l, r) \rangle P$ extracts the sub-probability distribution with support $\subseteq \{l, \dots, r\}$. Indeed, for any $i \in \{l, \dots, r\}$ $\langle \mathsf{ext}\,(l, r) \rangle P$ reduces to $\underline{i}$ with probability p_i where p_i is the probability of P to reduce to $\underline{i}$..

We also introduce, for $\vec{n} = (n_0, \ldots, n_k)$ a sequence of $k+1$ natural numbers, a term $\mathsf{win}_i(\vec{n})$ of type $\iota \multimap \iota$ which extracts the sub-probability distribution whose support is in the ith window of length n_i for $0 \le i < k$:

$$
\begin{aligned}
\mathsf{win}_0(\vec{n}) &= \mathsf{ext}\,(0,\, n_0 - 1) \\
\mathsf{win}_{i+1}(\vec{n}) &= \mathsf{ext}\,(n_0 + \cdots + n_i,\, n_0 + \cdots + n_i + n_{i+1} - 1)\,.
\end{aligned}
$$

2.4. **On products and recursive definitions of general types.** This section has nothing to do with probabilities, so we consider the deterministic language Λ_{HP}, which is just $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ without the $\mathsf{coin}(p)$ construct. Our Λ_{HP} general types, which are similar to Levy's CBPV computation types [22] or to the SFPL general types [23], have $\multimap$ as only type constructor. This may seem weird when one compares our language with CBPV where products and recursive definitions are possible on computation types and are used for encoding CBN functional languages such as PCF extended with products.

For keeping our presentation reasonably short, we will not consider the corresponding extensions of Λ_{HP} in this paper. Instead, we will shortly argue that such a PCF language with products can be easily encoded in our Λ_{HP}.

Another reason for not introducing a product $\&$ at the level of general types is that this would introduce a useless redundancy in the language since a type like $!\sigma_1 \otimes !\sigma_2 \multimap \tau$ would then be represented equivalently as $!(\sigma_1 \,\&\, \sigma_2) \multimap \tau$.

Concerning recursive type definitions, it is true that adding them at the level of general types would allow to define interesting types such as $\mathsf{Rec}\,\zeta \cdot (!\zeta \multimap \zeta)$, yielding a straightforward encoding of the pure lambda-calculus in our language. This goal can nevertheless be reached (admittedly in a slightly less natural way) by using the positive recursive type definition $\varphi = \mathsf{Rec}\,\zeta \cdot !(\zeta \multimap \zeta)$. A pure term t will be translated into a Λ_{HP} term t^* such that $x_1 : \varphi_1, \ldots, x_n : \varphi \vdash t^* : \varphi \multimap \varphi$, where the list $x_1, \ldots, x_n$ contains all the free variables of t. This translation is defined inductively as follows: $x^* = \mathsf{der}\,x$, $((s)\,t)^* = \mathsf{der}\,(\langle s^* \rangle (t^*)^!)$ and $(\lambda x\,s)^* = \lambda x\,(s^*)^!$. The examples we provide in Section 2.3 also show that our recursive definitions restricted to positive types allow to introduce a lot of useful datatypes (integers, lists, trees, streams etc). So we do not see any real motivations for adding recursive general type definitions (and their addition would make the proof of adequacy in Section 4 even more complicated).

Coming back to the redundancy of products, consider the following grammar of types

$$A, B, \cdots := \mathsf{nat} \mid A \Rightarrow B \mid A \times B$$

and the following language of terms

$$s, t, u, \cdots := x \mid \underline{n} \mid \mathsf{suc}\,s \mid \mathsf{pred}\,s \mid \mathsf{ifz}(s, t, u) \mid \lambda x^A\,s \mid (s)\,t \mid (s, t) \mid \mathsf{pr}_\ell\,s \mid \mathsf{pr}_r\,s \mid \mathsf{fix}\,x^A \cdot s$$

We call this languages PCF as it is a straightforward extension of the original PCF of [27].

The typing rules are described in Figure 3. A typing context is a sequence $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ where the variables are pairwise distinct.

We explain now how we interpret this simple programming language in Λ_{HP}.

Translating PCF *types.* With any type A, we associate a *finite sequence* of general types $A^* = (A_1^*, \ldots, A_n^*)$ whose length $n = \mathsf{l}(A)$ is given by: $\mathsf{l}(\mathsf{nat}) = 1$, $\mathsf{l}(A \Rightarrow B) = \mathsf{l}(B)$ and $\mathsf{l}(A \times B) = \mathsf{l}(A) + \mathsf{l}(B)$.

Given a sequence $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of general types we define $!\vec{\sigma}$ by induction: $!() = \mathbf{1}$ and $!(\sigma, \vec{\sigma}) = !\sigma \otimes !\vec{\sigma}$. Given a positive type φ and a sequence $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of general types, we define $\varphi \multimap \vec{\sigma} = (\varphi \multimap \sigma_1, \ldots, \varphi \multimap \sigma_n)$.

Using these notations, we can now define A^* as follows:

- $\mathsf{nat}^* = (\iota)$ (a one element sequence) where ι is the type of integers introduced in Section 2.3, $\iota = \mathsf{Rec}\, \zeta \cdot (\mathbf{1} \oplus \zeta)$,
- $(A \Rightarrow B)^* = !A^* \multimap B^*$,
- $(A \times B)^* = A^* \cdot B^*$ (list concatenation).

Translating PCF *terms.* Let now s be a PCF term with typing judgment[7] $\Gamma \vdash s : A$. Let $n = \mathsf{l}(A)$, we define a sequence s^* of length n such that $!\Gamma \vdash s_i^* : A_i^*$ (for $i = 1, \ldots, n$) as follows (if $\Gamma = (x_1 : C_1, \ldots, x_k : C_k)$, then $!\Gamma = (x_1 : !C_1^*, \ldots, x_k : !C_k^*)$).

If $s = x_j$, so that $A = C_j$ (for some $j \in \{1, \ldots, k\}$), let $(\sigma_1, \ldots, \sigma_n) = A^*$. Since $!A^* = !\sigma_1 \otimes (!\sigma_2 \otimes \cdots (!\sigma_{n-1} \otimes (!\sigma_n \otimes \mathbf{1})) \cdots)$ we can set $x^* = (x_1^*, \ldots, x_n^*)$ where $x_i^* = \mathsf{der}\, \mathsf{pr}_\ell \mathsf{pr}_r \mathsf{pr}_r \cdots \mathsf{pr}_r x$ (with $i - 1$ occurrences of pr_r).

We set $\underline{n}^* = \mathsf{in}_r \cdots \mathsf{in}_r \mathsf{in}_\ell()$ (n occurrences of in_r), $(\mathsf{suc}\, s)^* = \mathsf{in}_r s^*$, $(\mathsf{pred}\, s)^* = \mathsf{case}(s^*, x \cdot \mathsf{in}_\ell(), x \cdot x)$. Assume that $s = \mathsf{ifz}(t, u, v)$ with $\Gamma \vdash t : \iota$, $\Gamma \vdash u : A$ and $\Gamma \vdash v : A$ for some PCF type A. Let $l = \mathsf{l}(A)$. By inductive hypothesis we have $!\Gamma^* \vdash t^* : \iota$, $!\Gamma^* \vdash u_i^* : A_i^*$ and $!\Gamma^* \vdash v_i^* : A_i^*$ for $i = 1, \ldots, l$. So we set $s^* = (\mathsf{case}(t^*, z \cdot u_i^*, z \cdot v_i^*))_{i=1}^l$ where z is a variable which does not occur free in u or v.

Assume now that $\Gamma, x : A \vdash s : B$, we set $(\lambda x^A s)^* = (\lambda x^{!A^*} s_i^*)_{i=1}^{\mathsf{l}(B)}$. Assume that $\Gamma \vdash s : A \Rightarrow B$ and $\Gamma \vdash t : A$. Then, setting $n = \mathsf{l}(B)$, we have $!\Gamma^* \vdash s_i^* : !A^* \multimap B_i^*$ for $i = 1, \ldots, n$ and $!\Gamma^* \vdash t_j^* : A_j^*$ for $j = 1, \ldots, m$ where $m = \mathsf{l}(A)$. Then, setting

$$N = \left((t_1{}^*)^!, \left(\cdots, \left((t_{m-1}{}^*)^!, \left((t_m{}^*)^!, () \right) \right) \cdots \right) \right)$$

we have $!\Gamma^* \vdash N : !A^*$ and we set $((s)\, t)^* = (\langle s_i^* \rangle N)_{i=1}^n$. Assume that $s = (s_1, s_2)$ with $\Gamma \vdash s_i : A_i$ for $i = 1, 2$ and $\Gamma \vdash s : A_1 \times A_2$. Then we set $s^* = s_1^* \cdot s_2^*$ (list concatenation). Assume that $\Gamma \vdash s : A_1 \times A_2$, with $n_i = \mathsf{l}(A_i)$ for $i = 1, 2$. Then we set $(\mathsf{pr}_\ell s)^* = (s_1^*, \ldots, s_{n_1}^*)$ and $(\mathsf{pr}_r s)^* = (s_{n_1+1}^*, \ldots, s_{n_1+n_2}^*)$.

Last assume that $s = \mathsf{fix}\, x^A \cdot t$ with $\Gamma, x : A \vdash t : A$ so that, setting $n = \mathsf{l}(A)$, we have $!\Gamma^*, x : !A^* \vdash t_i^* : A_i^*$ for $i = 1, \ldots, n$. Let $x_1, \ldots, x_n$ be pairwise distinct fresh variables, and set

$$M_i = t_i^* \left[(x_1, \cdots (x_{n-1}, (x_n, ())) \cdots)/x \right]$$

for $i = 1, \ldots, n$. We have $!\Gamma^*, x_1 : !A_1^*, \ldots, x_n : !A_n^* \vdash M_i : A_i^*$. We are in position of applying the usual trick for encoding mutual recursive definitions using fixpoints operators. For the sake of readability, assume that $n = 2$, so we have $!\Gamma^*, x_1 : !A_1^*, x_2 : !A_2^* \vdash M_i : A_i^*$ for $i = 1, 2$. Let $N_1 = \mathsf{fix}\, x_1^{!A_1^*} M_1$ so that $!\Gamma^*, x_2 : !A_2^* \vdash N_1 : A_1^*$. Then we have $!\Gamma^*, x_2 : !A_2^* \vdash M_2 \left[N_1^!/x_1 \right] : A_2^*$. Therefore we can set $s_2^* = \mathsf{fix}\, x_2^{!A_2^*} M_2 \left[N_1^!/x_1 \right]$ and we have $!\Gamma^* \vdash s_2^* : A_2^*$. Finally we set $s_1^* = N_1 \left[(s_2^*)^!/x_2 \right]$ with $!\Gamma^* \vdash s_1^* : A_1^*$.

---

[7]So our translation depends on the typing judgment and not only on the term; this is fairly standard and can be avoided by considering typed free variables.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{}{\Gamma \vdash \underline{n} : \mathsf{nat}} \qquad \frac{\Gamma \vdash s : \mathsf{nat}}{\Gamma \vdash \mathsf{suc}\, s : \mathsf{nat}} \qquad \frac{\Gamma \vdash s : \mathsf{nat}}{\Gamma \vdash \mathsf{pred}\, s : \mathsf{nat}}$$

$$\frac{\Gamma \vdash s : \mathsf{nat} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{ifz}(s, t, u) : A} \qquad \frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x^A\, s : A \Rightarrow B}$$

$$\frac{\Gamma \vdash s : A \Rightarrow B \qquad \Gamma \vdash t : A}{\Gamma \vdash (s)\, t : B} \qquad \frac{\Gamma \vdash s_\ell : A_\ell \qquad \Gamma \vdash s_r : A_r}{\Gamma \vdash (s_\ell, s_r) : A_\ell \times A_r} \qquad \frac{\Gamma \vdash s : A_\ell \times A_r}{\Gamma \vdash \mathsf{pr}_i\, s : A_i}\, i \in \{\ell, r\}$$

$$\frac{\Gamma, x : A \vdash M : A}{\Gamma \vdash \mathsf{fix}\, x^A \cdot M : A}$$

Figure 3: Typing rules for a simple call-by-name language with products, PCF

We should check now that this translation is compatible with the operational semantics of our extended PCF language. A simple way to do so would be to choose a simple model of linear logic (for instance, the relational model) and to prove that the semantics of a PCF term is equal to the semantics of its translation in $\Lambda_{\mathsf{HP}}^{\mathsf{p}}$ stripped from its probabilistic construct $\mathsf{coin}(p)$ (interpreting tuples of types using the "additive" cartesian product &). This is a long and boring exercise.

3. Probabilistic Coherent Spaces

3.1. **Semantics of LL, in a nutshell.** The kind of denotational models we are interested in, in this paper, are those induced by a model of LL, as explained in [5]. We remind the basic definitions and notations, referring to that paper for more details.

3.1.1. *Models of Linear Logic.* A model of LL consists of the following data.

A symmetric monoidal closed category $(\mathcal{L}, \otimes, 1, \lambda, \rho, \alpha, \sigma)$ where we use simple juxtaposition $g\,f$ to denote composition of morphisms $f \in \mathcal{L}(X, Y)$ and $g \in \mathcal{L}(Y, Z)$. We use $X \multimap Y$ for the object of linear morphisms from X to Y, $\mathsf{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$ for the evaluation morphism and $\mathsf{cur} \in \mathcal{L}(Z \otimes X, Y) \to \mathcal{L}(Z, X \multimap Y)$ for the linear curryfication map. For convenience, and because it is the case in the concrete models we consider (such as Scott Semantics [5] or Probabilistic Coherent Spaces here), we assume this SMCC to be a $*$-autonomous category with dualizing object $\bot$. We use $X^\bot$ for the object $X \multimap \bot$ of $\mathcal{L}$ (the dual, or linear negation, of X).

The category $\mathcal{L}$ is cartesian with terminal object $\top$, product &, projections pr_i. By $*$-autonomy $\mathcal{L}$ is co-cartesian with initial object 0, coproduct $\oplus$ and injections in_i. By monoidal closeness of $\mathcal{L}$, the tensor product $\otimes$ distributes over the coproduct $\oplus$.

We are given a comonad $!\_ : \mathcal{L} \to \mathcal{L}$ with co-unit $\mathsf{der}_X \in \mathcal{L}(!X, X)$ (*dereliction*) and co-multiplication $\mathsf{dig}_X \in \mathcal{L}(!X, !!X)$ (*digging*) together with a strong symmetric monoidal structure (Seely isos m^0 and m^2) for the functor $!\_$, from the symmetric monoidal category $(\mathcal{L}, \&)$ to the symmetric monoidal category $(\mathcal{L}, \otimes)$ satisfying an additional coherence condition wrt. dig.

We use $?\_$ for the "De Morgan dual" of $!\_$: $?X = (!(X^\bot))^\bot$ and similarly for morphisms. It is a monad on $\mathcal{L}$.

3.1.2. *The Eilenberg-Moore category.* It is then standard to define the category $\mathcal{L}^!$ of !-coalgebras. An object of this category is a pair $P = (\underline{P}, h_P)$ where $\underline{P} \in \mathsf{Obj}(\mathcal{L})$ and $h_P \in \mathcal{L}(\underline{P}, !\underline{P})$ is such that $\mathrm{der}_{\underline{P}}\, h_P = \mathrm{Id}$ and $\mathrm{dig}_{\underline{P}}\, h_P = !h_P\, h_P$. Then $f \in \mathcal{L}^!(P, Q)$ iff $f \in \mathcal{L}(\underline{P}, \underline{Q})$ such that $h_Q\, f = !f\, h_P$. The functor $!_-$ can be seen as a functor from $\mathcal{L}$ to $\mathcal{L}^!$ mapping X to $(!X, \mathrm{dig}_X)$ and $f \in \mathcal{L}(X, Y)$ to $!f$. It is right adjoint to the forgetful functor $\mathsf{U} : \mathcal{L}^! \to \mathcal{L}$. Given $f \in \mathcal{L}(\underline{P}, X)$, we use $f^! \in \mathcal{L}^!(P, !X)$ for the morphism associated with f by this adjunction, one has $f^! = !f\, h_P$. If $g \in \mathcal{L}^!(Q, P)$, we have $f^!\, g = (f\, g)^!$.

Then $\mathcal{L}^!$ is cartesian (with product of shape $P \otimes Q = (\underline{P} \otimes \underline{Q}, h_{P \otimes Q})$ and terminal object $(1, h_1)$, still denoted as 1). This category is also co-cartesian with coproduct of shape $P \oplus Q = (\underline{P} \oplus \underline{Q}, h_{P \oplus Q})$ and initial object $(0, h_0)$ still denoted as 0. The complete definitions can be found in [5]. We use $c_P \in \mathcal{L}^!(P, P \otimes P)$ (*contraction*) for the diagonal and $w_P \in \mathcal{L}^!(P, 1)$ (*weakening*) for the unique morphism to the terminal object.

We also consider occasionally the *Kleisli category*[8] $\mathcal{L}_!$ of the comonad !: its objects are those of $\mathcal{L}$ and $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$. The identity at X in this category is der_X and composition of $f \in \mathcal{L}_!(X, Y)$ and $g \in \mathcal{L}_!(Y, Z)$ is defined as

$$g \circ f = g\, !f\, \mathrm{dig}_X \ .$$

This category is cartesian closed but this fact will not play an essential role in this work.

3.1.3. *Fixpoints.* For any object X, we assume to be given $\mathrm{fix}_X \in \mathcal{L}(!(!X \multimap X), X)$, a morphism such that[9] $\mathrm{ev}\,(\mathrm{der}_{!X \multimap X} \otimes \mathrm{fix}_X^!) \circ c_{!(!X \multimap X)} = \mathrm{fix}_X$ which will allow to interpret term fixpoints.

In order to interpret fixpoints of types, we assume that the category $\mathcal{L}$ is equipped with a notion of embedding-retraction pairs, following a standard approach. We use $\mathcal{L}_{\subseteq}$ for the corresponding category. It is equipped with a functor $\mathsf{F} : \mathcal{L}_{\subseteq} \to \mathcal{L}^{\mathsf{op}} \times \mathcal{L}$ such that $\mathsf{F}(X) = (X, X)$ and for which we use the notation $(\varphi^-, \varphi^+) = \mathsf{F}(\varphi)$ and assume that $\varphi^-\, \varphi^+ = \mathrm{Id}_X$. We assume furthermore that $\mathcal{L}_{\subseteq}$ has all countable directed colimits and that the functor $\mathsf{E} = \mathrm{pr}_2\, \mathsf{F} : \mathcal{L}_{\subseteq} \to \mathcal{L}$ is continuous. We also assume that all the basic operations on objects ($\otimes$, $\oplus$, $(\_)^\perp$ and $!_-$) are continuous functors from $\mathcal{L}_{\subseteq}$ to itself[10].

Then it is easy to carry this notion of embedding-retraction pairs to $\mathcal{L}^!$, defining a category $\mathcal{L}^!_{\subseteq}$, to show that this category has all countable directed colimits and that the functors $\otimes$ and $\oplus$ are continuous on this category: $\mathcal{L}^!_{\subseteq}(P, Q)$ is the set of all $\varphi \in \mathcal{L}_{\subseteq}(\underline{P}, \underline{Q})$ such that $\varphi^+ \in \mathcal{L}^!(P, Q)$. One checks also that ! defines a continuous functor from $\mathcal{L}_{\subseteq}$ to $\mathcal{L}^!_{\subseteq}$. This allows to interpret recursive types, more details can be found in [5].

3.1.4. *Interpreting types.* Using straightforwardly the object 1 and the operations $\otimes$, $\oplus$, ! and $\multimap$ of the model $\mathcal{L}$ as well as the completeness and continuity properties explained in Section 3.1.3, we associate with any positive type φ and any repetition-free list $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_n)$ of type variables containing all free variables of φ a continuous functor $[\varphi]^!_{\overrightarrow{\zeta}}$:

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[8]It is the full subcategory of $\mathcal{L}^!$ of free coalgebras, see any introductory text on monads and co-monads.

[9]It might seem natural to require the stronger uniformity conditions of *Conway operator* [30]. This does not seem to be necessary as far as soundness of our semantics is concerned even if the fixpoint operators arising in concrete models satisfy these further properties.

[10]This is a rough statement; one has to say for instance that if $\varphi_i \in \mathcal{L}_{\subseteq}(X_i, Y_i)$ for $i = 1, 2$ then $(\varphi_1 \otimes \varphi_2)^- = \varphi_1^- \otimes \varphi_2^-$ etc. The details can be found in [5].

$(\mathcal{L}_{\subseteq}^!)^n \to \mathcal{L}_{\subseteq}^!$ and with any general type σ and any list $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)$ of pairwise distinct type variables containing all free variables of σ we associate a continuous functor $[\sigma]_{\vec{\zeta}}$: $(\mathcal{L}_{\subseteq}^!)^n \to \mathcal{L}_{\subseteq}$.

When we write $[\sigma]$ or $[\varphi]^!$ (without subscript), we assume implicitly that the types σ and φ have no free type variables. Then $[\sigma]$ is an object of $\mathcal{L}$ and $[\varphi]^!$ is an object of $\mathcal{L}^!$. We have $[\varphi] = \underline{[\varphi]^!}$ that is, considered as a generalized type, the semantics of a positive type φ is the carrier of the coalgebra $[\varphi]^!$.

Given a typing context $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$, we define $[\mathcal{P}] = [\varphi_1]^! \otimes \cdots \otimes [\varphi_k]^! \in \mathcal{L}^!$.

In the model or probabilistic coherence spaces considered in this paper, we define $\mathcal{L}_{\subseteq}$ in such a way that the only isos are the identity maps. This implies that the types $\mathsf{Rec}\, \zeta \cdot \varphi$ and $\varphi\,[(\mathsf{Rec}\, \zeta \cdot \varphi)/\zeta]$ are interpreted as *the same object* (or functor). Such definitions of $\mathcal{L}_{\subseteq}$ are possible in many other models (relational, coherence spaces, hypercoherences etc).

We postpone the description of term interpretation because this will require constructions specific to our probabilistic semantics, in addition to the generic categorical ingredients introduced so far.

3.2. The model of probabilistic coherence spaces.

Given a countable set I and $u, u' \in (\mathbb{R}^+)^I$, we set $\langle u, u' \rangle = \sum_{i \in I} u_i u_i'$. Given $\mathcal{F} \subseteq (\mathbb{R}^+)^I$, we set $\mathcal{F}^\perp = \{u' \in (\mathbb{R}^+)^I \mid \forall u \in \mathcal{F}\ \langle u, u' \rangle \leq 1\}$.

A *probabilistic coherence space* (PCS) is a pair $X = (|X|, \mathsf{P}X)$ where $|X|$ is a countable set and $\mathsf{P}X \subseteq (\mathbb{R}^+)^{|X|}$ satisfies

- $\mathsf{P}X^{\perp\perp} = \mathsf{P}X$ (equivalently, $\mathsf{P}X^{\perp\perp} \subseteq \mathsf{P}X$),
- for each $a \in |X|$ there exists $u \in \mathsf{P}X$ such that $u_a > 0$,
- for each $a \in |X|$ there exists $A > 0$ such that $\forall u \in \mathsf{P}X\ u_a \leq A$.

If only the first of these conditions holds, we say that X is a *pre-probabilistic coherence space* (pre-PCS).

The purpose of the second and third conditions is to prevent infinite coefficients to appear in the semantics. This property in turn will be essential for guaranteeing the morphisms interpreting proofs to be analytic functions, which will be the key property to prove full abstraction. So these conditions, though aesthetic at first sight, are important for our ultimate goal.

Lemma 1. *Let X be a pre-PCS. The following conditions are equivalent:*

- *X is a PCS,*
- *$\forall a \in |X|\ \exists u \in \mathsf{P}X\ \exists u' \in \mathsf{P}X^\perp\ u_a > 0$ and $u_a' > 0$,*
- *$\forall a \in |X|\ \exists A > 0\ \forall u \in \mathsf{P}X\ \forall u' \in \mathsf{P}X^\perp\ u_a \leq A$ and $u_a' \leq A$.*

The proof is straightforward.

