Module 2.2: Models of Programming Languages
Exam, parts II and III
(Correction)

There are 5 exercices gathered in 3 parts which are independent of each other. Part (a) and (b) will be corrected by Thomas Ehrhard, Part (c) will be corrected by Michele Pagani. Try to answer some questions in the three parts, but you are free to invest more time in one part than in the others, depending on your feeling and strengths.
We expect from you a **personal** work. You can use all the documents provided during the lecture (lecture notes, slides, exercise sheets). You can write in French and in English.
You must submit your solutions in an electronic format (pdf, jpeg, png etc) by email to both ehrhard@irif.fr and pagani@irif.fr, strictly **before 11 :50 am** this morning (Tue Mar 9th, 2021). The email must have as object "`MPRI EXAM : MODULE 2-02 [yourname]`".

a) Lists in the relational model of linear logic

Exercice 1 :
Remember that **Rel** is the category of sets and relations, which is a model of linear logic. All the objects and morphisms in this exercise are in **Rel**.
Let L be the set $\mathbb{N}^{<\omega}$ of finite sequences of integers and $\mathsf{N} = \mathbb{N}$, considered as objects of **Rel** (the category of sets and relations). We write $\langle n_1, \dots, n_k\rangle$ for an empty sequence of length k, $\langle\,\rangle$ for the empty sequence and we set $n@\langle n_1, \dots, n_k\rangle = \langle n, n_1, \dots, n_k\rangle$. Remember that, in **Rel**, the object 1 is the set $\{*\}$.

1. Let $\theta \in \mathbf{Rel}(\mathsf{L}, 1 \oplus (\mathsf{N} \otimes \mathsf{L}))$ be defined as
$$\theta = \{(\langle\,\rangle, (1, *))\} \cup \{(n@s, (2, (n, s))) \mid n \in \mathsf{N} \text{ and } s \in \mathsf{L}\}\,.$$
Prove that θ is an isomorphism in **Rel**, that is, that θ is (the graph of) a bijection.
2. Let E be a set and let $f \in \mathbf{Rel}(1 \oplus (\mathsf{N} \otimes E), E)$. We define a sequence f_k of elements of $\mathbf{Rel}(L, E)$ by induction on k as follows
$$\begin{aligned} f_0 &= \emptyset \\ f_{k+1} &= \{(\langle\,\rangle, e) \mid ((1, *), e) \in f\} \cup \{(n@s, e) \mid \exists e' \in E\ ((2, (n, e')), e) \in f \text{ and } (s, e') \in f_k\} \end{aligned}$$
Prove that $\forall k \in \mathbb{N}\ f_k \subseteq f_{k+1}$. We set $\tilde{f} = \bigcup_{k\in\mathbb{N}} f_k \in \mathbf{Rel}(\mathsf{L}, E)$.
3. Prove that the following diagram is commutative in **Rel**
$$\begin{tikzcd} \mathsf{L} \arrow[rr, "\tilde{f}"] \arrow[d, "\theta"'] & & E \\ 1 \oplus (\mathsf{N} \otimes \mathsf{L}) \arrow[rr, "f' = 1 \oplus (\mathsf{N} \otimes \tilde{f})"] & & 1 \oplus (\mathsf{N} \otimes E) \arrow[u, "f"'] \end{tikzcd}$$
where f' is obtained by applying the functor $1 \oplus (\mathsf{N} \otimes \_)$ to $\tilde{f}$, that is
$$f' = \{((1, *), (1, *))\} \cup \{((2, (n, s)), (2, (n, e))) \mid n \in \mathbb{N} \text{ and } (s, e) \in \tilde{f})\}\,.$$
4. Prove that $\tilde{f}$ is the only element of $\mathbf{Rel}(\mathsf{L}, E)$ such that the diagram above is commutative. In other word, prove that if $g \in \mathbf{Rel}(\mathsf{L}, E)$ satisfies
$$\begin{tikzcd} \mathsf{L} \arrow[rr, "g"] \arrow[d, "\theta"'] & & E \\ 1 \oplus (\mathsf{N} \otimes \mathsf{L}) \arrow[rr, "1 \oplus (\mathsf{N} \otimes g)"] & & 1 \oplus (\mathsf{N} \otimes E) \arrow[u, "f"'] \end{tikzcd}$$
then $g = \tilde{f}$. [*Hint :* assuming the commutation above, prove by induction on (the length of) $s \in \mathsf{L}$ that, for any $e \in E$, one has $(s, e) \in g$ iff $(s, e) \in \tilde{f}$.]

