Cartesian Coherent Differential Categories

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Abstract

We extend to general Cartesian categories the idea of Coherent Differentiation recently introduced by Ehrhard in the setting of categorical models of Linear Logic. The first ingredient is a summability structure which induces a partial left-additive structure on the category. Additional functoriality and naturality assumptions on this summability structure implement a differential calculus which can also be presented in a formalism close to Blute, Cockett and Seely's Cartesian differential categories. We show that a simple term language equipped with a natural notion of differentiation can easily be interpreted in such a category.

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Introduction

Linear Logic (LL) and its models [1] strongly suggest that differentiation of proofs should be a natural operation, extracting their best "local" linear approximation. Remember that if E, F are Banach spaces and if $f : E \to F$ and $x \in E$ then f is differentiable at x if there is a neighborhood U of 0 in E and a linear and continuous function $\phi : E \to F$ such that, for all $u \in U$

$$f(x+u) = f(x) + \phi(u) + o(\|u\|)\,. \tag{1}$$

When such a ϕ exists, it is unique and is often denoted as $f'(x)$, and called the differential (or derivative) of f at x.

When $f'(x)$ exists for all $x \in E$ in the domain of definition of f, the function $f' : E \to \mathcal{L}(E, F)$ where $\mathcal{L}(E, F)$ is the Banach space of linear and continuous functions $E \to F$ is also called the differential of f. This function can itself admit a differential *etc*, and when all these iterated differentials exist one says that f is *smooth* and the nth derivative of f is a function $f^{(n)} : E \to \mathcal{L}_n(E, F)$ where $\mathcal{L}_n(E, F)$ is the space of n-linear symmetric functions $E^n \to F$. It can even happen that f is locally (or even globally) expressed using its iterated derivatives by means of the *Taylor Formula* $f(x+u) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)(u, \ldots, u)$; when this holds locally at any point x, f is said *analytic*.

Based on categorical models of LL where all functions are analytic in a similar sense, the Differential λ-Calculus and Differential LL provide a logical and syntactical account of differentiation where a program of type $A \Rightarrow B$ is turned into a program of type $A \Rightarrow (A \multimap B)$, providing in particular a new approach of finite approximations of functions by a syntactical version of the Taylor Formula which has shown relevance in the study of the λ-calculus and of LL. One crucial feature of the differential calculus in general is its deep connection with the basic operation of *addition* as can already be seen in its definition Eq. (1). This connection appears most clearly when one writes the differential of $f : \mathbb{R}^n \to \mathbb{R}$ as a sum of *partial derivatives*:

$$f'(x_1, \ldots, x_n) \cdot (u_1, \ldots, u_n) = \sum_{i=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} u_i$$

and, of course, in the Taylor formula itself. For this reason, until recently, all categorical models of the Differential λ-Calculus and of Differential LL were using categories where hom-sets have a structure of commutative monoid (additive or left-additive categories), and both formalisms feature a formal and unrestricted addition operation on terms or proofs of the same type. The only available operational interpretation of such a sum being erratic choice, these formalisms are inherently non-deterministic.

Recently, the first author observed that, in a setting where all coefficients are non-negative, differentiation survives to strong restrictions on the use of addition. Consider for instance a function $[0,1] \to [0,1]$ which is smooth on $[0,1)$ and whose all derivatives are everywhere ≥ 0[1]. If $x, u \in [0,1]$ are such that $x + u \in [0,1]$ then $f(x) + f'(x)u \leq f(x+u) \in [0,1]$ (this makes sense even if $f'(1) = \infty$, which can happen: take $f(x) = 1 - \sqrt{1-x}$). So if S is the set of all such pairs (x, u) that we call *summable*, we can consider the function $\mathsf{D}(f) : (x, u) \to (f(x), f'(x)u)$ as a map $S \to S$. This basic observation is generalized in [2] to a wide range of categorical models $\mathcal{L}$ of LL including coherence spaces, probabilistic coherence spaces *etc.* where hom-sets have only a *partially defined* addition. In these *Summable Categories*, S becomes an endofunctor $\mathcal{L} \to \mathcal{L}$ equipped with an additional structure which allows to define summability and (partial) sums in a very general way and turns out to induce a monad. Differentiation is then axiomatized as a distributive law between this monad (similar to the tangent bundle monad of a tangent category [3]) and the resource comonad $!\_$ of the LL structure of the category[2] $\mathcal{L}$. Indeed, this distributive law allows to extend S to the Kleisli category of $!\_$ and this extension $\mathsf{D} : \mathcal{L}_! \to \mathcal{L}_!$ turns out to be a monad which has all the required properties of differentiation.

