A Dendroidal Process Calculus

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Abstract. The Dendroidal Process Calculus (DPC) is designed as a new theory for modeling concurrent and distributed computations and systems, equipped with a non-sequential and compositional semantics. In this theory, a parallel composition is parameterized by a graph at the vertices of which subprocesses are located. Communication is allowed only between subprocesses related by an edge in this graph. Moreover, an observational equivalence based on barbs as well as a weak bisimilarity equivalence are defined and an adequacy theorem relating these two notions is proved. DPC is shown to be a conservative extension of both top-down tree automata and of the process algebra CCS, and to endow CCS with a non-sequential semantics. The expressiveness of this theory looks promising to describe and analyze some phenomena arising in weak memory models and in network security. As an illustration of potential applications, an associated notion of tree shuffle is introduced and analyzed.

1 Introduction

There is no need to insist on the importance of tree automata [1] in modern theoretical and applied computer science: they are pervasive in logic, verification, rewriting, structured documents handling, constraint solving etc. Tree automata are similar to usual finite word automata with the difference that they recognize trees instead of words (sequences of letters). Let Σ be a ranked signature (Σ_n is the set of function symbols of arity n). A Σ-tree is just a term written with the signature Σ. A *top-down tree automaton* has a finite number of *states* and transitions labeled by elements of Σ: a transition labeled by $f \in \Sigma_n$ has one *source* and a sequence of n *targets* which all are states of the automaton. Thus, a tree automaton is an extension of a word automaton in the sens that a word automaton can be seen as a tree automaton over a signature Σ such that Σ_n is empty for all $n > 1$ and Σ_0 has a unique distinguished element $*$.

The definition of tree recognition by a top-down tree automaton A is quite simple: a tree $f(t_1, \ldots, t_n)$ is recognized by A at state X if A has an f-labeled transition whose source is X and target is $(X_1, \dots, X_n)$ and t_i is recognized by A at state X_i for each $i = 1, \dots, n$.

Automata feature a *dualist* vision of computation with an essential dichotomy between programs (automata) and data (words, trees), very much in the spirit of Turing machines (based on the machine/tape dichotomy). The process algebra CCS, introduced in the early 1980's by Milner [2], encompasses this restriction, extending finite automata with interactive capabilities. In CCS, finite automata (labeled with letters $a, b, \dots$) can typically interact with other automata (labeled with dual letters $\overline{a}, \overline{b}, \dots$), as soon as they are combined through a new binary operation: *parallel composition*. Much more general interaction scenarios are of course possible in CCS. This fundamental invention led to very fruitful lines of research in the theory of concurrent processes and to the introduction of new process algebra, among which the π-calculus [3] is not the less remarkable, with many important applications to cryptography, bioinformatics etc.

We propose a new process algebra, called Dendroidal Process Calculus (DPC), equipped with a non-sequential and compositional semantics. More specifically, we introduce an "interactive closure" of top-down tree automata which extends tree automata just as ordinary CCS extends word automata. To achieve this goal, the natural idea is of course to add a parallel composition operation on processes, but this requires some care. Indeed, when a prefixed process $f \cdot (P_1, \dots, P_n)$ — after a prefix $f \in \Sigma_n$, it is natural to have n subprocesses, and not only one, as explained in [4] — interacts with a dually prefixed one $\overline{f} \cdot (Q_1, \dots, Q_n)$, we should remove the prefixes (just as in CCS) and then authorize interaction between the subprocess P_i with all processes which could communicate with its father $f \cdot (P_1, \dots, P_n)$ as well as with Q_i, *but not with the Q_j's for $j \neq i$*; neither should the P_i's be allowed to communicate with each other in the resulting process. The same should hold of course for the Q_i's.

For this purpose we have to preserve carefully the distinction between the various sons of tree nodes, preventing sons which are not at similar positions to interact. Therefore we were *forced* to generalize parallel composition: it is now given as a *graph*, at the vertices of which subprocesses (which are guarded sums) are located; the edges of this graph specify which interactions are allowed. In Section 2, we introduce the syntax of this new process calculus DPC, restricting ourselves to a fragment where all sums are guarded; indeed, the corresponding fragment of CCS is known to be sensible and well behaved. Section 2.1 introduces the operational semantics for DPC by defining a single rewriting rule. This rule generalizes the $a/\overline{a}$ reduction of CCS to the case where a can be an n-ary function symbol and implements the idea of restricted communication capabilities explained above.

In order to define an operational equivalence on processes, we adapt in Section 5 the concept of *weak barbed congruence* [5, 6] which is a natural way of saying that two processes behave in the same way, in all possible contexts. As usual, this notion is quite difficult to handle as it involves a universal quantifications on contexts, and we thus introduce a notion of weak bisimilarity in Section 6 and prove that two weakly bisimilar processes are weakly barbed congruent in

Section 7. To this end, a labeled transition system on processes is defined, and the definition of its transitions involves crucially the locations (graph vertices). The notion of bisimulation itself takes locations carefully into account.

Section 3 shows that tree recognition can be expressed in DPC, using only the rewriting semantics. Though quite simple, this result uses in an essential way the restricted communication capabilities of DPC and shows that DPC is a conservative extension of top-down tree automata. We also argue that DPC is a conservative extension of CCS by isolating two fragments of DPC which are isomorphic to guarded CCS, as far as internal reduction is concerned. These fragments are defined by considering DPC processes where all parallel composition graphs are complete and where symbols have arity 1 (for the first fragment) and 2 (for the second one). As shown in Annex 8.8, the first fragment also coincides with CCS when weak bisimilarity is considered. It seems clear that we would get the same when considering barbed congruence instead of weak bisimilarity. But this is *not true* for the second fragment: $a \mid b$ and $a \cdot b + b \cdot a$ are not weak barbed congruent when considered as DPC processes in the second fragment (the equivalence induced on CCS processes in this way is larger than structural congruence however). This strongly suggests that DPC is an interesting candidate for defining a truly concurrent and interactive semantics of concurrency.

To explore further this idea, we analyze the notion of *tree shuffle* induced by DPC. Remember that CCS induces the canonical notion of shuffle of words as follows: any word $u = a_1 \dots a_n$ can be seen as the CCS process $a_1 \cdot a_2 \cdots a_n$ and v is a shuffle of words $u_1, \dots, u_k$ iff the CCS process $u_1 \mid \cdots \mid u_k \mid \overline{v}$ reduces (internally, in several steps) to the empty process. Similarly, any tree u can be seen as a DPC process. So, given trees $u_1, \dots, u_k$ and v, what does it mean that $u_1 \mid \cdots \mid u_k \mid \overline{v}$ reduces to the empty process in DPC? We prove in Section 4 that this holds exactly when there is a bijection between the internal nodes of the forest $u_1, \dots, u_n$ and the internal nodes of v, which satisfies mainly an acyclicity property. If we consider the vertical order of trees as a time order between elementary actions labeling nodes, this notion of shuffle describes all possibilities of performing the actions of $u_1, \dots, u_n$ in a parallel way, without creating cycles in time. We provide examples supporting this intuition. This condition is very reminiscent of time acyclicity constraints which are crucial in the formalization of weak memory models generalizing the SC setting [7].

Our results suggest that DPC is a sound and interesting extension of CCS. The main technical feature of DPC is that processes are located at the vertices of a graph which describes its internal interactions capabilities. Similar features can be found in various earlier works of several authors, see for instance [8] where processes are localized (and can interact when they are located at the same place) or [9, 10] where processes are graphical objects which evolve by graph rewriting. The precise connection between DPC and these related formalisms is not completely clear yet and will be explored in further work.

Most importantly we think that DPC might bring an useful contribution to the fundamental scientific programme explained in [11], whose main purpose is to endow process algebras with truly concurrent semantics.

We omit all proofs and some formal details for lack of space, but they can be found in the Annex.

2 Syntax of processes

We use letters $\boldsymbol{P}, \boldsymbol{Q}, \ldots$ to denote vectors $(P_1, \ldots, P_n)$, $(Q_1, \ldots, Q_n)$ etc. Let Loc be a countable set whose elements are called *locations* denoted with letters $p, q \ldots$ with or without subscripts or superscripts.

Graphs. Let E and F be disjoint sets and let $p \in E$. We set $E\,[F/p] = (E \setminus \{p\}) \cup F$. In other words, $E\,[F/p]$ is the set obtained from E by substituting the element p with the set F. By a Loc-*graph* (or simply graph) we mean a pair $G = (|G|, \frown_G)$, where $|G|$ is a finite subset of Loc and $\frown_G$ is a symmetric and antireflexive relation on $|G|$. Let G and H be graphs with $|G| \cap |H| = \emptyset$ and let $p \in |G|$. We define a graph $G\,[H/p]$ as follows: $|G\,[H/p]| = |G|\,[|H|/p]$ and, given $q, r \in |G\,[H/p]|$, we say that $q \frown_{G[H/p]} r$ if $q \frown_G r$ or $q \frown_H r$ or $q \frown_G p$ and $r \in |H|$ or $r \frown_G p$ and $q \in |H|$.

Processes. We set $\mathbf{N}^+ = \mathbf{N} \setminus \{0\}$. We assume to be given a countable set of *processes variables* $\mathcal{V}$, denoted with letters $X, Y, \ldots$ Let $\Sigma = (\Sigma_n)_{n \in \mathbf{N}}$ be a signature. For each $n \in \mathbf{N}^+$, we assume to be given a bijection $f \mapsto \bar{f}$ from Σ_n to Σ_n such that $\bar{f} \neq f$ and $\bar{\bar{f}} = f$ for each $f \in \Sigma_n$. If $f \in \Sigma_n$, we say that f is a symbol of arity n. In order to build finite trees, we need at least one symbol of arity 0: we decide to have only one such symbol, denoted as $*$. So we have $\Sigma_0 = \{*\}$, and $\bar{*}$ is undefined.

Definition 1. *The set of Σ-trees is the smallest set such that, if $t_1, \ldots, t_n$ are Σ-trees and $f \in \Sigma_n$ then $f(t_1, \ldots, t_n)$ is a Σ-tree.*

In the case $n = 0$ we must have $f = *$ and the corresponding Σ-tree is $*()$ also simply denoted as $*$ and called the *empty tree*. A Σ-tree of which all symbols (but $*$, the last one) have arity 1 is simply a sequence of such symbols, that is, a word. In that way, our formalism extends the usual presentation of words on an alphabet. We define the set of DPC processes as follows.

$$P := X \mid \mu X \cdot P \mid f \cdot (P_1, \ldots, P_n) \mid G\langle \Phi \rangle \mid 0 \mid P_1 + P_2 \mid P \setminus I$$

where $X \in \mathcal{V}$, $f \in \Sigma_n$, G is a finite Loc-graph and Φ is a function from $|G|$ to processes, and I is a finite subset of Σ. When we want to be specific about the signature used in a given version of DPC, we use $\mathsf{DPC}(\Sigma)$ to denote this version.

The notion of free and bound variable does not deserve further comments, μ being of course a binder of process variables. Similarly, process restriction $\_ \setminus I$ is a binding operation (names mentioned in I are bound in $P \setminus I$).

Definition 2 (α-equivalence for locations). *Two processes P and P' such that there exists a bijection $\varphi : |P| \to |P'|$ which is a graph isomorphism (that is $p \frown_P q \Leftrightarrow \varphi(p) \frown_{P'} \varphi(q)$) and $P'(\varphi(p)) = P(p)$ for all $p \in |P|$ are said to be* externally α-equivalent for locations.

General α-equivalence is defined by extending this relation to sub-processes in the obvious way. When we consider several processes $P_1, \ldots, P_n$ at the same time, we always assume that the sets $|P_1|, \ldots, |P_n|$ are pairwise disjoint.

If R and P are processes and $X \in \mathcal{V}$, then the process $R\,[P/X]$ is defined in the obvious way, substituting each occurrence of X in R with P. Of course, one has as usual to perform α-conversions (for process variables, names and locations) when needed during substitution.

We define now the notion of *canonical process*: it is a process where all sums are guarded. More precisely, we define by mutual induction three classes of objects: *canonical processes*, *canonical guarded sum* and *recursive canonical guarded sum*.

Definition 3 (canonical process, (recursive) canonical guarded sum).

- *If $X \in \mathcal{V}$ then X is a canonical process.*
- *If G is a finite* Loc*-graph, Φ is a function from $|G|$ to recursive canonical guarded sums and I is a finite subset of Σ, then $G\langle\Phi\rangle \setminus I$ is a canonical process (we identify $G\langle\Phi\rangle$ with $G\langle\Phi\rangle \setminus \emptyset$ so that we also consider $G\langle\Phi\rangle$ as a canonical process).*
- *A canonical guarded sum is either* 0 *or a process of the shape $f \cdot (P_1, \ldots, P_n) +$ S where $f \in \Sigma_n$, S is a canonical guarded sum and $P_1, \ldots, P_n$ are canonical processes.*
- *A recursive canonical guarded sum is either a canonical guarded sum or a process of shape $\mu X \cdot S$ where S is a recursive canonical guarded sum.*

For instance, the processes $G\langle\Phi\rangle + H\langle\Psi\rangle$ and $\mu X \cdot X$ are not canonical.

Lemma 1. *If R and P are canonical processes then so is $R\,[P/X]$. If R is a recursive canonical guarded sum, then so is $R\,[P/X]$. If R is a canonical guarded sum, then so is $R\,[P/X]$.*

With any *recursive canonical guarded sum S*, we associate a *canonical guarded sum* $\mathsf{cs}(S)$ as follows: $\mathsf{cs}(S)$ is S if S is a canonical guarded sum and is $\mathsf{cs}(T\,[S/X])$ if $S = \mu X \cdot T$. Using Lemma 1, one sees easily that this function is well defined and total and that canonical processes are closed under substitution.

All the processes we consider in this paper are canonical.

More notations. We use Proc for the set of all canonical processes. If $P = G\langle\Phi\rangle \setminus I$ is a canonical process, we set $|P| = |G|$. Also, for $p \in |P|$, we often write $P(p)$ instead of $\Phi(p)$, and we denote as $\frown_P$ the graph relation of G. Last, we use I_P for the set I of symbols.

Given two graphs G and H with disjoint webs, and a subset D of $|G| \times |H|$ we define a graph $K = G \oplus_D H$ by $|K| = |G| \cup |H|$ and, given $p, q \in |K|$, we stipulate that $p \frown_K q$ if $p \frown_G q$ or $p \frown_H q$ or $(p, q) \in D$ or $(q, p) \in D$. If $D = \emptyset$ then we set $G \oplus H = G \oplus_D H$.

Given processes $P = G\langle\Phi\rangle \setminus I$ and $Q = H\langle\Psi\rangle \setminus J$ and a relation $D \subseteq |P| \times |Q|$, we define the process $P \oplus_D Q$ as $(G \oplus_D H)\langle\Phi \cup \Psi\rangle \setminus (I \cup J)$, assuming the sets I

and J to be disjoint (this is always possible since restriction is a binder). When D is empty we simply denote this sum as $P \oplus Q$, and generally, we denote as $\oplus \boldsymbol{P}$ the sum $P_1 \oplus \cdots \oplus P_n$ of the processes $\boldsymbol{P} = (P_1, \ldots, P_n)$. When $D = |P| \times |Q|$, the process $P \oplus_D Q$ will be denoted as $P \mid Q$ and called the *full parallel composition* of P and Q. It corresponds to the standard parallel composition of process algebras, where all processes can freely interact with each other.