5. If m is a multiset and $k \in \mathbb{N}$, we set $km = \overbrace{m + \cdots + m}^{k}$. We define a morphism $a \in \mathbf{Rel}(1 \oplus (\mathsf{N} \otimes !\mathsf{L}), !\mathsf{L})$ by

$$a = \{((1,*), k[\langle\rangle]) \mid k \in \mathbb{N}\} \\ \cup \{((2,(n,[s_1,\dots,s_k]})), [n@s_1,\dots,n@s_k]) \mid n,k \in \mathbb{N} \text{ and } s_1,\dots,s_k \in \mathsf{L}\}\,.$$

By the construction above, there is a unique $h_{\mathsf{L}} = \tilde{a} \in \mathbf{Rel}(\mathsf{L}, !\mathsf{L})$ such that

$$\begin{tikzcd} \mathsf{L} \arrow[r, "h_{\mathsf{L}}"] \arrow[d, "\theta"'] & !\mathsf{L} \\ 1 \oplus (\mathsf{N} \otimes \mathsf{L}) \arrow[r, "1\oplus(\mathsf{N}\otimes h_{\mathsf{L}})"] & 1 \oplus (\mathsf{N} \otimes !\mathsf{L}) \arrow[u, "a"'] \end{tikzcd}$$

Prove that

$$h_L = \{(s, k[s]) \mid k \in \mathbb{N} \text{ and } s \in \mathsf{L}\}\,.$$

▷

1. *This amounts to proving that θ is (the graph of) a bijection. First, it is a total function defined by cases as follows : $\theta(\langle\rangle) = (1,*)$ and $\theta(n@s) = (2,(n,s))$. It is injective since, if $s, s' \in \mathsf{L}$ satisfy $\theta(s) = \theta(s')$ then either $\theta(s) = \theta(s') = (1,*)$, in which case $s = s' = \langle\rangle$, or $\theta(s) = \theta(s') = (2,(n,t))$ in which case we must have $s = s' = n@t$. Last θ is surjective since an element d of $1 \oplus (\mathsf{N} \otimes \mathsf{L})$ is either of shape $d = (1,*))$ in which case $d = \theta(\langle\rangle)$ or of shape $d = (2,(n,t))$ in which case $d = \theta(n@t)$.*
2. *Straightforward induction on k.*
3. *We prove first that $\tilde{f} \subseteq f\, f'\, \theta$, that is, we prove that for all $k \in \mathbb{N}$, $f_k \subseteq f\, f'\, \theta$. The proof is by induction on k. For $k = 0$ this is obvious since $f_0 = \emptyset$. So assume that $f_k \subseteq f\, f'\, \theta$ and let us prove that $f_{k+1} \subseteq f\, f'\, \theta$. Let $(s,e) \in f_{k+1}$. If $s = \langle\rangle$ we have that $((1,*),e) \in f$ by definition of f_{k+1}. We also have $(s,(1,*)) \in \theta$ and since $((1,*),(1,*)) \in f'$, we have $(s,e) \in f\, f'\, \theta$. Assume now that $s = n@t$. We have $(s,(2,(n,t))) \in \theta$. Moreover by definition of f_{k+1} we have that $(t,e') \in f_k$ and $((2,(n,e')),e) \in f$ for some $e' \in E$. We have $(t,e') \in \tilde{f}$ since $f_k \subseteq \tilde{f}$ and hence $((2,(n,t)),(2,(n,e'))) \in f'$. Therefore $(s,e) \in f\, f'\, \theta$.*
   We prove now that $f\, f'\, \theta \subseteq \tilde{f}$. So let $(s,e) \in f\, f'\, \theta$ and let $(d,r) \in f'$ be such that $(s,d) \in \theta$ and $(r,e) \in f$.
   - *If $s = \langle\rangle$ we have $d = (1,*)$ and hence $r = (1,*)$, therefore $((1,*),e) \in f$ so that $(s,e) \in f_1 \subseteq \tilde{f}$.*
   - *If $s = n@t$ then $d = (2,(n,t))$ and therefore, by definition of f', we have $r = (2,(n,e'))$ for some $e' \in E$ such that $(t,e') \in \tilde{f}$. Let $k \in \mathbb{N}$ be such that $(t,e') \in f_k$. Since $((2,(n,e')),e) \in f$ we have $(s,e) \in f_{k+1}$ by definition of f_{k+1} and hence $(s,e) \in \tilde{f}$.*
4. *We follow the* Hint. *Assume first $s = \langle\rangle$. If $(s,e) \in g = f\, g'\, \theta$ (where $g' = 1 \oplus (\mathsf{N} \otimes g)$), then we have $((1,*),e) \in f$ by definition of g' and hence $(s,e) \in \tilde{f}$. If $(s,e) \in \tilde{f}$ then by definition of $\tilde{f}$ we have $((1,*),e) \in f$ and hence $(s,e) \in f\, g'\theta$ by definition of g' so that $(s,e) \in g$. Assume now that $s = n@t$ and assume that $\forall e' \in E\ (t,e') \in g \Leftrightarrow (t,e') \in \tilde{f}$ (inductive hypothesis). Assume first $(s,e) \in g = f\, g'\, \theta$. By definition of θ and g', this means that there is $e' \in E$ such that $(t,e') \in g$ such that $((2,(n,e')),e) \in f$. By inductive hypothesis we have $(t,e') \in \tilde{f}$ so let $k \in \mathbb{N}$ be such that $(t,e') \in f_k$. Then by definition of f_{k+1} we have $(s,e) \in f_{k+1}$ and hence $(s,e) \in \tilde{f}$. Assume next that $(s,e) \in \tilde{f}$ and let $k \in \mathbb{N}$ be such that $(s,e) \in f_k$. This implies that $k \neq 0$ and since $s = n@t$ there must be $e' \in E$ such that $(t,e') \in f_{k-1}$ and $((2,(n,e')),e) \in f$. By inductive hypothesis we have $(t,e') \in g$ and hence $(s,e) \in f\, g'\, \theta = g$.*
5. *With the notations above we have $h_{\mathsf{L}} = \bigcup_{k\in\mathbb{N}} a_k$ where*