We equip $\mathsf{P}X$ with the most obvious partial order relation: $u \leq v$ if $\forall a \in |X|\ u_a \leq v_a$ (using the usual order relation on $\mathbb{R}$).

Theorem 2. *$\mathsf{P}X$ is an ω-continuous domain. Given $u, v \in \mathsf{P}X$ and $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha + \beta \leq 1$, one has $\alpha u + \beta v \in \mathsf{P}X$.*

This is an easy consequence of the hypothesis $\mathsf{P}X^{\perp\perp} \subseteq \mathsf{P}X$. See [4] for details; from this result, we will only use the closure properties: $\mathsf{P}X$ is closed under sub-probabilistic linear combinations and under lubs of monotonic sequences. Though the ω-continuity property

(and the associated way-below relation) does not play any technical role, it is an intuitively satisfactory fact[11] which means that the "size" of our domains remains bounded.

3.2.1. *Morphisms of PCSs.* Let X and Y be PCSs. Let $t \in (\mathbb{R}^+)^{|X| \times |Y|}$ (to be understood as a matrix). Given $u \in \mathsf{P}X$, we define $t\,u \in \overline{\mathbb{R}^+}^{|Y|}$ by $(t\,u)_b = \sum_{a \in |X|} t_{a,b} u_a$ (application of the matrix t to the vector u)[12]. We say that t is a *(linear) morphism* from X to Y if $\forall u \in \mathsf{P}X\ t\,u \in \mathsf{P}Y$, that is

$$\forall u \in \mathsf{P}X\ \forall v' \in \mathsf{P}Y^\perp \qquad \sum_{(a,b) \in |X| \times |Y|} t_{a,b} u_a v'_b \leq 1\,.$$

The diagonal matrix $\mathrm{Id} \in (\mathbb{R}^+)^{|X| \times |X|}$ given by $\mathrm{Id}_{a,b} = 1$ if $a = b$ and $\mathrm{Id}_{a,b} = 0$ otherwise is a morphism. In that way we have defined a category **Pcoh** whose objects are the PCSs and whose morphisms have just been defined. Composition of morphisms is defined as matrix multiplication: let $s \in \mathbf{Pcoh}(X, Y)$ and $t \in \mathbf{Pcoh}(Y, Z)$, we define $t\,s \in (\mathbb{R}^+)^{|X| \times |Z|}$ by

$$(t\,s)_{a,c} = \sum_{b \in |Y|} s_{a,b} t_{b,c}$$

and a simple computation shows that $t\,s \in \mathbf{Pcoh}(X, Z)$. More precisely, we use the fact that, given $u \in \mathsf{P}X$, one has $(t\,s)\,u = t\,(s\,u)$. Associativity of composition holds because matrix multiplication is associative. Id_X is the identity morphism at X.

Given $u \in \mathsf{P}X$, we define $\|u\|_X = \sup\{\langle u, u' \rangle \mid u' \in \mathsf{P}X^\perp\}$. By definition, we have $\|u\|_X \in [0, 1]$.

3.2.2. *Multiplicative constructs.* One sets $X^\perp = (|X|, \mathsf{P}X^\perp)$. It results straightforwardly from the definition of PCSs that $X^\perp$ is a PCS. Given $t \in \mathbf{Pcoh}(X, Y)$, one has $t^\perp \in \mathbf{Pcoh}(Y^\perp, X^\perp)$ if $t^\perp$ is the transpose of t, that is $(t^\perp)_{b,a} = t_{a,b}$.

One defines $X \otimes Y$ by $|X \otimes Y| = |X| \times |Y|$ and

$$\mathsf{P}(X \otimes Y) = \{u \otimes v \mid u \in \mathsf{P}X \text{ and } v \in \mathsf{P}Y\}^{\perp\perp}$$

where $(u \otimes v)_{(a,b)} = u_a v_b$. Then $X \otimes Y$ is a pre-PCS.

We have

$$\mathsf{P}(X \otimes Y^\perp)^\perp = \{u \otimes v' \mid u \in \mathsf{P}X \text{ and } v' \in \mathsf{P}Y^\perp\}^\perp = \mathbf{Pcoh}(X, Y)\,.$$

It follows that $X \multimap Y = (X \otimes Y^\perp)^\perp$ is a pre-PCS. Let $(a, b) \in |X| \times |Y|$. Since X and $Y^\perp$ are PCSs, there is $A > 0$ such that $u_a v'_b < A$ for all $u \in \mathsf{P}X$ and $v' \in \mathsf{P}Y^\perp$. Let $t \in (\mathbb{R}^+)^{|X \multimap Y|}$ be such that $t_{(a',b')} = 0$ for $(a', b') \neq (a, b)$ and $t_{(a,b)} = 1/A$, we have $t \in \mathsf{P}(X \multimap Y)$. This shows that $\exists t \in \mathsf{P}(X \multimap Y)$ such that $t_{(a,b)} > 0$. Similarly we can find $u \in \mathsf{P}X$ and $v' \in \mathsf{P}Y^\perp$ such that $\varepsilon = u_a v'_b > 0$. It follows that $\forall t \in \mathsf{P}(X \multimap Y)$ one has $t_{(a,b)} \leq 1/\varepsilon$. We conclude that $X \multimap Y$ is a PCS, and therefore $X \otimes Y$ is also a PCS.

Lemma 3. *Let X and Y be PCSs. One has $\mathsf{P}(X \multimap Y) = \mathbf{Pcoh}(X, Y)$. That is, given $t \in (\mathbb{R}^+)^{|X| \times |Y|}$, one has $t \in \mathsf{P}(X \multimap Y)$ iff for all $u \in \mathsf{P}X$, one has $t\,u \in \mathsf{P}Y$.*

This results immediately from the definition above of $X \multimap Y$.

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[11]similar to separability for topological vector spaces

[12]This is an unordered sum, which is infinite in general. It makes sense because all its terms are ≥ 0.

Lemma 4. *Let X_1, X_2 and Y be PCSs. Let $t \in (\mathbb{R}^+)^{|X_1 \otimes X_2 \multimap Y|}$. One has $t \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$ iff for all $u_1 \in \mathsf{P}X_1$ and $u_2 \in \mathsf{P}X_2$ one has $t\,(u_1 \otimes u_2) \in \mathsf{P}Y$.*

Proof. The condition stated by the lemma is clearly necessary. Let us prove that it is sufficient: under this condition, it suffices to prove that
$$t^\perp \in \mathbf{Pcoh}(Y^\perp, (X_1 \otimes X_2)^\perp)\,.$$
Let $v' \in \mathsf{P}Y^\perp$, it suffices to prove that $t^\perp v' \in \mathsf{P}(X_1 \otimes X_2)^\perp$. So let $u_1 \in \mathsf{P}X_1$ and $u_2 \in \mathsf{P}X_2$, it suffices to prove that $\langle t^\perp v', u_1 \otimes u_2 \rangle \le 1$, that is $\langle t\,(u_1 \otimes u_2), v' \rangle \le 1$, which follows from our assumption. $\square$

Let $s_i \in \mathbf{Pcoh}(X_i, Y_i)$ for $i = 1, 2$. Then one defines
$$s_1 \otimes s_2 \in (\mathbb{R}^+)^{|X_1 \otimes X_2 \multimap Y_1 \otimes Y_2|}$$
by $(s_1 \otimes s_2)_{((a_1,a_2),(b_1,b_2))} = (s_1)_{(a_1,b_1)}(s_2)_{(a_2,b_2)}$ and one must check that
$$s_1 \otimes s_2 \in \mathbf{Pcoh}(X_1 \otimes X_2, Y_1 \otimes Y_2)\,.$$
This follows directly from Lemma 4. Let $1 = (\{*\}, [0,1])$. There are obvious choices of natural isomorphisms
$$\lambda_X \in \mathbf{Pcoh}(1 \otimes X, X)$$
$$\rho_X \in \mathbf{Pcoh}(X \otimes 1, X)$$
$$\alpha_{X_1,X_2,X_3} \in \mathbf{Pcoh}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$$
$$\sigma_{X_1,X_2} \in \mathbf{Pcoh}(X_1 \otimes X_2, X_2 \otimes X_1)$$
which satisfy the standard coherence properties. This shows that $(\mathbf{Pcoh}, 1, \lambda, \rho, \alpha, \sigma)$ is a symmetric monoidal category.

3.2.3. *Internal linear hom.* Given PCSs X and Y, let us define $\mathrm{ev} \in (\mathbb{R}^+)^{|(X \multimap Y) \otimes X \multimap Y|}$ by
$$\mathrm{ev}_{(((a',b'),a),b)} = \begin{cases} 1 & \text{if } (a,b) = (a',b') \\ 0 & \text{otherwise.} \end{cases}$$
Then it is easy to see that $(X \multimap Y, \mathrm{ev})$ is an internal linear hom object in $\mathbf{Pcoh}$, showing that this SMCC is closed. If $t \in \mathbf{Pcoh}(Z \otimes X, Y)$, the corresponding linearly curryfied morphism $\mathrm{cur}(t) \in \mathbf{Pcoh}(Z, X \multimap Y)$ is given by $\mathrm{cur}(t)_{(c,(a,b))} = t_{((c,a),b)}$.

3.2.4. $*$-*autonomy.* Take $\perp = 1$, then one checks readily that $(\mathbf{Pcoh}, 1, \lambda, \rho, \alpha, \sigma, \perp)$ is a $*$-autonomous category. The duality functor $X \mapsto (X \multimap \perp)$ can be identified with the strictly involutive contravariant functor $X \mapsto X^\perp$.

3.2.5. *Additives.* Let $(X_i)_{i \in I}$ be a countable family of PCSs. We define a PCS $\&_{i \in I} X_i$ by $|\&_{i \in I} X_i| = \bigcup_{i \in I} \{i\} \times |X_i|$ and $u \in \mathsf{P}(\&_{i \in I} X_i)$ if, for all $i \in I$, the family $u(i) \in (\mathbb{R}^+)^{|X_i|}$ defined by $u(i)_a = u_{(i,a)}$ belongs to $\mathsf{P}X_i$.

Lemma 5. *Let $u' \in (\mathbb{R}^+)^{|\&_{i \in I} X_i|}$. One has $u' \in \mathsf{P}(\&_{i \in I} X_i)^{\perp}$ iff*
- $\forall i \in I \; u'(i) \in \mathsf{P}X_i^{\perp}$
- *and $\sum_{i \in I} \|u'(i)\|_{X_i^{\perp}} \leq 1$.*

The proof is quite easy. It follows that $\&_{i \in I} X_i$ is a PCS. Moreover we can define $\mathrm{pr}_i \in \mathbf{Pcoh}(\&_{j \in I} X_j, X_i)$ by

$$(\mathrm{pr}_i)_{(j,a),a'} = \begin{cases} 1 & \text{if } j = i \text{ and } a = a' \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\&_{i \in I} X_i, (\mathrm{pr}_i)_{i \in I})$ is the cartesian product of the family $(X_i)_{i \in I}$ in the category $\mathbf{Pcoh}$. The coproduct $(\oplus_{i \in I} X_i, (\mathrm{in}_i)_{i \in I})$ is the dual operation, so that

$$|\underset{i \in I}{\oplus} X_i| = \bigcup_{i \in I} \{i\} \times |X_i|$$

and $u \in \mathsf{P}(\oplus_{i \in I} X_i)$ if $\forall i \in I \; u(i) \in \mathsf{P}X_i$ and $\sum_{i \in I} \|u(i)\|_{X_i} \leq 1$. The injections $\mathrm{in}_j \in \mathbf{Pcoh}(X_j, \oplus_{i \in I} X_i)$ are given by

$$(\mathrm{in}_i)_{a',(j,a)} = \begin{cases} 1 & \text{if } j = i \text{ and } a = a' \\ 0 & \text{otherwise.} \end{cases}$$

Given morphisms $s_i \in \mathbf{Pcoh}(X_i, Y)$ (for each $i \in I$), then the unique morphism $s \in \mathbf{Pcoh}(\oplus_{i \in I} X_i, Y)$ is given by $s_{(i,a),b} = (s_i)_{a,b}$ and denoted as $\mathrm{case}_{i \in I} s_i$ (in the binary case, we use $\mathrm{case}(s_1, s_2)$).

3.2.6. *Exponentials.* Given a set I, a *finite multiset* of elements of I is a function $b : I \to \mathbb{N}$ whose *support* $\mathsf{supp}(b) = \{a \in I \mid b(a) \neq 0\}$ is finite. We use $\mathcal{M}_{\mathrm{fin}}(I)$ for the set of all finite multisets of elements of I. Given a finite family $a_1, \ldots, a_n$ of elements of I, we use $[a_1, \ldots, a_n]$ for the multiset b such that $b(a) = \#\{i \mid a_i = a\}$. We use additive notations for multiset unions: $\sum_{i=1}^{k} b_i$ is the multiset b such that $b(a) = \sum_{i=1}^{k} b_i(a)$. The empty multiset is denoted as 0 or $[]$. If $k \in \mathbb{N}$, the multiset kb maps a to $k\,b(a)$.

Let X be a PCS. Given $u \in \mathsf{P}X$ and $b \in \mathcal{M}_{\mathrm{fin}}(|X|)$, we define $u^b = \prod_{a \in |X|} u_a^{b(a)} \in \mathbb{R}^+$. Then we set $u^! = (u^b)_{b \in \mathcal{M}_{\mathrm{fin}}(|X|)}$ and finally

$$!X = (\mathcal{M}_{\mathrm{fin}}(|X|), \{u^! \mid u \in \mathsf{P}X\}^{\perp\perp})$$

which is a pre-PCS.

We check quickly that $!X$ so defined is a PCS. Let $b = [a_1, \ldots, a_n] \in \mathcal{M}_{\mathrm{fin}}(|X|)$. Because X is a PCS, and by Theorem 2, for each $i = 1, \ldots, n$ there is $u(i) \in \mathsf{P}X$ such that $u(i)_{a_i} > 0$. Let $(\alpha_i)_{i=1}^{n}$ be a family of strictly positive real numbers such that $\sum_{i=1}^{n} \alpha_i \leq 1$. Then $u = \sum_{i=1}^{n} \alpha_i u(i) \in \mathsf{P}X$ satisfies $u_{a_i} > 0$ for each $i = 1, \ldots, n$. Therefore $u_b^! = u^b > 0$. This shows that there is $U \in \mathsf{P}(!X)$ such that $U_b > 0$.

Let now $A \in \mathbb{R}^+$ be such that $\forall u \in \mathsf{P}X \; \forall i \in \{1, \ldots, n\} \; u_{a_i} \leq A$. For all $u \in \mathsf{P}X$ we have $u^b \leq A^n$. We have

$$(\mathsf{P}(!X))^{\perp} = \{u^! \mid u \in \mathsf{P}X\}^{\perp\perp\perp} = \{u^! \mid u \in \mathsf{P}X\}^{\perp}.$$

Let $t \in (\mathbb{R}^+)^{|!X|}$ be defined by $t_c = 0$ if $c \neq b$ and $t_b = A^{-n} > 0$; we have $t \in (\mathsf{P}(!X))^{\perp}$. We have exhibited an element t of $(\mathsf{P}(!X))^{\perp}$ such that $t_b > 0$. By Lemma 1 it follows that $!X$ is a PCS.

3.2.7. *Kleisli morphisms as functions.* Let $s \in (\mathbb{R}^+)^{|!X \multimap Y|}$. We define a function $\widehat{s} : \mathsf{P}X \to \overline{\mathbb{R}^+}^{|Y|}$ as follows. Given $u \in \mathsf{P}X$, we set

$$\widehat{s}(u) = s \, u^! = \left(\sum_{c \in |!X|} s_{c,b} u^c \right)_{b \in |Y|} .$$

Theorem 6. *One has $s \in \mathsf{P}(!X \multimap Y)$ iff, for all $u \in \mathsf{P}X$, one has $\widehat{s}(u) \in \mathsf{P}Y$.*

This is an immediate consequence of the definition.

Theorem 7. *Let $s \in \mathbf{Pcoh}(!X, Y)$. The function $\widehat{s}$ is Scott-continuous. Moreover, given $s, s' \in \mathbf{Pcoh}(!X, Y)$, one has $s = s'$ (as matrices) iff $\widehat{s} = \widehat{s'}$ (as functions $\mathsf{P}X \to \mathsf{P}Y$).*

This is an easy consequence of the fact that two polynomials of n variables with real coefficients are identical iff they are the same function on any open subset of $\mathbb{R}^n$.

Terminology. We say that $s \in \mathsf{P}(!X \multimap Y)$ is a *power series* whose monomial u^c has coefficient $s_{c,b}$. Since s is characterized by the function $\widehat{s}$ we sometimes say that $\widehat{s}$ is a power series.

We can consider the elements of $\mathbf{Pcoh}_!(X, Y)$ (the morphisms of the Kleisli category of the comonad $!\_$ on the category $\mathbf{Pcoh}$) as particular Scott continuous functions $\mathsf{P}X \to \mathsf{P}Y$ and this identification is compatible with the definition of identity maps and of composition in $\mathbf{Pcoh}_!$, see Section 3.1.2. Of course, not all Scott continuous function are morphisms in $\mathbf{Pcoh}_!$.

Theorem 8. *Let $s, s' \in \mathbf{Pcoh}_!(X, Y)$ be such that $s \leq s'$ (as elements of $\mathsf{P}(!X \multimap Y)$). Then $\forall u \in \mathsf{P}X \ \widehat{s}(u) \leq \widehat{s'}(u)$. Let $(s(i))_{i \in \mathbb{N}}$ be a monotone sequence of elements of $\mathbf{Pcoh}_!(X, Y)$ and let $s = \sup_{i \in \mathbb{N}} s(i)$. Then $\forall u \in \mathsf{P}X \ \widehat{s}(u) = \sup_{i \in I} \widehat{s_i}(u)$.*

The first statement is obvious. The second one results from the monotone convergence Theorem.

Given a multiset $b \in \mathcal{M}_{\mathrm{fin}}(I)$, we define its *factorial* $b! = \prod_{i \in I} b(i)!$ and its *multinomial coefficient* $\mathsf{mn}(b) = (\#b)!/b! \in \mathbb{N}^+$ where $\#b = \sum_{i \in I} b(i)$ is the cardinality of b. Remember that, given an I-indexed family $a = (a_i)_{i \in I}$ of elements of a commutative semi-ring, one has the multinomial formula

$$\left(\sum_{i \in I} a_i \right)^n = \sum_{b \in \mathcal{M}_n(I)} \mathsf{mn}(b) a^b$$

where $\mathcal{M}_n(I) = \{ b \in \mathcal{M}_{\mathrm{fin}}(I) \mid \#b = n \}$.

Given $c \in |!X|$ and $d \in |!Y|$ we define $\mathsf{L}(c, d)$ as the set of all multisets r in $\mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ such that

$$\forall a \in |X| \ \sum_{b \in |Y|} r(a, b) = c(a) \quad \text{and} \quad \forall b \in |Y| \ \sum_{a \in |X|} r(a, b) = d(b) .$$

Let $t \in \mathbf{Pcoh}(X, Y)$, we define $!t \in (\mathbb{R}^+)^{!X \multimap !Y}$ by

$$(!t)_{c,d} = \sum_{r \in \mathsf{L}(c,d)} \frac{d!}{r!} t^r \,.$$

Observe that the coefficients in this sum are all non-negative integers.

Lemma 9. *For all $u \in \mathsf{P}X$ one has $!t\, u^! = (t\, u)^!$.*

This results from a simple computation applying the multinomial formula.

Theorem 10. *For all $t \in \mathbf{Pcoh}(X, Y)$ one has $!t \in \mathbf{Pcoh}(!X, !Y)$ and the operation $t \mapsto !t$ is functorial.*

Immediate consequences of Lemma 9 and Theorem 7.

3.2.8. *Description of the exponential comonad.* We equip now this functor with a structure of comonad: let $\mathrm{der}_X \in (\mathbb{R}^+)^{|!X \multimap X|}$ be given by $(\mathrm{der}_X)_{b,a} = \delta_{[a],b}$ (the value of the Kronecker symbol $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise) and $\mathrm{dig}_X \in (\mathbb{R}^+)^{|!X \multimap !!X|}$ be given by $(\mathrm{dig}_X)_{b,[b_1,\ldots,b_n]} = \delta_{\sum_{i=1}^n b_i, b}$. Then we have $\mathrm{der}_X \in \mathbf{Pcoh}(!X, X)$ and $\mathrm{dig}_X \in \mathbf{Pcoh}(!X, !!X)$ simply because

$$\widehat{\mathrm{der}_X}(u) = u \quad \text{and} \quad \widehat{\mathrm{dig}_X}(u) = (u^!)^!$$

for all $u \in \mathsf{P}X$, as easily checked. Using these equations, one also checks easily the naturality of these morphisms, and the fact that $(!\_, \mathrm{der}, \mathrm{dig})$ is a comonad.

As to the monoidality of this comonad, we introduce $\mathrm{m}^0 \in (\mathbb{R}^+)^{|1 \multimap !\top|}$ by $\mathrm{m}^0_{*,[]} = 1$ and $\mathrm{m}^2_{X,Y} \in (\mathbb{R}^+)^{|!X \otimes !Y \multimap !(X \& Y)|}$ by $(\mathrm{m}^2_{X,Y})_{b,c,d} = \delta_{d, 1 \cdot b + 2 \cdot c}$ where $i \cdot [a_1, \ldots, a_n] = [(i, a_1), \ldots, (i, a_n)]$. It is easily checked that the required commutations hold (again, we refer to [24]).

3.2.9. *Fixpoints in $\mathbf{Pcoh}_!$.* For any object Y of $\mathbf{Pcoh}$, a morphism $t \in \mathbf{Pcoh}_!(Y, Y)$ defines a Scott-continuous function $f = \widehat{t} \colon \mathsf{P}(Y) \to \mathsf{P}(Y)$ which has a least fixpoint $\sup_{n \in \mathbb{N}} f^n(0)$. Let X be an object of $\mathbf{Pcoh}$ and set $Y = !(!X \multimap X) \multimap X$. Then we have a morphism $t = \mathrm{cur}\, s \in \mathbf{Pcoh}_!(Y, Y)$ where $s \in \mathbf{Pcoh}(!Y \otimes !(!X \multimap X), X)$ is defined as the following composition of morphisms in $\mathbf{Pcoh}$:

$$
\begin{array}{ccc}
!Y \otimes !(!X \multimap X) & \xrightarrow{\ !Y \otimes \mathrm{c}_{!X \multimap X}\ } & !Y \otimes !(!X \multimap X) \otimes !(!X \multimap X) \\
 & & \big\downarrow {\scriptstyle (\mathrm{ev}\,(\mathrm{der}_Y \otimes\, !(!X \multimap X)))^! \otimes\, \mathrm{der}_{(!X \multimap X)}} \\
 {\scriptstyle s} & & !X \otimes (!X \multimap X) \\
 & & \big\downarrow {\scriptstyle \mathrm{ev}\,\sigma} \\
 & & X
\end{array}
$$

Then $\widehat{t}$ is a Scott continuous function $\mathsf{P}Y \to \mathsf{P}Y$ whose least fixpoint is fix, considered as a morphism $\mathrm{fix} \in \mathbf{Pcoh}_!(!X \multimap X, X)$, satisfies $\widehat{\mathrm{fix}}(u) = \sup_{n=0}^\infty \widehat{u}^n(0)$.