Definition 4. *With the same notations as above, if $p \in |G|$, we denote as $P[Q/p]$ the process $G[H/p]\langle \Phi' \rangle \setminus (I_P \cup I_Q)$ where $\Phi'(p') = \Phi(p')$ if $p' \notin |H|$ and $\Phi'(p') = \Psi(p')$ if $p' \in |H|$, assuming the sets I_P and I_Q to be disjoint (again this is possible because restriction is a binding operation).*

2.1 Internal reduction

Let P and P' be processes. We say that P (internally) *reduces* to P' if there are $p, q \in |P|$ such that $p \frown_P q$, $\mathsf{cs}(P(p)) = f \cdot (P_1, \ldots, P_n) + S$, $\mathsf{cs}(P(q)) = \overline{f} \cdot (Q_1, \ldots, Q_n) + T$ (this implies that $n \geq 1$) and P' is defined as follows[1]: $|P'| = (|P| \setminus \{p, q\}) \cup \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $\frown_{P'}$ is the least symmetric relation on $|P'|$ such that, for any, $p', q' \in |P'|$, one has $p' \frown_{P'} q'$ in one of the following cases:

1. $p' \frown_{P_i} q'$ or $p' \frown_{Q_i} q'$ for some $i = 1, \ldots, n$
2. $p' \in |P_i|$ and $q' \in |Q_i|$ for some $i = 1, \ldots, n$ (*the same i for both*)
3. $\{p', q'\} \not\subseteq \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$ and $\lambda_1(p') \frown_P \lambda_1(q')$

where $\lambda_1 : |P'| \to |P|$ is the *residual function* $\lambda_1(p') = \begin{cases} p & \text{if } p' \in \bigcup_{i=1}^{n} |P_i| \\ q & \text{if } p' \in \bigcup_{i=1}^{n} |Q_i| \\ p' & \text{otherwise.} \end{cases}$

Observe that λ_1 is a surjection since $n \geq 1$.

We finish the definition of P' by saying that $P'(p') = P_i(p')$ if $p' \in |P_i|$, $P'(p') = Q_i(p')$ if $p' \in |Q_i|$ (for $i = 1, \ldots, n$) and $P'(p') = P(p')$ if $p' \notin \bigcup_{i=1}^{n} |P_i| \cup \bigcup_{i=1}^{n} |Q_i|$, and by stipulating that $I_{P'} = I_P \cup I_{P_1} \cup \cdots \cup I_{P_n} \cup I_{Q_1} \cup \cdots \cup I_{Q_n}$, assuming as usual this union to be a disjoint union.

Remark 1. This function λ_1 (the "1" stresses that we are considering a single reduction step) will be essential in the definition of the localized transition system as it allows to trace the localizations of sub-processes during the reduction.

This crucial definition of internal reduction deserves some explanations. The process P to be reduced has two subprocesses located at p and q, with dual prefixes: $f \cdot \boldsymbol{P}$ and $\overline{f} \cdot \boldsymbol{Q}$. The fact that p and q are connected in P (that is $p \frown_P q$) means that these processes can interact. This interaction consists in suppressing both prefixes and in replacing the vertex p of the graph G of P by the graph $G_1 \oplus \cdots \oplus G_n$ (where G_i is the graph of P_i) and the vertex q by the

[1] We strongly use the implicit hypothesis that, when several processes $P_1, \ldots, P_n$ are considered at the same time, the sets $|P_i|$ are pairwise disjoint.

graph $H_1 \oplus \cdots \oplus H_n$ (where H_i is the graph of Q_i). The connection between p and q in P is inherited by the vertices of G_i and H_i in P', but a process located on G_i (one of the components of P_i) cannot communicate with a process located on H_j with $j \neq i$. The connections between p and other vertices of P, distinct from q, are also inherited by the vertices of all G_i's and similarly for the H_i's.

Notice that in this reduction the restriction sets associated with the canonical processes P_i and Q_i are lifted in outermost position in order to make sure that the resulting process is canonical.

We denote with $\rightarrow$ the internal reduction relation and with $\rightarrow^*$ its reflexive and transitive closure. The edge (p, q) of the graph G is called the main edge of the reduction $P \rightarrow P'$.

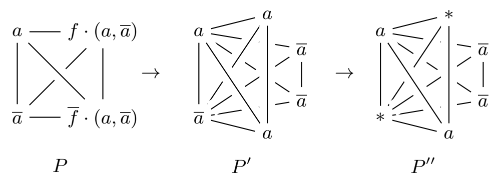

Fig. 1. A simple reduction sequence

Example 1. Let $a \in \Sigma_1$ and $f \in \Sigma_2$. Consider the process[2] $P = \overline{a} \mid a \mid f \cdot (a, \overline{a}) \mid \overline{f} \cdot (a, \overline{a})$. Figure 1 shows two reduction steps starting from P.

Example 2. Let $f \in \Sigma_2$, $P = \mu X \cdot f \cdot (X, X)$ and $\overline{P} = \mu X \cdot \overline{f} \cdot (X, X)$ and let $R = G\langle \Phi \rangle$ where $|G| = \{1, 2, 3\}$, $\Phi(1) = \Phi(2) = P$ and $\Phi(3) = \overline{P}$, $1 \frown_G 3$ and $2 \frown_G 3$. We show in Figure 2 three reduction steps starting from R.[3]

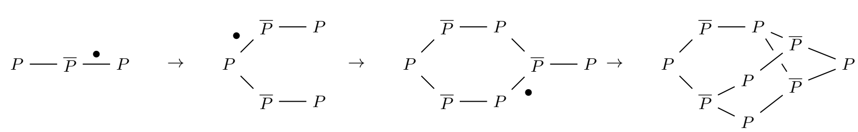

Fig. 2. Example of reduction, $P = \mu X \cdot f \cdot (X, X)$, fired edges labeled by $\bullet$

[2] We write simply "a" instead of $a \cdot * \cdot ()$

[3] To keep the picture simpler, we didn't put an edge between the two occurrences of P in R because this edge (and his residuals) would never be used during the reduction (P cannot interact with P since its only symbol is f).

3 Embedding tree automata and CCS processes in DPC

We will show that DPC is a natural extension of these two formlisms. For CCS, we exhibit two translations, one respecting its interleaving semantics, and the other featuring a non-sequential semantics of CCS.

Tree automata in DPC.

Definition 5 (Top-down tree automata). *A* top-down tree automaton *is a pair $A = (\mathcal{Q}, \mathcal{T})$ where $\mathcal{Q}$ is a finite subset of $\mathcal{V}$, whose elements are called* states*, and $\mathcal{T}$ is a finite set of triples $(X, f, (X_1, \dots, X_n))$ where $f \in \Sigma_n$ and $X_1, \dots, X_n \in \mathcal{Q}$ and whose elements are called* transitions*. The* language recognized by A at state $X \in \mathcal{Q}$*, denoted as $\mathsf{L}(A, X)$, is the least set of Σ-trees such that $f(t_1, \dots, t_n) \in \mathsf{L}(A, X)$ as soon as there are $X, X_1, \dots, X_n \in \mathcal{Q}$ such that $(X, f, (X_1, \dots, X_n)) \in \mathcal{T}$ and $t_i \in \mathsf{L}(A, X_i)$ for $i = 1, \dots, n$.*

We associate a process $\langle A \rangle_X$ with any pair (A, X) where $A = (\mathcal{Q}, \mathcal{T})$ is a tree automaton and $X \in \mathcal{Q}$. More generally we define $\langle A \rangle_X^{\mathcal{X}}$ where $\mathcal{X}$ is a finite subset of $\mathcal{V}$ (intuitively, $\mathcal{X}$ is the set of already defined processes), and then we set $\langle A \rangle_X = \langle A \rangle_X^{\emptyset}$.

- If $X \notin \mathcal{X}$, then $\langle A \rangle_X^{\mathcal{X}} = \mu X \cdot S$ where S is the sum of all prefixed processes $f \cdot (\langle A \rangle_{X_1}^{\mathcal{X} \cup \{X\}}, \dots, \langle A \rangle_{X_n}^{\mathcal{X} \cup \{X\}})$ with $(X, f, (X_1, \dots, X_n)) \in \mathcal{T}$,
- and if $X \in \mathcal{X}$, then $\langle A \rangle_X^{\mathcal{X}} = X$.

This inductive definition is well founded because the parameter $\mathcal{X}$ increases strictly at each inductive step, and remains included in the finite set $\mathcal{Q}$. Moreover, the invariant that all the free variables of $\langle A \rangle_X^{\mathcal{X}}$ belong to $\mathcal{X}$ is preserved by the inductive step, and hence $\langle A \rangle_X$ is closed.

With any Σ-tree $t = f(t_1, \dots, t_n)$, we associate a process $\mathsf{proc}(t)$ by $\mathsf{proc}(t) = f \cdot (\mathsf{proc}(t_1), \dots, \mathsf{proc}(t_n))$. We also associate with t a dual tree $\bar{t}$ by $\bar{t} = \bar{f}(\bar{t_1}, \dots, \bar{t_n})$ for $f \neq *$ and $\bar{t} = *()$ if $t = *()$.

Theorem 1. *Let $A = (\mathcal{Q}, \mathcal{T})$ be a tree automaton, let $X \in \mathcal{Q}$ and let t be a Σ-tree. Then $t \in \mathsf{L}(A, X)$ iff $(\langle A \rangle_X \mid \mathsf{proc}(\bar{t})) \to^* P_0$ where P_0 is an idle process.*

Sequentially embedding CCS in DPC. We assume here that $\Sigma_n = \emptyset$ for all $n > 1$. Then a Σ-tree is the same thing as a Σ_1-word, written $a_1 \dots a_p *$. We restrict our attention to processes in which all the graphs parameterizing parallel compositions are complete, so that any process is of shape $(S_1 \mid \cdots \mid S_p) \setminus I$, where each S_i is a recursive canonical guarded sum $\mu \boldsymbol{X} \cdot (a_1 \cdot P_1 + \cdots + a_m \cdot P_m)$: this restriction of our process algebra coincides with *guarded CCS*, with the additional requirement that all restrictions are pulled outside parallel compositions as much as possible (this operation on processes is part of Milner's basic transformations axiomatized in [12], Chapter 8).

Observe also that, if P is a process in this restricted setting (arities ≤ 1 and all parallel compositions are complete graphs), and if $P \to P'$, then P' belongs to the same restriction and the reduction $P \to P'$ is a standard τ-reduction of CCS. In that way we see that our process algebra is also a conservative extension of ordinary guarded CCS.

Through this embedding, DPC is a conservative extension of CCS for the notion of bisimulation we introduce later, this will be proved in an extended version of this paper (see also Annex 8.8).

Non-sequentially embedding CCS in DPC. Let Σ be such that $\Sigma_n = \emptyset$ for all $n > 1$. So the processes of $\mathsf{DPC}(\Sigma)$ where all graphs are complete can be identified with CCS processes built on the alphabet Σ_1. Let now Σ' be the signature such that $\Sigma'_n = \emptyset$ if $n = 1$ or $n > 2$ and $\Sigma'_2 = \Sigma_1$ (and $\Sigma'_0 = \{*\}$). Then we define a translation $P \mapsto P^\bullet$ from $\mathsf{DPC}(\Sigma)$ to $\mathsf{DPC}(\Sigma')$ as follows

$$X^\bullet = X \qquad (\mu X \cdot P)^\bullet = \mu X \cdot P^\bullet \qquad * \cdot ()^\bullet = * \cdot ()$$
$$(a \cdot P)^\bullet = a \cdot (P^\bullet, *) \qquad (G\langle \Phi \rangle)^\bullet = G\langle \Phi^\bullet \rangle \text{ where } \Phi^\bullet(p) = \Phi(p)^\bullet$$
$$(P_1 + P_2)^\bullet = P_1{}^\bullet + P_2{}^\bullet \qquad (P \setminus I)^\bullet = P^\bullet \setminus I$$

Observe that this translation maps canonical processes to canonical processes. We consider it only on processes of $\mathsf{DPC}(\Sigma)$ which are CCS processes, that is, whose all graphs are complete.

Proposition 1. *The map $P \mapsto P^\bullet$ is injective and is a strong bisimulation with respect to the internal reduction relation defined in Section 2.1: if $P \to P'$ then $P^\bullet \to (P')^\bullet$, and, if $P^\bullet \to Q'$, then there is a (uniquely defined) CCS process P' such that $P \to P'$ and $Q' = (P')^\bullet$.*

This translation is just another way to see DPC as a conservative extension of CCS, as far as internal reduction is concerned. However, we can now define a new equivalence relation on CCS processes using the barbed congruence $\cong$ that we shall define in Section 5: $P \sim_{\mathsf{b}} Q$ if $P^\bullet \cong Q^\bullet$. This relation $\sim_{\mathsf{b}}$ is a congruence on CCS processes which *does not satisfy Milner's Expansion Law* [12] as illustrated by the second part of Example 3 of Section 6.

However it is not hard to check that, for instance, $(f \cdot (a \mid b) \mid \bar{f}) \setminus \{f, \bar{f}\} \sim_{\mathsf{b}} ((f \cdot a \mid \bar{f}) \mid b) \setminus \{f, \bar{f}\}$ which shows that $\sim_{\mathsf{b}}$ is a non-trivial congruence on CCS processes. We think that it endows CCS with an interesting non-sequential semantics in the spirit of [11, 8], that we want to study in further work.

4 Application: a shuffle of trees

Just as words, trees can be shuffled. There are however several possible definitions for tree shuffles, all extending the standard shuffle of words (see for instance [13] and the tree shuffles occurring in the theory of dendroidal sets of Moerdijk and Weiss). Our formalism suggests a new one that we characterize combinatorially.

Given a set A, we use A^* for the set of finite sequences of elements of A. This set is equipped with the usual prefix order: $\alpha < \beta$ if there exists $\gamma \in A^* \setminus \{\langle\rangle\}$ such that $\beta = \alpha\gamma$. Assume that $A = \mathbf{N}^+$. Given a Σ-tree t, we define the

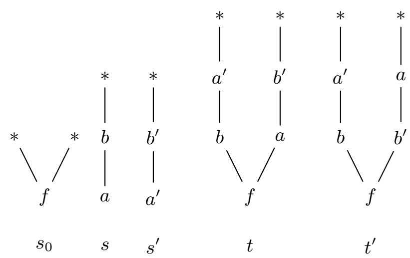

Fig. 3. t is a shuffle of s_0, s and s', while t' is not (the acyclicity condition is not satisfied).

domain of t as a set of finite sequences of natural numbers $|t|$. The definition is by induction on the size of trees: $|*| = \{\langle\rangle\}$ and $|f(t_1, \ldots, t_n)| = \{\langle\rangle\} \cup \{i\alpha \mid i = 1, \ldots, n \text{ and } \alpha \in |t_i|\}$. Given $\alpha \in |t|$, we define the subtree of t located at α: $t/\langle\rangle = t$ and $f(t_1, \ldots, t_n)/i\alpha = t_i/\alpha$ We use $\partial|s|$ for the set of maximal elements of $|s|$ (wrt. the prefix order). That is, $\partial|s| = \{\alpha \in |s| \mid s/\alpha = *\}$. We also use $|s|^\circ$ for the set of inner "addresses" in s, that is $|s|^\circ = |s| \setminus \partial|s|$. Let t be a Σ-tree and $\alpha \in |t|^\circ$. Then we define $t(\alpha) \in \Sigma \setminus \{*\}$, the symbol of t located at address α: $f(t_1, \ldots, t_n)(\langle\rangle) = f$ and $f(t_1, \ldots, t_n)(i\alpha) = t_i(\alpha)$.