$$\begin{aligned} a_0 &= \emptyset \\ a_{k+1} &= \{(\langle\rangle, m) \mid ((1,*),m) \in a\} \cup \{(n@t, m) \mid \exists m'((2,(n,m')),m) \in a \text{ and } (t,m') \in a_k\}\,. \end{aligned}$$

   We prove that $a_k = b_k$ where $b_k = \{(s, l[s]) \mid l \in \mathbb{N},\ s \in \mathsf{L}$ and $\mathsf{len}(s) < k\}$ where $\mathsf{len}(s)$ is the length of the sequence s. The proof is by induction on k. The base case $k = 0$ is obvious since $b_0 = \emptyset$. Assume that the equation holds for k and let us prove it for $k+1$. Let $(s,m) \in a_{k+1}$. If $s = \langle\rangle$ we must have $((1,*),m) \in a$ and hence $m = l[\langle\rangle]$ for some $l \in \mathbb{N}$ so that $(s,m) \in b_{k+1}$. If $s = n@t$ we must have $((2,(n,m')),m) \in a$ and $(t,m') \in a_k$ for some $m' \in !\mathsf{L}$. By the inductive hypothesis we have $a_k = b_k$ and hence $m' = l[t]$ where $l \in \mathbb{N}$ (this also shows that $\mathsf{len}(t) < k$). By definition of a we have $m = l[s]$ hence $(s,m) \in a_{k+1}$. Last let $(s,m) \in b_{k+1}$ which means that $m = l[s]$ for some $l \in \mathbb{N}$ and that $\mathsf{len}(s) \leqslant k$. If $s = \langle\rangle$ we have $((1,*),m) \in a$ by definition of a and hence $(s,m) \in a_{k+1}$. Assume that $s = n@t$. Let $m' = l[t]$, we have $\mathsf{len}(t) < k$ and hence $(t,m') \in b_k$ by definition of b_k and hence $(t,m') \in a_k$ by inductive hypothesis. By definition of a and a_{k+1} we get $(s,m) \in a_{k+1}$.

b) Computing the denotation of a probabilistic term

In this section, we consider the category $\mathbf{Pcoh}_!$ of probabilistic coherence spaces (PCS) and analytic maps between PCS. We recall that $\mathbf{Pcoh}_!$ is a model of probabilistic PCF and it is the Kleisli category associated with the ! comonad of the category **Pcoh** of PCS and linear morphisms between PCS.