We show in the present paper that the idea of Coherent Differentiation is very general and by no means limited to categorical models of LL: we define Coherent Differentiation in an *arbitrary category*, whose morphisms are intuitively considered as smooth. So we start from a category $\mathcal{C}$ that we assume to be equipped with a function[3] $\mathsf{D} : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{C})$ given together with morphisms $(\pi_0)_X, (\pi_1)_X, \sigma_X \in \mathcal{C}(\mathsf{D}(X), X)$ (for each $X \in \mathsf{Obj}(\mathcal{C})$), the intuition being that $\mathsf{D}(X)$ is the object of summable pairs of elements of X, that π_i are the obvious projections and that σ computes the sums. We assume π_0, π_1 to be jointly monic and this is sufficient to say when $f_0, f_1 \in \mathcal{C}(X, Y)$ are summable: this is when there is a necessarily unique $h \in \mathcal{C}(X, \mathsf{D}(Y))$ such that $\pi_i \circ h = f_i$ and when this holds we set $f_0 + f_1 = \sigma \circ h$. Under suitable assumptions this very light structure suffices to equip hom-sets of $\mathcal{C}$ with a structure of partial commutative monoid which is compatible with composition on the left[4] This structure is also a convenient setting for differentiation: it suffices to equip furthermore D with a functorial action on morphisms wrt. which some morphisms easily definable in terms of π_0, π_1, σ satisfy simple equational properties. This is the notion of *Coherent Differential Category* whose axioms are in one-to-one correspondence with those of a Cartesian Differential Category as presented in [4], a categorical notion of differentiation in additive categories. Just as in Tangent Categories [3, 5], our functor D can be equipped with a monad structure. Contrarily to the additive framework of [4], our differentiation functor D is not defined in terms of the Cartesian product so it is important to understand how it interacts with the Cartesian product when available: this is formalized by the concept of Cartesian Coherent Differential Category (CCDC). This compatibility can be expressed in terms of a strength with which D can be equipped, turning it into a commutative monad. This induces a satisfactory theory of partial derivatives. We provide a concrete example of such a category based on Probabilistic Coherence Spaces and illustrate our formalism by interpreting a simple term language equipped with a notion of differentiation in a CCDC.

[1] This actually implies that f is analytic.

[2] Which by the way needs not be a fully-fledged LL model.

[3] Or more precisely a functional class, which will become a functor and even a monad later.

[4] And not on the right in general since, intuitively, the morphisms of $\mathcal{C}$ are not assumed to be linear.

1 Summability Structure

We introduce in this section the notion of *Left Summability Structure* in order to generalize the notion of Summability Structure introduced in [2] to a setting where morphisms are not necessarily additive.

1.1 Pre-Summability Structures

Let $\mathcal{C}$ be a category with objects $\mathbf{Obj}(\mathcal{C})$ and hom-set $\mathcal{C}(X, Y)$ for any $X, Y \in \mathbf{Obj}(\mathcal{C})$. We assume that any hom-set $\mathcal{C}(X, Y)$ contains a morphism $0^{X,Y}$ (usually X and Y are kept implicit) such that for any $f \in \mathcal{C}(Z, X)$, $0^{X,Y} \circ f = 0^{Z,Y}$.

Definition 1. A *Summable Pairing Structure* on a category $\mathcal{C}$ is a tuple $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ such that:

- $\mathsf{D} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$ is a map (a functional class) on objects;
- $(\pi_{0,X})_{X \in \mathbf{Obj}(\mathcal{C})}, (\pi_{1,X})_{X \in \mathbf{Obj}(\mathcal{C})}$ and $(\sigma_X)_{X \in \mathbf{Obj}(\mathcal{C})}$ are collections of morphisms in $\mathcal{C}(\mathsf{D}X, X)$. The object X will usually be kept implicit;
- π_0, π_1 are jointly monic: for any $f, g \in \mathcal{C}(Y, \mathsf{D}X)$, if $\pi_0 \circ f = \pi_0 \circ g$ and $\pi_1 \circ f = \pi_1 \circ g$ then $f = g$.

We assume in what follows that $\mathcal{C}$ is equipped with a Summable Pairing Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$.

Definition 2. Two morphisms $f_0, f_1 \in \mathcal{C}(X, Y)$ are said to be *summable* if there exists $h \in \mathcal{C}(X, \mathsf{D}Y)$ such that $\pi_i \circ h = f_i$. The joint monicity of the π_i's ensures that when h exists, it is unique. We set $\langle\!\langle f_0, f_1 \rangle\!\rangle := h$, and we call it the *witness* of the sum. By definition, $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$. Then we set $f_0 + f_1 := \sigma \circ \langle\!\langle f_1, f_2 \rangle\!\rangle$.