Tree shuffles. We are now in position of defining our notion of tree shuffle.

Definition 6. *Let $s_1, \ldots, s_n$ (with $n \geq 1$) and t be trees. We say that t is a shuffle of $s_1, \ldots, s_n$ if there is a bijection $\varphi : \bigcup_{i=1}^n \{i\} \times |s_i|^\circ \to |t|^\circ$ satisfying:*

1. *labeling condition: $\forall i \in \{1, \ldots, n\}\, \forall \alpha \in |s_i|^\circ\; t(\varphi(i, \alpha)) = s_i(\alpha)$*
2. *branching condition: $\forall i \in \{1, \ldots, n\}\, \forall l \in \mathbf{N}^+\, \forall \alpha, \alpha' \in |s_i|^\circ$ $(\alpha l \leq \alpha'$ and $\varphi(i, \alpha) \leq \varphi(i, \alpha')) \Rightarrow \varphi(i, \alpha)l \leq \varphi(i, \alpha')$.*
3. *acyclicity condition: the binary relation $\prec_\varphi$ defined on $|t|^\circ$ as follows has no cycles: $\beta \prec_\varphi \beta'$ if $\exists l \in \mathbf{N}^+\; \beta' = \beta l$ or there are $i \in \{1, \ldots, n\}$, $l \in \mathbf{N}^+$ and $\alpha \in |s_i|^\circ$ such that $\alpha l \in |s_i|^\circ$, $\beta = \varphi(i, \alpha)$ and $\beta' = \varphi(i, \alpha l)$.*

And we say that φ is a witness *of t being a shuffle of $s_1, \ldots, s_n$.*

The first condition means that φ must respect the symbols attached to the internal nodes of the trees. The second means that, when φ respects the immediate descendant order, it must also respect the sibling order. The last condition means that φ can break the descendant order, provided it creates no cycles.

Figures 3 and 4 provide examples of tree shuffles and of non tree shuffles. The tree t' of Figure 3 is not a shuffle of s_0, s and s' because the (unique possible) embedding φ induces a relation $\prec_\varphi$ which has a cycle, namely $b \prec_\varphi a' \prec_\varphi b' \prec_\varphi a \prec_\varphi b$. The tree t' of Figure 4 is not a shuffle of s_1 and s_2 because, in s_1, b appears in the left sub-tree spanned by f whereas, in t', it appears in the right sub-tree spanned by f. Remember: P is *idle* if for each $p \in |P|$, $P(p) = *$.

Theorem 2. *Let $s_1, \ldots, s_n$ and t be trees. Then t is a shuffle of $s_1, \ldots, s_n$ if and only if $(\mathsf{proc}(s_1) \oplus \cdots \oplus \mathsf{proc}(s_n)) \mid \mathsf{proc}(\bar{t})$ reduces to an idle process.*

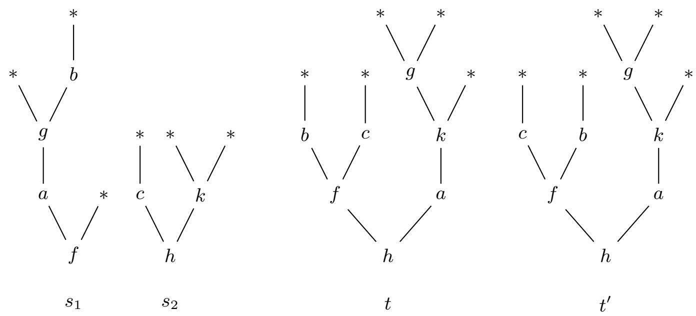

Fig. 4. t is a shuffle of s_1 and s_2, while t' is not (b is ill-placed wrt. f, the branching condition is not satisfied)

5 Observational equivalence

We define now an observational equivalence for DPC, using the concept of *barb*.

Weak barbed bisimilarity. Let $f \in \Sigma$ and let P be a canonical process. We say that f is a *barb* of P, and write $P \downarrow_f$, if $f \notin I_P$ and there exists $p \in |P|$ such that $\mathsf{cs}(P(p))$ is of shape $f \cdot (P_1, \ldots, P_n) + S$.

Definition 7. *A relation $\mathcal{B} \subseteq \mathsf{Proc}^2$ is a* weak barbed bisimulation *if it is symmetric and satisfies the following conditions. For any $P, Q \in \mathsf{Proc}$ such that $P \mathrel{\mathcal{B}} Q$,*

– *for any $P' \in \mathsf{Proc}$, if $P \rightarrow^* P'$, then there exists $Q' \in \mathsf{Proc}$ such that $Q \rightarrow^* Q'$ and $P' \mathrel{\mathcal{B}} Q'$ ($\mathcal{B}$ is a* weak reduction bisimulation*);*
– *for any $P' \in \mathsf{Proc}$ and any $f \in \Sigma$, if $P \rightarrow^* P'$ and $P' \downarrow_f$, then there exists $Q' \in \mathsf{Proc}$ such that $Q \rightarrow^* Q'$ and $Q' \downarrow_f$ ($\mathcal{B}$ is* weak barb preserving*).*

The diagonal relation $\{(P, P) \mid P \in \mathsf{Proc}\}$ is a weak barbed bisimulation, and if $\mathcal{B}$ and $\mathcal{B}'$ are weak barbed bisimulations, then so are $\mathcal{B}' \circ \mathcal{B}$ and $\mathcal{B} \cup \mathcal{B}'$. We say that $P, Q \in \mathsf{Proc}$ are *weakly barbed bisimilar* if there exists a weak barbed bisimulation $\mathcal{B}$ such that $P \mathrel{\mathcal{B}} Q$. Notation: $P \stackrel{\bullet}{\approx} Q$.

Lemma 2. *Weak barbed bisimilarity is an equivalence relation.*

Weak barbed congruence. Let Y be a variable; a *Y-context* is a process R which contains exactly one free occurrence of Y, which does not occur in a subprocess of R of the shape $\mu X \cdot R'$ (in other words, Y must really occur only once in R). If R and S are Y-contexts, so is $R\,[S/Y]$.

A relation $\mathcal{R} \subseteq \mathsf{Proc}^2$ is a *congruence* if it is reflexive and such that, for any Y-context R, one has $P \mathrel{\mathcal{R}} Q \Rightarrow R\,[P/Y] \mathrel{\mathcal{R}} R\,[Q/Y]$.

Proposition 2. *For any reflexive relation $\mathcal{R} \subseteq \mathsf{Proc}^2$, there is a largest congruence $\overline{\mathcal{R}}$ contained in $\mathcal{R}$, characterized by: $P \mathrel{\overline{\mathcal{R}}} Q$ iff for any Y-context R one has $R\,[P/Y] \mathrel{\mathcal{R}} R\,[Q/Y]$. Moreover, if $\mathcal{R}$ is an equivalence relation, so is $\overline{\mathcal{R}}$.*

The largest congruence contained in $\stackrel{\bullet}{\approx}$, written $\cong$, is called *weak barbed congruence*: it is our main notion of operational equivalence on processes and is ensured to be an equivalence relation by the proposition above and by Lemma 2. Moreover,

$$P \cong Q \text{ iff, for any } Y\text{-context } R,\ R\,[P/Y] \stackrel{\bullet}{\approx} R\,[Q/Y]\ .$$

6 Localized transition systems of processes

As in CCS, it is difficult to prove that two processes are weak barbed congruent, because of the universal quantification on contexts used in the definition. To prove weak barbed congruence of processes, one needs more convenient tools. A canonical tool is *weak bisimilarity*, an equivalence relation which means that two processes manifest the same interaction capabilities along their internal reductions. This equivalence relation is defined as the union of all *weak bisimulations*.

Weak bisimilarity is a congruence, this is the main ingredient in the proof that two weakly bisimilar processes are weakly barbed congruent. To prove this fact, one associates with each weak bisimulation $\mathcal{R}$ a new one, $\mathcal{R}'$, called its *parallel extension*. In CCS, $U\ \mathcal{R}'\ V$ if $U = P \mid S$ and $V = Q \mid S$ with $P\ \mathcal{R}\ Q$ and S is a process. The main step is of course to show that $\mathcal{R}'$ is a weak bisimulation.

In DPC however, we cannot simply speak of "the parallel composition" U of P and S, we have to specify a relation $C \subseteq |P| \times |S|$, and then we set $U = P \oplus_C S$. Similarly we must say that $V = Q \oplus_D S$ for some relation $D \subseteq |Q| \times |S|$, and that $P\ \mathcal{R}\ Q$. These relations C and D must fulfill some requirement. Our bisimulations cannot be simple relations between processes: when two processes $P = G\langle\Phi\rangle$ and $Q = H\langle\Psi\rangle$ are bisimilar, we must say which subprocess $\Phi(p)$ of P should be in bisimulation with which subprocesses $\Psi(q)$ of Q. For instance, if $P = f \cdot (P_1, P_2)$ and $Q = f \cdot (Q_1, Q_2)$ (with $|P| = |Q| = \{1\}$) are related by a bisimulation $\mathcal{R}$, then (after performing the action f on both sides), the processes $P_1 \oplus P_2$ and $Q_1 \oplus Q_2$ (with $|P_1 \oplus P_2| = |Q_1 \oplus Q_2| = \{1, 2\}$, and P_i and Q_i located at i for $i = 1, 2$) should be related by $\mathcal{R}$. But this cannot be achieved by saying that $P_1\ \mathcal{R}\ Q_2$ for instance: if P_1 manifests some interaction capability a, the same interaction capability a should be manifested by Q_1.

We enforce this discipline by saying that a bisimulation is a set of triples (P, E, Q) where P and Q are processes and $E \subseteq |P| \times |Q|$. In the example above, we start with $(P, \{(1, 1)\}, Q) \in \mathcal{R}$ (where 1 is the location of $f \cdot (P_1, P_2)$ in P and similarly for Q), and then, after having performed the action f on both sides, we arrive to $(P_1 \oplus P_2, \{(1, 1), (2, 2)\}, Q_1 \oplus Q_2) \in \mathcal{R}$.

Let us come back to the parallel extension of a bisimulation $\mathcal{R}$, which is a set of triples (P, E, Q) as explained above. We say that $(U, F, V) \in \mathcal{R}'$ when we can find a process S and two relations $C \subseteq |P| \times |S|$ and $D \subseteq |Q| \times |S|$ with $U = P \oplus_C S$ and $V = Q \oplus_D S$. There must also be a relation E such that $(P, E, Q) \in \mathcal{R}$ and $F = E \cup \mathrm{Id}_{|S|}$ and we require C and D to be "equivalent up to E": if $(p, q) \in E$, we have $(p, s) \in C$ iff $(q, s) \in D$.

Bisimulations are usually defined in terms of a *transition system*, a very general and flexible concept. Due to our more complex definition of bisimulations, it is not clear how to use transition systems in DPC; at least should we generalize them to take localization into account. An abstract notion of *localized transition system* might be of general interest, but we focus here on DPC and define one particular localized transition system of processes. Its states are processes. There are τ-transitions $P \xrightarrow[\rho]{\tau} P'$ corresponding to one internal reduction and $\rho : |P'| \to |P|$ allows to trace the "locative history" of the reduction. Labeled transition have shape $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$ where $p \in |P|$, $\boldsymbol{L} = (L_1, \dots, L_n)$ with $L_i \subseteq |P'|$ and $\lambda_1 : |P'| \to |P|$ keep track of the locative history of the reduction.

Localized transitions. We define now this localized transition system. Let P and P' be processes. We write $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$ if $p \in |P|$, $\mathsf{cs}(P(p)) = f \cdot (P_1, \dots, P_n) + S$ with $P' = P\,[\oplus \boldsymbol{P}/p]$ (see Definition 4), $L_1 = |P_1|, \dots, L_n = |P_n|$, $f \notin I_P$, and $\lambda_1 : |P'| \to |P|$ is the residual function defined by $\lambda_1(p') = p$ if $p' \in \bigcup_{i=1}^n L_i$ and $\lambda_1(p') = p'$ otherwise[4].

We write $P \xrightarrow[\lambda_1]{\tau} P'$ if $P \to P'$ in the sense of Section 2.1 and, with the notations of that section, $\lambda_1 : |P'| \to |P|$ is the residual function defined by $\lambda_1(p') = p$ if $p' \in \bigcup_i |P_i|$, $\lambda_1(p') = q$ if $p' \in \bigcup_i |Q_i|$, and $\lambda_1(p') = p'$ otherwise.

We define the reflexive-transitive closure $\xrightarrow[\lambda]{\tau *}$ as follows. We say that $P \xrightarrow[\lambda]{\tau *} P'$ if there are $n \geq 1$, processes $P_1, \dots, P_n$ and functions $\lambda_1, \dots, \lambda_{n-1}$ such that $P = P_1$, $P_n = P'$ and $P_i \xrightarrow[\lambda_i]{\tau} P_{i+1}$ for $i = 1, \dots, n-1$, and $\lambda = \lambda_1 \circ \cdots \circ \lambda_{n-1}$.

We write $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$ if there are processes P_1 and P_1' such that $P \xrightarrow[\lambda]{\tau *} P_1 \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P_1' \xrightarrow[\lambda']{\tau *} P'$.

Localized weak bisimilarity. The definition is coinductive and is based on a concept of bisimulation which strongly uses locations. A *localized relation* (on processes) is a set $\mathcal{R} \subseteq \mathsf{Proc} \times \mathcal{P}(\mathsf{Loc}^2) \times \mathsf{Proc}$ such that, if $(P, E, Q) \in \mathcal{R}$ then $E \subseteq |P| \times |Q|$. Such a relation $\mathcal{R}$ is *symmetric* if $(P, E, Q) \in \mathcal{R}$ implies $(Q, {}^{\mathrm{t}}E, P) \in \mathcal{R}$ where ${}^{\mathrm{t}}E = \{(q, p) \mid (p, q) \in E\}$.