Exercice 2 :
Consider the following PCF terms :

$$T = \mathsf{if}(x, \mathsf{if}(x, y, z \cdot \underline{0}), z \cdot \mathsf{if}(x, \underline{1}, w \cdot y))$$
$$U = \lambda x^{\iota}\, \mathsf{fix}(\lambda y^{\iota}\, T)$$

1. Give a type derivation of $\vdash U : \iota \Rightarrow \iota$.
2. Suppose $v, u \in \mathsf{PN}$, compute the value of $\widehat{[\![T]\!]_{x:\iota,y:\iota}}(v, u)$. (It can be convenient to use the notation $v_{>0}$ for the scalar $\sum_{n=1}^{\infty} v_n$).
3. Let $\varphi_v = \widehat{[\![U]\!]}(v)$. Prove that $\varphi_v = \widehat{[\![T]\!]_{x:\iota,y:\iota}}(v, \varphi_v)$.
4. Suppose $v_0 + v_{>0} = 1$ and $v_0 v_{>0} > 0$. By using the recursive equation above, compute $\widehat{[\![U]\!]}(v)$.
5. In the case $v_0 = 1$ or $v_{>0} = 1$ what is the value of $\widehat{[\![U]\!]}(v)$?
6. Deduce a specification for the operational behaviour of the term U.

▷

1. *See Figure 2.*
2. $\widehat{[\![T]\!]_{x:\iota,y:\iota}}(v, u) = v_0 v_{>0}(e_0 + e_1) + (v_0^2 + v_{>0}^2)u$
3. *We have :*

$$\begin{aligned}
\varphi_v &= \sup_{n=0}^{\infty} \left(\left(\widehat{[\![\lambda y^{\iota}\, T]\!]_{x:\iota}}(v)\right)^n(0) \right) \\
&= \sup_{n=1}^{\infty} \left(\widehat{[\![T]\!]_{x:\iota,y:\iota}}(v, (\widehat{[\![\lambda y^{\iota}\, T]\!]_{x:\iota}}(v))^{n-1}(0)) \right) \\
&= \widehat{[\![T]\!]_{x:\iota,y:\iota}} \left(v, \sup_{n=0}^{\infty} \left((\widehat{[\![\lambda y^{\iota}\, T]\!]_{x:\iota}}(v))^n(0) \right) \right) \\
&= \widehat{[\![T]\!]_{x:\iota,y:\iota}} \left(v, \varphi_v \right)
\end{aligned}$$

4. *By 2 and 3, we have* $\varphi_v = v_0 v_{>0}(e_0 + e_1) + (v_0^2 + v_{>0}^2)\varphi_v$, *so that :* $\varphi_v = \frac{v_0 v_{>0}}{1 - v_0^2 - v_{>0}^2}(e_0 + e_1)$. *By hypothesis* $v_{>0} = 1 - v_0$, *so* $\varphi_v = \frac{v_0 - v_0^2}{2v_0 - 2v_0^2}(e_0 + e_1) = \frac{1}{2}(e_0 + e_1)$.
5. *If one between* v_0 *or* $v_{>0}$ *is* 1, *so the other one is* 0, *we have the recursive equation* $\varphi_v = \varphi_v$, *of which the smallest solution is* 0, *so* $\widehat{[\![U]\!]}(v) = 0$.
6. *By the Adequacy Theorem, we deduce that* U *returns the uniform distribution over* $\underline{0}, \underline{1}$ *whenever applied to a probabilistic distribution having* $0 < v_0 < 1$. *In the cases* $v_0 = 0, 1$, U *diverges.*

c) Extending pPCF with a type for lists

We recall that $\mathbb{N}^{<\omega}$ is the set of finite sequences of natural numbers, the writing $\langle n_1, \ldots, n_k \rangle$ denoting a sequence of length k, and $\langle\,\rangle$ being the empty sequence. The metavariable s will always range over $\mathbb{N}^{<\omega}$. Given $n \in \mathbb{N}$ and $\langle n_1, \ldots, n_k \rangle \in \mathbb{N}^{<\omega}$, we denote by $n@\langle n_1, \ldots, n_k \rangle$ the sequence $\langle n, n_1, \ldots, n_k \rangle$.