3.2.10. *The partially ordered class of probabilistic coherence spaces.* We define the category **Pcoh**$_\subseteq$. This category is actually a partially ordered class whose objects are those of **Pcoh**. The order relation, denoted as $\subseteq$, is defined as follows: $X \subseteq Y$ if $|X| \subseteq |Y|$ and the matrices $\eta^+_{X,Y}$ and $\eta^-_{X,Y}$ defined, for $a \in |X|$ and $b \in |Y|$, by $(\eta^+_{X,Y})_{a,b} = (\eta^-_{X,Y})_{b,a} = \delta_{a,b}$ satisfy $\eta^+_{X,Y} \in \mathbf{Pcoh}(X,Y)$ and $\eta^-_{X,Y} \in \mathbf{Pcoh}(Y,X)$. In other words: given $u \in \mathsf{P}X$, the element $\eta^+_{X,Y} u$ of $(\mathbb{R}^+)^{|Y|}$ obtained by extending u with 0's outside $|X|$ belongs to $\mathsf{P}Y$. And conversely, given $v \in \mathsf{P}Y$, the element $\eta^-_{X,Y} v$ of $(\mathbb{R}^+)^{|X|}$ obtained by restricting v to $|X|$ belongs to $\mathsf{P}X$. Considering **Pcoh**$_\subseteq$ as a category, $\eta_{X,Y}$ is a notation for the unique element of **Pcoh**$_\subseteq(X,Y)$ when $X \subseteq Y$, in accordance with the notations of Paragraph 3.1.3.

Lemma 11. *If $X \subseteq Y$ then $X^\perp \subseteq Y^\perp$, $\eta^+_{X^\perp,Y^\perp} = (\eta^-_{X,Y})^\perp$ and $\eta^-_{X^\perp,Y^\perp} = (\eta^+_{X,Y})^\perp$.*

The proof is a straightforward verification.

We contend that **Pcoh**$_\subseteq$ is directed co-complete. Let $(X_\gamma)_{\gamma \in \Gamma}$ be a countable directed family in **Pcoh**$_\subseteq$ (so Γ is a countable directed poset and $\gamma \le \gamma' \Rightarrow X_\gamma \subseteq X_{\gamma'}$), we have to check that this family has a least upper bound X. We set $|X| = \bigcup_{\gamma \in \Gamma} |X_\gamma|$ and $\mathsf{P}X = \{w \in (\mathbb{R}^+)^{|X|} \mid \forall \gamma \in \Gamma \; \eta^-_{X,Y} w \in \mathsf{P}X_\gamma\}$. This defines an object of **Pcoh** which satisfies $\mathsf{P}X = \{\eta^+_{X_\gamma,X} u \mid \gamma \in \Gamma \text{ and } u \in \mathsf{P}X_\gamma\}^{\perp\perp}$ and is therefore the lub of the family $(X_\gamma)_{\gamma \in \Gamma}$ in **Pcoh**$_\subseteq$. This object X is denoted $\bigcup_{\gamma \in \Gamma} X_\gamma$. One checks easily that $(\bigcup_{\gamma \in \Gamma} X_\gamma)^\perp = \bigcup_{\gamma \in \Gamma} X_\gamma^\perp$.

Then the functor $\mathsf{E} : \mathbf{Pcoh}_\subseteq \to \mathbf{Pcoh}$ defined by $\mathsf{E}(X) = X$ and $\mathsf{E}(\eta_{X,Y}) = \eta^+_{X,Y}$ is continuous: given a directed family $(X_\gamma)_{\gamma \in \Gamma}$ whose lub is X and given a collection of morphisms $t_\gamma \in \mathbf{Pcoh}(X_\gamma, Y)$ such that $t_{\gamma'} \eta^+_{X_\gamma,X_{\gamma'}} = t_\gamma$ for any $\gamma, \gamma' \in \Gamma$ such that $\gamma \le \gamma'$, there is exactly one morphism $t \in \mathbf{Pcoh}(X,Y)$ such that $t \eta^+_{X_\gamma,X} = t_\gamma$ for each $\gamma \in \Gamma$. Given $a \in |X|$ and $b \in |Y|$, $t_{a,b} = (t_\gamma)_{a,b}$ for any γ such that $a \in |X_\gamma|$ (our hypothesis on the t_γ's means that $(t_\gamma)_{a,b}$ does not depend on the choice of γ).

All the operations of Linear Logic define monotone continuous functionals on **Pcoh**$_\subseteq$ which moreover commute with the functor F. This means for instance that if $X \subseteq Y$ then $!X \subseteq !Y$, $\eta^+_{!X,!Y} = !(\eta^+_{X,Y})$, $\eta^-_{!X,!Y} = !(\eta^-_{X,Y})$ and $!(\bigcup_{\gamma \in \Gamma} X_\gamma) = \bigcup_{\gamma \in \Gamma} !X_\gamma$ and similarly for $\otimes$ and $\oplus$. As a consequence, and as a consequence of Lemma 11, if $X_i \subseteq Y_i$ for $i = 1, 2$ then $X_1 \multimap X_2 \subseteq Y_1 \multimap Y_2$, $\eta^+_{X_1 \multimap X_2, Y_1 \multimap Y_2} = \eta^-_{X_1,Y_1} \multimap \eta^+_{X_1,Y_1}$ and $\eta^-_{X_1 \multimap X_2, Y_1 \multimap Y_2} = \eta^+_{X_1,Y_1} \multimap \eta^-_{X_1,Y_1}$ and $\multimap$ commutes with directed colimits in **Pcoh**$_\subseteq$.

This notion of inclusion on probabilistic coherence spaces extends to coalgebras as outlined in Section 3.1.3 (again, we refer to [5] for more details). We describe briefly this notion of inclusion in the present concrete setting.

Let P and Q be object of **Pcoh**$^!$, we have $P \subseteq Q$ in **Pcoh**$^!_\subseteq$ if $\underline{P} \subseteq \underline{Q}$ and $\mathrm{h}_Q \, \eta^+_{\underline{P},\underline{Q}} = !(\eta^+_{\underline{P},\underline{Q}}) \, \mathrm{h}_P$. The lub of a directed family $(P_\gamma)_{\gamma \in \Gamma}$ of coalgebras (for this notion of substructure) is the coalgebra $P = \bigcup_{\gamma \in \Gamma} P_\gamma$ defined by $\underline{P} = \bigcup_{\gamma \in \Gamma} \underline{P_\gamma}$ and h_P is characterized by the equation $\mathrm{h}_P \, \eta^+_{\underline{P_\gamma},\underline{P}} = !\eta^+_{\underline{P_\gamma},\underline{P}} \, \mathrm{h}_{P_\gamma}$ which holds for each $\gamma \in \Gamma$.

As outlined in Section 3.1.4, this allows to interpret any type σ as an object $[\sigma]$ of **Pcoh** and any positive type φ as an object $[\varphi]^!$ such that $\underline{[\varphi]^!} = [\varphi]$, in such a way that the coalgebras $[\mathsf{Rec}\,\zeta \cdot \varphi]^!$ and $[\varphi\,[\mathsf{Rec}\,\zeta \cdot \varphi/\varphi]]^!$ are *exactly the same objects of* **Pcoh**$^!$. We use h_φ for $\mathrm{h}_{[\varphi]^!}$.

3.2.11. *Dense coalgebras.* Let P be an object of $\mathbf{Pcoh}^!$, so that $P = (\underline{P}, \mathrm{h}_P)$ where $\underline{P}$ is a probabilistic coherence space and $\mathrm{h}_P \in \mathbf{Pcoh}(\underline{P}, !\underline{P})$ satisfies $\mathrm{dig}_{\underline{P}} \mathrm{h}_P = !\mathrm{h}_P \mathrm{h}_P$. Given coalgebras P and Q, a morphism $t \in \mathbf{Pcoh}(\underline{P}, \underline{Q})$ is coalgebraic (that is $t \in \mathbf{Pcoh}^!(P, Q)$) if $\mathrm{h}_Q t = !t \mathrm{h}_P$. In particular, we say that $u \in \mathsf{P}(\underline{P})$ is coalgebraic if, considered as a morphism from 1 to $\underline{P}$, u belongs to $\mathbf{Pcoh}^!(1, P)$. This means that $u^! = \mathrm{h}_P u$.

Definition 12. Given an object P of $\mathbf{Pcoh}^!$, we use $\mathsf{P}^!(P)$ for the set of coalgebraic elements of $\mathsf{P}(\underline{P})$.

The following lemma is useful in the sequel and holds in any model of Linear Logic.

Lemma 13. *Let X be a probabilistic coherence space, one has $\mathsf{P}^!(!X) = \{u^! \mid u \in \mathsf{P}X\}$. Let P_ℓ and P_r be objects of $\mathbf{Pcoh}^!$.*

$P_\ell \otimes P_r$ is the cartesian product of P_ℓ and P_r in $\mathbf{Pcoh}^!$. The function $\mathsf{P}^!(P_\ell) \times \mathsf{P}^!(P_r) \to \mathsf{P}^!(P_\ell \otimes P_r)$ which maps (u, v) to $u \otimes v$ is a bijection. The projections $\mathrm{pr}_i^\otimes \in \mathbf{Pcoh}^!(P_\ell \otimes P_r, P_i)$ are characterized by $\mathrm{pr}_i^\otimes(u_\ell \otimes u_r) = u_i$.

The function $\{\ell\} \times \mathsf{P}^!(P_\ell) \cup \{r\} \times \mathsf{P}^!(P_r) \to \mathsf{P}^!(P_\ell \oplus P_r)$ which maps (i, u) to $\mathrm{in}_i(u)$ is a bijection. The injection $u \mapsto \mathrm{in}_i(u)$ has a left inverse $\mathrm{pr}_i \in \mathbf{Pcoh}(\underline{P_\ell} \oplus \underline{P_r}, \underline{P_i})$ defined by $(\mathrm{pr}_i)_{(j,a),b} = \delta_{i,j} \delta_{a,b}$, which is not a coalgebra morphism in general.

Proof. Let $v \in \mathsf{P}^!(!X)$, we have $v^! = \mathrm{h}_{!X} v = \mathrm{dig}_X v$ hence $(\mathrm{der}_X v)^! = !\mathrm{der}_X v^! = !\mathrm{der}_X \mathrm{dig}_X v = v$. The other properties result from the fact that the Eilenberg-Moore category $\mathbf{Pcoh}^!$ is cartesian and co-cartesian with $\otimes$ and $\oplus$ as product and co-product, see [24] for more details. $\square$

Because of these properties we write sometimes (u_ℓ, u_r) instead of $u_\ell \otimes u_r$ when $u_i \in \mathsf{P}^! P_i$ for $i \in \{\ell, r\}$.

Definition 14. An object P of $\mathbf{Pcoh}^!$ is *dense* if, for any object Y of $\mathbf{Pcoh}$ and any two morphisms $t, t' \in \mathbf{Pcoh}(\underline{P}, Y)$, if $t u = t' u$ for all $u \in \mathsf{P}^!(P)$, then $t = t'$.

Theorem 15. *For any probabilistic coherence space X, $!X$ is a dense coalgebra. If P_ℓ and P_r are dense coalgebras then $P_\ell \otimes P_r$ and $P_\ell \oplus P_r$ are dense. The colimit in $(\mathbf{Pcoh}^!)_\subseteq$ of a directed family of dense coalgebras is dense.*

Proof. Let X be an object of $\mathbf{Pcoh}$, one has $\mathsf{P}^!(!X) = \{u^! \mid u \in \mathsf{P}X\}$ by Lemma 13. It follows that $!X$ is a dense coalgebra by Theorem 7. Assume that P_ℓ and P_r are dense coalgebras. Let $t, t' \in \mathbf{Pcoh}(\underline{P_\ell} \otimes \underline{P_r}, Y)$ be such that $t w = t' w$ for all $w \in \mathsf{P}^!(P_\ell \otimes P_r)$. We have $\mathrm{cur}(t), \mathrm{cur}(t') \in \mathbf{Pcoh}(\underline{P_\ell}, \underline{P_r} \multimap Y)$ so, using the density of P_ℓ, it suffices to prove that $\mathrm{cur}(t) u_\ell = \mathrm{cur}(t') u_\ell$ for each $u_\ell \in \mathsf{P}^!(P_\ell)$. So let $u_\ell \in \mathsf{P}^!(P_\ell)$ and let $s = \mathrm{cur}(t) u_\ell$ and $s' = \mathrm{cur}(t') u_\ell$. Let $u_r \in \mathsf{P}^!(P_r)$, we have $s u_r = t(u_\ell \otimes u_r) = t'(u_\ell \otimes u_r) = s' u_r$ since $u_\ell \otimes u_r \in \mathsf{P}^!(P_\ell \otimes P_r)$ and therefore $s = s'$ since P_r is dense. Let now $t, t' \in \mathbf{Pcoh}(\underline{P_\ell} \oplus \underline{P_r}, Y)$ be such that $t w = t' w$ for all $w \in \mathsf{P}^!(P_\ell \oplus P_r)$. To prove that $t = t'$, it suffices to prove that $t \mathrm{in}_i = t' \mathrm{in}_i$ for $i \in \{\ell, r\}$. Since P_i is dense, it suffices to prove that $t \mathrm{in}_i u = t' \mathrm{in}_i u$ for each $u \in \mathsf{P}^!(P_i)$ which follows from the fact that $\mathrm{in}_i u \in \mathsf{P}^! P_i$. Last let $(P_\gamma)_{\gamma \in \Gamma}$ be a directed family of dense coalgebras (in $\mathbf{Pcoh}^!_\subseteq$) and let $P = \bigcup_{\gamma \in \Gamma} P_\gamma$, and let $t, t' \in \mathbf{Pcoh}^!(\underline{P}, Y)$ be such that $t w = t' w$ for all $w \in \mathsf{P}^!(P)$. It suffices to prove that, for each $\gamma \in \Gamma$, one has $t \eta^+_{\underline{P_\gamma}, \underline{P}} = t' \eta^+_{\underline{P_\gamma}, \underline{P}}$ and this results from the fact that P_γ is dense and $\eta^+_{\underline{P_\gamma}, \underline{P}}$ is a coalgebra morphisms (and therefore maps $\mathsf{P}^!(P_\gamma)$ to $\mathsf{P}^!(P)$). $\square$

The sub-category $\mathbf{Pcoh}^!$ of dense coalgebras is cartesian and co-cartesian and is well-pointed by Theorem 15. We use $\mathbf{Pcoh}^!_{\mathsf{den}}$ for this sub-category and $(\mathbf{Pcoh}^!_{\mathsf{den}})_\subseteq$ for the sub-class of $\mathbf{Pcoh}^!_\subseteq$ whose objects are the dense coalgebras (with the same order relation).

3.2.12. *Interpreting types and terms in* $\mathbf{Pcoh}$. Given a type σ with free type variables contained in the repetition-free list $\vec{\zeta}$, and given a sequence $\vec{P}$ of length n of objects of $\mathbf{Pcoh}^!$, we define $[\sigma]_{\vec{\zeta}}(\vec{P})$ as an object of $\mathbf{Pcoh}$ and when φ is a positive type (whose free variables are contained in $\vec{\zeta}$) we define $[\varphi]^!_{\vec{\zeta}}(\vec{P})$ as an object of $\mathbf{Pcoh}^!$. These operations are continuous and their definition follows the general pattern described in Section 3.1.4.

Theorem 16. *Let φ be a positive type and let $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be a repetition-free list of type variables which contains all the free variables of φ. Let $\vec{P}$ be a sequence of n dense coalgebras. Then $[\varphi]^!_{\vec{\zeta}}(\vec{P})$ is a dense coalgebra. In particular, when φ is closed, the coalgebra $[\varphi]^!$ is dense.*

This is an immediate consequence of the definition of $[\varphi]^!$ and of Theorem 15

Remark. This means that we have *only one model*, where positive types are interpreted as objects of $\mathbf{Pcoh}^!$, and not two models as one could expect (one where positive types are interpreted in $\mathbf{Pcoh}^!$, and one where they are interpreted in $\mathbf{Pcoh}^!_{\mathsf{den}}$); it simply turns out that the interpretation of positive types in the model $\mathbf{Pcoh}/\mathbf{Pcoh}^!$ are dense coalgebras. This is mainly due to the fact that the colimit of a directed family of dense coalgebras *in the partially ordered class* $\mathbf{Pcoh}^!_\subseteq$ is dense, see Theorem 15. From the viewpoint of Levy's CBPV [22], whose semantics is described in terms of adjunctions, we are using a resolution of the comonad "!" through the category $\mathbf{Pcoh}^!$ (or, equivalently, through the category $\mathbf{Pcoh}^!_{\mathsf{den}}$). As pointed out to us by one of the referees and already mentioned in the introduction, there is another resolution through a category of families and introduced in [1], which is initial among all resolutions that model CBPV. This other option will be explored in further work.

Then $o = \mathbf{1} \oplus \mathbf{1}$ satisfies $|[o]| = \{(\ell, *), (r, *)\}$ and $u \in (\mathbb{R}^+)^{|[o]|}$ satisfies $u \in \mathsf{P}[o]$ iff $u_{(\ell, *)} + u_{(r, *)} \leq 1$. The coalgebraic structure of this object is given by

$$(\mathrm{h}_o)_{(j,*),[(j_1,*),\ldots,(j_k,*)]} = \begin{cases} 1 & \text{if } j = j_1 = \cdots = j_k \\ 0 & \text{otherwise.} \end{cases}$$

The object $\mathsf{N} = [\iota]$ satisfies $\mathsf{N} = 1 \oplus \mathsf{N}$ so that $|\mathsf{N}| = \{(\ell, *), (r, (\ell, *)), (r, (r, (\ell, *))), \ldots\}$ and we use $\overline{n}$ for the element of $|\mathsf{N}|$ which has n occurrences of r. Given $u \in (\mathbb{R}^+)^{|\mathsf{N}|}$, we use $l(u)$ for the element of $(\mathbb{R}^+)^{|\mathsf{N}|}$ defined by $l(u)_{\overline{n}} = u_{\overline{n+1}}$. By definition of N, we have $u \in \mathsf{PN}$ iff $u_{\overline{0}} + \|l(u)\|_{\mathsf{N}} \leq 1$, and then $\|u\|_{\mathsf{N}} = u_{\overline{0}} + \|l(u)\|_{\mathsf{N}}$. It follows that $u \in \mathsf{PN}$ iff $\sum_{n=0}^{\infty} u_{\overline{n}} \leq 1$ and of course $\|u\|_{\mathsf{N}} = \sum_{n=0}^{\infty} u_{\overline{n}}$. Then the coalgebraic structure h_ι is defined exactly as h_o above. In the sequel, we identify $|\mathsf{N}|$ with $\mathbb{N}$.

Given a typing context $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$, a type σ and a term M such that $\mathcal{P} \vdash M : \sigma$, M is interpreted as a morphism $[M]^{\mathcal{P}} \in \mathbf{Pcoh}([\mathcal{P}], [\sigma])$. For all constructs of the language but probabilistic choice, this interpretation uses the generic structures of the model described in Section 3.1, the description of this interpretation can be found in [5]. We set $[\mathsf{coin}(p)] = p e_{(\ell,*)} + (1-p) e_{(r,*)}$.

If $x_1 : \varphi_1, \ldots, x_k : \varphi_k \vdash M : \sigma$, the morphism $[M]^{\mathcal{P}}$ is completely characterized by its values on $(u_1, \ldots, u_k) \in \mathsf{P}^!([\mathcal{P}]^!)$. We describe now the interpretation of terms using this observation.

- $[()] = 1 \in \mathsf{P}1 = [0, 1]$.
- $[x_i]^{\mathcal{P}} (u_1, \ldots, u_k) = u_i$.
- $[N^!]^{\mathcal{P}} (u_1, \ldots, u_k) = ([N]^{\mathcal{P}} (u_1, \ldots, u_k))^!$.
- $[(M_\ell, M_r)]^{\mathcal{P}} (u_1, \ldots, u_k) = [M_\ell]^{\mathcal{P}} (u_1, \ldots, u_k) \otimes [M_r]^{\mathcal{P}} (u_1, \ldots, u_k)$.
- $[\mathsf{in}_i N]^{\mathcal{P}} (u_1, \ldots, u_k) = \mathsf{in}_i([N]^{\mathcal{P}} (u_1, \ldots, u_k))$, $i \in \{\ell, r\}$.
- $[\mathsf{der}\, N]^{\mathcal{P}} (u_1, \ldots, u_k) = \mathsf{der}_{[\sigma]}([N]^{\mathcal{P}} (u_1, \ldots, u_k))$, assuming that $\mathcal{P} \vdash N : !\sigma$.
- If $\mathcal{P} \vdash N : \varphi \multimap \sigma$ and $\mathcal{P} \vdash R : \varphi$ then $[N]^{\mathcal{P}} (u_1, \ldots, u_k) \in \mathsf{P}([\varphi] \multimap [\sigma])$, and $[R]^{\mathcal{P}} (u_1, \ldots, u_k) \in \mathsf{P}([\varphi])$ and using the application of a matrix to a vector we have $[\langle N \rangle R]^{\mathcal{P}} (u_1, \ldots, u_k) = [N]^{\mathcal{P}} (u_1, \ldots, u_k)\, [R]^{\mathcal{P}} (u_1, \ldots, u_k)$.
- If $\mathcal{P}, x : \varphi \vdash N : \sigma$ then $[\lambda x^\varphi N]^{\mathcal{P}} (u_1, \ldots, u_k) \in \mathsf{P}([\varphi] \multimap [\sigma])$ is completely described by the fact that, for all $u \in \mathsf{P}^!([\varphi]^!)$, one has $[\lambda x^\varphi N]^{\mathcal{P}} (u_1, \ldots, u_k)\, u = [N]^{\mathcal{P}, x:\varphi} (u_1, \ldots, u_k, u)$. This is a complete characterization of this interpretation by Theorem 16.
- If $\mathcal{P} \vdash N : \varphi_\ell \oplus \varphi_r$ and $\mathcal{P}, y_i : \varphi_i \vdash R_i : \sigma$ for $i \in \{\ell, r\}$, then
  $[\mathsf{case}(N, y_\ell \cdot R_\ell, y_r \cdot R_r)]^{\mathcal{P}} (u_1, \ldots, u_k) = [R_\ell]^{\mathcal{P}, y_\ell : \varphi_\ell} (u_1, \ldots, u_k, \mathrm{pr}_\ell([N]^{\mathcal{P}} (u_1, \ldots, u_k))) + [R_r]^{\mathcal{P}, y_r : \varphi_r} (u_1, \ldots, u_k, \mathrm{pr}_r([N]^{\mathcal{P}} (u_1, \ldots, u_k)))$ where $\mathrm{pr}_i \in \mathbf{Pcoh}(P_\ell \oplus P_r, P_i)$ is the ith "projection" introduced in 3.2.11, left inverse for in_i.
- If $\mathcal{P}, x : !\sigma \vdash N : \sigma$ then $[N]^{\mathcal{P}, x:!\sigma} \in \mathbf{Pcoh}([\mathcal{P}] \otimes ![\sigma], [\sigma])$ and $[\mathsf{fix}\, x^{!\sigma} N]^{\mathcal{P}} (u_1, \ldots, u_k) = \sup_{n=0}^{\infty} f^n(0)$ where $f : \mathsf{P}[\sigma] \to \mathsf{P}[\sigma]$ is the Scott-continuous function given by $f(u) = [N]^{\mathcal{P}, x:!\sigma} (u_1, \ldots, u_k, u^!)$.
- If $\mathcal{P} \vdash N : \psi\,[\mathsf{Rec}\, \zeta \cdot \psi / \zeta]$ then $[\mathsf{fold}(N)]^{\mathcal{P}} = [N]^{\mathcal{P}}$ which makes sense since $[\psi\,[\mathsf{Rec}\, \zeta \cdot \psi / \zeta]] = [\mathsf{Rec}\, \zeta \cdot \psi]$.
- If $\mathcal{P} \vdash N : \mathsf{Rec}\, \zeta \cdot \psi$ then $[\mathsf{unfold}(N)]^{\mathcal{P}} = [N]^{\mathcal{P}}$.

Theorem 17 (Soundness). *If M satisfies $\mathcal{P} \vdash M : \sigma$ then*

$$[M]^{\mathcal{P}} = \sum_{\mathcal{P} \vdash M' : \sigma} \mathsf{Red}_{M, M'}\, [M']^{\mathcal{P}}$$

The proof is done by induction and is a straightforward verification.

Corollary 18. *Let M be a term such that $\vdash M : 1$ so that $[M] \in [0, 1]$. Then $[M] \geq \mathsf{Red}_{M, ()}^{\infty}$.*

This is an immediate consequence of Theorem 17 and of the definition of Red^{∞}, see Section 2.2.

3.3. **Examples of term interpretations.** We give the interpretation of terms that we gave as examples in Subsection 2.3.