Definition 8. *A* (localized) weak bisimulation *is a symmetric localized relation such that*

- *if* $(P, E, Q) \in \mathcal{R}$ *and* $P \xrightarrow[\lambda_1]{\tau} P'$ *then* $Q \xrightarrow[\rho]{\tau *} Q'$ *with* $(P', E', Q') \in \mathcal{R}$ *for some* $E' \subseteq |P'| \times |Q'|$ *such that, if* $(p', q') \in E'$ *then* $(\lambda_1(p'), \rho(q')) \in E$ *(this latter condition will be called* condition on residuals*)*
- *if* $(P, E, Q) \in \mathcal{R}$ *and* $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$ *then* $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ *with* $(p, \rho(q)) \in E$ *and* $(P', E', Q') \in \mathcal{R}$ *for some* $E' \subseteq |P'| \times |Q'|$ *such that if* $(p', q') \in E'$ *then* $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$*, and, moreover, if* $n \geq 2$*, then either* $(p', \rho'(q')) \in$

[4] There are redundancies in these notations, for instance λ_1 is completely determined by the data p, $\boldsymbol{L}$. This redundancy will be useful in the sequel.

$\bigcup_{i=1}^{n}(L_i \times M_i)$ *or* $p' \notin \bigcup_{i=1}^{n} L_i$ *and* $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$ *(this condition is called* condition on residuals*).*

This dichotomy, according to whether $n = 1$ or $n \geq 2$ yields three effects which seem impossible to conciliate otherwise: weak bisimilarity must be transitive, imply weak barbed congruence and extend the standard weak bisimilarity of CCS (seen as a subsystem of DPC as in Section 3). Now we give a symmetric characterization of weak bisimulation.

Lemma 3. *A symmetric localized relation* $\mathcal{R} \subseteq \mathsf{Proc} \times \mathcal{P}(\mathsf{Loc}^2) \times \mathsf{Proc}$ *is a weak bisimulation iff the following properties hold.*

- *If* $(P, E, Q) \in \mathcal{R}$ *and* $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$*, then* $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ *with* $(\lambda(p), \rho(q)) \in E$ *and* $(P', E', Q') \in \mathcal{R}$ *for some* $E' \subseteq |P'| \times |Q'|$ *such that if* $(p', q') \in E'$ *then* $(\lambda\lambda_1\lambda'(p'), \rho\rho_1\rho'(q')) \in E)$ *and, moreover, if* $n \geq 2$*, either* $(\lambda'(p'), \rho'(q')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$ *or* $\lambda'(p') \notin \bigcup_{i=1}^{n} L_i$ *and* $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$*.*
- *If* $(P, E, Q) \in \mathcal{R}$ *and* $P \xrightarrow[\lambda]{\tau *} P'$*, then* $Q \xrightarrow[\rho]{\tau *} Q'$ *with* $(P', E', Q') \in \mathcal{R}$ *for some* $E' \subseteq |P'| \times |Q'|$ *such that if* $(p', q') \in E'$ *then* $(\lambda(p'), \rho(q')) \in E$*.*

Lemma 4 (Reflexivity). *Let* $\mathcal{I}$ *be the localized relation defined by:* $(P, E, Q) \in \mathcal{I}$ *if* $P = Q$ *and* $E = \mathrm{Id}_{|P|}$*. Then* $\mathcal{I}$ *is a weak bisimulation.*

Let $\mathcal{R}$ and $\mathcal{S}$ be localized relations. We define a localized relation $\mathcal{S} \circ \mathcal{R}$ as follows: $(P, H, R) \in \mathcal{S} \circ \mathcal{R}$ if $H \subseteq |P| \times |R|$ and there exist Q, E and F such that $(P, E, Q) \in \mathcal{R}$, $(Q, F, R) \in \mathcal{S}$ and $F \circ E \subseteq H$.

Lemma 5 (Transitivity and union). *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are weak bisimulations, then so are* $\mathcal{S} \circ \mathcal{R}$ *and* $\mathcal{R} \cup \mathcal{S}$*.*

Definition 9. *P and Q are* weakly bisimilar *(notation* $P \approx Q$*) if there is a weak bisimulation* $\mathcal{R}$ *and a relation* $E \subseteq |P| \times |Q|$ *such that* $(P, E, Q) \in \mathcal{R}$*.*

Proposition 3. *The relation* $\approx$ *is an equivalence relation on processes.*

Proposition 4. *If* $P \approx Q$ *then* $P \overset{\bullet}{\approx} Q$*.*

We want now to prove that weak bisimilarity implies weak barbed congruence (and not just weak barbed bisimilarity). This boils down to proving that weak bisimilarity is a congruence. Let us first illustrate this implication.

Example 3. Let first Σ be such that $\Sigma_1 = \{a, b\}$ and $\Sigma_i = \emptyset$ if $i > 1$. Then it is easy to see that $a \cdot * \mid b \cdot *$ and $a \cdot b \cdot * + b \cdot a \cdot *$ are weakly bisimilar just as in usual CCS and so they are weak barbed congruent.

Let now Σ be such that $\Sigma_1 = \{a\}$, $\Sigma_2 = \{f, g\}$ and $\Sigma_i = \emptyset$ for $i > 2$. Let $P = f \cdot (g \cdot (*, *), *) + g \cdot (f \cdot (*, *), *)$ and $Q = f \cdot (*, *) \mid g \cdot (*, *)$. Then we cannot prove that P and Q are weakly bisimilar (because, in the definition of a localized bisimulation, we are in the case $n > 1$). And indeed, surprisingly, P and Q are not weak barbed congruent. Actually, let $R = \overline{f} \cdot (*, \overline{g} \cdot (a \cdot *, *)))$. Then $Q \mid R \to^* Q_0$ with $Q_0 \downarrow_a$ (Q_0 is a process such that $Q_0(p) = *$ for all $p \in |Q_0|$ but for $p = p_0$, and $Q_0(p_0) = a \cdot *$), see Figure 5. And there is no process M such that $P \mid R \to^* M$ with $M \downarrow_a$.

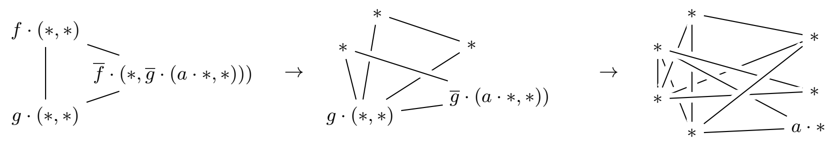

Fig. 5. A process featuring an interaction capability a after some internal reductions

7 Weak bisimilarity is a congruence

As for CCS, the main step for proving that weak bisimilarity is a congruence consists in extending a localized relation $\mathcal{R}$ on processes into another localized relation $\mathcal{R}'$ which is, intuitively, a congruence wrt. "parallel composition".

Definition 10 (Adapted triples of relations). *A triple of relations (D, D', E) with $D \subseteq A \times B$, $D' \subseteq A \times B'$ and $E \subseteq B \times B'$ is* adapted, *if, for any $(a, b, b') \in A \times B \times B'$, with $(b, b') \in E$, one has $(a, b) \in D$ iff $(a, b') \in D'$.*

Definition 11 (Parallel extension of a localized relation). *Let $\mathcal{R}$ be a localized relation on processes. The* parallel extension *of $\mathcal{R}$ is the relation $\mathcal{R}'$ such that $(U, F, V) \in \mathcal{R}'$ if there is a process S, a triple $(P, E, Q) \in \mathcal{R}$ and two relations $C \subseteq |S| \times |P|$ and $D \subseteq |S| \times |Q|$ such that $U = S \oplus_C P$, $V = S \oplus_D Q$ (see Section 2), the triple (C, D, E) is adapted and $F = \mathrm{Id}_{|S|} \cup E \subseteq |U| \times |V|$.*

Intuitively, we express here that U is the parallel composition of S and P, with connections between the processes of S and those of P specified by C. And similarly for V, defined as the parallel composition of S and Q through the relation D. The hypothesis that (C, D, E) should be adapted means that C and D specify the same connections between processes up to E.

Lemma 6. *If $\mathcal{R}$ is symmetric, then so is its parallel extension $\mathcal{R}'$.*

The next result is essential to prove that weak bisimulation is a congruence.

Proposition 5. *If $\mathcal{R}$ is a weak bisimulation, so is its parallel extension $\mathcal{R}'$.*

Now we are in position of proving that weak bisimilarity is a congruence, a result which is interesting *per se* and will be essential for proving Theorem 4.

Theorem 3. *The weak bisimilarity relation $\approx$ is a congruence.*

We can prove now the main theorem of this section.

Theorem 4. *Let P and Q be processes. If $P \approx Q$ (P and Q are weakly bisimilar) then $P \cong Q$ (P and Q are weakly barb congruent).*

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8 Annex: proofs of statements

8.1 Proof of Theorem 1

We use the notations introduced in Section 3.

Lemma 7. $\mathsf{cs}(\langle A\rangle_Y)$ *is the sum of all prefixed processes* $f \cdot (\langle A\rangle_{Y_1}, \ldots, \langle A\rangle_{Y_n})$ *where* $(Y, f, (Y_1, \ldots, Y_n)) \in \mathcal{T}$.

Proof. More generally,

$$\mathsf{cs}(\langle A\rangle_X^{\{X_1,\ldots,X_p\}}\left[\langle A\rangle_{X_1}/X_1, \ldots, \langle A\rangle_{X_p}/X_p\right])$$

is equal to the sum above, for any subset $\{X_1, \ldots, X_p\}$ of $\mathcal{Q}$ (with the X_i's pairwise distinct). The proof is a simple induction on $q - p$, where q is the cardinality of $\mathcal{Q}$. □

The proof of the Theorem is straightforward, once observed that, if $t = f(t_1, \ldots, t_n)$ and if $(X, f, (X_1, \ldots, X_n)) \in \mathcal{T}$, one has

$$\langle A\rangle_X \mid \mathsf{proc}(\bar{t}) \to (\langle A\rangle_{X_1} \mid \mathsf{proc}(\bar{t_1})) \oplus \cdots \oplus (\langle A\rangle_{X_n} \mid \mathsf{proc}(\bar{t_n}))\,,$$

thanks to Lemma 7. Observe then that $(\langle A\rangle_{X_1} \mid \mathsf{proc}(\bar{t_1})) \oplus \cdots \oplus (\langle A\rangle_{X_n} \mid \mathsf{proc}(\bar{t_n}))$ reduces to an idle process iff each process $\langle A\rangle_{X_i} \mid \mathsf{proc}(\bar{t_i})$ reduces to an idle process since these processes cannot interact with each other. Finally, f $\mathcal{T}$ has no element of the shape $(X, f, (X_1, \ldots, X_n))$, then the process $\langle A\rangle_X \mid \mathsf{proc}(\bar{t})$ does not reduce.

8.2 Proof of Theorem 2

We shall need the following observation, using the notations of the beginning of Section 2.1.

Lemma 8. *If* $p', q' \in |P'|$ *and* $p' \frown_{P'} q'$ *then* $\lambda_1(p') \frown_P \lambda_1(q')$ *or* $\lambda_1(p') = \lambda_1(q')$.

For the proof of the Theorem, we use the notations of its statement.

Let $J \subseteq |t|^\circ$. We say that J is a *bar* of t if, for any $\beta \in \partial|t|$ there is exactly one $\beta_0 \in J$ such that $\beta_0 < \beta$. Similarly, a bar of the forest $(s_k)_{k=1}^n$ is a subset I of $\bigcup_{k=1}^n \{k\} \times |s_k|^\circ$ such that, for any $(i, \alpha) \in \bigcup_{k=1}^n \{k\} \times \partial|s_k|$, there is exactly one $\alpha_0 \in |s_i|^\circ$ such that $(i, \alpha_0) \in I$ and $\alpha_0 < \alpha$. Observe that a bar is an antichain (for the prefix order) because of the uniqueness condition[5].

In this proof, a *state* is a triple $P = (I, J, G)$ where I is a bar of $(s_i)_{i=1}^n$, J is a bar of t and $G \subseteq I \times J$ is such that $\pi_1(G) = I$ and $\pi_2(G) = J$. Any state $P = (I, J, G)$ can be seen as a DPC process, also denoted as P to simplify notations. This process is given by: $|P| = I \cup J$ (I and J are disjoint sets), $p \frown_P q$ if $(p, q) \in G$ or $(q, p) \in G$, $P(i, \alpha) = \mathsf{proc}(s_i/\alpha)$ and $P(\beta) = \mathsf{proc}(\bar{t}/\beta)$.

[5] $\bigcup_{k=1}^n \{k\} \times |s_k|^\circ$ is equipped with the following prefix order: $(i, \alpha) \le (i', \alpha')$ if $i = i'$ and $\alpha \le \alpha'$.

The *initial state* is $P_0 = (I_0, J_0, G_0)$ where $I_0 = \{(1, \langle\rangle), \ldots, (n, \langle\rangle)\}$, $J_0 = \{\langle\rangle\}$ and $G_0 = I_0 \times J_0$. Seen as a process, it is clear that P_0 coincides with the process $(\mathsf{proc}(s_1) \oplus \cdots \oplus \mathsf{proc}(s_n)) \mid \mathsf{proc}(\overline{t})$ of the theorem we are proving.

Assume that t is a shuffle of $s_1, \ldots, s_n$ and let φ be a witness of this shuffle. We say that a state $P = (I, J, G)$ is *compatible* with φ if the following properties hold.

- By restriction, φ induces a bijection between $\uparrow I \subseteq \bigcup_{k=1}^{n} \{k\} \times |s_k|^\circ$ and $\uparrow J \subseteq |t|^\circ$ (where $\uparrow J$ and $\uparrow I$ are upward closures defined as usual, wrt. the prefix order).
- If $(i, \alpha) \in \uparrow I$ and $\beta \in \uparrow J$ is defined by $\beta = \varphi(i, \alpha)$, then the unique pair of sequences $\alpha_0 \leq \alpha$ and $\beta_0 \leq \beta$ such that $(i, \alpha_0) \in I$ and $\beta_0 \in J$ satisfy $((i, \alpha_0), \beta_0) \in G$.

Assume $P = (I, J, G)$ is a state compatible with φ.

If $J = \emptyset$ then $I = \emptyset$ by the first condition in the definition of compatibility and therefore P is an idle state.

Assume that $J \neq \emptyset$ and let $\beta \in \uparrow J$ be minimal for the (transitive closure of the) $\prec_\varphi$ relation; such an element must exist since $\prec_\varphi$ has no cycles. Let β_0 be the unique element of J such that $\beta_0 \leq \beta$, we must have $\beta_0 = \beta$ since otherwise $\beta = \beta' l$ for a $\beta' \geq \beta_0$ which belongs to $\uparrow J$, contradicting the $\prec_\varphi$ minimality of β. Let (i, α) be the unique element of $\uparrow I$ such that $\varphi(i, \alpha) = \beta$. If $(i, \alpha) \notin I$ then $\alpha = \alpha' l$ with $(i, \alpha') \in \uparrow I$. Let $\beta' = \varphi(i, \alpha')$, we have $\beta' \in \uparrow J$ and $\beta' \prec_\varphi \beta$, contradicting the $\prec_\varphi$-minimality of β. Hence we have $(i, \alpha) \in I$.