Consider the extension of pPCF with the ground type List for the set of finite sequences of natural numbers and the new operators presented in Figure 1 with the associated typing rules 1a and operational semantics 1b. In particular, $\underline{s}$ is the constant of pPCF associated with a sequence s, the writing :: denotes the append operation over pPCF terms of suitable type and a further conditional ifl is introduced, allowing a pattern matching and a decomposition for non-empty sequences. Notice that the definition of the stochastic matrix Red (and hence of Red^{∞}) can be also extended to encompass these new operations by following the rules of Figure 1b.

$$\frac{s \in \mathbb{N}^{<\omega}}{\Gamma \vdash \underline{s} : \mathsf{List}} \qquad \frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash N : \mathsf{List}}{\Gamma \vdash (M :: N) : \mathsf{List}}$$

$$\frac{\Gamma \vdash P : \mathsf{List} \qquad \Gamma \vdash Q : A \qquad \Gamma, x : \iota, y : \mathsf{List} \vdash R : A}{\Gamma \vdash \mathsf{ifl}(P, Q, x \cdot y \cdot R) : A}$$

(a) The new typing rules : notice that $x \cdot y \cdot R$ is a binder in $\mathsf{ifl}(P, Q, x \cdot y \cdot R)$ for the free variables x, y of R.

$$\frac{}{(\underline{n} :: \underline{s}) \xrightarrow{1} \underline{n@s}} \qquad \frac{}{\mathsf{ifl}(\underline{\langle\,\rangle}, P, x \cdot y \cdot R) \xrightarrow{1} P} \qquad \frac{}{\mathsf{ifl}(\underline{n@s}, P, x \cdot y \cdot R) \xrightarrow{1} R[\underline{n}/x, \underline{s}/y]}$$

$$\frac{M \xrightarrow{p} N}{(M :: P) \xrightarrow{p} (N :: P)} \qquad \frac{M \xrightarrow{p} N}{(\underline{n} :: M) \xrightarrow{p} (\underline{n} :: N)} \qquad \frac{M \xrightarrow{p} N}{\mathsf{ifl}(M, P, x \cdot y \cdot R) \xrightarrow{p} \mathsf{ifl}(N, P, x \cdot y \cdot R)}$$

(b) The new reduction rules extending the pPCF reduction relation.

$$\llbracket \mathsf{List} \rrbracket = (|\mathsf{L}|, \mathsf{PL})$$

$$\widehat{\llbracket \underline{s} \rrbracket_\Gamma}(\vec{v}) = e_s$$

$$\widehat{\llbracket (M :: N) \rrbracket_\Gamma}(\vec{v}) = \sum_{n \in \mathbb{N}} \sum_{s \in \mathbb{N}^{<\omega}} \widehat{\llbracket M \rrbracket_\Gamma}(\vec{v})_n \widehat{\llbracket N \rrbracket_\Gamma}(\vec{v})_s e_{n@s}$$

$$\widehat{\llbracket \mathsf{ifl}(P, Q, x \cdot y \cdot R) \rrbracket_\Gamma}(\vec{v}) = \widehat{\llbracket P \rrbracket_\Gamma}(\vec{v})_{\langle\rangle} \widehat{\llbracket Q \rrbracket_\Gamma}(\vec{v}) + \sum_{n \in \mathbb{N}} \sum_{s \in \mathbb{N}^{<\omega}} \widehat{\llbracket P \rrbracket_\Gamma}(\vec{v})_{n@s} \widehat{\llbracket R \rrbracket_{\Gamma, x:\iota, y:\mathsf{List}}}(\vec{v}, e_n, e_s)$$

(c) The extension of the $\mathbf{Pcoh}_!$ denotational model of pPCF for modelling the new primitives, where $\vec{v} \in \mathsf{P}\llbracket \Gamma \rrbracket$.

FIGURE 1 – The extension of pPCF with new primitives manipulating finite sequences of natural numbers.

The new ground type List is interpreted in $\mathbf{Pcoh}_!$ by endowing the **Rel** object L of finite sequences of natural numbers with the PCS of subprobability distributions, that is :

$$|\mathsf{L}| = \mathbb{N}^{<\omega}, \qquad \mathsf{PL} = \left\{ u \in [0,1]^{\mathbb{N}^{<\omega}} \, ; \, \sum_{s \in \mathbb{N}^{<\omega}} u_s \leqslant 1 \right\}$$

Figure 1c gives the functional characterisation of the denotation of the new primitives of pPCF in $\mathbf{Pcoh}_!$, where we recall that, for any sequence $s \in \mathbb{N}^{<\omega}$, e_s is the vector in PL giving 1 to s and zero to any other sequence.