- $[\Omega^\sigma] = [\mathsf{fix}\, x^{!\sigma}\, \mathsf{der}\, x] = 0$
- $[\mathbf{t}] = e_{(\ell, *)}$ and $[\mathbf{f}] = e_{(r, *)}$
- $[\mathsf{if}(M, N_\ell, N_r)]^{\mathcal{P}} (u_1, \ldots, u_k) = [M]_{(\ell, *)}^{\mathcal{P}} (u_1, \ldots, u_k)\, [N_\ell]^{\mathcal{P}} (u_1, \ldots, u_k) + [M]_{(r, *)}^{\mathcal{P}} (u_1, \ldots, u_k)\, [N_r]^{\mathcal{P}} (u_1, \ldots, u_k)$
- $[\mathsf{dice}_p(M_\ell, M_r)]^{\mathcal{P}} (u_1, \ldots, u_k) = p\, [M_\ell]^{\mathcal{P}} (u_1, \ldots, u_k) + (1 - p)\, [M_r]^{\mathcal{P}} (u_1, \ldots, u_k)$

- $[\underline{n}] = \overline{n}$ for $n \in \mathbb{N}$
- $[\text{suc}(M)]^{\mathcal{P}}_{n+1}(u_1, \ldots, u_k) = [M]_n(u_1, \ldots, u_k)$
- $[\text{ifz}(M, N_\ell, x \cdot N_r)]^{\mathcal{P}}(u_1, \ldots, u_k) = [M]^{\mathcal{P}}_0(u_1, \ldots, u_k)[N_\ell]^{\mathcal{P}}(u_1, \ldots, u_k)$
  $+ \sum_{n=0}^{\infty} [M]^{\mathcal{P}}_{n+1}(u_1, \ldots, u_k)[N_r]^{\mathcal{P}}(u_1, \ldots, u_k)(\overline{n})$
- $[\text{ran}(\overrightarrow{p})] = \sum_{i=1}^{n} p_i e_{\overline{i}}$
- $[\langle \text{eq}_\ell \rangle M]^{\mathcal{P}}(u_1, \ldots, u_k) = [M]^{\mathcal{P}}(u_1, \ldots, u_k)_\ell e_*$
- $[M_0 \cdot N]^{\mathcal{P}}(u_1, \ldots, u_k) = [M_0]^{\mathcal{P}}(u_1, \ldots, u_k)[N]^{\mathcal{P}}(u_1, \ldots, u_k)$
- $[M_0 \wedge \cdots \wedge M_l]^{\mathcal{P}}(u_1, \ldots, u_k) = \prod_{i=0}^{l} [M_i]^{\mathcal{P}}(u_1, \ldots, u_k)$
- $\left[\langle \text{choose}^{\sigma}_{l+1}(N_0, \ldots, N_l)\rangle P\right]^{\mathcal{P}}(u_1, \ldots, u_k) = \sum_{i=0}^{l} [P]^{\mathcal{P}}_i(u_1, \ldots, u_k) \cdot [N_i]^{\mathcal{P}}(u_1, \ldots, u_k)$
- $\forall u \in \mathsf{P}([\iota]), \ [\text{ext}(l, r)](u) = \sum_{i=l}^{r} u_i e_{\overline{i}}$ and

$$[\text{win}_l(\overrightarrow{n})](u) = \sum_{i=n_1+\cdots+n_{l-1}+1}^{n_1+\cdots+n_l} u_i e_{\overline{i}}$$

4. Adequacy

Our goal is to prove the converse of Corollary 18: for any closed term M such that $\vdash M : \mathbf{1}$, the probability that M reduces to () is larger than or equal to $[M] \in \mathsf{P}[\mathbf{1}] \simeq [0, 1]$, so that we shall know that these two numbers are actually equal.

In spite of its very simple statement, the proof of this property is rather long mainly because we have to deal with the recursive type definitions allowed by our syntax. As usual, the proof is based on the definition of a logical relations between terms and elements of the model (more precisely, given any type σ, we have to define a relation between closed terms of types σ and elements of $\mathsf{P}[\sigma]$; let us call such a relation a σ-relation).

Since we have no positivity restrictions on the occurrence of type variables wrt. which recursive types are defined so that types are neither covariant nor contravariant wrt. these type variables, we use a very powerful technique introduced in [26] for defining this logical relation.

Indeed a type variable ζ can have positive and negative occurrences in a positive[13] type φ, consider for instance the case $\varphi = !(\zeta \multimap \zeta)$ where the type variable ζ has a positive (on the right of the $\multimap$) and a negative occurrence (on the left). To define the logical relation associated with $\text{Rec}\,\zeta \cdot \varphi$, we have to find a fixpoint for the operation which maps a $(\text{Rec}\,\zeta \cdot \varphi)$-relation $\mathcal{R}$ to the relation $\Phi(\mathcal{R}) = !(\mathcal{R} \multimap \mathcal{R})$ (which can be defined using $\mathcal{R}$ as a "logical relation" in a fairly standard way). Relations are naturally ordered by inclusion, and this strongly suggests to define the above fixpoint using this order relation by e.g. Tarski's Fixpoint Theorem. The problem however is that Φ is neither a monotone nor an anti-monotone operation on relations, due to the fact that ζ has a positive and a negative occurrence in φ.

It is here that Pitts's trick comes in: we replace the relations $\mathcal{R}$ with pairs of relations $\mathcal{R} = (\mathcal{R}^-, \mathcal{R}^+)$ ordered as follows: $\mathcal{R} \sqsubseteq \mathcal{S}$ if $\mathcal{R}^+ \subseteq \mathcal{S}^+$ and $\mathcal{S}^- \subseteq \mathcal{R}^-$. Then we define accordingly $\Phi(\mathcal{R})$ as a pair of relations by $\Phi(\mathcal{R})^- = !(\mathcal{R}^+ \multimap \mathcal{R}^-)$ and $\Phi(\mathcal{R})^+ = !(\mathcal{R}^- \multimap \mathcal{R}^+)$. Now the operation Φ is monotone wrt. the $\sqsubseteq$ relation and it becomes possible to apply Tarski's Fixpoint Theorem to Φ and get a pair of relations $\mathcal{R}$ such that $\mathcal{R} = \Phi(\mathcal{R})$. The next step consists in proving that $\mathcal{R}^- = \mathcal{R}^+$. This is obtained by means of an analysis of

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[13]Warning: the word "positive" has two different meanings here!

- Let $\mathcal{R} \in \mathsf{Rel}(\sigma)$, we define $!\mathcal{R} \in \mathsf{Rel}^{\vee}(!\sigma)$ by: $!\mathcal{R}^{\varepsilon} = \{(M^!, u^!) \mid (M, u) \in \mathcal{R}^{\varepsilon}\}$ for $\varepsilon \in \{-, +\}$.
- Let $\mathcal{V}_i \in \mathsf{Rel}^{\vee}(\varphi_i)$ for $i \in \{\ell, r\}$. We define $(\mathcal{V}_{\ell} \otimes \mathcal{V}_r)^{\varepsilon} = \{((V_{\ell}, V_r), v_{\ell} \otimes v_r)\} \mid (V_i, v_i) \in \mathcal{V}_i^{\varepsilon}\}$ for $\varepsilon \in \{-, +\}$, so that $\mathcal{V}_{\ell} \otimes \mathcal{V}_r \in \mathsf{Rel}^{\vee}(\varphi_{\ell} \otimes \varphi_r)$.
- Let $\mathcal{V}_i \in \mathsf{Rel}^{\vee}(\varphi_i)$ for $i \in \{\ell, r\}$. We define $(\mathcal{V}_{\ell} \oplus \mathcal{V}_r)^{\varepsilon} = \{(\mathsf{in}_i V, \mathsf{in}_i(v))\} \mid i \in \{\ell, r\}$ and $(V, v) \in \mathcal{V}_i^{\varepsilon}\}$ for $\varepsilon \in \{-, +\}$, so that $\mathcal{V}_{\ell} \oplus \mathcal{V}_r \in \mathsf{Rel}^{\vee}(\varphi_{\ell} \oplus \varphi_r)$.
- Let $\mathcal{V} \in \mathsf{Rel}^{\vee}(\varphi)$ and $\mathcal{R} \in \mathsf{Rel}(\sigma)$. We define $\mathcal{V} \multimap \mathcal{R} \in \mathsf{Rel}(\varphi \multimap \sigma)$ as follows: $(\mathcal{V} \multimap \mathcal{R})^{\varepsilon} = \{(M, u) \mid \vdash M : \varphi \multimap \sigma,\ u \in \mathsf{P}[\varphi \multimap \sigma]$ and $\forall (V, v) \in \mathcal{V}^{-\varepsilon}\ (\langle M \rangle V, uv) \in \mathcal{R}^{\varepsilon}\}$.
- Last, given $\mathcal{V} \in \mathsf{Rel}^{\vee}(\varphi)$, we define $\overline{\mathcal{V}} \in \mathsf{Rel}(\varphi)$ as follows: $\overline{\mathcal{V}}^{\varepsilon}$ is the set of all (M, u) such that $\vdash M : \varphi$, $u \in \mathsf{P}[\varphi]$ and, for all $(T, t) \in (\mathcal{V} \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, one has $(\langle T \rangle M, t\, u) \in \mathcal{R}(\mathbf{1})$.

Figure 4: Logical operations for pairs of relations

the definition of the interpretation of fixpoints of types as colimits in the category $\mathbf{Pcoh}_{\subseteq}$. One is finally in position of proving a fairly standard "Logical Relation Lemma" from which adequacy follows straightforwardly.

In this short description of our adequacy proof, many technicalities have obviously been hidden, the most important one being that values are handled in a special way so that we actually consider two kinds of pairs of relations. Also, a kind of "biorthogonality closure" plays an essential role in the handling of positive types, no surprise for the readers acquainted with Linear Logic, see for instance the proof of normalization in [12].

4.1. **Pairs of relations and basic operations.** Given a *closed* type σ, we define $\mathsf{Rel}(\sigma)$ as the set of all pairs of relations $\mathcal{R} = (\mathcal{R}^-, \mathcal{R}^+)$ such that, for $\varepsilon \in \{+, -\}$, each element of $\mathcal{R}^{\varepsilon}$ is a pair (M, u) where $\vdash M : \sigma$ and $u \in \mathsf{P}[\sigma]$. For a *closed* positive type φ, we also define $\mathsf{Rel}^{\vee}(\varphi)$ as the set of all pairs of relations $\mathcal{V} = (\mathcal{V}^-, \mathcal{V}^+)$ such that, for $\varepsilon \in \{+, -\}$, each element of $\mathcal{V}^{\varepsilon}$ is a pair (V, v) where $\vdash V : \varphi$ is a value and $v \in \mathsf{P}^![\varphi]$.

Given $\mathcal{R}, \mathcal{S} \in \mathsf{Rel}(\sigma)$, we write $\mathcal{R} \sqsubseteq \mathcal{S}$ if $\mathcal{R}^+ \subseteq \mathcal{S}^+$ and $\mathcal{S}^- \subseteq \mathcal{R}^-$. We define similarly $\mathcal{V} \sqsubseteq \mathcal{W}$ for $\mathcal{V}, \mathcal{W} \in \mathsf{Rel}^{\vee}(\varphi)$. Then $\mathsf{Rel}(\sigma)$ is a complete meet-lattice, the infimum of a collection $(\mathcal{R}_i)_{i \in I}$ being $\prod_{i \in I} \mathcal{R}_i = (\bigcup_{i \in I} \mathcal{R}_i^-, \bigcap_{i \in I} \mathcal{R}_i^+)$. The same holds of course for $\mathsf{Rel}^{\vee}(\varphi)$ and we use the same notations.

We define $\mathcal{R}(\mathbf{1})$ as the set of all pairs (M, p) such that $\vdash M : \mathbf{1}$, $p \in [0, 1]$ and $\mathsf{Red}^{\infty}_{M, ()} \geq p$.

We define in Figure 4 logical operations on these pairs of relations. The last one is the aforementioned biorthogonality closure operation on pairs of relations.

Observe that all these operations are monotone wrt. $\sqsubseteq$. For instance $\mathcal{V} \sqsubseteq \mathcal{W} \wedge \mathcal{R} \sqsubseteq \mathcal{S} \Rightarrow (\mathcal{V} \multimap \mathcal{R}) \sqsubseteq (\mathcal{W} \multimap \mathcal{S})$, and $\mathcal{V} \sqsubseteq \mathcal{W} \Rightarrow \overline{\mathcal{V}} \sqsubseteq \overline{\mathcal{W}}$.

4.2. **Fixpoints of pairs of relations.** To deal with fixpoint types $\mathsf{Rec}\, \zeta \cdot \varphi$, we need to consider types parameterized by relations as follows.

Let σ be a type and let $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be a list of type variables without repetitions and which contains all free variables of σ. For all list $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$ of *closed* positive types, we define

$$\mathcal{R}(\sigma)_{\vec{\zeta}} : \prod_{i=1}^{n} \mathsf{Rel}^{\vee}(\varphi_i) \to \mathsf{Rel}\left(\sigma\left[\vec{\varphi}/\vec{\zeta}\right]\right).$$

Let also φ be a positive type whose free variables are contained in $\overrightarrow{\zeta}$, we define

$$\mathcal{V}(\varphi)_{\overrightarrow{\zeta}} : \prod_{i=1}^{n} \mathsf{Rel}^{\vee}(\varphi_i) \to \mathsf{Rel}^{\vee}(\varphi\left[\overrightarrow{\varphi}/\overrightarrow{\zeta}\right]).$$

The definition is by simultaneous induction on σ and φ. All cases but one consist in applying straightforwardly the above defined logical operations on pairs of relations, for instance

$$\mathcal{R}(\varphi \multimap \tau)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}}) = \mathcal{V}(\varphi)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}}) \multimap \mathcal{R}(\sigma)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}}) \quad \text{and} \quad \mathcal{R}(\varphi)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}}) = \overline{\mathcal{V}(\varphi)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}})}.$$

We are left with the case of recursive definitions of types, so assume that $\varphi = \mathsf{Rec}\,\zeta \cdot \psi$. Let $\overrightarrow{\varphi} = (\varphi_1, \dots, \varphi_n)$ be a list of closed positive types and let $\overrightarrow{\mathcal{V}} \in \prod_{i=1}^{n} \mathsf{Rel}^{\vee}(\varphi_i)$, we set

$$\mathcal{V}(\varphi)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}}) = \bigsqcap\{\mathcal{V} \in \mathsf{Rel}^{\vee}(\varphi\left[\overrightarrow{\varphi}/\overrightarrow{\zeta}\right]) \mid \mathsf{fold}(\mathcal{V}(\psi)_{\overrightarrow{\zeta},\zeta}(\overrightarrow{\mathcal{V}}, \mathcal{V})) \sqsubseteq \mathcal{V}\} \tag{4.1}$$

where we use the following notation: given $\mathcal{W} \in \mathsf{Rel}^{\vee}(\psi\left[\overrightarrow{\varphi}/\overrightarrow{\zeta}, \varphi\left[\overrightarrow{\varphi}/\overrightarrow{\zeta}\right]/\zeta\right])$, $\mathsf{fold}(\mathcal{W}) \in \mathsf{Rel}^{\vee}(\varphi\left[\overrightarrow{\varphi}/\overrightarrow{\zeta}\right])$ is given by $\mathsf{fold}(\mathcal{W})^{\varepsilon} = \{(\mathsf{fold}(W), v) \mid (W, v) \in \mathcal{W}^{\varepsilon}\}$ for $\varepsilon \in \{+, -\}$.

We recall the statement of Tarski's fixpoint theorem.

Theorem 19. *Let S and T be complete meet semi-lattices and let $f : S \times T \to T$ be a monotone function. For $x \in S$, let $g(x)$ be the meet of the set $\{y \in T \mid f(x, y) \leq y\}$. Then the function g is monotone and satisfies $f(x, g(x)) = g(x)$ for each $x \in S$.*

Applying this theorem we obtain, by induction on types, the following property.

Lemma 20. *For any type σ and any positive type φ, the maps $\mathcal{R}(\sigma)_{\overrightarrow{\zeta}}$ and $\mathcal{V}(\varphi)_{\overrightarrow{\zeta}}$ are monotone wrt. the $\sqsubseteq$ order relation. If ψ is a positive type, $\overrightarrow{\zeta} = (\zeta_1, \dots, \zeta_n, \zeta)$ is a repetition-free list of type variables containing all the free variables of ψ and $\varphi = \mathsf{Rec}\,\zeta \cdot \psi$ and $\overrightarrow{\mathcal{V}} = (\mathcal{V}_1, \dots, \mathcal{V}_n)$ is a list of pairs of relations such that $\mathcal{V}_i \in \mathsf{Rel}^{\vee}(\varphi_i)$ for each i, then $\mathcal{V} = \mathcal{R}(\varphi)_{\overrightarrow{\zeta}}(\overrightarrow{\mathcal{V}})$ satisfies $\mathcal{V} = \mathsf{fold}(\mathcal{R}(\psi)_{\overrightarrow{\zeta},\zeta})(\overrightarrow{\mathcal{V}}, \mathcal{V})$.*

4.3. **Some useful closeness lemmas.** We state and prove a series of lemmas expressing that our pairs of relations are closed under various syntactic and semantic operations.

Lemma 21. *Let M and M' be terms such that $\vdash M : \mathbf{1}$ and $\vdash M' : \mathbf{1}$. If $M \to_{\mathsf{w}} M'$ then $\mathsf{Red}^{\infty}_{M,()} = \mathsf{Red}^{\infty}_{M',()}$.*

This is straightforward since any reduction path from M to $()$ must start with the step $M \to_{\mathsf{w}} M'$, and this is a probability 1 step.

Lemma 22. *Let φ be a closed positive type and let σ be a closed type. Let $(M, u) \in \mathcal{R}(\varphi)^{-\varepsilon}$ and $(R, r) \in \mathcal{R}(\varphi \multimap \sigma)^{\varepsilon}$. Then $(\langle R \rangle M, r\, u) \in \mathcal{R}(\sigma)^{\varepsilon}$.*

Proof. We can write $\sigma = \varphi_1 \multimap \cdots \multimap \varphi_n \multimap \psi$ for some n and $\varphi_1, \dots, \varphi_n, \psi$ positive and closed. Given $(V_i, v_i) \in \mathcal{V}(\varphi_i)^{-\varepsilon}$ for $i = 1, \dots, n$, we have to prove that

$$(\langle R \rangle M\, V_1 \cdots V_n, r\, u\, v_1 \cdots v_n) \in (\overline{\mathcal{V}(\psi)})^{\varepsilon}$$

so let $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, we have to prove that

$$(\langle T \rangle(\langle R \rangle M\, V_1 \cdots V_n), t(r\, u\, v_1 \cdots v_n)) \in \mathcal{R}(\mathbf{1}).$$

Let $S = \lambda x^\varphi \langle T \rangle (\langle R \rangle x \, V_1 \cdots V_n)$ so that $\vdash S : \varphi \multimap \mathbf{1}$. Similarly let $s \in \mathsf{P}[\varphi \multimap \mathbf{1}]$ be the linear morphism defined by $s \, u' = t(r \, u' \, v_1 \cdots v_n)$ (the fact that s so defined is actually a morphism in $\mathbf{Pcoh}$ results from the symmetric monoidal closeness of that category and from the fact that r and t are morphisms in $\mathbf{Pcoh}$). Let $(V, v) \in \mathcal{V}(\varphi)^{-\varepsilon}$, we have $(\langle R \rangle V, r \, v) \in \mathcal{R}(\sigma)^\varepsilon$ and hence $(\langle R \rangle V \, V_1 \cdots V_n), r \, v \, v_1 \cdots v_n \in \mathcal{R}(\psi)^\varepsilon$. Therefore

$$(\langle T \rangle (\langle R \rangle V \, V_1 \cdots V_n)), t(r \, v \, v_1 \cdots v_n) \in \mathcal{R}(\mathbf{1})$$

since we have assumed that $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$. Since $t(r \, v \, v_1 \cdots v_n) = s \, v$, and by Lemma 21, it follows that $(\langle S \rangle V, s \, v) \in \mathcal{R}(\mathbf{1})$. Hence $(S, s) \in \mathcal{R}(\varphi \multimap \mathbf{1})^\varepsilon$ and therefore $(\langle S \rangle M, s \, u) \in \mathcal{R}(\mathbf{1})$ since we have $(M, u) \in \mathcal{R}(\varphi)^{-\varepsilon}$.

We finish the proof by observing that $s \, u = t(r \, u \, v_1 \cdots v_n)$ and by showing that

$$\mathsf{Red}^\infty_{\langle T \rangle \langle R \rangle M \, V_1 \cdots V_n, ()} = \mathsf{Red}^\infty_{\langle S \rangle M, ()}$$

For this it suffices to observe (by inspection of the reduction rules) that each reduction path π from $\langle T \rangle (\langle R \rangle M \, V_1 \cdots V_n)$ to $()$ is of shape $\pi = \lambda \rho$ where

- λ is a reduction path

$$\langle T \rangle (\langle R \rangle M_1 \, V_1 \cdots V_n) \xrightarrow{p_1} \langle T \rangle (\langle R \rangle M_2 \, V_1 \cdots V_n) \xrightarrow{p_2} \cdots \xrightarrow{p_k} \langle T \rangle (\langle R \rangle M_{k+1} \, V_1 \cdots V_n)$$

where $M_1 = M$, M_{k+1} is a value V and $M = M_1 \xrightarrow{p_1} M_2 \xrightarrow{p_2} \cdots \xrightarrow{p_k} M_{k+1} = V$
- and ρ is a reduction path from $\langle T \rangle (\langle R \rangle V \, V_1 \cdots V_n)$ to $()$.