By the second property in the definition of compatibility, we have $((i, \alpha), \beta) \in G$ and $s_i(\alpha) = t(\beta) = f \in \Sigma_r$ for $r \geq 1$. Therefore we have a reduction $P \to P' = (I', J', G')$ with $((i, \alpha), \beta)$ as main edge, so that

$$
\begin{aligned}
I' &= I\left[\{(i, \alpha l) \mid l = 1, \ldots, r\}/(i, \alpha)\right] \\
J' &= J\left[\{\beta l \mid l = 1, \ldots, r\}/\beta\right] \\
G' &= \cup\{((i, \alpha l), \beta l) \mid l = 1, \ldots, r\} \\
&\quad \cup \{((i, \alpha l), \beta') \mid ((i, \alpha), \beta') \in G,\ \beta' \neq \beta \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } l = 1, \ldots, r\} \\
&\quad \cup \{(((k, \alpha')), \beta l) \mid ((k, \alpha'), \beta) \in G\ (k, \alpha') \neq (i, \alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } l = 1, \ldots, r\} \\
&\quad \cup \{((k, \alpha'), \beta') \in G \mid (k, \alpha') \neq (i, \alpha) \text{ and } \beta' \neq \beta\}
\end{aligned}
$$

It is clear that I' and J' are bars in $(s_i)_{i=1}^{n}$ and t, respectively. Observe that $\uparrow I' = \uparrow I \setminus \{(i, \alpha)\}$ and $\uparrow J' = \uparrow J \setminus \{\beta\}$. It follows that φ restricts to a bijection from $\uparrow I'$ to $\uparrow J'$.

Now we show that $P' = (I', J', G')$ satisfies the second condition of compatibility with φ. Let $((i', \alpha'), \beta') \in G'$ be such that $\beta' = \varphi(i', \alpha')$ and let α'_0 and β'_0 uniquely determined by the fact that $(i', \alpha'_0) \in I'$, $\beta'_0 \in J'$, $\alpha'_0 \leq \alpha'$ and $\beta'_0 \leq \beta'$. We must check that $((i', \alpha'_0), \beta'_0) \in G'$. We have $((i', \alpha'), \beta') \in \uparrow I \times \uparrow J \supseteq I' \times J' \supseteq G'$ and hence the sequences α_1 and β_1 uniquely defined by the fact that $(i', \alpha_1) \in I$, $\beta_1 \in J$, $\alpha_1 \leq \alpha'$ and $\beta_1 \leq \beta'$ satisfy $((i', \alpha_1), \beta_1) \in G$: this is due to our hypothesis that P is compatible with φ.

Moreover, we have $\alpha_1 \leq \alpha'_0$ and $\beta_1 \leq \beta'_0$ because $I' \subseteq {\uparrow}I$ and $J' \subseteq {\uparrow}J$. By the definition of I' and J', the following cases can occur. It is helpful to remember that ${\uparrow}I' = {\uparrow}I \setminus \{(i, \alpha)\}$ and ${\uparrow}J' = {\uparrow}J \setminus \{\beta\}$.

- $\alpha'_0 = \alpha_1$ and $\beta'_0 = \beta_1$. Then we must have $(i', \alpha'_0) \neq (i, \alpha)$ and $\beta'_0 \neq \beta$ and hence $((i, \alpha'_0), \beta'_0) \in G'$.
- $\alpha'_0 = \alpha_1$ and $\beta_1 < \beta'_0$. Then we must have $\beta_1 = \beta$ and $\beta'_0 = \beta l$ for some $l \in \{1, \ldots, r\}$. We know therefore that $((i', \alpha_1), \beta) \in G$ and therefore $((i', \alpha'_0), \beta l) \in G'$ by definition of G', as required.
- The case where $\alpha_1 < \alpha'_0$ and $\beta_1 = \beta'_0$ is similar (we must have $i' = i$, $\alpha_1 = \alpha$, $\alpha'_0 = \alpha l$ for some l, and $\beta'_0 = \beta'_1 \neq \beta$).
- In the last case we have $\alpha_1 < \alpha'_0$ and $\beta_1 < \beta'_0$. Then we must have $i' = i$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ and there must be $l, m \in \{1, \ldots, r\}$ such that $\alpha'_0 = \alpha l$ and $\beta'_0 = \beta m$. We know that $\varphi(i, \alpha') = \beta' \geq \beta'_0 = \beta m = \varphi(i, \alpha) m$, but $\alpha' \geq \alpha l$ and therefore $l = m$ by our assumptions that φ is the witness of the shuffle. Hence $((i', \alpha'_0), \beta l) \in G'$ by definition of G', as required.

We have shown that P' is a state which is compatible with φ. It is also clear that the state P_0 is compatible with φ. Altogether, this shows that $P \to^* Q$ where Q is an idle state.

Conversely, assume that $P_0 \to^* Q$ where Q is an idle state and let us show that t is a shuffle of $(s_i)_{i=1}^n$. So we have a sequence $(P_q)_{q=0}^N$ with $P_N = Q$ and $P_q \to P_{q+1}$ for $q = 0, \ldots, N-1$. We have $P_q = (I_q, J_q, G_q)$ and we know that, for each $q < N$, there is a unique $((i_q, \alpha_q), \beta_q) \in G_q$ which is the main edge of the reduction $G_q \to G_{q+1}$. Let

$$\varphi = \{((i_q, \alpha_q), \beta_q) \mid q = 1, \ldots, N\}.$$

We prove first that φ is a bijection $\bigcup_{i=1}^n \{i\} \times |s_i|^\circ \to |t|^\circ$.

Let $i \in \{1, \ldots, n\}$ and $\alpha \in |s_i|^\circ$. We have $(i, \alpha) \in {\uparrow}I_0 \supset {\uparrow}I_1 \supset \cdots \supset {\uparrow}I_N = \emptyset$ so let q be such that $(i, \alpha) \in {\uparrow}I_q$ and $(i, \alpha) \notin {\uparrow}I_{q+1}$. This means that $(i, \alpha) \in I_q$ and that there is $\beta \in J_q$ such that $((i, \alpha), \beta)$ is the main edge of the reduction $P_q \to P_{q+1}$, that is, $((i, \alpha), \beta) = ((i_q, \alpha_q), \beta_q)$ hence $((i, \alpha), \beta) \in \varphi$. If $q' \in \{1, \ldots, N\}$ and $(i_{q'}, \alpha_{q'}) = (i, \alpha)$ we must have $q' \leq q$ because $(i, \alpha) \notin {\uparrow}I_{q+1}$. But since $((i_{q'}, \alpha_{q'}), \beta_{q'})$ is the main edge of the reduction $P_{q'} \to P_{q'+1}$, we must have $(i, \alpha) \notin {\uparrow}I_{q'+1}$ and therefore $q' = q$. Hence q is the unique element of $\{1, \ldots, N\}$ such that $(i_q, \alpha_q) = (i, \alpha)$ and hence there is only one β (namely $\beta = \beta_q$) such that $((i, \alpha), \beta) \in \varphi$. Hence φ is a function, and a symmetric reasoning shows that φ is a bijection.

Now we show that φ satisfies the three conditions of being a witness that t is a shuffle of the s_i's.

The first condition obviously holds because if $\beta = \varphi(i, \alpha)$, then $((i, \alpha), \beta)$ is the main edge of a reduction $G_q \to G_{q+1}$.

As to the second condition, let $\alpha, \alpha' \in |s_i|^\circ$ with $\alpha l \leq \alpha'$ for some $l \in \mathbf{N}^+$. Assume that $\varphi(i, \alpha') > \varphi(i, \alpha)$, we must prove that $\varphi(i, \alpha') \geq \varphi(i, \alpha) l$. We have $(i, \alpha) = (i_q, \alpha_q)$ and $(i, \alpha') = (i_{q'}, \alpha_{q'})$ with $q < q'$ and $\beta_q < \beta_{q'}$ by our assumption so that $\beta_q m \leq \beta_{q'}$ for some m. We have $((i, \alpha_q m), \beta_q m) \in G_{q+1}$ by definition of the reduction of DPC processes. By iterating Lemma **??** along the reductions $P_{q+1} \to \cdots \to P_{q'}$ we see that $((i, \alpha_q l), \beta_q m) \in G_{q+1}$ (using the fact that $((i, \alpha'), \beta') \in G_{q'}$). Therefore we have $l = m$ and this shows that $\varphi(i, \alpha') \geq \varphi(i, \alpha) l$.

To prove the third condition, consider now $\beta, \beta' \in |t|^\circ$ such that $\beta \prec_\varphi \beta'$. Let (i, α) and (i', α') be such that $\alpha \in |s_i|^\circ$, $\alpha' \in |s_{i'}|^\circ$, $\varphi(i, \alpha) = \beta$ and $\varphi(i', \alpha') = \beta'$. Let q and q' be such that $((i, \alpha), \beta)$ (resp. $((i', \alpha'), \beta')$) is the main edge of the reduction $P_q \to P_{q+1}$ (resp. of the reduction $P_{q'} \to P_{q'+1}$). We have $q < q'$. Indeed, since $q \neq q'$, we must have otherwise $q' < q$. Then there are only two possibilities as to the fact that $\beta \prec_\varphi \beta'$:

- If $\beta' = \beta l$ for some l, then we have a contradiction because $\beta \in J_q$ and $\beta' \in J_{q'}$ and, for all $\gamma \in J_q$ there is $\gamma' \in J_{q'}$ such that $\gamma' \leq \gamma$. Therefore there must be $\gamma' \in J_{q'}$ such that $\gamma' \leq \beta$ and we also have $\beta l \in J_{q'}$ hence $\gamma' < \beta l$ and these two sequences belong to $J_{q'}$, contradicting the fact that this set is an antichain for the prefix order.
- Otherwise, we have $i = i'$ and $\alpha' = \alpha l$. We reason similarly, using the fact that $I_{q'}$ is an antichain in the forest $\bigcup_{j=1}^n \{i\} \times |s_i|^\circ$, and the fact that, for any $(j, \delta) \in I_q$ there exists $(j', \delta') \in I_{q'}$ such that $j' = j$ and $\delta' \leq \delta$.

It follows that the relation $\prec_\varphi$ has no cycle, as contended.

8.3 Proof of Lemma 3

The proof uses the following results.

Lemma 9. *Let $\mathcal{R}$ be a weak bisimulation. If $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda]{\tau *} P'$, then $Q \xrightarrow[\rho]{\tau *} Q'$ with $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(\lambda(p'), \rho(q')) \in E$.*

Proof. Simple induction on the length of the sequence of reductions $P \xrightarrow[\lambda]{\tau *} P'$. □

Lemma 10. *If $P \xrightarrow[\lambda]{\tau *} P_1$, $P_1 \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda_1,\lambda_2,\lambda'_1}{\Longrightarrow}} P'_1$ and $P'_1 \xrightarrow[\lambda']{\tau *} P'$ then $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda\lambda_1,\lambda_2,\lambda'_1\lambda'}{\Longrightarrow}} P'$.*

Proof. Results immediately from the definitions. □

Assume that $(P, E, Q) \in \mathcal{R}$ and $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$, that is $P \xrightarrow[\lambda]{\tau *} P_1 \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'_1 \xrightarrow[\lambda']{\tau *} P'$. By Lemma 9 one has $Q \xrightarrow[\rho]{\tau *} Q_1$ with $(P_1, E_1, Q_1) \in \mathcal{R}$ where E_1 is such that $(p_1, q_1) \in E_1$ implies $(\lambda(p_1), \rho(q_1)) \in E$.

Since $P_1 \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'_1$ and $(P_1, E_1, Q_1) \in \mathcal{R}$, one has $Q_1 \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho_1,\rho_2,\rho'_1}{\Longrightarrow}} Q'_1$ with $(p, \rho_1(q)) \in E_1$ and $(P'_1, E'_1, Q'_1) \in \mathcal{R}$ where E'_1 is such that if $(p'_1, q'_1) \in E'_1$ then

$(\lambda_1(p'_1), \rho_1\rho_2\rho'_1(q'_1)) \in E_1$ and, if $n \geq 2$, then either $(p'_1, \rho'_1(q'_1)) \in \bigcup_{i=1}^n (L_i \times M_i)$, or $p'_1 \notin \bigcup_{i=1}^n L_i$ and $\rho'_1(q'_1) \notin \bigcup_{i=1}^n M_i$. Since $P'_1 \xrightarrow[\lambda']{\tau *} P'$ and $(P'_1, E'_1, Q'_1) \in \mathcal{R}$, we can apply Lemma 9 again which shows that $Q'_1 \xrightarrow[\rho']{\tau *} Q'$ with $(P', E', Q') \in \mathcal{R}$ where E' is such that $(p', q') \in E'$ implies $(\lambda'(p'), \rho'(q')) \in E'_1$. By Lemma 10, we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho\rho_1,\rho_2,\rho'_1\rho'}{\Longrightarrow}} Q'$ and remember that $(P', E', Q') \in \mathcal{R}$. We have $(p, \rho_1(q)) \in E_1$ and hence $(\lambda(p), \rho\rho_1(q)) \in E$ by definition of E_1. Last, the condition on residuals obviously holds.

8.4 Proof of Lemma 5

Let $(P, H, R) \in \mathcal{S} \circ \mathcal{R}$. Let Q, E and F be such that $(P, E, Q) \in \mathcal{R}$, $(Q, F, R) \in \mathcal{S}$ and $F \circ E \subseteq H$.

► Assume first that $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$. Then we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(\lambda(p), \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ with E' such that if $(p', q') \in E'$ then $(\lambda\lambda_1\lambda'(p'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$ then $(\lambda'(p'), \rho'(q')) \in \bigcup_{i=1}^n (L_i \times M_i)$ or $\lambda'(p') \notin \bigcup_{i=1}^n L_i$ and $\rho'(q') \notin \bigcup_{i=1}^n M_i$. Therefore we have $R \overset{r:f\cdot(\boldsymbol{N})}{\underset{\sigma,\sigma_1,\sigma'}{\Longrightarrow}} R'$ with $(\rho(q), \sigma(r)) \in F$ and $(Q', F', R') \in \mathcal{S}$ with F' such that if $(q', r') \in F'$ then $(\rho\rho_1\rho'(q'), \sigma\sigma_1\sigma'(r')) \in F)$ and, if $n \geq 2$ then $(\rho'(q'), \sigma'(r')) \in \bigcup_{i=1}^n (M_i \times N_i)$ or $\rho'(q') \notin \bigcup_{i=1}^n M_i$ and $\sigma'(r') \notin \bigcup_{i=1}^n N_i$. So we have $(\lambda(p), \sigma(r)) \in F \circ E \subseteq H$. Let

$$
\begin{aligned}
H' = \{(p', r') \in |P'| \times |R'| \mid \; & (\lambda\lambda_1\lambda'(p'), \sigma\sigma_1\sigma'(r')) \in H \\
& \text{and if } n \geq 2 \text{ then} \\
& (\lambda'(p'), \sigma'(r')) \in \bigcup_{i=1}^n (L_i \times N_i) \\
& \text{or } \lambda'(p') \notin \bigcup_{i=1}^n L_i \text{ and } \sigma'(r') \notin \bigcup_{i=1}^n N_i\}
\end{aligned}
$$

By definition of H', the triple (P', H', R') satisfies the conditions on residuals, and we are left with proving that $F' \circ E' \subseteq H'$ which will show that $(P', H', R') \in \mathcal{S} \circ \mathcal{R}$. Let $(p', r') \in F' \circ E'$, there exists q' such that $(p', q') \in E'$ and $(q', r') \in F'$.