Exercice 3 :

Consider the hom-set $\mathbf{Pcoh}(\mathsf{L}, 1 \oplus (\mathsf{N} \otimes \mathsf{L}))$ of linear morphisms between the PCSs L and $1 \oplus (\mathsf{N} \otimes \mathsf{L})$. Prove that the matrix $\mathsf{mat}(\theta)$ generated by the relational isomorphism θ discussed in Exercise 1 is an isomorphism in $\mathbf{Pcoh}(\mathsf{L}, 1 \oplus (\mathsf{N} \otimes \mathsf{L}))$.

▷ *By definition we have :*

$$\mathsf{mat}(\theta)_{s,(1,\star)} = \begin{cases} 1 & \text{if } s = \langle\,\rangle \\ 0 & \text{otherwise} \end{cases} \qquad \mathsf{mat}(\theta)_{s,(2,(n,s'))} = \begin{cases} 1 & \text{if } s = n@s' \\ 0 & \text{otherwise} \end{cases}$$

One way to prove that $\mathsf{mat}(\theta) \in \mathbf{Pcoh}([\![\mathsf{List}]\!], 1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]))$ *is to check that for every* $x \in \mathsf{P}[\![\mathsf{List}]\!]$, $\mathsf{mat}(\theta) \cdot x \in \mathsf{P}(1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]))$. *Notice that :*

$$\mathsf{mat}(\theta) \cdot x = x_{\langle\,\rangle} e_{(1,\star)} + \sum_{n@s \in \mathbb{N}^{<\omega}} x_{n@s} e_{(2,(n,s))}$$

Notice also that $e_{(1,\star)}, e_{(2,(n,s))} \in \mathsf{P}(1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]))$, *so* $\mathsf{mat}(\theta) \cdot x \in \mathsf{P}(1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]))$ *as a barycentric combination of vectors in* $\mathsf{P}(1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]))$.
In order to prove that $\mathsf{mat}(\theta^{-1}) \in \mathbf{Pcoh}(1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]), [\![\mathsf{List}]\!])$. *Consider the matrices* $f \in \mathbb{R}_{\geqslant 0}^{\{\star\} \times |[\![\mathsf{List}]\!]|}$ *and* $g \in \mathbb{R}_{\geqslant 0}^{|\mathsf{N} \times [\![\mathsf{List}]\!]| \times |[\![\mathsf{List}]\!]|}$

$$f_{\star,s} = \delta_{s,\langle\,\rangle} \qquad g_{(n,s),s'} = \delta_{s',n@s}$$

Notice that they are morphisms in **Pcoh**, *in fact for* g *we have, for any* $u \in \mathsf{PN}$ *and* $v \in \mathsf{P}[\![\mathsf{List}]\!]$, $g(u \otimes v) = \sum_n u_n \sum_s v_s e_{n@s}$, *this latter being in* $\mathsf{P}[\![\mathsf{List}]\!]$ *as a barycentric combination of vectors in* $\mathsf{P}[\![\mathsf{List}]\!]$. *By the density lemma for the monoidal product this is enough to conclude that* $g \in \mathbf{Pcoh}(\mathsf{N} \otimes [\![\mathsf{List}]\!], [\![\mathsf{List}]\!])$.
Finally, we can conclude that $\mathsf{mat}(\theta^{-1}) \in \mathbf{Pcoh}(1 \oplus (\mathsf{N} \otimes [\![\mathsf{List}]\!]), [\![\mathsf{List}]\!])$ *as* $\mathsf{mat}(\theta^{-1})$ *is the copairing of* f *and* g.

The goal of the next exercices is to prove the adequacy Theorem of $\mathbf{Pcoh}_!$ for this extension of pPCF with List. The idea is to adapt the technique of logical relations for standard pPCF. We first extend the definition of logical relation we have considered in the lecture notes with the relation $\mathcal{R}_{\mathsf{List}} \subseteq \mathsf{PL} \times \Lambda_{\emptyset}^{\mathsf{List}}$:

$$u \, \mathcal{R}_{\mathsf{List}} \, M \ \text{ iff } \ \forall s \in \mathbb{N}^{<\omega}, u_s \leqslant \mathsf{Red}(\mathsf{List})^{\infty}_{M,s}$$

Exercice 4 :

An auxiliary lemma convenient for the logical relation technique is the following statement [1] :

($\star$) For every closed terms M of type ι and P of type List, we have :

$$\mathsf{Red}(\iota)^{\infty}_{M,\underline{n}} \mathsf{Red}(\mathsf{List})^{\infty}_{P,\underline{s}} \leqslant \mathsf{Red}(\mathsf{List})^{\infty}_{(M::P),\underline{n@s}}$$

Prove the above inequality. [*Hint : one can prove that for any* $k, h \in \mathbb{N}$, $\mathsf{Red}(\iota)^{k}_{M,\underline{n}} \mathsf{Red}(\mathsf{List})^{h}_{P,\underline{s}} \leqslant \mathsf{Red}(\mathsf{List})^{\infty}_{(M::P),\underline{n@s}}$. *The proof can be developed by induction on* $k+h$.]