Then we have $\langle S \rangle M = \langle S \rangle M_1 \xrightarrow{p_1} \langle S \rangle M_2 \xrightarrow{p_2} \cdots \xrightarrow{p_k} \langle S \rangle V \xrightarrow{1} \langle T \rangle (\langle R \rangle V \, V_1 \cdots V_n)$, the last step resulting from the definition of S. In that way, we have defined a probability preserving bijection between the reduction paths from $\langle T \rangle (\langle R \rangle M_1 \, V_1 \cdots V_n)$ to $()$ and the reduction paths from $\langle S \rangle M$ to $()$, proving our contention. $\square$

Lemma 23. *Let φ_i be closed positive types and $(M_i, u_i) \in \mathcal{R}(\varphi_i)^\varepsilon$ for $i \in \{\ell, r\}$. Then $((M_\ell, M_r), u_\ell \otimes u_r) \in \mathcal{R}(\varphi_\ell \otimes \varphi_r)^\varepsilon$.*

Proof. Let $(T, t) \in (\mathcal{V}(\varphi_\ell \otimes \varphi_r) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, we must prove that $(\langle T \rangle (M_\ell, M_r), t(u_\ell \otimes u_r)) \in \mathcal{R}(\mathbf{1})$. Let $S = \lambda x_\ell^{\varphi_\ell} \lambda x_r^{\varphi_r} \langle T \rangle (x_\ell, x_r)$ and $s \in \mathsf{P}[\varphi_\ell \multimap (\varphi_r \multimap \mathbf{1})]$ be defined by $s \, u_\ell \, u_r = t(u_\ell \otimes u_r)$ (again, s is a morphism in $\mathbf{Pcoh}$ by symmetric monoidal closeness of that category). It is clear that $(S, s) \in (\mathcal{V}(\varphi_\ell) \multimap (\mathcal{V}(\varphi_r) \multimap \mathcal{R}(\mathbf{1})))^{-\varepsilon}$. By Lemma 22 we get $(\langle S \rangle M_\ell, s \, u_\ell) \in (\mathcal{V}(\varphi_r) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ and then $(\langle S \rangle M_\ell \, M_r, t(u_\ell \otimes u_r) \in \mathcal{R}(\mathbf{1})$. Observing that there is a probability preserving bijection between the reduction paths from $\langle S \rangle M_\ell \, M_r$ to $()$ and the reduction paths from $\langle T \rangle (M_\ell, M_r)$ to $()$, we conclude that $(\langle T \rangle (M_\ell, M_r), t(u_\ell \otimes u_r)) \in \mathcal{R}(\mathbf{1})$ as contended (in both terms one has to reduce first M_ℓ and then M_r to a value). $\square$

Lemma 24. *Let φ_ℓ and φ_r be closed positive types. If $(M, u) \in \mathcal{R}(\varphi_\ell \otimes \varphi_r)^\varepsilon$ then $(\mathsf{pr}_i M, \mathsf{pr}_i \, u) \in \mathcal{R}(\varphi_i)^\varepsilon$.*

Proof. Let $(T, t) \in (\mathcal{V}(\varphi_i) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, we have to prove that $(\langle T \rangle \mathsf{pr}_i M, t \, (\mathsf{pr}_i \, u)) \in \mathcal{R}(\mathbf{1})$. Let $S = \lambda x^{\varphi_\ell \otimes \varphi_r} \langle T \rangle \mathsf{pr}_i x$ and $s \in \mathsf{P}[\varphi_\ell \otimes \varphi_r \multimap \mathbf{1}]$ be defined by $s \, u_0 = t \, (\mathsf{pr}_i \, u_0)$ for all $u_0 \in \mathsf{P}[\varphi_\ell \otimes \varphi_r]$. Let $(W, w) \in \mathcal{V}(\varphi_\ell \otimes \varphi_r)^\varepsilon$, which means that $W = (V_\ell, V_r)$ and $w = v_\ell \otimes v_r$ with $(V_j, v_j) \in \mathcal{V}(\varphi_j)^\varepsilon$ for $j \in \{\ell, r\}$. We have $\langle S \rangle W \to_\mathsf{w} \langle T \rangle V_i$ and $s \, w = t \, v_i$ and we know that $(\langle T \rangle V_i, t \, v_i) \in \mathcal{R}(\mathbf{1})$ from which it follows by Lemma 21 that $(\langle S \rangle W, s \, w) \in \mathcal{R}(\mathbf{1})$. So we have proven that $(S, s) \in (\mathcal{V}(\varphi_\ell \otimes \varphi_r) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ and hence $(\langle S \rangle M, s \, u) \in \mathcal{R}(\mathbf{1})$. We have $s \, u = t \, (\mathsf{pr}_i \, u)$. Moreover we have a probability preserving bijection between the reduction paths from $\langle T \rangle \mathsf{pr}_i M$ to $()$ and the reduction paths from $\langle S \rangle M$ to $()$, and hence we have $(\langle T \rangle \mathsf{pr}_i M, t \, (\mathsf{pr}_i \, u)) \in \mathcal{R}(\mathbf{1})$ as contended.

Indeed, any reduction path π from $\langle T \rangle \mathsf{pr}_i M$ to $()$ has shape $\pi = \lambda\rho$ where λ is a reduction path $\langle T \rangle \mathsf{pr}_i M = \langle T \rangle \mathsf{pr}_i M_\ell \overset{p_1}{\to} \langle T \rangle \mathsf{pr}_i M_r \overset{p_2}{\to} \cdots \overset{p_k}{\to} \langle T \rangle \mathsf{pr}_i W \overset{1}{\to} \langle T \rangle V_i$ (with $W = (V_\ell, V_r)$) and ρ is a reduction path from $\langle T \rangle V_i$ to $()$. The first steps λ of this reduction are determined by the reduction path $M = M_\ell \overset{p_1}{\to} \cdots \overset{p_k}{\to} W$ from M to the value W. This reduction path determines uniquely the reduction path $\langle S \rangle M = \langle S \rangle M_\ell \overset{p_1}{\to} \cdots \overset{p_k}{\to} \langle S \rangle W \overset{1}{\to} \langle T \rangle \mathsf{pr}_i W \overset{1}{\to} \langle T \rangle V_i$ followed by the reduction ρ from $\langle T \rangle V_i$ to $()$ by ρ. $\qquad\square$

Lemma 25. *Let φ_ℓ and φ_r be closed positive types and let $(M, u) \in \mathcal{R}(\varphi_i)^\varepsilon$ for $i = \ell$ or $i = r$. Then $(\mathsf{in}_i M, \mathsf{in}_i u) \in \mathcal{R}(\varphi_\ell \oplus \varphi_r)^\varepsilon$.*

Proof. Let $(T, t) \in (\mathcal{V}(\varphi_\ell \oplus \varphi_r) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, we must prove that $(\langle T \rangle \mathsf{in}_i M, t\,(\mathsf{in}_i u)) \in \mathcal{R}(\mathbf{1})$. Let $S = \lambda x^{\varphi_i} \langle t \rangle \mathsf{in}_i(x)$ and let $s \in \mathsf{P}[\varphi_i \multimap \mathbf{1}]$. It is clear that $(S, s) \in (\mathcal{V}(\varphi_i) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ and it follows that $(\langle S \rangle M, s\,u) \in \mathcal{R}(\mathbf{1})$ which implies $(\langle T \rangle \mathsf{in}_i M, t\,(\mathsf{in}_i u)) \in \mathcal{R}(\mathbf{1})$ by the usual bijective and probability preserving bijection on reductions. $\qquad\square$

The next lemma uses notations introduced in Section 3.2.5.

Lemma 26. *Let φ_ℓ and φ_r be closed positive type and σ be a closed type. Let $(M, u) \in \mathcal{R}(\varphi_\ell \oplus \varphi_r)^\varepsilon$. For $i \in \{\ell, r\}$, let R_i be a term such that $y_i : \varphi_i \vdash R_i : \sigma$ and assume that $(\lambda x_i^{\varphi_i} R_i, r_i) \in \mathcal{R}(\varphi_i \multimap \sigma)^{-\varepsilon}$. Then $(\mathsf{case}(M, y_\ell \cdot R_\ell, y_r \cdot R_r), \mathsf{case}(r_\ell, r_r)\,u) \in \mathcal{R}(\sigma)^{-\varepsilon}$.*

Proof. We can write $\sigma = \psi_1 \multimap \cdots \multimap \psi_k \multimap \psi$ where ψ and the ψ_j's are closed and positive types. Given $(W_j, w_j) \in \mathcal{V}(\psi_j)^\varepsilon$ for $j = 1, \ldots, k$, we have to prove that

$$(\langle \mathsf{case}(M, y_\ell \cdot R_\ell, y_r \cdot R_r) \rangle \overrightarrow{W}, \mathsf{case}(r_\ell, r_r)\,u\,\overrightarrow{w}) \in \mathcal{V}(\psi)^{-\varepsilon} \tag{4.2}$$

so let $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$, our goal is to prove that

$$(\langle T \rangle \langle \mathsf{case}(M, y_\ell \cdot R_\ell, y_r \cdot R_r) \rangle \overrightarrow{W}, t\,(\mathsf{case}(r_\ell, r_r)\,u\,\overrightarrow{w})) \in \mathcal{R}(\mathbf{1}) . \tag{4.3}$$

Let $S = \lambda x^{\varphi_\ell \oplus \varphi_r} \langle T \rangle \langle \mathsf{case}(x, y_\ell \cdot R_\ell, y_r \cdot R_r) \rangle \overrightarrow{W}$ and $s \in \mathsf{P}[\varphi_\ell \oplus \varphi_r \multimap \mathbf{1}]$ be defined by $s\,u_0 = t\,(\mathsf{case}(r_\ell, r_r)\,u_0\,\overrightarrow{w})$ for each $u_0 \in \mathsf{P}[\varphi_\ell \oplus \varphi_r]$. Then we have $(S, s) \in (\mathcal{V}(\varphi_\ell \oplus \varphi_r) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$. Let indeed $i \in \{\ell, r\}$ and let $(V, v) \in \mathcal{V}(\varphi_i)^{-\varepsilon}$ so that $(\mathsf{in}_i V, \mathsf{in}_i v) \in \mathcal{V}(\varphi_\ell \oplus \varphi_r)^{-\varepsilon}$. We have $\langle S \rangle \mathsf{in}_i V \to_{\mathsf{w}}^* \langle T \rangle \langle R_i\,[V/y_i] \rangle \overrightarrow{W}$ and $s\,(\mathsf{in}_i v) = t\,(r_i\,v\,\overrightarrow{w})$ and, by our assumptions and Lemma 21, $(R_i\,[V/y_i], r_i\,v) \in \mathcal{R}(\sigma)^{-\varepsilon}$ and hence $(\langle R_i\,[V/y_i] \rangle \overrightarrow{W}, r_i\,v\,\overrightarrow{w}) \in \mathcal{V}(\psi)^{-\varepsilon}$. By Lemma 21 it follows that $(\langle S \rangle \mathsf{in}_i V, s\,(\mathsf{in}_i v)) \in \mathcal{R}(\mathbf{1})$ and hence $(S, s) \in (\mathcal{V}(\varphi_\ell \oplus \varphi_r) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$ as contended.

Therefore $(\langle S \rangle M, s\,u) \in \mathcal{R}(\mathbf{1})$. There is a bijective and probability preserving correspondence between the reductions from $\langle S \rangle M$ to $()$ and the reductions from $\langle T \rangle \langle \mathsf{case}(M, x_\ell \cdot \langle R_\ell \rangle x_\ell, x_r \cdot \langle R_r \rangle x_r) \rangle \overrightarrow{W}$ to $()$: as usual such reductions start with a reduction $M = M_1 \overset{p_1}{\to} M_2 \overset{p_2}{\to} \cdots \overset{p_k}{\to} M_k = \mathsf{in}_i V$ where $i \in \{\ell, r\}$ and V is a value of type φ_i and (after a few $\to_{\mathsf{w}}$-steps) continue with a reduction from $\langle T \rangle \langle R_i \rangle V\,\overrightarrow{W}$ to $()$. Therefore (4.3) holds and hence we have (4.2), this ends the proof of the lemma. $\qquad\square$

Lemma 27. *Let σ be a closed type and let $(M, u) \in \mathcal{R}(!\sigma)^\varepsilon$. We have $(\mathsf{der}\,M, \mathsf{der}\,u) \in \mathcal{R}(\sigma)^\varepsilon$.*

Proof. We can write $\sigma = \psi_1 \multimap \cdots \multimap \psi_k \multimap \psi$ where ψ and the ψ_j's are closed and positive types. Given $(W_j, w_j) \in \mathcal{V}(\psi_i)^{-\varepsilon}$, we have to prove that

$$(\langle \mathsf{der}\,M \rangle \overrightarrow{W}, \mathsf{der}\,u\,\overrightarrow{w}) \in \mathcal{V}(\psi)^\varepsilon \tag{4.4}$$

so let $(T, t) \in (\mathcal{V}(\psi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, our goal is to prove that

$$(\langle T \rangle \langle \mathsf{der}\, M \rangle \overrightarrow{W}, t\, (\mathsf{der}\, u\, \vec{w})) \in \mathcal{R}(\mathbf{1}) \,. \tag{4.5}$$

We set $S = \lambda x^{!\sigma}\, \langle T \rangle \langle \mathsf{der}\, x \rangle \overrightarrow{W}$ and we define $s \in \mathsf{P}[!\sigma \multimap \mathbf{1}]$ by $s\, u_0 = t\, (\mathsf{der}\, u_0\, \vec{w})$ for all $u_0 \in \mathsf{P}[!\sigma]$, and we prove that $(S, s) \in (\mathcal{V}(!\sigma) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$ as in the proof of Lemme 26 (for instance). We finish the proof in the same way, showing (4.5) by establishing a bijective and probability preserving correspondence between reductions. Our contention (4.4) follows. $\square$

Lemma 28. *Let φ be a closed positive type of shape $\varphi = \mathbf{Rec}\,\zeta \cdot \psi$. If $(M, u) \in \mathcal{R}(\varphi)^\varepsilon$ then $(\mathsf{unfold}(M), u) \in \mathcal{R}(\psi\,[\varphi/\zeta])^\varepsilon$.*

Proof. Let $(T, t) \in (\mathcal{V}(\psi\,[\varphi/\zeta]) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, we must prove that $(\langle T \rangle \mathsf{unfold}(M), u) \in \mathcal{R}(\mathbf{1})$. As usual one defines $S = \lambda x^\varphi\, \langle T \rangle \mathsf{unfold}(x)$ and one proves that $(S, t) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$. This results from the fact that if $(V, v) \in \mathcal{V}(\varphi)^\varepsilon$ then $V = \mathsf{fold}(W)$ with $(W, v) \in \mathcal{V}(\psi\,[\varphi/\zeta])^\varepsilon$, from Lemma 21 and from the fact that $\mathsf{unfold}(\mathsf{fold}(W)) \to_{\mathsf{w}} W$ (and of course from our assumption on (T, t)). It follows that $(\langle S \rangle M, t\, u) \in \mathcal{R}(\mathbf{1})$ from which we deduce $(\langle T \rangle \mathsf{unfold}(M), u) \in \mathcal{R}(\mathbf{1})$ by the usual reasoning involving a bijective probability preserving correspondence on reductions. $\square$

Lemma 29. *Let φ be a closed positive type of shape $\varphi = \mathbf{Rec}\,\zeta \cdot \psi$. If $(M, u) \in \mathcal{R}(\psi\,[\varphi/\zeta])^\varepsilon$ then $(\mathsf{fold}(M), u) \in \mathcal{R}(\varphi)^\varepsilon$.*

Proof. Let $(T, t) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$, we must prove that $(\langle T \rangle \mathsf{fold}(M), u) \in \mathcal{R}(\mathbf{1})$. As usual one defines $S = \lambda x^{\psi\,[\varphi/\zeta]}\, \langle T \rangle \mathsf{fold}(x)$ and one proves that $(S, t) \in (\mathcal{V}(\psi\,[\varphi/\zeta]) \multimap \mathcal{R}(\mathbf{1}))^{-\varepsilon}$. This results easily from the fact that, if $(V, v) \in \mathcal{V}(\psi\,[\varphi/\zeta])^\varepsilon$ then $(\mathsf{fold}(V), v) \in \mathcal{V}(\varphi)^\varepsilon$, from Lemma 21 and from our assumption about (T, t). Therefore we have $(\langle S \rangle M, t\, u) \in \mathcal{R}(\mathbf{1}))$ from which we deduce $(\langle T \rangle \mathsf{fold}(M), u) \in \mathcal{R}(\mathbf{1})$ by the usual reasoning. $\square$

Lemma 30. *Let σ be a closed type and let M be a closed term of type σ. Then $(M, 0) \in \mathcal{R}(\sigma)^\varepsilon$ and, if $D \subseteq \mathsf{P}[\sigma]$ is directed and satisfies $\forall u \in D\ (M, u) \in \mathcal{R}(\sigma)^\varepsilon$ then $(M, \sup D) \in \mathcal{R}(\sigma)^\varepsilon$. Last, if $(M, u) \in \mathcal{R}(\sigma)^\varepsilon$ and $u' \leq u$ then $(M, u') \in \mathcal{R}(\sigma)^\varepsilon$.*

Proof. We can write $\sigma = \varphi_1 \multimap \cdots \multimap \varphi_n \multimap \psi$ for some n and $\varphi_1, \ldots, \varphi_n, \psi$ positive and closed. Let us prove the second statement. For $i = 1, \ldots, n$, let $(V_i, v_i) \in \mathcal{V}(\varphi_i)^{-\varepsilon}$, we must prove that $(\langle M \rangle V_1 \cdots V_n, (\sup D)\, v_1 \cdots v_n) \in \overline{\mathcal{V}(\psi)}^\varepsilon$, knowing that

$$\forall u \in D\quad (\langle M \rangle V_1 \cdots V_n, u\, v_1 \cdots v_n) \in \overline{\mathcal{V}(\psi)}^\varepsilon\,.$$

This results from the fact that, given $t \in \mathsf{P}[\psi \multimap \mathbf{1}]$, the map $u \mapsto t\, (u\, v_1 \cdots v_n)$ is Scott continuous from $\mathsf{P}[\varphi]$ to $[0, 1]$. The first statement of the lemma results from the fact that this function maps 0 to 0. The last one results from the fact that this function is monotone. $\square$

4.4. **Uniqueness of the relation.** With any closed type σ we have associated a pair of relations $\mathcal{R}(\sigma)$. We need now to prove that this pair satisfies $\mathcal{R}(\sigma)^+ = \mathcal{R}(\sigma)^-$ so that we have actually associated a unique relation with any type.

To this end we prove first that $\mathcal{R}(\sigma)^+ \subseteq \mathcal{R}(\sigma)^-$. Defining, for any pair of relations $\mathcal{R}$, the relation $\mathcal{R}^{\mathsf{op}}$ as $(\mathcal{R}^+, \mathcal{R}^-)$, this amounts to proving that $\mathcal{R}(\sigma) \sqsubseteq \mathcal{R}(\sigma)^{\mathsf{op}}$. We use the same notation for the elements of $\mathsf{Rel}^{\mathsf{v}}(\varphi)$ for φ positive.

For the next lemma, we use the same notational conventions as above.

Lemma 31. *Let $\vec{\mathcal{V}}$ be a list of pairs of relations such that $\mathcal{V}_i \in \mathsf{Rel}^{\vee}(\varphi_i)$ and $\mathcal{V}_i \sqsubseteq \mathcal{V}_i{}^{\mathsf{op}}$ for each i. Then $\mathcal{R}(\sigma)_{\vec{\zeta}}(\vec{\mathcal{V}}) \sqsubseteq \mathcal{R}(\sigma)_{\vec{\zeta}}(\vec{\mathcal{V}})^{\mathsf{op}}$ and $\mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}}) \sqsubseteq \mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}})^{\mathsf{op}}$.*

Proof. The proof is by induction on types. All cases result straightforwardly from the monotony of the logical operations on pairs of relations, but the case of fixpoints of types. So assume that $\varphi = \mathsf{Rec}\, \zeta \cdot \psi$, let $\mathcal{V} = \mathcal{V}(\varphi)_{\vec{\zeta}}(\vec{\mathcal{V}})$ and let us prove that $\mathcal{V} \sqsubseteq \mathcal{V}^{\mathsf{op}}$. For this, because of the definition of $\mathcal{V}$ as a glb (4.1), it suffices to show that

$$\mathsf{fold}(\mathcal{V}(\psi)_{\vec{\zeta},\zeta}(\vec{\mathcal{V}}, \mathcal{V}^{\mathsf{op}})) \sqsubseteq \mathcal{V}^{\mathsf{op}}\,.$$

By the first statement of Lemma 20 and our assumption on the $\mathcal{V}_i$'s we have

$$\mathsf{fold}(\mathcal{V}(\psi)_{\vec{\zeta},\zeta}(\vec{\mathcal{V}}, \mathcal{V}^{\mathsf{op}})) \sqsubseteq \mathsf{fold}(\mathcal{V}(\psi)_{\vec{\zeta},\zeta}(\vec{\mathcal{V}}^{\mathsf{op}}, \mathcal{V}^{\mathsf{op}}))\,.$$

By inductive hypothesis we have $\mathsf{fold}(\mathcal{V}(\psi)_{\vec{\zeta},\zeta}(\vec{\mathcal{V}}^{\mathsf{op}}, \mathcal{V}^{\mathsf{op}})) \sqsubseteq \mathsf{fold}(\mathcal{V}(\psi)_{\vec{\zeta},\zeta}(\vec{\mathcal{V}}, \mathcal{V}))^{\mathsf{op}} = \mathcal{V}^{\mathsf{op}}$ since $\mathcal{V} = \mathsf{fold}(\mathcal{V}(\psi)_{\vec{\zeta},\zeta}(\vec{\mathcal{V}}, \mathcal{V}))$ by Lemma 20. $\square$

We are left with proving the converse property, namely that $\mathcal{R}(\sigma)^{\mathsf{op}} \sqsubseteq \mathcal{R}(\sigma)$ for each closed type σ. This requires a bit more work, and is based on a notion of "finite" approximation of elements of the model, that we define by syntactic means as follows.

4.4.1. *Restriction operators.* We define[14] closed terms $\mathsf{p}^{\mathsf{g}}(n, \sigma)$ and $\mathsf{p}^{\mathsf{v}}(n, \varphi)$ (for $n \in \mathbb{N}$, σ a type and φ a positive type) typed as follows: $\vdash \mathsf{p}^{\mathsf{g}}(n, \sigma) : {!\sigma} \multimap \sigma$ and $\vdash \mathsf{p}^{\mathsf{v}}(n, \varphi) : \varphi \multimap \varphi$.

$$\mathsf{p}^{\mathsf{g}}(n, \varphi) = \lambda x^{!\varphi} \, \langle \mathsf{p}^{\mathsf{v}}(n, \varphi) \rangle \mathsf{der}\, x$$

$$\mathsf{p}^{\mathsf{g}}(n, \varphi \multimap \sigma) = \lambda f^{!(\varphi \multimap \sigma)} \, \lambda x^{\varphi} \, \langle \mathsf{p}^{\mathsf{g}}(n, \sigma) \rangle (\langle \mathsf{der}\, f \rangle \langle \mathsf{p}^{\mathsf{v}}(n, \varphi) \rangle x)^{!}$$

$$\mathsf{p}^{\mathsf{v}}(n, \mathbf{1}) = \lambda x^{\mathbf{1}} \, x$$

$$\mathsf{p}^{\mathsf{v}}(n, !\sigma) = \lambda x^{!\sigma} \, (\langle \mathsf{p}^{\mathsf{g}}(n, \sigma) \rangle x)^{!}$$

$$\mathsf{p}^{\mathsf{v}}(n, \varphi_\ell \otimes \varphi_r) = \lambda x^{\varphi_\ell \otimes \varphi_r} \, (\langle \mathsf{p}^{\mathsf{v}}(n, \varphi_\ell) \rangle \mathsf{pr}_\ell x, \langle \mathsf{p}^{\mathsf{v}}(n, \varphi_r) \rangle \mathsf{pr}_r x)$$

$$\mathsf{p}^{\mathsf{v}}(n, \varphi_\ell \oplus \varphi_r) = \lambda x^{\varphi_\ell \oplus \varphi_r} \, \mathsf{case}(x, x_\ell \cdot \mathsf{in}_\ell \langle \mathsf{p}^{\mathsf{v}}(n, \varphi_\ell) \rangle x_\ell, x_r \cdot \mathsf{in}_r \langle \mathsf{p}^{\mathsf{v}}(n, \varphi_r) \rangle x_r)$$

$$\mathsf{p}^{\mathsf{v}}(0, \mathsf{Rec}\, \zeta \cdot \varphi) = \lambda x^{\mathsf{Rec}\, \zeta \cdot \varphi} \, \Omega^{\mathsf{Rec}\, \zeta \cdot \varphi}$$

$$\mathsf{p}^{\mathsf{v}}(n+1, \mathsf{Rec}\, \zeta \cdot \varphi) = \lambda x^{\mathsf{Rec}\, \zeta \cdot \varphi} \, \mathsf{fold}(\langle \mathsf{p}^{\mathsf{v}}(n, \varphi\, [\mathsf{Rec}\, \zeta \cdot \varphi/\zeta]) \rangle \mathsf{unfold}(x))$$

This is a well-founded lexicographic inductive definition on triples (n, σ, l) (where $l \in \{\mathsf{v}, \mathsf{g}\}$) if we order the symbols v and g by $\mathsf{v} < \mathsf{g}$. We consider actually only triples (n, σ, l) such that σ is positive if $l = \mathsf{v}$.

We describe similarly the interpretation of these terms: we give an explicit description of the matrices $[\mathsf{p}^{\mathsf{g}}(n, \sigma)]$ and $[\mathsf{p}^{\mathsf{v}}(n, \varphi)]$. To this end, we define a family of sets $\mathsf{I}^l(n, \sigma) \subseteq |[\sigma]|$ by induction on (n, σ, l) (where $n \in \mathbb{N}$, $l \in \{\mathsf{v}, \mathsf{g}\}$ and σ is a closed type which is positive if $l = \mathsf{v}$).