Remark 1. A more standard approach to notations would be to write π_1 and π_2 instead of π_0 and π_1. The reason we proceed that way is that Equation (1) will be formalized in our setting with the use of a pair $\langle\!\langle f(x), f'(x) \cdot u \rangle\!\rangle$. That is, the left element of this pair is of order 0, and the right element is of order 1.

Notations 1. We write $f_0 \boxplus f_1$ for the property that f_0 and f_1 are summable. We say that an algebraic expression containing binary sums is *well defined* if each pair of morphisms involved in these sums is summable. For example, $(f_0 + f_1) + f_2$ is well defined if $f_0 \boxplus f_1$ and $f_0 + f_1 \boxplus f_2$.

Proposition 1. *The morphisms* π_0, π_1 *are summable of witness* $\langle\!\langle \pi_0, \pi_1 \rangle\!\rangle = \mathsf{id}$ *and sum* $\pi_0 + \pi_1 = \sigma$.

Proof. $\pi_i \circ \mathsf{id} = \pi_i$ so by definition, π_0, π_1 are summable of witness id and sum $\sigma \circ \mathsf{id} = \sigma$. □

Proposition 2 (Left compatibility of sum)**.** *For any* $f_0, f_1 \in \mathcal{C}(Y, Z)$ *and* $g \in \mathcal{C}(X, Y)$*, if* $f_0 \boxplus f_1$*, then* $f_0 \circ g \boxplus f_1 \circ g$ *with witness* $\langle\!\langle f_0 \circ g, f_1 \circ g \rangle\!\rangle = \langle\!\langle f_0, f_1 \rangle\!\rangle \circ g$. *Moreover,* $f_0 \circ g + f_1 \circ g = (f_0 + f_1) \circ g$.

Proof. Let $w = \langle\!\langle f_0, f_1 \rangle\!\rangle \circ g$. Then $\pi_i \circ w = f_i \circ g$ so w is a witness for the summability of $f_0 \circ g$ and $f_1 \circ g$. And $f_0 \circ g + f_1 \circ g := \sigma \circ w = (f_0 + f_1) \circ g$. □

An important class of morphisms is that of additive morphisms, for which addition is compatible with composition on the right.

Definition 3. A morphism $h \in \mathcal{C}(Y, Z)$ is *additive* if $h \circ 0 = 0$ and if for any $f_0, f_1 \in \mathcal{C}(X, Y)$, if $f_0 \boxplus f_1$ then $h \circ f_0 \boxplus h \circ f_1$ and $h \circ (f_0 + f_1) = h \circ f_0 + h \circ f_1$. Note that id is additive and that the composition of two additive morphisms is an additive morphism.

Proposition 3. *A morphism h is additive if and only if $h \circ 0 = 0$ and $h \circ \pi_0 \boxplus h \circ \pi_1$ of sum $h \circ \sigma$.*

Proof. For the forward implication, remember that $\pi_0 \boxplus \pi_1$. Thus by assumption, $h \circ \pi_0 \boxplus h \circ \pi_1$ of sum $h \circ (\pi_0 + \pi_1) = h \circ \sigma$.

For the reverse implication, assume that $f_0 \boxplus f_1$. Since $h \circ \pi_0 \boxplus h \circ \pi_1$, Proposition 2 ensures that $h \circ f_0 = h \circ \pi_0 \circ \langle\!\langle f_0, f_1 \rangle\!\rangle$ and $h \circ f_1 = h \circ \pi_1 \circ \langle\!\langle f_0, f_1 \rangle\!\rangle$ are summable, of sum $(h \circ \pi_0 + h \circ \pi_1) \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = h \circ \sigma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = h \circ (f_0 + f_1)$. □

Definition 4. The Summable Pairing Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ is a *Left Pre-Summability Structure* if π_0, π_1 and σ are additive

The additivity of the projections implies that the structure behaves well with respect to the summability witness operation $\langle\!\langle \_, \_ \rangle\!\rangle$ itself.