▷ *We have to prove that for any* $k, h \in \mathbb{N}$, $\mathsf{Red}(\iota)^{k}_{M,\underline{n}} \mathsf{Red}(\mathsf{List})^{h}_{P,\underline{s}} \leqslant \mathsf{Red}(\mathsf{List})^{\infty}_{(M::P),\underline{n@s}}$. *The proof is by induction on* $k+h$.

- *For* $k+h=0$, *then the left-hand side of the claimed inequality is non-zero only for* $M = \underline{n}$ *and* $P = \underline{s}$, *in which case the right-hand side values* 1.

1. Actually also the inverse inequality of ($\star$) holds, but it is not necessary for the proof of the adequacy.

– *For $k = 0$, $h > 0$, then for a similar reasoning as above, we can suppose $M = \underline{n}$, in which case we have :*

$$\begin{aligned}
\mathsf{Red}(\iota)^0_{M,\underline{n}}\mathsf{Red}(\mathsf{List})^h_{P,\underline{s}} &= \sum_L \mathsf{Red}(\mathsf{List})_{P,L}\mathsf{Red}(\iota)^0_{\underline{n},\underline{n}}\mathsf{Red}(\mathsf{List})^{h-1}_{L,\underline{s}} \\
&\leqslant \sum_L \mathsf{Red}(\mathsf{List})_{P,L}\mathsf{Red}(\mathsf{List})^\infty_{(\underline{n}::L),\underline{n@s}} && \textit{by ind. hyp.} \\
&= \sum_L \mathsf{Red}(\mathsf{List})_{(\underline{n}::P),(\underline{n}::L)}\mathsf{Red}(\mathsf{List})^\infty_{(\underline{n}::L),\underline{n@s}} && \textit{by the contextual rules of Figure 1b} \\
&= \mathsf{Red}(\mathsf{List})^\infty_{(\underline{n}::P),\underline{n@s}}
\end{aligned}$$

– *For $k > 0$, we have :*

$$\begin{aligned}
\mathsf{Red}(\iota)^k_{M,\underline{n}}\mathsf{Red}(\mathsf{List})^h_{P,\underline{s}} &= \sum_L \mathsf{Red}(\iota)_{M,L}\mathsf{Red}(\iota)^{k-1}_{L,\underline{n}}\mathsf{Red}(\mathsf{List})^h_{P,\underline{s}} \\
&\leqslant \sum_L \mathsf{Red}(\mathsf{List})_{M,L}\mathsf{Red}(\mathsf{List})^\infty_{(L::P),\underline{n@s}} && \textit{by ind. hyp.} \\
&= \sum_L \mathsf{Red}(\mathsf{List})_{(M::P),(L::P)}\mathsf{Red}(\mathsf{List})^\infty_{(L::P),\underline{n@s}} && \textit{by the contextual rules of Figure 1b} \\
&= \mathsf{Red}(\mathsf{List})^\infty_{(M::P),\underline{n@s}}
\end{aligned}$$

Exercice 5 :

The key lemma of a logical relation is the so-called interpretation Lemma, stating that for all $\Gamma \vdash M : A$, with $\Gamma = x_1 : A_1, \ldots, x_k : A_k$, for all closed terms N_i of type A_i, for all vectors $u_i \in \mathsf{P}([\![A]\!])$ such that $u_i\, \mathcal{R}_{A_i}\, N_i$ for $i = 1, \ldots k$, one has :

$$\widehat{[\![M]\!]_\Gamma}(u_1, \ldots, u_k)\; \mathcal{R}_A\; M[N_1/x_1, \ldots, N_k/x_k]. \tag{1}$$

The proof of this lemma is by structural induction on the type derivation of $\Gamma \vdash M : A$. Detail the cases of this inductive proof for the three new typing rules of Figure 1a.
In addition to the inequality $(\star)$ of Exercise 4 you can also use (without proving it) the following inequality, for any type judgments $\vdash M : \mathsf{List}$, $\vdash P : A$ and $x : \iota, y : \mathsf{List} \vdash R : A$:

$(\star\star)$ for all closed value V of type A,

$$\mathsf{Red}(\mathsf{List})^\infty_{M,\underline{\langle\rangle}}\mathsf{Red}(A)^\infty_{P,V} + \sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{N}^{<\omega}} \mathsf{Red}(\mathsf{List})^\infty_{M,\underline{n@s}}\mathsf{Red}(A)^\infty_{R[\underline{n}/x,\underline{s}/y],V} \leqslant \mathsf{Red}(A)^\infty_{\mathsf{ifl}(M,P,x\cdot y\cdot R),V}$$

▷

– *If $M = \underline{s}$, then the proof is trivial, as $\widehat{[\![\underline{s}]\!]_\Gamma}(\vec{u}) = e_s = (\mathsf{Red}(\mathsf{List})^\infty_{M[\vec{N}/\vec{x}],\underline{s'}})_{s'}$.*

– *If $M = (P :: Q)$, then we have $\Gamma \vdash P : \iota$, $\Gamma \vdash Q : \mathsf{List}$, and A is the ground type List. We should then prove that for any $n \in \mathbb{N}$, $s \in \mathbb{N}^{<\omega}$, $\widehat{[\![M]\!]_\Gamma}(\vec{u})_{n@s} \leqslant \mathsf{Red}(\mathsf{List})_{M[\vec{N}/\vec{x}],\underline{n@s}}$ (the case of the empty list being trivial as the left-hand side of the inequality is null). We have :*

$$\begin{aligned}
\widehat{[\![M]\!]_\Gamma}(\vec{u})_{n@s} &= \widehat{[\![P]\!]_\Gamma}(\vec{u})_n\widehat{[\![Q]\!]_\Gamma}(\vec{u})_s \\
&\leqslant \mathsf{Red}(\iota)^\infty_{P[\vec{N}/\vec{x}],\underline{n}}\mathsf{Red}(\mathsf{List})^\infty_{Q[\vec{N}/\vec{x}],\underline{s}} && \textit{by ind. hyp.} \\
&\leqslant \mathsf{Red}(\mathsf{List})^\infty_{M[\vec{N}/\vec{x}],\underline{n@s}} && \textit{by } (\star)
\end{aligned}$$

– *If $M = \mathsf{ifl}(P, Q, x \cdot y \cdot R)$. Then $A = B_1 \Rightarrow \cdots \Rightarrow B_q \Rightarrow G$, for some $q \in \mathbb{N}$ and $G \in \{\iota, \mathsf{List}\}$. For every $j \leqslant q$, let $v_j\, \mathcal{R}_{B_j}\, H_j$, we have to prove that (with a bit of abuse of notation), for every w value of type G, $\widehat{[\![M]\!]_\Gamma}(\vec{u}, \vec{v})_w \leqslant \mathsf{Red}(G)^\infty_{M[\vec{N}/\vec{x}]\vec{H},\underline{w}}$. In fact, we have (remarking that $e_n\, \mathcal{R}_\iota\, \underline{n}$ and $e_s\, \mathcal{R}_{\mathsf{List}}\, \underline{n}$) :*

$$\begin{aligned}
\widehat{[\![M]\!]_\Gamma}(\vec{u}, \vec{v})_w &= \widehat{[\![P]\!]_\Gamma}(\vec{u})_{\langle\rangle}\widehat{[\![Q]\!]_\Gamma}(\vec{u}, \vec{v})_w + \sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{N}^{<\omega}} \widehat{[\![P]\!]_\Gamma}(\vec{u})_{n@s}\widehat{[\![R]\!]_{\Gamma,x:\iota,y:\mathsf{List}}}(\vec{u}, e_n, e_s, \vec{v})_w \\
&\leqslant \mathsf{Red}(\iota)^\infty_{P[\vec{N}/\vec{x}],\underline{\langle\rangle}}\mathsf{Red}(G)^\infty_{Q[\vec{N}/\vec{x}]\vec{H},\underline{w}} + \sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{N}^{<\omega}} \mathsf{Red}(\iota)^\infty_{P[\vec{N}/\vec{x}],\underline{n@s}}\mathsf{Red}(G)^\infty_{R[\vec{N}/\vec{x},\underline{n}/x,\underline{s}/y]\vec{H},\underline{w}} && \textit{by ind.hyp.} \\
&\leqslant \mathsf{Red}(G)^\infty_{M[\vec{N}/\vec{x}]\vec{H},\underline{w}} && \textit{by } (\star\star)
\end{aligned}$$

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