We know that $(\lambda\lambda_1\lambda'(p'), \rho\rho_1\rho'(q')) \in E$ and $(\rho\rho_1\rho'(q'), \sigma\sigma_1\sigma'(r)) \in F$ and therefore $(\lambda\lambda_1\lambda'(p'), \sigma\sigma_1\sigma'(r)) \in F \circ E \subseteq H$. So assume now that $n \geq 2$. We must prove that if $\lambda'(p') \in \bigcup_{i=1}^n L_i$ or $\sigma'(r') \in \bigcup_{i=1}^n N_i$ then $(\lambda'(p'), \sigma'(r')) \in L_i \times N_i$ for some i. Without loss of generality, we can assume that $\lambda'(p') \in \bigcup_{i=1}^n L_i$ (because the situation is symmetric). Then by the condition on residuals for E' we know that $(\lambda'(p'), \rho'(q')) \in L_j \times M_j$ for some $j \in \{1, \ldots, n\}$, because $n \geq 2$. Therefore $(\rho'(q'), \sigma'(r')) \in M_i \times N_i$ by the conditions on residuals satisfied by F'. It follows that $(\lambda'(p'), \sigma'(r')) \in L_i \times N_i$ as required.

► Assume now that $P \xrightarrow[\lambda]{\tau*} P'$. Since $(P,E,Q) \in \mathcal{R}$ we have $Q \xrightarrow[\rho]{\tau*} Q'$ and there exists E' such that $(P',E',Q') \in \mathcal{R}$ and, if $(p',q') \in E'$, then $(\lambda(p'),\rho(q')) \in E$. Since $(Q,F,R) \in \mathcal{S}$, we have $R \xrightarrow[\sigma]{\tau*} R'$ and there exists F' such that $(Q',F',R') \in \mathcal{S}$ and for any $(q',r') \in F'$, one has $(\rho(q'),\sigma(r')) \in F$. We have $(P',F' \circ E',Q') \in \mathcal{S} \circ \mathcal{R}$ and it is obvious that $F' \circ E'$ satisfies the condition on residuals.

8.5 Proof of Proposition 4

Let $\mathcal{R}$ be a weak bisimulation. Let $\mathcal{B}$ be the binary relation on processes defined by: $(P,Q) \in \mathcal{B}$ if there exists $E \subseteq |P| \times |Q|$ such that $(P,E,Q) \in \mathcal{R}$. We contend that $\mathcal{B}$ is a weak barbed bisimulation, and this will prove the proposition. First observe that $\mathcal{B}$ is symmetric because $\mathcal{R}$ is a symmetric localized relation.

► Let $(P,Q) \in \mathcal{B}$ and assume first that $P \to^* P'$, that is $P \xrightarrow[\lambda]{\tau*} P'$ for some residual function λ. Let $E \subseteq |P| \times |Q|$ be such that $(P,E,Q) \in \mathcal{R}$. By Lemma 3, one has $Q \xrightarrow[\rho]{\tau*} Q'$ for some residual function ρ, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P',E',Q') \in \mathcal{R}$ and therefore $(P',Q') \in \mathcal{B}$ as required; this shows that $\mathcal{B}$ is a weak reduction bisimulation.

► Assume now that $(P,Q) \in \mathcal{B}$ and that $P \to^* P'$ with $P' \downarrow_f$ (with $f \in \Sigma$ of arity n), meaning that $P' \xrightarrow[\lambda'_1]{p':f\cdot(\boldsymbol{L})} P''$ for some $p' \in |P'|$, some sequence of sets of locations $\boldsymbol{L}$ and some residual function λ'_1.

Let $E \subseteq |P| \times |Q|$ be such that $(P,E,Q) \in \mathcal{R}$. By Lemma 3, one has $Q \xrightarrow[\rho]{\tau*} Q'$ for some residual function ρ, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P',E',Q') \in \mathcal{R}$. Since $\mathcal{R}$ is a weak bisimulation we have therefore $Q' \overset{q':f\cdot(\boldsymbol{M})}{\underset{\rho',\rho_1,\rho''}{\Longrightarrow}} Q''$ and hence $Q' \to^* Q'_1$ with $Q'_1 \downarrow_f$. This shows that $\mathcal{B}$ is weak barb preserving since $Q \to^* Q'_1$ and $Q'_1 \downarrow_f$.

8.6 Proof of Proposition 5

Symmetry of $\mathcal{R}'$ results from the symmetry of $\mathcal{R}$ and from Lemma 6.

Let $(U,F,V) \in \mathcal{R}'$ with $U = S \oplus_C P$, $V = S \oplus_D Q$, $(P,E,Q) \in \mathcal{R}$, (C,D,E) adapted and $F = \mathrm{Id}_{|S|} \cup E$.

Case of a τ-transition. Assume that $U \xrightarrow[\lambda]{\tau} U'$. We must show that $V \xrightarrow[\rho]{\tau*} V'$ with $(U',F',V') \in \mathcal{R}'$ and $(\lambda(u'),\rho(v')) \in F$ for each $(u',v') \in F'$ (condition on residuals). There are three cases as to the locations of the two guarded sums involved in that reduction.

► Assume first that they are located in S, in other words there are $s,t \in |S|$ with $s \frown_S t$, $\mathsf{cs}(S(s)) = f \cdot \boldsymbol{S} + \tilde{S}$ ($\tilde{S}$ is a guarded sum) and $\mathsf{cs}(S(t)) = \overline{f} \cdot \boldsymbol{T} + \tilde{T}$ ($\tilde{T}$ is a guarded sum), and we have $S \xrightarrow[\mu]{\tau} S'$ with

- $I_{S'} = I_S \cup \bigcup_{i=1}^n I_{S_i} \cup \bigcup_{i=1}^n I_{T_i}$
- $|S'| = (|S| \setminus \{s,t\}) \cup \bigcup_{i=1}^n |S_i| \cup \bigcup_{i=1}^n |T_i|$
- and $\frown_{S'}$ is the least symmetric relation on $|S'|$ such that $s' \frown_{S'} t'$ if $s' \frown_{S_i} t'$, or $s' \frown_{T_i} t'$, or $(s',t') \in |S_i| \times |T_i|$ for some $i \in \{1,\dots,n\}$, or $\{s',t'\} \not\subseteq \bigcup_{i=1}^n |S_i| \cup \bigcup_{i=1}^n |T_i|$ and $\mu(s') \frown_S \mu(t')$

where n is the arity of f so that $\boldsymbol{S} = (S_1,\dots,S_n)$ and $\boldsymbol{T} = (T_1,\dots,T_n)$.

Remember that the residual function μ is given by $\mu(s') = s$ if $s' \in \bigcup_{i=1}^n |S_i|$, $\mu(s') = t$ if $s' \in \bigcup_{i=1}^n |T_i|$ and $\mu(s') = s'$ otherwise. We have $U' = S' \oplus_{C'} P$ where $C' = \{(s',p) \in |S'| \times |P| \mid (\mu(s'),p) \in C\}$ and $\lambda = \mu \cup \mathrm{Id}_{|P|}$.

Then we have similarly $V = S \oplus_D Q \xrightarrow[\rho]{\tau} V' = S' \oplus_{D'} Q$ with $\rho = \mu \cup \mathrm{Id}_{|Q|}$, and $D' = \{(s',q) \in |S'| \times |Q| \mid (\mu(s'),q) \in D\}$.

The triple (C',D',E) is adapted: let $s' \in |S'|$, $p \in |P|$ and $q \in |Q|$ be such that $(p,q) \in E$. If $(s',p) \in C'$ then $(\mu(s'),p) \in C$ and hence $(\mu(s'),q) \in D$ since (C,D,E) is adapted, that is $(s',q) \in D'$, and similarly for the converse implication.

Coming back to the definition of $\mathcal{R}'$, we see that $(U',F',V') \in \mathcal{R}'$ where $F' = \mathrm{Id}_{|S'|} \cup E$. Moreover, the condition on residuals is satisfied, since, given $(u',v') \in F'$, we have either $u' = v' \in |S'|$ and then $\lambda(u') = \rho(v') \in |S|$ or $(u',v') \in E$ and $(\lambda(u'),\rho(v')) = (u',v') \in E$. In both cases $(\lambda(u'),\rho(v')) \in F$.

► Assume next that they are located in P, in other words there are $p,r \in |P|$ with $\mathsf{cs}(P(p)) = f \cdot \boldsymbol{P} + \tilde{P}$ (where $\tilde{P}$ is a guarded sum) and $\mathsf{cs}(P(r)) = \overline{f} \cdot \boldsymbol{R} + \tilde{R}$ (where $\tilde{R}$ is a guarded sum), and we have $P \xrightarrow[\mu]{\tau} P'$ with

- $I_{P'} = I_P \cup \bigcup_{i=1}^n I_{P_i} \cup \bigcup_{i=1}^n I_{R_i}$
- $|P'| = (|P| \setminus \{p,r\}) \cup \bigcup_{i=1}^n |P_i| \cup \bigcup_{i=1}^n |R_i|$
- and $\frown_{P'}$ is the least symmetric relation on $|P'|$ such that $p' \frown_{P_i} r'$ or $p' \frown_{R_i} r'$ or $(p',r') \in |P_i| \times |R_i|$ for some $i \in \{1,\dots,n\}$, or $\{p',r'\} \not\subseteq \bigcup_{i=1}^n |P_i| \cup \bigcup_{i=1}^n |R_i|$ and $\mu(p') \frown_P \mu(r')$.

We recall that the residual function μ is given by $\mu(p') = p$ if $p' \in \bigcup_{i=1}^n |P_i|$, $\mu(p') = r$ if $p' \in \bigcup_{i=1}^n |R_i|$ and $\mu(p') = p'$ otherwise. With these notations, the process U' is $U' = S \oplus_{C'} P'$ where $C' = \{(s,p') \in |S| \times |P'| \mid (s,\mu(p')) \in C\}$ and the residual function λ is defined as $\lambda = \mathrm{Id}_{|S|} \cup \mu$. Since $(P,E,Q) \in \mathcal{R}$ and $P \xrightarrow[\mu]{\tau} P'$, one has $Q \xrightarrow[\nu]{\tau*} Q'$ with $(P',E',Q') \in \mathcal{R}$ where $E' \subseteq |P'| \times |Q'|$ satisfies the condition on residuals $(p',q') \in E'$ implies $(\mu(p'),\nu(q')) \in E$. Let $D' = \{(s,q') \in |S| \times |Q'| \mid (s,\nu(q')) \in D\}$. Setting $V' = S \oplus_{D'} Q'$, we have $V \xrightarrow[\rho]{\tau*} V'$ where $\rho = \mathrm{Id}_{|S|} \cup \nu$.

The triple (C',D',E') is adapted: let $(p',q') \in E'$ and let $s \in |S|$. If $(s,p') \in C'$, we have $(s,\mu(p')) \in C$. Since $(\mu(p'),\nu(q')) \in E$ (by definition of E'), we have $(s,\nu(q')) \in D$ because (C,D,E) is adapted. That is $(s,q') \in D'$. The converse implication is proved similarly.

Let $F' = \mathrm{Id}_{|S|} \cup E' \subseteq |U'| \times |V'|$, we have therefore $(U',F',V') \in \mathcal{R}'$ (by definition $\mathcal{R}'$). Last we check the condition on residuals. Let $(u',v') \in F'$, then either $u' = v' \in |S|$ and then $\lambda(u') = u' = v' = \rho(v')$ or $u' \in |P'|$, $v' \in |Q'|$ and $(u', v') \in E'$ and then $(\lambda(u'), \rho(v')) = (\mu(u'), \nu(v')) \in E$ by the condition on residuals satisfied by E.

► Assume last that one of the involved guarded sums is located in S and that the other one is located in P, this is of course the most interesting case in this first part of the proof.

By definition of internal reduction (see Section 2.1) we have $s \in |S|$, $p \in |P|$ with $(s, p) \in C$, $\mathsf{cs}(S(s)) = \overline{f} \cdot \boldsymbol{S} + \tilde{S}$ and $\overline{f} \notin I_S$, and $\mathsf{cs}(P(p)) = f \cdot \boldsymbol{P} + \tilde{P}$ and $f \notin I_P$ with the usual notational conventions, and $U' = S' \oplus_{C'} P'$ where $S' = S\,[\oplus\boldsymbol{S}/s]$, $P' = P\,[\oplus\boldsymbol{P}/p]$, and $C' \subseteq |S'| \times |P'|$ is defined as follows: $(s', p') \in C'$ if

- $(s', p') \in |S_i| \times |P_i|$ for some i,
- or $(s', p') \notin (\bigcup_{i=1}^n |S_i|) \times (\bigcup_{i=1}^n |P_i|)$ and $(\lambda(s'), \lambda(p')) \in C$,

where the residual map $\lambda : |U'| = |S'| \cup |P'| \to |U| = |S| \cup |P|$ is defined by $\lambda(u') = u'$ if $u' \in (|S'| \setminus \bigcup_{i=1}^n |S_i|) \cup (|P'| \setminus \bigcup_{i=1}^n |P_i|)$, $\lambda(s') = s$ if $s' \in \bigcup_{i=1}^n |S_i|$ and $\lambda(p') = p$ if $p' \in \bigcup_{i=1}^n |P_i|$.

The crucial property that $\overline{f} \notin I_S$ and $f \notin I_P$ is due to the fact that, in S, the names bound in I_S are defined up to α-conversion and similarly for P.

Because $f \notin I_P$ we have $P \xrightarrow[\lambda]{p:f\cdot(\boldsymbol{L})} P'$ (where $L_i = |P_i|$ for each $i = 1, \ldots, n$) and hence, since we have assumed that $(P, E, Q) \in \mathcal{R}$, we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ where E' is such that if $(p', q') \in E'$ then $(\lambda(p'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$, then either $(p', \rho'(q')) \in L_i \times M_i$ for some i, or $p' \notin \bigcup_{i=1}^n L_i$ and $\rho'(q') \notin \bigcup M_i$.

We can decompose this transition as follows

$$Q \xrightarrow[\rho]{\tau*} Q_1 \xrightarrow[\rho_1]{q:f\cdot(\boldsymbol{M})} Q_1' \xrightarrow[\rho']{\tau*} Q'\,.$$

With these notations we have $V \xrightarrow[\mu]{\tau*} V_1$ with $V_1 = S \oplus_{D_1} Q_1$ where $D_1 = \{(s, q_1) \in |S| \times |Q_1| \mid (s, \rho(q_1)) \in D\}$, and $\mu = \mathrm{Id}_{|S|} \cup \rho$.