- $\mathsf{I}^{\mathsf{v}}(n, !\sigma) = \mathcal{M}_{\mathrm{fin}}(\mathsf{I}^{\mathsf{g}}(n, \sigma))$
- $\mathsf{I}^{\mathsf{v}}(n, \varphi_\ell \otimes \varphi_r) = \mathsf{I}^{\mathsf{v}}(n, \varphi_\ell) \times \mathsf{I}^{\mathsf{v}}(n, \varphi_r)$
- $\mathsf{I}^{\mathsf{v}}(n, \varphi_\ell \oplus \varphi_r) = \{\ell\} \times \mathsf{I}^{\mathsf{v}}(n, \varphi_\ell) \cup \{r\} \times \mathsf{I}^{\mathsf{v}}(n, \varphi_r)$
- $\mathsf{I}^{\mathsf{v}}(0, \mathsf{Rec}\, \zeta \cdot \psi) = \emptyset$
- $\mathsf{I}^{\mathsf{v}}(n+1, \mathsf{Rec}\, \zeta \cdot \psi) = \mathsf{I}^{\mathsf{v}}(n, \varphi\, [\mathsf{Rec}\, \zeta \cdot \psi/\zeta])$

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[14]This definition as well as our reasoning below features some similarities with *step-indexing* that we would like to understand better.

- $\mathsf{I}^{\mathsf{g}}(n,\varphi) = \mathsf{I}^{\mathsf{v}}(n,\varphi)$
- $\mathsf{I}^{\mathsf{g}}(n,\varphi \multimap \sigma) = \mathsf{I}^{\mathsf{v}}(n,\varphi) \times \mathsf{I}^{\mathsf{g}}(n,\sigma)$.

Lemma 32. *Let $n \in \mathbb{N}$, φ be a closed positive type and σ be a closed type. One has*

$$[\mathsf{p}^{\mathsf{v}}(n,\varphi)]_{(a,b)} = \begin{cases} 1 & \text{if } a = b \in \mathsf{I}^{\mathsf{v}}(n,\varphi) \\ 0 & \text{otherwise.} \end{cases} \qquad [\mathsf{p}^{\mathsf{g}}(n,\sigma)]_{(c,b)} = \begin{cases} 1 & \text{if } c = [b] \text{ and } b \in \mathsf{I}^{\mathsf{g}}(n,\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 16, for a closed positive type φ and for $u \in \mathsf{P}^!{[\varphi]}^!$, it suffices to prove that

$$[\mathsf{p}^{\mathsf{v}}(n,\varphi)](u)_a = \begin{cases} u_a & \text{if } a \in \mathsf{I}^{\mathsf{v}}(n,\varphi) \\ 0 & \text{otherwise} \end{cases}$$

And for a closed type σ and for $u \in \mathsf{P}[\sigma]$, it suffices to prove

$$[\mathsf{p}^{\mathsf{g}}(n,\sigma)](u^!)_a = \begin{cases} u_a & \text{if } a \in \mathsf{I}^{\mathsf{g}}(n,\sigma) \\ 0 & \text{otherwise} \end{cases}$$

Both statements are easily proved by induction. $\qquad\square$

We need now to prove that, given $u \in \mathsf{P}[\sigma]$, the sequence $[\mathsf{p}^{\mathsf{g}}(n,\sigma)](u^!)$ is monotone and has u as lub.

Lemma 33. *For any triple (n,σ,l) where σ is positive if $l = \mathsf{v}$, one has $\mathsf{I}^l(n,\sigma) \subseteq \mathsf{I}^l(n+1,\sigma)$. Moreover*

$$\bigcup_{n=0}^{\infty} \mathsf{I}^l(n,\sigma) = ||[\sigma]||.$$

Proof. The first statement is straightforward, by induction on (n,σ,l). For the second statement, we only have to prove the right-to-left inclusions. We define the *height* $\mathsf{h}(a)$ of an element a of $||[\sigma]||$ as follows.

- $\mathsf{h}(*) = 1$
- $\mathsf{h}(a_1,a_2) = 1 + \max(\mathsf{h}(a_1),\mathsf{h}(a_2))$ (this definition is used when φ is a tensor and when σ is a linear implication)
- $\mathsf{h}(i,a) = 1 + \mathsf{h}(a)$
- $\mathsf{h}([a_1,\ldots,a_k]) = 1 + \max(\mathsf{h}(a_1),\ldots,\mathsf{h}(a_k))$

Then by induction on $\mathsf{h}(a)$ one proves that

$$\forall a \in ||[\sigma]|| \; \exists n \in \mathbb{N} \; a \in \mathsf{I}^l(n,\sigma)$$

We deal only with the statement relative to $\mathsf{I}^{\mathsf{v}}(n,\varphi)$. The closed positive type φ is of shape

$$\varphi = \mathsf{Rec}\,\zeta_1 \cdot \ldots \cdot \mathsf{Rec}\,\zeta_k \cdot \psi$$

where ψ is not of shape $\mathsf{Rec}\,\zeta \cdot \rho$. We introduce auxiliary closed types $\varphi_1,\ldots,\varphi_k$ as follows:

$$\varphi_1 = \varphi = \mathsf{Rec}\,\zeta_1 \cdot \ldots \cdot \mathsf{Rec}\,\zeta_k \cdot \psi$$
$$\varphi_2 = \mathsf{Rec}\,\zeta_2 \cdot \ldots \cdot \mathsf{Rec}\,\zeta_k \cdot \psi\,[\varphi_1/\zeta_1]$$
$$\vdots$$
$$\varphi_{k+1} = \psi\,[\varphi_1/\zeta_1, \varphi_2/\zeta_2, \ldots, \varphi_k/\zeta_k]$$

and all these types have the same interpretation in **Pcoh**$^!$. The type ψ cannot be one of the type variables ζ_i as otherwise we would have $||[\varphi]|| = \emptyset$, contradicting our assumption that a belongs to this set. Assume that $\psi = !\sigma$ so that we must have $a = [b_1, \ldots, b_l]$ with $b_i \in ||[\sigma']||$ (where $\sigma' = \sigma\,[\varphi_1/\zeta_1, \varphi_2/\zeta_2, \ldots, \varphi_k/\zeta_k]$) for each $i = 1, \ldots, l$. We have $\mathsf{h}(b_i) < \mathsf{h}(a)$ for each i so that we can apply the inductive hypothesis: for each i there is n_i such that $b_i \in \mathsf{l}^{\mathsf{g}}(n_i, \sigma')$. Using the monotonicity property (first statement of the lemma) and setting $n = \max(n_1, \ldots, n_l)$ we have $b_i \in \mathsf{l}^{\mathsf{g}}(n, \sigma')$ and hence $a \in \mathsf{l}^{\mathsf{v}}(n, !\sigma')$. Therefore $a \in \mathsf{l}^{\mathsf{v}}(n + k, \varphi)$ (coming back to the definition of this set). The other cases are dealt with similarly. $\quad\square$

Lemma 34. *Let σ be a closed type and let φ be a closed positive type. If $u \in \mathsf{P}[\sigma]$ then the sequence $([\mathsf{p}^{\mathsf{g}}(n, \sigma)]\, u^!)_{n \in \mathbb{N}}$ is monotone (in $\mathsf{P}[\sigma]$) and has u as lub. If $u \in \mathsf{P}[\varphi]$ then the sequence $([\mathsf{p}^{\mathsf{v}}(n, \varphi)]\, u)_{n \in \mathbb{N}}$ is monotone and has u as lub.*

Proof. Immediate consequence of Lemmas 32 and 33. $\quad\square$

4.4.2. *Main Inclusion Lemma.* Now we are in position of proving the key lemma in the proof of the uniqueness of relations.

Lemma 35. *Let σ be a closed type, $n \in \mathbb{N}$ and $l \in \{\mathsf{v}, \mathsf{g}\}$. If $l = \mathsf{g}$ and $(M, u) \in \mathcal{R}(\sigma)^-$ then $(M, [\mathsf{p}^{\mathsf{g}}(n, \sigma)]\, u^!) \in \mathcal{R}(\sigma)^+$. If $l = \mathsf{v}$ and σ is a closed positive type φ then $(V, v) \in \mathcal{V}(\varphi)^- \Rightarrow (V, [\mathsf{p}^{\mathsf{v}}(n, \varphi)]\, v) \in (\overline{\mathcal{V}(\varphi)})^+ = \mathcal{R}(\varphi)^+$.*

Proof. By lexicographic induction on triples (n, σ, l) (with σ positive when $l = \mathsf{v}$).

Until further notice, we assume that $l = \mathsf{v}$.

The only case where "n decreases" in this induction is when $\varphi = \mathsf{Rec}\,\zeta \cdot \psi$, we start with this case.

Assume that $\varphi = \mathsf{Rec}\,\zeta \cdot \psi$ and that $(V, v) \in \mathcal{V}(\varphi)^-$. If $n = 0$ we have $[\mathsf{p}^{\mathsf{v}}(n, \varphi)]\, v = 0$ and hence $(V, [\mathsf{p}^{\mathsf{v}}(n, \varphi)]\, v) \in \mathcal{R}(\varphi)^+$ by Lemma 30. Assume that the implication holds for n and let us prove it for $n + 1$. Let $(V, v) \in \mathcal{V}(\varphi)^-$, that is $V = \mathsf{fold}(W)$ with $(W, v) \in \mathcal{V}(\psi\,[\varphi/\zeta])^-$. We have $[\mathsf{p}^{\mathsf{v}}(n + 1, \mathsf{Rec}\,\zeta \cdot \psi)]\, v = [\mathsf{p}^{\mathsf{v}}(n, \psi\,[\varphi/\zeta])]\, v$ by definition. By inductive hypothesis we have

$$(W, [\mathsf{p}^{\mathsf{v}}(n, \psi\,[\varphi/\zeta])]\, v) \in \mathcal{R}(\psi\,[\varphi/\zeta])^+ \tag{4.6}$$

and we must prove that $(\mathsf{fold}(W), [\mathsf{p}^{\mathsf{v}}(n, \psi\,[\varphi/\zeta])]\, v) \in \mathcal{R}(\varphi)^+$. Let $(T, t) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\mathbf{1}))^-$, we must prove that $(\langle T \rangle \mathsf{fold}(W), t([\mathsf{p}^{\mathsf{v}}(n, \psi\,[\varphi/\zeta])]\, v)) \in \mathcal{R}(\mathbf{1})$. Let $S = \lambda x^{\psi[\varphi/\zeta]}\, \langle T \rangle \mathsf{fold}(x)$, we have $(S, t) \in (\mathcal{V}(\psi\,[\varphi/\zeta]) \multimap \mathcal{R}(\mathbf{1}))^-$ by Lemma 21 and therefore

$$(\langle S \rangle W, t([\mathsf{p}^{\mathsf{v}}(n, \psi\,[\varphi/\zeta])]\, v)) \in \mathcal{R}(\mathbf{1})$$

by (4.6) and Lemma 22 and this implies $(\langle T \rangle \mathsf{fold}(W), t([\mathsf{p}^{\mathsf{v}}(n, \psi\,[\varphi/\zeta])]\, v)) \in \mathcal{R}(\mathbf{1})$ by Lemma 21.

Assume that $\varphi = !\sigma$ and that $(V, v) \in \mathcal{V}(!\sigma)^-$, that is $V = M^!$ and $v = u^!$ with $(M, u) \in \mathcal{R}(\sigma)^-$. By inductive hypothesis we have $(M, [\mathsf{p}^{\mathsf{g}}(n, \sigma)]\, u^!) \in \mathcal{R}(\sigma)^+$ and hence $(M^!, ([\mathsf{p}^{\mathsf{g}}(n, \sigma)]\, u^!)^!) \in \mathcal{V}(!\sigma)^+$ and since $([\mathsf{p}^{\mathsf{g}}(n, \sigma)]\, u^!)^! = [\mathsf{p}^{\mathsf{v}}(n, !\sigma)]\, u^!$ we get

$$(V, [\mathsf{p}^{\mathsf{v}}(n, !\sigma)]\, v) \in \mathcal{V}(!\sigma)^+ \subseteq \mathcal{R}(!\sigma)^+$$

as expected.

Assume that $\varphi = \varphi_\ell \otimes \varphi_r$ and that $(V, v) \in \mathcal{V}(\varphi_\ell \otimes \varphi_r)^-$, that is $V = (V_\ell, V_r)$ and $v = v_\ell \otimes v_r$ with $(V_i, v_i) \in \mathcal{V}(\varphi_i)^-$ for $i \in \{\ell, r\}$. By inductive hypothesis we have $(V_i, [\mathsf{p}^{\mathsf{g}}(n, \varphi_i)]\, v_i) \in$

$\mathcal{R}(\varphi_i)^+$. By Lemma 23 we get $((V_\ell, V_r), [\mathsf{p}^\mathsf{g}(n, \varphi_\ell)]\, v_\ell \otimes [\mathsf{p}^\mathsf{g}(n, \varphi_r)]\, v_r) \in \mathcal{R}(\varphi_\ell \otimes \varphi_r)^+$, that is $((V_\ell, V_r), [\mathsf{p}^\mathsf{g}(n, \varphi_\ell \otimes \varphi_r)]\, (v_\ell \otimes v_r)) \in \mathcal{R}(\varphi_\ell \otimes \varphi_r)^+$.

Assume that $\varphi = \varphi_\ell \oplus \varphi_r$ and $(V, v) \in \mathcal{V}(\varphi_\ell \oplus \varphi_r)^-$. This means that for some $i \in \{\ell, r\}$, one has $V = \mathsf{in}_i W$ and $v = \mathsf{in}_i w$ for $(W, w) \in \mathcal{V}(\varphi_i)^-$. By inductive hypothesis we have $(W, [\mathsf{p}^\mathsf{g}(n, \varphi_i)]\, w) \in \mathcal{R}(\varphi_i)^+$ and hence $(\mathsf{in}_i W, \mathsf{in}_i([\mathsf{p}^\mathsf{g}(n, \varphi_i)]\, w)) \in \mathcal{R}(\varphi_\ell \oplus \varphi_r)^+$ by Lemma 25, that is $(\mathsf{in}_i W, [\mathsf{p}^\mathsf{g}(n, \varphi_\ell \oplus \varphi_r)]\, w) \in \mathcal{R}(\varphi_\ell \oplus \varphi_r)^+$.

We assume now that $l = \mathsf{g}$.

If σ is a closed positive type φ and let $(M, u) \in \mathcal{R}(\sigma)^-$, we must prove that

$$(M, [\mathsf{p}^\mathsf{g}(n, \sigma)]\, u^!) \in \mathcal{R}(\sigma)^+$$

which follows directly from the definition of $\mathsf{p}^\mathsf{g}(n, \varphi)$ and from the inductive hypothesis applied to (n, φ, v).

Assume last that $\sigma = \varphi \multimap \tau$ and that $(M, u) \in \mathcal{R}(\varphi \multimap \tau)^-$, we must prove that $(M, [\mathsf{p}^\mathsf{g}(n, \varphi \multimap \tau)]\, u^!) \in \mathcal{R}(\varphi \Rightarrow \tau)^+$. Let $(V, v) \in \mathcal{V}(\varphi)^-$, we must prove that

$$(\langle M \rangle V, [\mathsf{p}^\mathsf{g}(n, \varphi \multimap \tau)]\, u^!\, v) \in \mathcal{R}(\tau)^+ \qquad (4.7)$$

which follows from the fact that $[\mathsf{p}^\mathsf{g}(n, \varphi \multimap \tau)]\, u^!\, v = [\mathsf{p}^\mathsf{g}(n, \tau)]\, (u([\mathsf{p}^\mathsf{v}(n, \varphi)]\, v))^!$. Indeed the inductive hypothesis applied to (n, φ) yields $(V, [\mathsf{p}^\mathsf{v}(n, \varphi)]\, v) \in \mathcal{R}(\varphi)^+$ and hence $(\langle M \rangle V, u([\mathsf{p}^\mathsf{v}(n, \varphi)]\, v)) \in \mathcal{R}(\tau)^+$ by Lemma 22, from which we derive (4.7) by Lemma 34 and Lemma 30. $\qquad \square$

Lemma 36. *For any closed type σ one has $\mathcal{R}(\sigma)^- = \mathcal{R}(\sigma)^+$.*

Proof. Immediate consequence of lemmas 30, 34 and 35. $\qquad \square$

From now on we simply use the notation $\mathcal{R}(\sigma)$ instead of $\mathcal{R}(\sigma)^-$ and $\mathcal{R}(\sigma)^+$.

4.5. **Logical relation lemma.** We can prove now the main result of this section.

Theorem 37 (Logical Relation Lemma). *Assume that $x_1 : \varphi_1, \ldots, x_k : \varphi_k \vdash M : \sigma$ and let $(V_i, v_i) \in \mathcal{R}(\varphi_i)$ (where V_i is a value and $v_i \in \mathsf{P}^![\varphi_i]$) for $i = 1, \ldots, k$. Then $(M\,[V_1/x_1, \ldots, V_k/x_k], [M]^{x_1, \ldots, x_k}\, \vec{v}) \in \mathcal{R}(\sigma)$ where $\vec{v} = (v_1, \ldots, v_k)$.*

Remark. One would expect to have rather assumptions of the shape "$(V_i, v_i) \in \mathcal{V}(\varphi_i)$"; the problem is that we don't know whether $\mathcal{V}(\varphi_i)^+ = \mathcal{V}(\varphi_i)^-$.

Proof. By induction on the typing derivation of M, that is, on M. We set $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$ and, given a term R, we use R' for the term $R\,[V_1/x_1, \ldots, V_k/x_k]$. We also use $\vec{v}$ for the sequence $v_1, \ldots, v_k$ and $\vec{x}$ for the sequence $x_1, \ldots, x_k$.

The case $M = x_i$ is straightforward.

Assume that $M = N^!$ and that $\varphi = !\sigma$ with $\mathcal{P} \vdash N : \sigma$. By inductive hypothesis we have $(N', [N]^{\vec{x}}\, \vec{v}) \in \mathcal{R}(\sigma)$. Therefore $((N')^!, ([N]^{\vec{x}}\, \vec{v})^!) \in \mathcal{V}(!\sigma)^\varepsilon$ (for $\varepsilon = +$ or $\varepsilon = -$)[15]. We have $\mathcal{V}(!\sigma)^\varepsilon \subseteq \overline{\mathcal{V}(!\sigma)^\varepsilon} = \mathcal{R}(!\sigma)^\varepsilon = \mathcal{R}(!\sigma)$ and hence $(M', [M]^{\vec{x}}\, \vec{v})) \in \mathcal{R}(!\sigma)$ as contended.

Assume that $M = (N_\ell, N_r)$ and $\sigma = \psi_\ell \otimes \psi_r$ with $\mathcal{P} \vdash N_i : \psi_i$ for $i \in \{\ell, r\}$. By inductive hypothesis we have $(N_i', [N_i]^{\vec{x}}\, \vec{v}) \in \mathcal{R}(\psi_i)$. By Lemma 23 we get

$$((N_\ell', N_r'), [(N_\ell, N_r)]^{\vec{x}}\, \vec{v}) \in \mathcal{R}(\psi_\ell \otimes \psi_r)$$

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[15]It is not clear whether $\mathcal{V}(\varphi)^- = \mathcal{V}(\varphi)^+$ for any closed positive type φ, but we don't need this property in this proof, so we leave this technical question unanswered.

as contended, since $[(N_\ell, N_r)]^{\overrightarrow{x}} \, \overrightarrow{v} = [N_\ell]^{\overrightarrow{x}} \, \overrightarrow{v} \otimes [N_r]^{\overrightarrow{x}} \, \overrightarrow{v}$.

The case $M = \mathsf{in}_i N$ (and $\sigma = \psi_\ell \oplus \psi_r$) is handled similarly, using Lemma 25.

Assume that $M = \mathsf{fold}(N)$ and $\sigma = \varphi = \mathsf{Rec}\,\zeta \cdot \psi$ with $\mathcal{P} \vdash N : \psi\,[\varphi/\zeta]$. By inductive hypothesis we have $(N', [N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\psi\,[\varphi/\zeta])$ which implies $(\mathsf{fold}(N'), [N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\varphi)$ by Lemma 29.

Assume that $M = \lambda x^\varphi \, N$ and $\sigma = \varphi \multimap \tau$, with $\mathcal{P}, x : \varphi \vdash N : \tau$. We must prove that $(\lambda x^\varphi \, N', [\lambda x^\varphi \, N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in (\mathcal{V}(\varphi) \multimap \mathcal{R}(\tau))^\varepsilon$ for an arbitrary $\varepsilon \in \{-, +\}$. So let $(V, v) \in \mathcal{V}(\varphi)^{-\varepsilon}$. Since $\mathcal{V}(\varphi)^{-\varepsilon} \subseteq \mathcal{R}(\varphi)$, we have $(N'\,[V/x], [N]^{\overrightarrow{x}, x}\,(\overrightarrow{v}, v)) \in \mathcal{R}(\tau)$ by inductive hypothesis. It follows that $(\langle \lambda x^\varphi \, N' \rangle V, [\lambda x^\varphi \, N]^{\overrightarrow{x}} \, \overrightarrow{v} \, v) \in \mathcal{R}(\tau)$ by Lemma 21, proving our contention.

Assume that $M = \langle R \rangle N$ with $\mathcal{P} \vdash R : \varphi \multimap \sigma$ and $\mathcal{P} \vdash N : \varphi$ where φ is a closed positive type. By inductive hypothesis we have $(R', [R]^{\overrightarrow{x}}\,(\overrightarrow{v})) \in \mathcal{R}(\varphi \multimap \sigma)$ and $(N', [N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\varphi)$ and hence $(\langle R' \rangle N', [R]^{\overrightarrow{x}} \, \overrightarrow{v}\,([N]^{\overrightarrow{x}} \, \overrightarrow{v})) \in \mathcal{R}(\sigma)$ by Lemma 22, that is $(M', [M]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\sigma)$.

Assume that $M = \mathsf{fix}\,x^{!\sigma}\,N$ with $\mathcal{P}, x : !\sigma \vdash N : \sigma$. The function $f : \mathsf{P}[\sigma] \to \mathsf{P}[\sigma]$ defined by

$$f(u) = [N]^{\overrightarrow{x}, x}\,(\overrightarrow{v}, u^!)$$

is Scott continuous and we have $[M]^{\overrightarrow{x}} \, \overrightarrow{v} = \sup_{k \in \mathbb{N}} f^k(0)$. By induction on k, we prove that

$$\forall k \in \mathbb{N} \quad (M', f^k(0)) \in \mathcal{R}(\sigma)\,. \tag{4.8}$$

The base case is proven by Lemma 30. Assume that $(M', f^k(0)) \in \mathcal{R}(\sigma)$. Choosing an arbitrary ε, we have $((M')^!, (f^k(0))^!) \in \mathcal{V}(!\sigma)^\varepsilon \subseteq \mathcal{R}(!\sigma)$ and hence by our "outermost" inductive hypothesis we have $(N'\,[(M')^!/x], f^{k+1}(0)) \in \mathcal{R}(\sigma)$ from which we get $(M', f^{k+1}(0)) \in \mathcal{R}(\sigma)$ by Lemma 21 and this ends the proof of (4.8). We conclude that $(M', [M]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\sigma)$ by Lemma 30.

Assume that $M = \mathsf{der}\,N$ with $\mathcal{P} \vdash N : !\sigma$. By inductive hypothesis we have $(N', [N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(!\sigma)$ which implies $(\mathsf{der}\,N', \mathsf{der}([N]^{\overrightarrow{x}} \, \overrightarrow{v})) \in \mathcal{R}(\sigma)$ by Lemma 27, that is $(M', [M]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\sigma)$.

Assume that $M = \mathsf{pr}_j N$ with $j \in \{\ell, r\}$, $\sigma = \varphi_\ell \otimes \varphi_r$ and $\mathcal{P} \vdash M : \varphi_\ell \otimes \varphi_r$. By inductive hypothesis we have $(N', [N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\varphi_\ell \otimes \varphi_r)$ and hence $(\mathsf{pr}_j N', \mathrm{pr}_j\,([N]^{\overrightarrow{x}} \, \overrightarrow{v})) \in \mathcal{R}(\varphi_j)$ by Lemma 24 that is $(M', [M]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\varphi_j)$.