Proposition 4. *For any $f_0, f_1, g_0, g_1 \in \mathcal{C}(X, Y)$. If $f_0 \boxplus f_1$, $g_0 \boxplus g_1$ and $\langle\!\langle f_0, f_1 \rangle\!\rangle \boxplus \langle\!\langle g_0, g_1 \rangle\!\rangle$, then $f_0 \boxplus g_0$, $f_1 \boxplus g_1$ $f_0 + g_0 \boxplus f_1 + g_1$ and $\langle\!\langle f_0, f_1 \rangle\!\rangle + \langle\!\langle g_0, g_1 \rangle\!\rangle = \langle\!\langle f_0 + g_0, f_1 + g_1 \rangle\!\rangle$.*

Proof. By additivity of π_i, $\pi_i \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_i$ and $\pi_i \circ \langle\!\langle g_0, g_1 \rangle\!\rangle = g_i$ are summable of sum $f_i + g_i = \pi_i \circ (\langle\!\langle f_0, f_1 \rangle\!\rangle + \langle\!\langle g_0, g_1 \rangle\!\rangle)$. Since $\pi_0 \boxplus \pi_1$ this entails by Proposition 2 that $f_0 + g_0$, $f_1 + g_1$ are summable of witness $\langle\!\langle f_0, f_1 \rangle\!\rangle + \langle\!\langle g_0, g_1 \rangle\!\rangle$. □

The additivity of σ means that whenever $\langle\!\langle f_0, f_1 \rangle\!\rangle \boxplus \langle\!\langle g_0, f_1 \rangle\!\rangle$, one has $\sigma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle \boxplus \sigma \circ \langle\!\langle g_0, g_1 \rangle\!\rangle$ and

$$\sigma \circ (\langle\!\langle f_0, f_1 \rangle\!\rangle + \langle\!\langle g_0, g_1 \rangle\!\rangle) = (\sigma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle) + (\sigma \circ \langle\!\langle g_0, g_1 \rangle\!\rangle)$$

Assuming the additivity of the projections, it means that whenever $\langle\!\langle \langle\!\langle f_0, f_1 \rangle\!\rangle, \langle\!\langle g_0, g_1 \rangle\!\rangle \rangle\!\rangle$ exists, the two sums below are well defined (see Notations 1) and

$$(f_0 + g_0) + (f_1 + g_1) = (f_0 + f_1) + (g_0 + g_1)\,. \tag{2}$$

Proposition 5. *The morphisms 0 and 0 are summable of witness 0 and sum 0. In particular, 0 is additive.*

Proof. On the one hand, $\pi_i \circ 0 = 0$ by additivity of π_i, so $0 \boxplus 0$ of witness 0. On the other hand, $\sigma \circ 0 = 0$ by additivity of σ so $0 + 0 = 0$. In particular, 0 is additive thanks to Proposition 3 because $0 \circ \pi_0 = 0$ and $0 \circ \pi_1 = 0$ are summable of witness 0 and sum $0 = 0 \circ \sigma$. □

1.2 Axioms on the addition

We consider a category $\mathcal{C}$ equipped with a Left Pre-Summability Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$. The goal of this section is to make $(\mathcal{C}(X, Y), +, 0)$ a *partial* commutative monoid. Similar structures appear in [6] or more recently in [7], in a setting where sums can be infinitary. Our partial monoids have only finite sums[5]. More crucially, the categorical notion of summability defined above is essential for us whereas it is not categorically formalized in these works.

[5]Although the extension of the finite sum to an infinitary operations will have to be considered when dealing with fixpoints.

Definition 5. The Left Pre-Summability Structure is *commutative* if for any object X, $\pi_1, \pi_0 \in \mathcal{C}(\mathsf{D}X, X)$ are summable of sum σ. Then we set $\gamma = \langle\!\langle \pi_1, \pi_0 \rangle\!\rangle \in \mathcal{C}(\mathsf{D}X, \mathsf{D}X)$ so that $\pi_i \circ \gamma = \pi_{1-i}$. This property is called (D-com).

Proposition 6 (Commutativity)**.** *The Left Pre-Summability Structure is commutative if and only if for any $f_0, f_1 \in \mathcal{C}(X, Y)$, if $f_0 \boxplus f_1$ then $f_1 \boxplus f_0$ and $f_0 + f_1 = f_1 + f_0$.*

Proof. For the direct implication, assume that $f_0 \boxplus f_1$. Then $\pi_i \circ \gamma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = \pi_{1-i} \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_{1-i}$ so $f_1 \boxplus f_0$ of witness $\gamma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle$. Furthermore, $f_1 + f_0 = \sigma \circ \gamma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = \sigma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle = f_0 + f_1$. Conversely, $\pi_0 \boxplus \pi_1$ so by commutativity $\pi_1 \boxplus \pi_0$ and $\pi_1 + \pi_0 = \pi_0 + \pi_1 = \sigma$. $\square$

Definition 6. The Left Pre-Summability Structure *has* 0 *as a neutral element* if for any object X, $\mathsf{id}_X \boxplus 0$ and $0 \boxplus \mathsf{id}_X$ with sums equal to id_X. We call this property (D-zero). We define $\iota_0, \iota_1 \in \mathcal{C}(X, \mathsf{D}X)$ as $\iota_0 := \langle\!\langle \mathsf{id}_X, 0 \rangle\!\rangle$ and $\iota_1 := \langle\!\langle 0, \mathsf{id}_X \rangle\!\rangle$.