We have $q \in |Q_1|$ with $\mathsf{cs}(Q_1(q)) = f \cdot \boldsymbol{R} + \tilde{R}$, $f \notin I_{Q_1}$ and $|R_i| = M_i$ for $i = 1, \ldots, n$. Moreover, since $(p, \rho(q)) \in E$ and $(s, p) \in C$, and since (C, D, E) is adapted, we have $(s, \rho(q)) \in D$, that is $(s, q) \in D_1$. Therefore, since $\mathsf{cs}(S(s)) = \overline{f} \cdot \boldsymbol{S} + \tilde{S}$ and $\overline{f} \notin I_S$, we have $V_1 \xrightarrow[\theta]{\tau} V_1' = S' \oplus_{D_1'} Q_1'$ where $D_1' \subseteq |S'| \times |Q_1'|$ is defined as follows: given $(s', q_1') \in |S'| \times |Q_1'|$, we have $(s', q_1') \in D_1'$

- if $s' \in |S_i|$ and $q_1' \in |R_i|$ for some $i = 1, \ldots, n$
- or $s' \notin \bigcup_{i=1}^n |S_i|$ or $q_1' \notin \bigcup_{i=1}^n |R_i|$ and $(\theta(s'), \theta(q_1')) \in D_1$ (that is $(\theta(s'), \rho\theta(q_1')) \in D$),

and the residual function θ is defined by $\theta(v_1') = v_1'$ if $v_1' \in (|S'| \setminus \bigcup_{i=1}^n |S_i|) \cup (|Q_1'| \setminus \bigcup_{i=1}^n |R_i|)$, $\theta(s') = s$ if $s' \in \bigcup_{i=1}^n |S_i|$ and $\theta(q_1') = q$ if $q_1' \in \bigcup_{i=1}^n |R_i|$.

Observe that $\theta(q_1') = \rho_1(q_1')$ for all $q_1' \in |Q_1'|$.

Since $Q'_1 \xrightarrow[\rho']{\tau*} Q'$, we have $V'_1 = S' \oplus_{D'_1} Q'_1 \xrightarrow[\mu']{\tau*} V' = S' \oplus_{D'} Q'$ where $\mu' = \mathrm{Id}_{|S'|} \cup \rho'$ and $D' = \{(s', q') \in |S'| \times |Q'| \mid (s', \rho'(q')) \in D'_1\}$. So we have $V \xrightarrow[\mu\theta\mu']{\tau*} V'$. Let $F' \subseteq |U'| \times |V'|$ be defined by $F' = \mathrm{Id}_{|S'|} \cup E'$. It is clear then that $(u', v') \in F'$ implies $(\lambda(u'), \mu\theta\mu'(v')) \in F$ because $(p', q') \in E'$ implies $(\lambda(p'), \rho\rho_1\rho'(q')) \in E$ and θ and ρ_1 coincide on $|Q'_1|$.

To finish, we must prove that $(U', F', V') \in \mathcal{R}'$ and to this end it suffices to show that the triple of relations (C', D', E') is adapted. So let $s' \in |S'|$, $p' \in |P'|$ and $q' \in |Q'|$ with $(p', q') \in E'$ (so that in particular $(\lambda(p'), \rho\theta\rho'(q')) \in E$).

Assume first that $(s', p') \in C'$ and let us show that $(s', q') \in D'$, that is $(s', \rho'(q')) \in D'_1$. Coming back to the definition of C', we can reduce our analysis to three cases.

- First case: $(s', p') \in |S_i| \times |P_i|$ for some i. We distinguish two cases as to the value of n (the arity of f). Assume first that $n \geq 2$. Since $p' \in |P_i| = L_i$, we must have $\rho'(q') \in M_i = |R_i|$ because $(p', q') \in E'$ and then $(s', \rho'(q')) \in D'_1$ as required. Assume now $n = 1$. If $\rho'(q') \in M_1$ we reason as above, so assume that $\rho'(q') \notin M_1 = \bigcup_{i=1}^n |R_i|$. Coming back to the definition of D'_1, it suffices to prove that $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$. Since $(p', q') \in E'$ we have $(\lambda(p'), \rho\theta\rho'(q')) = (p, \rho\rho'(q')) \in E$. We also have $(s, p) \in C$, and hence $(s, \rho\rho'(q')) \in D$ as required, since (C, D, E) is adapted.
- Second case: $s' \notin \bigcup_{i=1}^n |S_i|$. In order to prove $(s', q') \in D'$, it suffices to prove that $(\theta(s'), \rho\theta\rho'(q')) = (s', \rho\theta\rho'(q')) \in D$. But we have $(s', p') \in C'$ and $s' \notin \bigcup_{i=1}^n |S_i|$, hence $(\lambda(s'), \lambda(p')) = (s', \lambda(p')) \in C$. Since $(p', q') \in E'$, we have $(\lambda(p'), \rho\theta\rho'(q')) \in E$ and hence $(s', \rho\theta\rho'(q')) \in D$ since (C, D, E) is adapted.
- Third case: $s' \in \bigcup_{i=1}^n |S_i|$ and $p' \notin \bigcup_{i=1}^n |P_i|$ so that we have $(s, p') \in C$ (by definition of C' and because $(s', p') \in C'$). Assume first that $n \geq 2$. Since $(p', q') \in E'$, we must have $\rho'(q') \notin \bigcup_{i=1}^n M_i$. To prove that $(s', \rho'(q')) \in D'_1$, it suffices therefore to check that $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$. This property holds because (C, D, E) is adapted, $(s, p') \in C$ and $(p', \rho\rho'(q')) \in E$. The latter holds because $(p', q') \in E'$. Assume now that $n = 1$. If $\rho'(q') \notin \bigcup_{i=1}^n M_i = M_1$, we can reason as above, so assume that $\rho'(q') \in M_1$. Then we have $(s', \rho'(q')) \in |S_1| \times M_1$ and hence $(s', \rho'(q')) \in D'_1$.

Let us prove now the converse implication, assuming that $(s', \rho'(q')) \in D'_1$; we contend that $(s', p') \in C'$. Again, we consider three cases.

- First case: $s' \in |S_i|$ and $\rho'(q') \in M_i = |R_i|$ for some $i \in \{1, \ldots, n\}$. If $n \geq 2$ the fact that $(p', q') \in E'$ implies that $p' \in L_i = |P_i|$ and hence $(s', p') \in C'$ as required. Assume that $n = 1$. If $p' \in L_1$ we have $(s', p') \in C'$ since $(s', p') \in |S_1| \times |P_1|$. So assume that $p' \notin L_1$. We know that $(s', \rho'(q')) \in D'_1$ and hence $(s, \rho(q)) \in D$. We have $(p', \rho(q)) = (\lambda(p'), \rho\theta\rho'(q'))$ since $p' \notin L_1$, and hence $(p', \rho(q)) \in E$ since $(p', q') \in E'$ by assumption. As (C, D, E) is adapted we have $(s, p') = (\lambda(s'), \lambda(p')) \in C$, hence $(s', p') \in C'$ because $p' \notin L_1$ (come back to the definition of C').

– Second case: $s' \notin \bigcup_{i=1}^{n} |S_i|$. In view of the definition of C', it suffices to prove that $(\lambda(s'), \lambda(p')) = (s', \lambda(p')) \in C$. Since $(s', \rho'(q')) \in D'_1$ and $s' \notin \bigcup_{i=1}^{n} |S_i|$, we have $(\theta(s'), \rho\theta\rho'(q')) = (s', \rho\theta\rho'(q')) \in D$. And since $(p', q') \in E'$ we have $(\lambda(p'), \rho\theta\rho'(q')) \in E$, and hence $(s', \lambda(p')) \in C$ because (C, D, E) is adapted. Therefore $(s', p') \in C'$ as required.

– Third case: $s' \in |S_i|$ for some $i \in \{1, \ldots, n\}$ and $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$. If $n \geq 2$, we must have $p' \notin \bigcup_{i=1}^{n} L_i$ because $(p', q') \in E'$. Therefore, to check that $(s', p') \in C'$, it suffices to prove that $(\lambda(s'), \lambda(p')) = (s, p') \in C$. We have $(s', \rho'(q')) \in D'_1$ and hence $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$. Since $(p', q') \in E'$ we have $(\lambda(p'), \rho\theta\rho'(q')) = (p', \rho\rho'(q')) \in E$ and hence $(s, p') \in C$ because (C, D, E) is adapted. Assume now that $n = 1$. If $p' \in L_1$ we have $(s', p') \in C'$ since $(s', p') \in |S_1| \times |P_1|$. So assume that $p' \notin L_1$. Since then $p' \notin \bigcup_{i=1}^{n} |P_i|$, it suffices to prove that $(\lambda(s'), \lambda(p')) = (s, p') \in C$ (by definition of C'). We have $(p', \rho\theta\rho'(q')) = (p', \rho\rho'(q')) \in E$ because $(p', q') \in E'$ and $p' \notin L_1$. Moreover $(s, \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$ because $(s', \rho'(q')) \in D'_1$ and $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$. It follows that $(s, p') \in C$ as required.

This ends the first part of the proof.

Case of a labeled transition. We assume now that $U \xrightarrow[\mu_1]{r:f\cdot(\boldsymbol{L})} U'$. Since $U = S \oplus_C P$, we consider two cases as to the location of r. Observe that $f \notin I_U = I_S \cup I_P$ (this two sets being assumed to be disjoint which is possible by α-conversion).

► If $r \in |S|$ then we have $\mathsf{cs}(S(r)) = f \cdot \boldsymbol{S} + \tilde{S}$ and $S \xrightarrow[\sigma_1]{r:f\cdot(\boldsymbol{L})} S'$ where $S' = S\,[\oplus \boldsymbol{S}/r]$ (so that $L_i = |S_i|$ for each $i = 1, \ldots, n$), and $U' = S' \oplus_{C'} P$ where $C' = \{(s', p) \in |S'| \times |P| \mid (\sigma_1(s'), p) \in C\}$. Let $D' = \{(s', q) \in |S'| \times |Q| \mid (\sigma_1(s'), q) \in D\}$. We have $\mu_1 = \sigma_1 \cup \mathrm{Id}_{|P|}$. It is clear that (C', D', E) is adapted, since (C, D, E) is adapted.

Let $V' = S' \oplus_{D'} Q$, we have just seen that $(U', F', V') \in \mathcal{R}'$ where $F' = \mathrm{Id}_{|S'|} \cup E$. We have $(r, r) \in F$, $V \xrightarrow[\nu_1]{r:f\cdot(\boldsymbol{L})} V'$ (with $\nu_1 = \sigma_1 \cup \mathrm{Id}_{|Q|}$; observe indeed that we can assume that $f \notin I_Q$ up to α-conversion of Q) and, given $(u', v') \in F'$, we have either $(u', v') \in \bigcup_{i=1}^{n}(L_i \times L_i)$ (and actually $u' = v'$) or $u' \notin \bigcup_{i=1}^{n} L_i$, $v' \notin \bigcup_{i=1}^{n} L_i$ and $(u', v') \in F$ as easily checked. Therefore the condition on residuals is satisfied.

► The last case to consider is when $r = p \in |P|$. We have $P(p) = f \cdot \boldsymbol{P} + \tilde{P}$ and $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$. Then $U' = S \oplus_{C'} P'$ where $C' = \{(s, p') \in |S| \times |P'| \mid (s, \lambda_1(p')) \in C\}$.

Since $(P, E, Q) \in \mathcal{R}$ we have $Q \xRightarrow[\rho, \rho_1, \rho']{q:f\cdot(\boldsymbol{M})} Q'$ with $(p, \rho(q)) \in E$ and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$ and, for any $(p', q') \in E'$, $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$, either $(p', \rho'(q')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$, or $p' \notin \bigcup_{i=1}^{n} L_i$ and $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$.

Therefore we have $V \overset{q:f\cdot(\boldsymbol{M})}{\underset{\nu,\nu_1,\nu'}{\Longrightarrow}} V'$ (again, up to α-conversion we have $f \notin I_V$) where $V' = S \oplus_{D'} Q'$ with $D' = \{(s,q') \in |S| \times |Q'| \mid (s, \rho\rho_1\rho'(q')) \in D\}$. Moreover $\nu = \mathrm{Id}_{|S|} \cup \rho$, $\nu_1 = \mathrm{Id}_{|S|} \cup \rho_1$ and $\nu' = \mathrm{Id}_{|S|} \cup \rho'$.

Let $F' \subseteq |U'| \times |V'|$ be defined by $F' = \mathrm{Id}_{|S|} \cup E'$. Let $(u',v') \in F'$. If $u' \in |S|$ or $v' \in |S|$, we must have $u' = v'$. If $u' \notin |S|$ and $v' \notin |S|$ then we have $(u',v') \in E'$ and hence $(\mu_1(u'), \nu\nu_1\nu'(v')) = (\lambda_1(u'), \rho\rho_1\rho'(v')) \in E$ and, if $n \geq 2$, either there exists i such that $u' \in L_i$ and $\nu'(v') = \rho'(v') \in M_i$, or $u' \notin \bigcup_{i=1}^n L_i$ and $\nu'(v') = \rho'(v') \notin \bigcup_{i=1}^n M_i$.

Moreover, the triple (C', D', E') is adapted: let $(p', q') \in E'$ and $s \in |S|$. We have $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$. We have $(s, p') \in C'$ iff $(s, \lambda_1(p')) \in C$ iff $(s, \rho\rho_1\rho'(q')) \in D$ iff $(s, q') \in D'$.

8.7 Proof of Theorem 3

Let $\mathcal{R}$ be a weak bisimulation. Let R be a Y-context. We define a new localized relation denoted as $R\,[\mathcal{R}/Y]$ as follows:

- if $R = Y$ then $R\,[\mathcal{R}/Y] = \mathcal{R}$;
- if $R \neq Y$ then we stipulate that $(P', E', Q') \in R\,[\mathcal{R}/Y]$ if there exists $(P, E, Q) \in \mathcal{R}$ and if $E' = \mathrm{Id}_{|R|}$, $P' = R\,[P/Y]$ and $Q' = R\,[Q/Y]$ (observe that $|P'| = |Q'| = |R|$ because $R \neq Y$).

We define a localized relation $\mathcal{R}^+$ as the union of $\mathcal{I}$ (the set of all triples (U, E, U) where U is any process and $E = \mathrm{Id}_{|U|}$), of the parallel extension $\mathcal{R}'$ of $\mathcal{R}$ (see Proposition 5) and of all the relations of the shape $R\,[\mathcal{R}/Y]$ for all Y-contexts R.

We prove that $\mathcal{R}^+$ is a weak bisimulation and the theorem will follow easily.

Let $(U, F, V) \in \mathcal{R}^+$ and assume that we are in one of the two following situations

- $U \xrightarrow[\mu]{\tau} U'$ (called case (**1**) in the sequel)
- or $U \xrightarrow[\mu_1]{p:f\cdot(\boldsymbol{L})} U'$ (called case (**2**) in the sequel).

We describe explicitly our goals.