Assume that $M = \mathsf{case}(N, y_\ell \cdot R_\ell, y_r \cdot R_r)$ with $\mathcal{P} \vdash N : \varphi_\ell \oplus \varphi_r$ and $\mathcal{P}, y_j : \varphi_j \vdash R_j : \sigma$ for $j \in \{\ell, r\}$. By inductive hypothesis we have $(N', [N]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\varphi_\ell \oplus \varphi_r)$ and $(\lambda y_j^{\varphi_j}\,R_j', [\lambda y_j^{\varphi_j}\,R_j]^{\overrightarrow{x}} \, \overrightarrow{v}) \in \mathcal{R}(\varphi_j \multimap \sigma)$ for $j \in \{\ell, r\}$ (to prove this latter fact, one chooses $\varepsilon \in \{-, +\}$ and considers an arbitrary $(V, v) \in \mathcal{V}(\varphi_j)^{-\varepsilon}$, we have $(V, v) \in \mathcal{R}(\varphi_j)$ and hence $(R_j'\,[V/y_j], [R_j]^{\overrightarrow{x}, y_j}\,(\overrightarrow{v}, v)) \in \mathcal{R}(\sigma)$ by inductive hypothesis, which implies

$$(\langle \lambda y_j^{\varphi_j}\,R_j' \rangle V, [\lambda y_j^{\varphi_j}\,R_j]^{\overrightarrow{x}} \, \overrightarrow{v} \, v) \in \mathcal{R}(\sigma)$$

by Lemma 21). By Lemma 26 we get

$$(\mathsf{case}(N', y_\ell \cdot R_\ell', y_r \cdot R_r'), \mathrm{case}([\lambda y_\ell^{\varphi_\ell}\,R_\ell]^{\overrightarrow{x}} \, \overrightarrow{v}), [\lambda y_r^{\varphi_r}\,R_r]^{\overrightarrow{x}} \, \overrightarrow{v}))([N]^{\overrightarrow{x}} \, \overrightarrow{v})) \in \mathcal{R}(\sigma)$$

that is $(M', [M]^{\overrightarrow{x}}\,(\overrightarrow{v})) \in \mathcal{R}(\sigma)$, by Lemma 21.

Assume that $M = \mathsf{unfold}(N)$ where $\mathcal{P} \vdash N : \varphi$ with $\varphi = \mathsf{Rec}\,\zeta \cdot \psi$. We apply Lemma 28 straightforwardly.

Assume that $M = ()$ and the typing derivation consists of the axiom $\mathcal{P} \vdash () : \mathbf{1}$ so that $\sigma = \mathbf{1}$. We have $(M, [M]) \in \mathcal{R}(\mathbf{1})$ by definition since $\mathsf{Red}^\infty_{M,()} = 1 = [M]$.

Assume last that $M = \mathsf{coin}(p)$ for some $p \in [0,1] \cap \mathbb{Q}$ and the typing derivation consists of the axiom $\mathcal{P} \vdash \mathsf{coin}(p) : \mathbf{1} \oplus \mathbf{1}$ so that $\sigma = \mathbf{1} \oplus \mathbf{1}$. We must prove that $(\mathsf{coin}(p), [\mathsf{coin}(p)]) \in \mathcal{R}(\mathbf{1} \oplus \mathbf{1})$. Remember that $[\mathsf{coin}(p)] = pe_{(\ell,*)} + (1-p)e_{(r,*)}$. Let $\varepsilon \in \{-,+\}$ and let $(T,t) \in (\mathcal{V}(\mathbf{1} \oplus \mathbf{1}) \multimap \mathcal{R}(\mathbf{1}))^\varepsilon$, we must prove that $(\langle T\rangle \mathsf{coin}(p), t(pe_{(\ell,*)} + (1-p)e_{(r,*)})) \in \mathcal{R}(\mathbf{1})$. We have

$$\mathsf{Red}^\infty_{\langle T\rangle \mathsf{coin}(p),()} = p\mathsf{Red}^\infty_{\langle T\rangle \mathsf{in}_\ell(),()} + (1-p)\mathsf{Red}^\infty_{\langle T\rangle \mathsf{in}_r(),()}$$

since the first reduction step must be $\mathsf{coin}(p) \overset{p}{\to} \mathsf{in}_\ell()$ or $\mathsf{coin}(p) \overset{1-p}{\to} \mathsf{in}_r()$. By our assumption on (T,t) we have $\mathsf{Red}^\infty_{\langle T\rangle \mathsf{in}_i()} \geq t(e_{(i,*)})$ and hence $\mathsf{Red}^\infty_{\langle T\rangle \mathsf{coin}(p),()} \geq t(pe_{(\ell,*)} + (1-p)e_{(r,*)})$ as contended, by linearity of t. $\qquad\square$

Theorem 38 (Adequacy). *Let M be a closed term such that $\vdash M : \mathbf{1}$. Then $[M] = \mathsf{Red}^\infty_{M,()}$.*

By Corollary 18 and Theorem 37.

5. Full Abstraction

We prove now the Full Abstraction Theorem 41, that is the converse of the Adequacy Theorem.

5.1. **Outline of the proof.** We reason by contradiction and assume that two closed terms M_1 and M_2 have different semantics. Remember from Section 3.2, that a closed term of type σ is interpreted as a vector with indices in the web $|[\sigma]|$, so that there is $a \in |[\sigma]|$ such that $[M_1]_a \neq [M_2]_a$. We want to design a term that will separate M_1 and M_2 observationally.

We define a *testing term* $\vdash a^- : !\iota \multimap (!\sigma \multimap \mathbf{1})$ that will depend only on the structure of the element a of the web. We then use properties of the semantics (namely that terms of type $!\iota \multimap \tau$ can be seen as power series) to find reals $\vec{p}$ such that $\langle a^-\rangle \mathsf{ran}(\vec{p})^!$ separates M_1 and M_2:

$$\mathsf{Red}^\infty_{\langle\langle a^-\rangle \mathsf{ran}(\vec{p})^!\rangle M_1^!,()} \neq \mathsf{Red}^\infty_{\langle\langle a^-\rangle \mathsf{ran}(\vec{p})^!\rangle M_2^!,()}$$

Let us detail the key points of the proof. Remember from Section 3.2.7 that, because $\vdash a^- : !\iota \multimap (!\sigma \multimap \mathbf{1})$, its interpretation $[a^-]$ can be seen as a power series

$$\widehat{[a^-]}(\vec{\zeta}) = [a^-]\,\vec{\zeta}^{\,!} = \left(\sum_{[k_1,\ldots,k_n]\in|!\iota|} [a^-]_{[k_1,\ldots,k_n],b} \prod_{i=1}^n \zeta_{k_i}\right)_{b\in|!\sigma\multimap\mathbf{1}|}$$

with infinitely many parameters $\vec{\zeta} = (\zeta_0, \ldots, \zeta_n, \ldots)$. Moreover, if $\sum_{i=0}^\infty \zeta_i \leq 1$, then $\vec{\zeta} \in \mathsf{P}[\iota]$ and $\widehat{[a^-]}(\vec{\zeta}) \in \mathsf{P}[!\sigma \multimap \mathbf{1}]$ (see Theorem 6).

The first key point is to remark that the testing term a^- is defined in such a way that $[a^-]$ has actually only finitely many parameters $\zeta_0, \ldots, \zeta_{|a|^-}$ (meaning that if the support of the multiset c is not included in $\{0, \ldots, |a|^-\}$, then $[a^-]_{(c,b)} = 0$). Now, for any $u \in \mathsf{P}[\sigma]$, $\left[\lambda y^{!\sigma} \lambda x^{!\iota} \langle\langle a^-\rangle x\rangle y\right] u^! \in \mathsf{P}[!\iota \multimap \mathbf{1}]$. It is also a power series that depends on the same finitely many parameters $\zeta_0, \ldots, \zeta_{|a|^-}$.

The second key point is a separation property of $[a^-]$: we prove that, in the power series $\left[\lambda y^{!\sigma}\,\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle y\right]u^!$, the coefficient of the unitary monomial[16] $\prod_{k=0}^{|a|^-}\zeta_k$ is equal to $\mathsf{m}^-(a)\,u_a$ with a coefficient $\mathsf{m}^-(a)\neq 0$ which depends only on a. Now, by assumption, $[M_1]_a$ and $[M_2]_a$ have different coefficient. For $i=1,2$, we have $\left[\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle M_i^!\right]=\left[\lambda y^{!\sigma}\,\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle y\right][M_i]^!$. Thus, the power series $\left[\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle M_i^!\right]$ (for $i=1,2$) have *different coefficients*.

The last key point uses classical analysis: if two power series with non-negative real coefficients and finitely many parameters have different coefficients, then they differ on non-negative arguments $\vec{p}$ close enough to zero: $\sum p_i\leq 1$, so that $\vec{p}\in\mathsf{P}[\iota]$ and $\left[\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle M_1^!\right]\vec{p}^{\,!}\neq\left[\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle M_2^!\right]\vec{p}^{\,!}$.

Finally, in order to substitute in a^- the parameters $\vec{\zeta}$ with the reals $\vec{p}$, we use $\mathsf{ran}(\vec{p})$ as introduced in Paragraph 2.3. Indeed, $\left[\langle\langle a^-\rangle\mathsf{ran}(\vec{p})^!\rangle M_i^!\right]=\left[\lambda x^{!\iota}\,\widehat{\langle\langle a^-\rangle x\rangle M_i^!}\right]\vec{p}^{\,!}$. We conclude thanks to the Adequacy Theorem 38 that ensures that

$$\mathsf{Red}^\infty_{\langle\langle a^-\rangle\mathsf{ran}(\vec{p})^!\rangle M_1^!,()}\neq\mathsf{Red}^\infty_{\langle\langle a^-\rangle\mathsf{ran}(\vec{p})^!\rangle M_2^!,()}$$

5.2. **Notations.** In order to define the testing term a^-, we will reason by induction and we will need to associate three kinds of *testing terms* with the points of the webs. More precisely:

- Given a positive type φ and $a\in|[\varphi]|$, we define a term a^0 such that
$$\vdash a^0:{!\iota}\multimap\varphi\multimap\mathbf{1}.$$

- Given a general type σ and $a\in|[\sigma]|$, we define terms a^+ and a^- such that
$$\vdash a^+:{!\iota}\multimap\sigma\qquad\qquad\vdash a^-:{!\iota}\multimap{!\sigma}\multimap\mathbf{1}.$$

We also introduce natural numbers $|a|^0$, $|a|^-$ and $|a|^+$ depending only on a. They represent the finite numbers of parameters on which the power series $\widehat{[a^0]}$, $\widehat{[a^-]}$ and $\widehat{[a^+]}$ depend respectively.

We denote as $\mathsf{m}^0(a)$, $\mathsf{m}^-(a)$ and $\mathsf{m}^+(a)$ natural numbers depending only on a and that will appear as the coefficient of the unitary monomial $\prod_{k=0}^{|a|^0}\zeta_k$, $\prod_{k=0}^{|a|^-}\zeta_k$ and $\prod_{k=0}^{|a|^+}\zeta_k$ respectively of the corresponding power series. These numbers are all >0.

We use the terms introduced in the probabilistic tests paragraph of Subsection 2.3 and whose semantics are given in Subsection 3.3:

- $\mathsf{ran}(\vec{p})$ which reduces to $\underline{i}$ with probability p_i for $\sum_{i=0}^{k-1}p_i\leq 1$,
- $M_0\cdot N$ which reduces to V with probability $p\,q$ if M_0 reduces to () with probability p and N reduces to V with probability q,
- $M_0\wedge\ldots\wedge M_{k-1}$ which reduces to () with probability $\prod_{i=0}^{k-1}p_i$ if M_i reduces to () with probability p_i,
- $\langle\mathsf{choose}_k^\sigma(M_0,\ldots,M_{k-1})\rangle P$ which reduces to M_i with probability p_i if p_i is the probability of P to reduce to $\underline{i}$,
- $\langle\mathsf{ext}\,(l,r)\rangle\mathsf{der}\,Z$ and $\langle\mathsf{win}_k(\vec{n})\rangle\mathsf{der}\,Z$ to partition the parameters $\mathsf{der}\,Z$. Indeed Z will denote a variable of type $!\iota$ and $\mathsf{der}\,Z$ has to be considered as the sequence of parameters $\vec{\zeta}$ of the power series interpreting testing terms. $|a|^0$, $|a|^-$ and $|a|^+$ represent the number of parameters on which the respective testing terms

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[16]That is, the monomial where each exponent is equal to one.

depend. We use $\langle \mathsf{win}_i(\vec{n}) \rangle \mathsf{der}\, Z$ to extract subsequences of $\vec{\zeta}$ that will be given as arguments to subterms in the inductive definition of the testing terms. Remember that $\langle \mathsf{win}_i(n_0, \ldots, n_k) \rangle \mathsf{der}\, Z \xrightarrow{p_l} \underline{l}$ if l is in the ith window of size n_i, that is $n_0 + \cdots + n_{i-1} \le l \le n_0 + \cdots + n_i - 1$ and p_l is the probability that $\mathsf{der}\, Z$ reduces to $\underline{l}$, that is the non-negative real parameter ζ_l. This is a key ingredient in the computation of the coefficient of the unitary monomial of the interpretation of testing terms by induction on type and on the structure of a.

5.3. Testing terms. We define the terms a^0, a^+ and a^- and the associated natural numbers $|a|^0$, $|a|^+$ and $|a|^-$, by induction on the structure of the point a.

Let φ be a positive type and $a \in |[\varphi]|$. We define a^0 and a^+ by induction on the size of a using the structure of φ

Let $\varphi = !\tau$ and $a = [b_0, \ldots, b_{k-1}]$ with $b_i \in |[\tau]|$. By inductive hypothesis, we have built terms $\vdash b_i^- : !\iota \multimap !\tau \multimap \mathbf{1}$ and $\vdash b_i^+ : !\iota \multimap \tau$. Then we set

$$a^0 = \lambda Z^{!\iota}\, \lambda x^{!\tau} \, \langle \langle b_0^- \rangle \big(\langle \mathsf{win}_0(|b_0|^-, \ldots, |b_{k-1}|^-) \rangle \mathsf{der}\, Z \big)^! \rangle x \wedge \cdots$$
$$\wedge \, \langle \langle b_{k-1}^- \rangle \big(\langle \mathsf{win}_{k-1}(|b_0|^-, \ldots, |b_{k-1}|^-) \rangle \mathsf{der}\, Z \big)^! \rangle x,$$
$$\mathsf{m}^0(a) = \prod_{i=0}^{k-1} \mathsf{m}^-(b_i), \quad \text{and} \quad |a|^0 = |b_0|^- + \cdots + |b_{k-1}|^-.$$

$$a^+ = \lambda Z^{!\iota} \, \Big(\langle \mathsf{choose}_k^\tau \big(\langle b_0^+ \rangle \big(\langle \mathsf{win}_1(k, |b_0|^+, \cdots, |b_{k-1}|^+) \rangle \mathsf{der}\, Z \big)^!, \ldots,$$
$$\langle b_{k-1}^+ \rangle \big(\langle \mathsf{win}_k(k, |b_0|^+, \cdots, |b_{k-1}|^+) \rangle \mathsf{der}\, Z \big)^! \big) \rangle \mathsf{der}\, Z \Big)^!,$$
$$\mathsf{m}^+(a) = a! \prod_{i=0}^{k-1} \mathsf{m}^+(b_i), \quad \text{and} \quad |a|^+ = k + |b_0|^+ + \cdots + |b_{k-1}|^+.$$

Remember that the factorial $a!$ of a multiset a has been defined in Paragraph 3.2.7 as the number of permutations that fix a.

If $\varphi = \varphi_\ell \otimes \varphi_r$ and $a = (b_\ell, b_r)$ with $b_i \in |[\varphi_i]|$ for $i \in \{\ell, r\}$, then we set

$$a^0 = \lambda Z^{!\iota} \, \lambda x^\varphi \, \langle \langle b_\ell^0 \rangle \big(\langle \mathsf{win}_0(|b_\ell|^0, |b_r|^0) \rangle \mathsf{der}\, Z \big)^! \rangle \mathsf{pr}_\ell x \wedge \langle \langle b_r^0 \rangle \big(\langle \mathsf{win}_1(|b_\ell|^0, |b_r|^0) \rangle \mathsf{der}\, Z \big)^! \rangle \mathsf{pr}_r x,$$
$$\mathsf{m}^0(a) = \mathsf{m}^0(b_\ell) \, \mathsf{m}^0(b_r), \quad \text{and} \quad |a|^0 = |b_\ell|^0 + |b_r|^0.$$

$$a^+ = \lambda Z^{!\iota} \, \Big(\langle b_\ell^+ \rangle \big(\langle \mathsf{win}_0(|b_\ell|^+, |b_r|^+) \rangle \mathsf{der}\, Z \big)^!, \langle b_r^+ \rangle \big(\langle \mathsf{win}_1(|b_\ell|^+, |b_r|^+) \rangle \mathsf{der}\, Z \big)^! \Big),$$
$$\mathsf{m}^+(a) = \mathsf{m}^+(b_\ell) \, \mathsf{m}^+(b_r) \quad \text{and} \quad |a|^+ = |b_\ell|^+ + |b_r|^+.$$

If $\varphi = \varphi_\ell \oplus \varphi_r$ and $a = (\ell, a_\ell)$ with $b \in ||[\varphi_\ell]||$ (the case $a = (r, a_r)$ is similar), then we set
$$a^0 = \lambda Z^{!\iota}\, \lambda x^{\varphi_\ell \oplus \varphi_r}\, \mathsf{case}(x, y_\ell \cdot \langle\langle a_\ell^0 \rangle Z\rangle y_\ell, y_r \cdot \Omega^{\mathbf{1}}), \quad \mathfrak{m}^0(a) = \mathfrak{m}^0(a_\ell) \quad \text{and} \quad |a|^0 = |a_\ell|^0\,.$$

$$a^+ = \lambda Z^{!\iota}\, \mathsf{in}_\ell\langle a_\ell^+ \rangle Z, \quad \mathfrak{m}^+(a) = \mathfrak{m}^+(a_\ell) \quad \text{and} \quad |a|^+ = |a_\ell|^+\,.$$

Finally, for a general type σ and $a \in ||[\sigma]||$, we define a^+ and a^-.

If $\sigma = \varphi$ is positive, then we have already defined a^+.
Let us now define a^-. This term does not depend on the structure of φ:
$$a^- = \lambda Z^{!\iota}\, \lambda x^{!\varphi}\, \langle\langle a^0 \rangle Z\rangle \mathsf{der}\, x, \quad \mathfrak{m}^-(a) = \mathfrak{m}^0(a) \quad \text{and} \quad |a|^- = |a|^0\,.$$

If $\sigma = \varphi \multimap \tau$ and $a = (b, c)$ with $b \in ||[\varphi]||$ and $c \in ||[\tau]||$, then we set
$$a^+ = \lambda Z^{!\iota}\, \lambda x^\varphi\, \langle\langle b^0 \rangle \left(\langle \mathsf{win}_0(|b|^0, |c|^+)\rangle \mathsf{der}\, Z\right)^!\rangle x \cdot \langle c^+ \rangle \left(\langle \mathsf{win}_1(|b|^0, |c|^+)\rangle \mathsf{der}\, Z\right)^!,$$
$$\mathfrak{m}^+(a) = \mathfrak{m}^0(b)\, \mathfrak{m}^+(c), \quad \text{and} \quad |a|^+ = |b|^0 + |c|^+\,.$$

$$a^- = \lambda Z^{!\iota}\, \lambda f^{!(\varphi \multimap \tau)}\, \langle\langle c^- \rangle \left(\langle \mathsf{win}_1(|b|^+, |c|^-)\rangle \mathsf{der}\, Z\right)^!\rangle \left(\langle \mathsf{der}\, f\rangle\langle b^+ \rangle \left(\langle \mathsf{win}_0(|b|^+, |c|^-)\rangle \mathsf{der}\, Z\right)^!\right)^!$$
$$\mathfrak{m}^-(a) = \mathfrak{m}^+(b)\, \mathfrak{m}^-(c), \quad \text{and} \quad |a|^- = |b|^+ + |c|^-\,.$$

It is easy to check that these terms satisfy the announced typing judgments. It is also clear that $\mathfrak{m}^0(a)$, $\mathfrak{m}^+(a)$ and $\mathfrak{m}^-(a)$ are non zero natural numbers.

We will now tackle the proof of the main observation: that is that the semantics of a^- is a power series with finitely many parameters and whose coefficient of the unitary monomial can be seen as a morphism in $\mathsf{P}[!\sigma \multimap \mathbf{1}]$.

Lemma 39 introduces notations for the unitary monomials and provides useful properties for proving the key Lemma 40 which gives the coefficients of these monomials.

Lemma 39. *Let σ be a general type and $t \in \mathsf{P}[!\iota \multimap \sigma]$.*

(1) *Assume that there is $k \in \mathbb{N}$ such that for any $c \in ||[\sigma]||$, the power series $\widehat{t}_c$ over $\mathsf{P}[\iota]$ depends on the k first parameters. For any $c \in ||[\sigma]||$, let us denote as $\mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right)_c$ the coefficient of the monomial $\zeta_0 \ldots \zeta_{k-1}$ of $\widehat{t}_c$. Then, $k^{-k}\, \mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right) \in \mathsf{P}[\sigma]$.*

(2) *Assume moreover that $\sigma = \varphi \multimap \tau$ where φ is a positive type and τ a general type. Let $m \in \mathsf{P}[\tau]$ and $a \in ||[\varphi]||$. If $\forall u \in \mathsf{P}^!{[\varphi]}^!\, \mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right) u = m u_a$ then $\forall u \in \mathsf{P}[\varphi]\, \mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right) u = m u_a$.*

Proof. We prove (1).

First notice that $\forall c \in ||[\sigma]||$, the coefficient of the monomial $\prod_{i=0}^{k-1} \zeta_i$ is $\mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right)_c = t_{([0,\ldots,k-1],c)}$. Now, let $\overrightarrow{\frac{1}{k}}$ be the sequence of k coefficients all equal to $\frac{1}{k}$:
$$\widehat{t}_c\left(\overrightarrow{\tfrac{1}{k}}\right) = \sum_{\substack{\mu \in ||[!\iota]|| \\ \mathsf{supp}(\mu) \subseteq \{0,\ldots,k-1\}}} t_{(\mu,c)} \tfrac{1}{k}^{\#\mu}$$

so that $\mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right)_c k^{-k} \leq \widehat{t}(\overrightarrow{\tfrac{1}{k}})_c$. Since $\overrightarrow{\frac{1}{k}} \in \mathsf{P}[\iota]$, $\widehat{t}(\overrightarrow{\tfrac{1}{k}}) \in \mathsf{P}[\sigma]$ which is downward closed, we have that $k^{-k}\, \mathsf{c}_{\overrightarrow{\zeta}}^{\mathbb{1}}\left(\widehat{t}\right) \in \mathsf{P}[\sigma]$.

Now we prove (2).

For any $a \in |[\varphi]|$, there is A such that $u_a \leq A$ for any $u \in \mathsf{P}[\varphi]$ (see Subsection 3.2). Hence, for any $u \in \mathsf{P}[\varphi]$, $\frac{u_a}{A} m \in \mathsf{P}[\tau]$ and we deduce thanks to Lemma 3 that $u \mapsto u_a \, m \, A^{-1}$ is in $\mathsf{P}[\varphi \multimap \tau]$. Without loss of generality, we can choose $A \geq k^k$ so that $u \mapsto u_a \, m \, A^{-1}$ and $u \mapsto A^{-1} \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}(\hat{t}) \, u$ are both in $\mathsf{P}[\varphi \multimap \tau]$. Now, since $[\varphi]^!$ is dense (see Theorem 16), if $u \mapsto A^{-1} \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}(\hat{t}) \, u$ and $u \mapsto A^{-1} u_a \, m$ are equal on all coalgebraic points $u \in \mathsf{P}^![\varphi]^!$, they are equal (see Definition 12). Thus, for all $u \in \mathsf{P}[\varphi]$, $u_a \, m = \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}(\hat{t}) \, u$. $\qquad \square$

We are now ready to prove that the coefficient of the unitary monomial of a testing term associated with a point a of the web allows to extract the a-coefficient of an argument, up to a non-zero coefficient depending only on a.