- In case (**1**) we must show that $V \xrightarrow[\nu]{\tau *} V'$ with $(U', F', V') \in \mathcal{R}^+$ for some $F' \subseteq |U'| \times |V'|$ such that for any $(u', v') \in F'$, one has $(\mu(u'), \nu(v')) \in F$.
- In case (**2**) we must show that $V \overset{q:f\cdot(\boldsymbol{M})}{\underset{\nu,\nu_1,\nu'}{\Longrightarrow}} V'$ with $(p, \nu(q))) \in F$ and $(U', F', V') \in \mathcal{R}^+$, for some $F' \subseteq |U'| \times |V'|$ such that, for any $(u', v') \in F'$, one has $(\mu_1(u'), \nu\nu_1\nu'(v')) \in F$ and, if $n \geq 2$, then one has either $(u', \nu'(v')) \in \bigcup_{i=1}^n (L_i \times M_i)$ or $u' \notin \bigcup_{i=1}^n L_i$ and $\nu'(v') \notin \bigcup_{i=1}^n M_i$.

The case where $(U, F, V) \in \mathcal{I}$ is trivial.

If $(U, F, V) \in \mathcal{R}'$ we apply directly Proposition 5 in both cases.

Assume now that $(U, F, V) \in R[\mathcal{R}/Y]$ for some Y-context R, so that $U = R[P/Y]$, $V = R[Q/Y]$ with $(P, E, Q) \in \mathcal{R}$ and $F = E$ if $R = Y$ and $F = \mathrm{Id}_{|R|}$ otherwise. If $R = Y$ we use directly the fact that $\mathcal{R}$ is a weak bisimulation to exhibit V' and F' satisfying the required conditions.

So we assume from now on that $R \neq Y$ and therefore $F = \mathrm{Id}_{|R|}$.

By definition of Y-contexts, there is exactly one $r \in |R|$ such that Y occurs free in $R(r)$. Then $R(r)$ can be written uniquely as $R(r) = g \cdot \boldsymbol{R} + \tilde{R}$ where Y does not occur in $\tilde{R}$ and occurs in exactly one of the processes $R_1, \dots, R_n$; without loss of generality we can assume that R_1 is a Y-context and that Y does not occur free in $R_2, \dots, R_n$.

Assume first that $R_1 \neq Y$. In both cases (**1**) and (**2**), we have $U' = R'[P/Y]$ with $R \xrightarrow[\mu]{\tau} R'$ (case (**1**)) or $R \xrightarrow[\mu_1]{p:f\cdot(\boldsymbol{L})} R'$ (case (**2**)). Let $V' = R'[Q/Y]$. In case (**1**), we have $V \xrightarrow[\mu]{\tau} V'$ and in case (**2**) we have $V \xrightarrow[\mu_1]{q:f\cdot(\boldsymbol{L})} V'$, and since $R' \neq Y$ (by our hypothesis on R_1), we have $(U', \mathrm{Id}_{|R'|}, V') \in \mathcal{R}^+$ because $(P, E, Q) \in \mathcal{R}$. The condition on residuals is obviously satisfied in both cases.

Assume now that $R_1 = Y$.

► Suppose first that we are in case (**1**). There are two cases to consider as to the locations $s, t \in |U|$ of the sub-processes involved in the transition $U \xrightarrow[\mu]{\tau} U'$. The case where $s \neq r$ and $t \neq r$ is similar to the case above where $R_1 \neq Y$. By symmetry we are left with the case where $s = r$ (and hence $t \neq r$).

So $U(t) = R(t) = \overline{f} \cdot \boldsymbol{T} + \tilde{T}$ and the guarded sum $R(r)$ has an unique summand which is involved in the transition $U \xrightarrow[\mu]{\tau} U'$ (called *active summand* in the sequel), and this summand is of the shape $f \cdot \boldsymbol{S}$.

If the active summand is $g\cdot\boldsymbol{R}$[6] (so that $g = f$) then $U(r) = f\cdot(P, R_2, \dots, R_n) + \tilde{S}$ and U' can be written $U' = R' \oplus_C P$ for some process R' which can be defined using only R, and $C \subseteq |R'| \times |P|$. Explicitly, R' is defined as follows:

– $I_{R'} = I_R \cup \bigcup_{i=2}^n I_{R_i} \cup \bigcup_{i=1}^n I_{T_i}$
– $|R'| = (|R| \setminus \{r, t\}) \cup \bigcup_{i=2}^n |R_i| \cup \bigcup_{i=1}^n |T_i|$
– and $\frown_{R'}$ is the least symmetric relation on $|R'|$ such that $r' \frown_{R'} t'$ if $r' \frown_{R_i} t'$ for some $i = 2, \dots, n$ or $r' \frown_{T_i} t'$ for some $i = 1, \dots, n$, or $(r', t') \in |R_i| \times |T_i|$ for some $i \in \{2, \dots, n\}$, or $r' \notin \bigcup_{i=2}^n |R_i|$ or $t' \notin \bigcup_{i=1}^n |T_i|$ and $r' \frown_R t$ and $\mu(r') \frown_R \mu(t')$

where the residual function $\mu : |U'| \to |U|$ is given by $\mu(r') = r$ if $r' \in |P| \cup \bigcup_{i=2}^n |R_i|$, $\mu(r') = t$ if $r' \in \bigcup_{i=1}^n |T_i|$ and $\mu(r') = r'$ when r' belongs to none of these two sets.

The relation C is defined as follows: given $(r', p) \in |R'| \times |P|$, one has $(r', p) \in C$ if $r' \in |T_1|$, or $r' \notin \bigcup_{i=2}^n |R_i| \cup \bigcup_{i=1}^n |T_i|$ and $r' \frown_R r$.

Let $V' = R' \oplus_D Q$, where $D \subseteq |R'| \times |Q|$ is defined exactly like C (just replace P with Q in the definition). Then (C, D, E) is adapted (because the property for $(r', p) \in |R'| \times |P|$ of belonging or not to C depends only on r', and does not

[6] Remember that $g \cdot \boldsymbol{R}$ is the unique summand of $R(r)$ which contains Y.

depend on p, and similarly for D). We can mimic that reduction on V, so that $V \xrightarrow[\nu]{\tau} V'$ for the residual function ν which is defined like μ (replacing P with Q). We have $(U', F', V') \in \mathcal{R}' \subseteq \mathcal{R}^+$ where $F' = \mathrm{Id}_{|R'|} \cup E$. Given $(u', v') \in F'$, we have $\mu(u') = \nu(v')$, that is $(\mu(u'), \nu(v')) \in F$ so that the condition on residuals holds[7].

Assume now that the active summand is not $g \cdot \boldsymbol{R}$. In that case we also have $V \xrightarrow[\mu]{\tau} U'$ (both P and Q vanish in the corresponding reductions), and we are done because $(U', \mathrm{Id}_{|U'|}, U') \in \mathcal{I} \subseteq \mathcal{R}^+$.

► We suppose now that we are in case (**2**). Assume first that $p \neq r$. In that case we have $R \xrightarrow[\theta_1]{p:f\cdot(\boldsymbol{L})} R'$ and $U' = R'[P/Y]$ and we also have $V \xrightarrow[\theta_1]{p:f\cdot(\boldsymbol{L})} V' = R'[Q/Y]$ so $(U', \mathrm{Id}_{|R'|}, V') \in R'[\mathcal{R}/Y] \subseteq \mathcal{R}^+$, and the condition on residuals is obvious.

Assume now that $p = r$. Then exactly one of the summands of the guarded sum $R(r)$ is the prefixed process performing the action f in the considered transition on U (again, this summand is called the active summand in the sequel). We also know that $f \notin I_R$.

The case where the active summand is not $g \cdot (P, R_2, \dots, R_n)$ is completely similar to the previous one (P vanishes in the transition).

Assume that the active summand is $g \cdot (P, R_2, \dots, R_n)$ (so that $g = f$), then $U' = R' \oplus_C P$ where R' is defined by

- $I_{R'} = I_R \cup \bigcup_{i=2}^n I_{R_i}$
- $|R'| = (|R| \setminus \{r\}) \cup \bigcup_{i=2}^n |R_i|$ and $\frown_{R'}$ is the least symmetric relation on $|R'|$ such that $r' \frown_{R'} t'$ if $r' \frown_{R_i} t'$ for some $i = 2, \dots, n$ or $\theta_1(r') \frown_R \theta_1(t')$
- the relation $C \subseteq |R'| \times |P|$ is defined by $(r', q) \in C$ if $r' \notin \bigcup_{i=2}^n |R_i|$ and $r' \frown_R r$ (this does not depend on q).

Then we have $V = R[Q/Y] \xrightarrow[\varphi_1]{p:f\cdot(\boldsymbol{M})} V'$ (with $M_1 = |Q|$ and $M_i = L_i = |R_i|$ for $i = 2, \dots, n$; remember also that $f \notin I_R$) with $V' = R' \oplus_D Q$ where D is defined like C (replacing P with Q in the definition). Then we have $(U', F', V') \in \mathcal{R}' \subseteq \mathcal{R}^+$ where $F' = \mathrm{Id}_{|R'|} \cup E$ since (C, D, E) is obviously adapted (as above). Moreover the condition on residuals is obviously satisfied.

This ends the proof of the fact that $\mathcal{R}^+$ is a weak bisimulation.

We can now prove that $\approx$ is a congruence. Assume that $P \approx Q$ and let R be a Y-context. Let $E \subseteq |P| \times |Q|$ and let $\mathcal{R}$ be a weak bisimulation such that $(P, E, Q) \in \mathcal{R}$. Then we have $(R[P/Y], \mathrm{Id}_{|R|}, R[Q/Y]) \in R[\mathcal{R}/Y] \subseteq \mathcal{R}^+$ and hence $R[P/Y] \approx R[Q/Y]$ since $\mathcal{R}^+$ is a weak bisimulation. □

8.8 Weak bisimilarity on CCS

We assume in this section that $\Sigma_n = \emptyset$ if $n > 1$ (see the end of Section 3). All processes P considered in this section are CCS processes built on Σ, meaning

[7] It is in this part of the proof that one understand the importance of adapted triples of relations in the definition of the parallel extension of a weak bisimulation.

that, in any subprocess of P which is of shape $G\langle \Phi \rangle$, the graph G is a complete graph (for all $p, q \in |G|$, if $p \neq q$ then $p \frown_G q$).

We answer here a very natural question: when restricted to ordinary CCS, does our weak localized bisimilarity coincide with standard weak bisimilarity?

Let $\mathcal{R}$ be a localized weak bisimulation. Let $\mathcal{R}^0$ be the following relation on CCS processes: $P \mathrel{\mathcal{R}^0} Q$ if $(P, E, Q) \in \mathcal{R}$ for some $E \subseteq |P| \times |Q|$. We prove that $\mathcal{R}^0$ is a weak bisimulation on CCS processes.

Lemma 11. *Let $\mathcal{R}$ be a localized weak bisimulation. Then $\mathcal{R}^0$ is a weak bisimulation on CCS processes.*

Proof. Let P and Q be CCS processes such that $P \mathrel{\mathcal{R}^0} Q$. Let $E \subseteq |P| \times |Q|$ be such that $(P, E, Q) \in \mathcal{R}$.

Assume first that $P \xrightarrow{\tau} P'$. Let $p_1, p_2 \in |P|$ with $\mathsf{cs}(P(p_1)) = a \cdot P_1 + S_1$ and $\mathsf{cs}(P(p_2)) = \overline{a} \cdot P_2 + S_2$ (the two sub-processes involved in this reduction). Then, by definition of the internal reduction in DPC, $P' = G\langle \Phi \rangle$ where G is the complete graph on $|G| = |P| \setminus \{p_1, p_2\} \cup |P_1| \cup |P_2|$ and $\Phi(r) = P(r)$ if $r \in |P|$, $\Phi(r) = P_i(r)$ if $r \in |P_i|$ for $i = 1, 2$. In other words $P' = P\,[P_1/p_1, P_2/p_2]$

Let $\lambda_1 : |P'| \to |P|$ be the corresponding residual map ($\lambda_1(r) = r$ if $r \in |P|$ and $\lambda_1(r) = p_i$ if $r \in |P_i|$), we have $P \xrightarrow[\lambda_1]{\tau} P'$ and therefore there is a DPC process Q' such that $(P', E', Q') \in \mathcal{R}$ for some relation $E' \subseteq |P'| \times |Q'|$, and a function $\rho : |Q'| \to |Q|$ with $Q \xrightarrow[\rho]{\tau *} Q'$ and $(p', q') \in E'$ implies $(\lambda_1(p'), \rho(q')) \in E$. Therefore we have $P' \mathrel{\mathcal{R}^0} Q'$ as required.

Assume now that $P \xrightarrow{a} P'$. Let $p \in |P|$ with $\mathsf{cs}(P(p)) = a \cdot P_1 + S_1$ and $P' = P\,[P_1/p]$, we also have $a \notin I_P$. Then we have $P \xrightarrow[\lambda_1]{p:a \cdot (L)} P'$ where $L = |P_1|$ and $\lambda_1 : |P'| \to |P|$ is given by $\lambda_1(r) = p$ if $r \in |P_1|$ and $\lambda_1(r) = r$ otherwise. Since $(P, E, Q) \in \mathcal{R}$, we have $Q \overset{q:a \cdot (M)}{\underset{\rho, \rho_1, \rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$, and $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$ for each $(p', q') \in E'$. In particular $P' \mathrel{\mathcal{R}^0} Q'$.

Since $\mathcal{R}$ is a localized bisimulation, the relation $\mathcal{R}^0$ is symmetric and is therefore a bisimulation on CCS processes. $\square$

We need now to prove the converse. Let $\mathcal{U}$ be a binary relation on CCS processes. Let $\widehat{\mathcal{U}}$ be the set of all triples (P, E, Q) where P and Q are CCS processes such that $P \mathrel{\mathcal{U}} Q$ and $E = |P| \times |Q|$.

Lemma 12. *If $\mathcal{U}$ is a bisimulation, then $\widehat{\mathcal{U}}$ is a localized bisimulation.*

Proof. Let P and Q be CCS processes and let E be such that $(P, E, Q) \in \widehat{\mathcal{U}}$, so that $E = |P| \times |Q|$ and $P \mathrel{\mathcal{U}} Q$.

Assume first that $P \xrightarrow[\lambda_1]{\tau} P'$ so that $P \xrightarrow{\tau} P'$ (in CCS) and hence there exists Q' such that $Q \xrightarrow{\tau *} Q'$ and $P' \mathrel{\mathcal{U}} Q'$. Then there is a function $\rho : |Q'| \to |Q|$ such that $Q \xrightarrow[\rho]{\tau *} Q'$ and we have $(P', E', Q') \in \widehat{\mathcal{U}}$. The condition on residuals holds obviously, by definition of E.

The case of a labeled transition is completely similar and the condition on residuals holds again by definition of $\widehat{\mathcal{U}}$ and because we are in the case where $n = 1$ (all function symbols are of arity 1).

$\square$

So we can conclude that, when restricted to CCS processes, our notion of weak bisimilarity coincides with the usual one.

Theorem 5. *Two CCS processes are weakly bisimilar (in the usual CCS sense) iff they are weakly bisimilar in the localized sense.*