This is central in the proof of Full Abstraction. Let us first introduce some notations that will be used along this proof.

Intuitively, for $s \in \mathsf{P}[!\iota \multimap \sigma]$, we reason on power series $\widehat{s}$ with values in $\mathsf{P}[\sigma]$. But formally, we reason on the non-negative real power series $\widehat{s}_{a'}$ defined for each $a' \in |[\sigma]|$ and for all parameters[17] $\vec{\zeta} \in \mathsf{P}[\iota]$ as $\widehat{s}_{a'}(\vec{\zeta}) = (\widehat{s}(\vec{\zeta}))_{a'} = (s \, \vec{\zeta}^{\,!})_{a'}$ (see Paragraph 3.2.7).

We want to compute the unitary monomial of $\widehat{s}$ which will be in $\mathsf{P}[\sigma]$. We define it for each $a' \in |[\sigma]|$ as $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}(s)_{a'} = \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}(\widehat{s}_{a'})$.

We will also use the fact that a morphism $t \in \mathsf{P}[\varphi \multimap \mathbf{1}]$ is defined by the collection of $t_{(a',*)}$ for $a' \in |[\varphi]|$ and is extensionally characterized by its values $t \, u$ on every $u \in \mathsf{P}[\varphi]$. Indeed, for any $a' \in |[\varphi]|$, there is $\varepsilon > 0$ such that $\varepsilon e_{a'} \in \mathsf{P}[\varphi]$ and $t_{(a',*)} = \frac{1}{\varepsilon} (t \, \varepsilon e_{a'})_*$ by linearity of matrix multiplication.

Lemma 40. *Let σ be a type and $a \in |[\sigma]|$.*

(1) *Assume that $\sigma = \varphi$ is positive. If $a' \in |[\varphi]|$, then $\widehat{[a^0]}_{(a',*)}$ is a power series over $\mathsf{P}[\iota]$ depending on $|a|^0$ parameters, so we define $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^0])_{(a',*)} = \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}\left(\widehat{[a^0]}_{(a',*)}\right)$ and check that $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^0]) \in \mathsf{P}[\varphi \multimap \mathbf{1}]$ and that for any $u \in \mathsf{P}([\varphi])$, $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^0]) \, u = \mathsf{m}^0(a) \, u_a$.*

(2) *Assume that σ is a general type. For any $a' \in |[\sigma]|$, $\widehat{[a^+]}_{a'}$ is a power series over $\mathsf{P}[\iota]$ depending on $|a|^+$ parameters, so we define $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^+])_{a'} = \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}\left(\widehat{[a^+]}_{a'}\right)$ and check that $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^+]) \in \mathsf{P}[\sigma]$ and that $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^+]) = \mathsf{m}^+(a) \, e_a$ where e_a is the base vector such that $(e_a)_{a'} = \delta_{a',a}$ for $a' \in |[\sigma]|$.*

(3) *Let σ be a general type. For any $a' \in |[!\sigma]|$, $\widehat{[a^-]}_{(a',*)}$ is a power series over $\mathsf{P}[\iota]$ depending on $|a|^-$ parameters, so we define $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^-])_{(a',*)} = \mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}\left(\widehat{[a^-]}_{(a',*)}\right)$ and check that $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^-]) \in \mathsf{P}[!\sigma \multimap \mathbf{1}]$ and that for any $u \in \mathsf{P}[\sigma]$, $\mathsf{c}_{\vec{\zeta}}^{\mathbb{1}}([a^-]) \, u^! = \mathsf{m}^-(a) \, u_a$.*

Proof. Let us argue by mutual induction on the size of a and the structure of φ.

Let φ be a positive type and $a \in |[\varphi]|$. We prove (2) and (1) by induction on the structure of φ

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[17]We follow the common mathematical practice of using the same notation $\vec{\zeta} = (\zeta_0, \dots, \zeta_n)$ to refer to the formal parameters of a power series and to real arguments of the corresponding function.

Assume that $\varphi = {!}\tau$ and that $a = [b_0, \ldots, b_{k-1}]$. with $b_i \in |[\tau]|$.

We prove (1). Let $a' = [b'_0, \ldots, b'_{k'-1}] \in [{!}\tau]$ with $b'_j \in |[\tau]|$ and $\vec{\zeta} \in \mathsf{P}[\iota]$ be the concatenation of the finite sequences[18] $\vec{\zeta^i} \in \mathsf{P}[\iota]$ such that the length of $\vec{\zeta^i}$ is $|b_i|^-$.

By Theorem 16, $\widehat{[a^0]}_{(a',*)}(\vec{\zeta}) = ([a^0]\ \vec{\zeta}^!)_{(a',*)} \in \mathsf{P}[{!}\tau \multimap \mathbf{1}]$ is completely determined by the function $u \mapsto [a^0]\ \vec{\zeta}^!\ u^!$ defined on $\mathsf{P}[\tau]$. By inductive hypothesis, $\widehat{[a^0]}_{(a',*)}$ depends on finitely many parameters $|a|^0 = |b_0|^- + \cdots + |b_{k-1}|^-$, since

$$[a^0]\ \vec{\zeta}^!\ u^! = \prod_{i=0}^{k-1} [b_i^-]\ \vec{\zeta^i}^!\ u^! \qquad \text{and therefore} \qquad \mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}\left([a^0]\right) u^! = \prod_{i=0}^{k-1} \mathbb{c}_{\vec{\zeta^i}}^{\mathbb{1}}\left([b_i^-]\right) u^!.$$

Again, by inductive hypothesis it follows that

$$\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}\left([a^0]\right) u^! = \prod_{i=0}^{k-1} \mathbb{m}^-(b_i)\, u_{b_i} = \mathbb{m}^0(a)\, (u^!)_a.$$

We can apply (2) of Lemma 39, so that we have $\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}\left([a^0]\right) u = \mathbb{m}^0(a)\, u_a$ for all $u \in \mathsf{P}[{!}\tau]$.

We prove (2). Let $a' = [b'_0, \ldots, b'_{k'-1}] \in [{!}\tau]$ and $\vec{\zeta} \in \mathsf{P}[\iota]$ be the concatenation of the finite sequences $\vec{\zeta^*}, \vec{\zeta^0}, \ldots, \vec{\zeta^{k-1}} \in \mathsf{P}[\iota]$ such that the length of $\vec{\zeta^*}$ is k and the length of $\vec{\zeta^i}$ is $|b_i|^+$ for $i \geq 0$. By inductive hypothesis, $\widehat{[a^+]}_{a'}$ depends on finitely many parameters $|a|^0 = k + |b_0|^+ + \cdots + |b_{k-1}|^+$, since

$$\widehat{[a^+]}_{a'}(\vec{\zeta}) = ([a^+]\ \vec{\zeta}^!)_{a'} = \left(\sum_{i=0}^{k-1} \zeta_i^*\ [b_i^+]\ \vec{\zeta^i}^!\right)^!_{a'} = \prod_{j=0}^{k'-1} \left(\sum_{i=0}^{k-1} \zeta_i^*\ [b_i^+]\ \vec{\zeta^i}^!\right)_{b'_j}. \tag{5.1}$$

We want to compute the coefficient of the unitary monomial, which contains exactly one copy of each parameter of each $\vec{\zeta^i}$. If $k' \neq k$ then expression (5.1) contains no monomial where each parameter of $\vec{\zeta^*}$ appears exactly once, so that $\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}([a^+])_{a'} = 0$ in that case. If $k' = k$ and $\mathfrak{S}_k$ is the set of permutations over k, then by using the fact that factorial $a! = \#\{\rho \in \mathfrak{S}_k \mid \forall i\, b_i = b_{\rho(i)}\}$, by denoting the Kronecker symbol as $\delta_{a,a'}$ and by the inductive hypothesis, we get:

$$\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}\left([a^+]\right)_{a'} = \sum_{\rho \in \mathfrak{S}_k} \prod_{i=0}^{k-1} \mathbb{c}_{\vec{\zeta^i}}^{\mathbb{1}}\left([b_i^+]\right)_{b'_{\rho(i)}} = a!\ \prod_{i=0}^{k-1} \mathbb{m}^+(b_i) \delta_{b_i, b'_i} = \mathbb{m}^+(a)\, \delta_{a,a'} = \mathbb{m}^+(a)\, (e_a)_{a'}.$$

Assume that $\varphi = \varphi_\ell \otimes \varphi_r$ and that $a = (b_\ell, b_r)$ with $a_i \in |[\varphi_i]|$. Let $\vec{\zeta} \in \mathsf{P}[\iota]$ be the concatenation of the finite sequences $\vec{\zeta^\ell}, \vec{\zeta^r} \in \mathsf{P}[\iota]$ such that the length of $\vec{\zeta^i}$ is $|b_i|^+$.

We prove (1). Let $a' = (b'_\ell, b'_r) \in |[\varphi_\ell \otimes \varphi_r]|$. By Theorem 16, $\widehat{[a^0]}_{(a',*)}(\vec{\zeta}) = ([a^0]\ \vec{\zeta}^!)_{(a',*)} \in$ $\mathsf{P}[\varphi_\ell \otimes \varphi_r \multimap \mathbf{1}]$ is completely determined by the function $u \mapsto [a^0]\ \vec{\zeta}^!\ u^!$ defined on $\mathsf{P}[\varphi_\ell \otimes \varphi_r]$. Besides, if $u \in \mathsf{P}^!([\varphi_\ell \otimes \varphi_r]^!)$, then $u = u_\ell \otimes u_r$ where $u_i = \mathrm{pr}_i^\otimes(u) \in \mathsf{P}^!([\varphi_i]^!)$ for

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[18]We assume that the support of indices of the sequences are disjoint even if this requires some renaming.

$i \in \{\ell, r\}$ (see Lemma 13). Therefore, by inductive hypothesis, $\widehat{[a^0]}_{(a',*)}$ depends on finitely many parameters $|a|^0 = |b_\ell|^0 + |b_r|^0$, since

$$\left[a^0\right] \vec{\zeta}^! u = \left[b_\ell^0\right] \vec{\zeta}^{\ell !} u_\ell \left[b_r^0\right] \vec{\zeta}^{r !} u_r \quad \text{and therefore} \quad c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^0\right]\right) u = c_{\vec{\zeta}^\ell}^{\mathbb{1}}\left(\left[b_\ell^0\right]\right) u_\ell \, c_{\vec{\zeta}^r}^{\mathbb{1}}\left(\left[b_r^0\right]\right) u_r.$$

Hence, by inductive hypothesis $c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^0\right]\right) u = \mathfrak{m}^0(b_\ell)(u_\ell)_{b_\ell} \, \mathfrak{m}^0(b_r)(u_r)_{b_r} = \mathfrak{m}^0(a)u_a$ for $u \in \mathsf{P}^![\varphi]^!$. We conclude by Lemma 39, that this holds also for $u \in \mathsf{P}[\varphi]$.

We prove (2). Let $a' = (b'_\ell, b'_r) \in |[\varphi_\ell \otimes \varphi_r]|$. By inductive hypothesis, $\widehat{[a^+]}_{a'}$ depends on finitely many parameters $|a|^+ = |b_\ell|^+ + |b_r|^+$, since

$$\widehat{[a^+]}_{a'}(\vec{\zeta}) = (\left[a^+\right] \vec{\zeta}^!)_{a'} = (\left[b_\ell^+\right] \vec{\zeta}^{\ell !})_{b'_\ell} \otimes (\left[b_r^+\right] \vec{\zeta}^{r !})_{b'_r}$$

We deduce using inductive hypothesis that

$$c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^+\right]\right)_{a'} = c_{\vec{\zeta}^\ell}^{\mathbb{1}}\left(\left[b_\ell^+\right]\right)_{b'_\ell} \, c_{\vec{\zeta}^r}^{\mathbb{1}}\left(\left[b_r^+\right]\right)_{b'_r} = \mathfrak{m}^+(b_\ell)\,\delta_{b_\ell,b'_\ell}\,\mathfrak{m}^+(b_r)\,\delta_{b_r,b'_r} = \mathfrak{m}^+(a)\,\delta_{a,a'}.$$

Assume that $\varphi = \varphi_\ell \oplus \varphi_r$ and that $a = (\ell, a_\ell)$ with $a_\ell \in |[\varphi_\ell]|$. (the case $a = (r, a_r)$ is similar).

We prove (1). Let $a' = (i, a'_i) \in |[\varphi_\ell \oplus \varphi_r]|$. By Theorem 16, $\widehat{[a^0]}_{(a',*)}(\vec{\zeta}) = (\left[a^0\right] \vec{\zeta}^!)_{(a',*)} \in \mathsf{P}[\varphi_\ell \oplus \varphi_r \multimap \mathbf{1}]$ is completely determined by the function $u \mapsto \left[a^0\right] \vec{\zeta}^! u$ defined on $\mathsf{P}^![\varphi_\ell \oplus \varphi_r]^!$. Besides, if $u \in \mathsf{P}^!([\varphi_\ell \oplus \varphi_r]^!)$, then there is $i \in \{\ell, r\}$ such that $u = \mathsf{in}_i u_i$ with $u_i \in \mathsf{P}^!([\varphi_i]^!)$ (see Lemma 13). Therefore, by inductive hypothesis, $\widehat{[a^0]}_{(a',*)}$ depends on finitely many parameters $|a|^0 = |a_\ell|^0$, since if $i = \ell$, then

$$\left[a^0\right] \vec{\zeta}^! u = \left[\lambda x^{\varphi_\ell \oplus \varphi_r} \mathsf{case}(x, y_\ell \cdot \langle a_\ell^0 \rangle y_\ell, y_r \cdot \Omega^{\mathbf{1}})\right] \vec{\zeta}^! u = \left[a_\ell^0\right] \vec{\zeta}^! u_\ell$$

and if $i = r$ then $\left[a^0\right] \vec{\zeta}^! u = \left[\Omega^1\right] = 0$. So we can compute that $c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^0\right]\right) u = \mathfrak{m}^0(a_\ell)(u_\ell)_{a_\ell} = \mathfrak{m}^0(a)\,u_a$ for $u \in \mathsf{P}^![\varphi]^!$ and this still holds for $u \in \mathsf{P}[\varphi]$ by Lemma 39.

We prove (2). Let $a' = (i, a'_i) \in |[\varphi_\ell \oplus \varphi_r]|$. By inductive hypothesis, $\widehat{[a^+]}_{a'}$ depends on finitely many parameters $|a|^+ = |a_\ell|^+$, since

$$\widehat{[a^+]}_{a'}(\vec{\zeta}) = (\left[a^+\right] \vec{\zeta}^!)_{a'} = (\left[\mathsf{in}_\ell a_\ell^+\right] \vec{\zeta}^!)_{(i,a'_i)} = \mathsf{in}^\ell(\left[a_\ell^+\right] \vec{\zeta}^!)_{i,a'_i} = \delta_{\ell,i}(\left[a_\ell^+\right] \vec{\zeta}^!)_{a'_i}.$$

We can therefore compute $c_{\vec{\zeta}}^{\mathbb{1}}([a^+])_{(i,a'_i)} = \delta_{\ell,i}\, c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a_\ell^+\right]\right)_{a'_i} = \mathfrak{m}^+(a)\,\delta_{a,(i,a'_i)}$ by inductive hypothesis.

Finally, for a general type σ and $a \in |[\sigma]|$, we prove (2) and (3).

If $\sigma = \varphi$ is positive, we have already proved (2). Let us prove (3). Let $a \in |[\varphi]|$. Let $a' \in |[!\varphi]|$. By Theorem 16 and Lemma 13, $\widehat{[a^-]}_{(a',*)}(\vec{\zeta}) = (\left[a^-\right] \vec{\zeta}^!)_{(a',*)} \in \mathsf{P}[!\tau \multimap \mathbf{1}]$ is completely determined by $u \mapsto \left[a^-\right] \vec{\zeta}^! u^!$ defined on $\mathsf{P}[\tau]$. Therefore, by inductive hypothesis, $\widehat{[a^-]}_{(a',*)}$ depends on finitely many parameters $|a|^- = |a|^0$, since

$$\left[a^-\right] \vec{\zeta}^! u^! = \left[a^0\right] \vec{\zeta}^! u \quad \text{and therefore} \quad c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^-\right]\right) u^! = c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^0\right]\right) u$$

By inductive hypothesis, $c_{\vec{\zeta}}^{\mathbb{1}}\left(\left[a^0\right]\right) u = \mathfrak{m}^0(a)\,u_a = \mathfrak{m}^-(a)\,u_a.$

Last, let $\sigma = \varphi \multimap \tau$. Let $a = (b,c) \in ||\sigma||$.

We prove (2). Let $a' = (b',c') \in ||\sigma||$. Let $\vec{\zeta} \in \mathsf{P}[\iota]$ be the concatenation of the finite sequences $\vec{\zeta}^1, \vec{\zeta}^2 \in \mathsf{P}[\iota]$ such that the length of $\vec{\zeta}^1$ is $|b|^0$ and the length of $\vec{\zeta}^2$ is $|c|^+$. By inductive hypothesis, for any $u \in \mathsf{P}[\varphi]$,

$$\widehat{[a^+]}(\vec{\zeta})u = [a^+]\,\vec{\zeta}^!\,u = ([b^0]\,\vec{\zeta}^{1\,!}\,u)_* ([c^+]\,\vec{\zeta}^{2\,!}).$$

Now, let $\varepsilon > 0$ such that $\varepsilon e_{b'} \in \mathsf{P}[\varphi]$, we compute

$$\widehat{[a^+]}_{(b',c')}(\vec{\zeta}^!) = \tfrac{1}{\varepsilon}([a^+]\,\vec{\zeta}^!\,\varepsilon e_{b'})_{c'} = \tfrac{1}{\varepsilon}([b^0]\,\vec{\zeta}^{1\,!}\,\varepsilon e_{b'})_* ([c^+]\,\vec{\zeta}^{2\,!})_{c'} = \widehat{[b^0]}_{(b',*)}(\vec{\zeta}^1)\,\widehat{[c^+]}_{c'}(\vec{\zeta}^2).$$

Therefore, by inductive hypothesis, $\widehat{[a^+]}_{(b',c')}$ depends on finitely many coefficients $|a|^+ = |b|^0 + |c|^+$. We compute using inductive hypothesis that

$$\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}\left([a^+]\right)_{(b',c')} = \mathbb{c}_{\vec{\zeta}^1}^{\mathbb{1}}\left([b^0]\right)_{(b',*)}\,\mathbb{c}_{\vec{\zeta}^2}^{\mathbb{1}}\left([c^+]\right)_{c'} = \mathsf{m}^0(b)\,\delta_{b,b'}\,\mathsf{m}^+(c)\,\delta_{c,c'}.$$

We conclude that $\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}([a^+]) = \mathsf{m}^0(b)\,\mathsf{m}^+(c)\,e_{b,c} = \mathsf{m}^+(a)\,e_a$.

We prove (3). Let $a' = [(b'_0,c'_0),\ldots,(b'_{k-1},c'_{k-1})]) \in ||!\sigma||$. Let $\vec{\zeta} \in \mathsf{P}[\iota]$ be the concatenation of the finite sequences $\vec{\zeta}^1, \vec{\zeta}^2 \in \mathsf{P}[\iota]$ such that the length of $\vec{\zeta}^1$ is $|b|^+$ and the length of $\vec{\zeta}^2$ is $|c|^-$. For any $w \in \mathsf{P}[!(\varphi \multimap \tau)]$, we have:

$$[a^-]\,\vec{\zeta}^!\,w = [c^-]\,\vec{\zeta}^{2\,!}\left(\mathsf{der}_{[\varphi\multimap\tau]}(w)\,[b^+]\,\vec{\zeta}^{1\,!}\right)^!.$$

Let $\varepsilon > 0$ such that $\varepsilon e_{a'} \in \mathsf{P}[!(\varphi \multimap \tau)]$, then

$$\widehat{[a^-]}_{(a',*)}(\vec{\zeta}) = \tfrac{1}{\varepsilon}([a^-]\,\vec{\zeta}^!\,\varepsilon e_{a'})_* = \tfrac{1}{\varepsilon}([c^-]\,\vec{\zeta}^{2\,!}\left(\mathsf{der}_{[\varphi\multimap\tau]}(\varepsilon e_{a'})\,[b^+]\,\vec{\zeta}^{1\,!}\right)^!)_*.$$

By inductive hypothesis, we get that $\widehat{[a^-]}_{(a',*)}$ depends on $|a|^- = |b|^0 + |c|^-$ coefficients.

Let now $u \in \mathsf{P}[\varphi \multimap \tau]$, then by Lemma 13 $\mathsf{der}_{[(\varphi\multimap\tau)]}(u^!) = u$ and we compute:

$$\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}\left([a^-]\right)u^! = \mathbb{c}_{\vec{\zeta}^1}^{\mathbb{1}}\left(\mathbb{c}_{\vec{\zeta}^2}^{\mathbb{1}}\left([c^-]\right)\left(u\,[b^+]\,\vec{\zeta}^{1\,!}\right)^!\right).$$

By inductive hypothesis, we have $\mathbb{c}_{\vec{\zeta}^2}^{\mathbb{1}}\left([c^-]\right)(u\,[b^+]\,\vec{\zeta}^{1\,!})^! = \mathsf{m}^-(c)\,(u\,[b^+]\,\vec{\zeta}^{1\,!})_c$. Moreover, notice that $u \in \mathsf{P}[\varphi \multimap \tau]$, seen as a morphism in $\mathbf{Pcoh}([\varphi],[\tau])$ is linear, and there is $\varepsilon > 0$ such that $\varepsilon e_b \in \mathsf{P}[\varphi]$), so that we can apply u to e_b. Now, by using inductive hypothesis, we get that $\mathbb{c}_{\vec{\zeta}^1}^{\mathbb{1}}(u\,[b^+])_c = (u\,\mathbb{c}_{\vec{\zeta}^1}^{\mathbb{1}}([b^+]))_c = (u\,\mathsf{m}^+(b)\,e_b)_c = \mathsf{m}^+(b)\,u_{(b,c)}$. Therefore, we have $\mathbb{c}_{\vec{\zeta}}^{\mathbb{1}}([a^-])\,u^! = \mathsf{m}^+(b)\,\mathsf{m}^-(c)\,u_{(b,c)} = \mathsf{m}^-(a)\,u_a$. $\square$

Theorem 41 (Full Abstraction).
  If $\vdash M_1 : \sigma$ and $\vdash M_2 : \sigma$ satisfy $M_1 \sim M_2$ then $[M_1] = [M_2]$.

Proof. Towards a contradiction, assume that $[M_1] \neq [M_2]$. There is $a \in ||\sigma||$ such that $[M_1]_a \neq [M_2]_a$. Then by Lemma 40, $[\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle M_i^!]$, for $i \in \{1,2\}$, are power series with different coefficients, namely the coefficients of the monomial $\zeta_0 \ldots \zeta_{|a|^--1}$ are $\mathsf{m}^-(a)\,[M_i]_a$ for $i \in \{1,2\}$ as $[\lambda x^{!\iota}\,\langle\langle a^-\rangle x\rangle M_i^!]\,\vec{\zeta}^! = [a^-]\,\vec{\zeta}^![M_i]^!$. There is $\vec{\zeta} = (\zeta_0,\ldots,\zeta_{|a|^--1}) \in \mathsf{P}[\iota]$ with $\zeta_i \in \mathbb{Q}\cap[0,1]$ such that $[a^-]\,\vec{\zeta}^![M_1]^! \neq [a^-]\,\vec{\zeta}^![M_2]^!$. Yet, $[\langle\langle a^-\rangle\mathsf{ran}(\vec{\zeta})^!\rangle M_i^!] = [a^-]\,\vec{\zeta}^!([M_i])$. By Theorem 38, we get that $\langle\langle a^-\rangle\mathsf{ran}(\vec{\zeta})^!\rangle M_1^!$ and $\langle\langle a^-\rangle\mathsf{ran}(\vec{\zeta})^!\rangle M_2^!$ converge to $()$ with different probabilities. It follows that $M_1 \not\sim M_2$. $\square$

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