Proposition 7 (Neutrality of 0)**.** *The Left Pre-Summability Structure has 0 as a neutral element if and only if for any morphism $f \in \mathcal{C}(X, Y)$, $0 \boxplus f$, $f \boxplus 0$ and $f + 0 = 0 + f = f$.*

Proof. By definition of ι_0, $\pi_0 \circ \iota_0 \circ f = \mathsf{id} \circ f = f$ and $\pi_1 \circ \iota_0 \circ f = 0 \circ f = 0$. So $f \boxplus 0$ of witness $\iota_0 \circ f$ and $f + 0 = \sigma \circ \iota_0 \circ f = \mathsf{id} \circ f = f$. We do the same for $0 + f$ with ι_1. Conversely, we apply the neutrality of 0 on id to get that $\mathsf{id} \boxplus 0$ and $0 \boxplus \mathsf{id}$, of sum id. $\square$

Associativity is not that straightforward, as there are two possible notions in the partial monoid setting. The situation is similar in the infinitary setting of [7] with the distinction between Weak Partition Associativity and Partition Associativity.

Definition 7 (Weak Associativity)**.** The operation $+$ is called *weakly associative* if whenever $(f_0+f_1)+f_2$ and $f_0+(f_1+f_2)$ are well defined (recall Notations 1), we have $(f_0+f_1)+f_2 = f_0 + (f_1 + f_2)$.

Definition 8 (Associativity)**.** The operation $+$ is called *associative* if whenever $(f_0+f_1)+f_2$ or $f_0+(f_1+f_2)$ is well defined, the other expression is also well defined and $(f_0+f_1)+f_2 = f_0 + (f_1 + f_2)$.

We need to work in a partial setting in which addition is associative: this is required for instance in Section 2.1 to define $\theta = \langle\!\langle \pi_0 \circ \pi_0, \pi_0 \circ \pi_1 + \pi_1 \circ \pi_0 \rangle\!\rangle$. This associativity seems related to a kind of positivity of the morphisms.

Example 1. Let $x, y \in [-1, 1]$ be summable when $|x| + |y| \leq 1$, with $x + y$ as sum. Then $+$ is weakly associative, but is not associative. Indeed, take $x_0 = -\frac{1}{2}, x_1 = \frac{1}{2}, y_1 = 1$. Then $(x_0 + x_1) + y_1$ is defined, but $x_0 + (x_1 + y_1)$ is not since $|x_1| + |y_1| = \frac{3}{2} > 1$. However, the same definition on $[0, 1]$ yields an associative operation.

Recall from Eq. Equation (2) that whenever $\langle\!\langle \langle\!\langle f_0, f_1 \rangle\!\rangle, \langle\!\langle g_0, g_1 \rangle\!\rangle \rangle\!\rangle$ exists, the expressions $(f_0 + g_0) + (f_1 + g_1)$ and $(f_0 + f_1) + (g_0 + g_1)$ are well defined and equal. Taking $g_0 = 0$ and assuming (D-zero), this means that whenever $\langle\!\langle \langle\!\langle f_0, f_1 \rangle\!\rangle, \langle\!\langle 0, g_1 \rangle\!\rangle \rangle\!\rangle$ exists, $(f_0 + f_1) + g_1$ and $f_0 + (f_1 + g_1)$ are well defined and equal. Taking $f_1 = 0$ and assuming (D-zero), whenever $\langle\!\langle \langle\!\langle f_0, 0 \rangle\!\rangle, \langle\!\langle g_0, g_1 \rangle\!\rangle \rangle\!\rangle$ exist, $f_0 + (g_0 + g_1)$ and $(f_0 + g_0) + g_1$ are well defined and equal. Thus associativity holds if (D-zero) and whenever $(f_0+f_1)+g_1$ is defined (respectively $f_0 + (g_0 + g_1)$ is defined), then $\langle\!\langle \langle\!\langle f_0, f_1 \rangle\!\rangle, \langle\!\langle 0, g_1 \rangle\!\rangle \rangle\!\rangle$ exists (respectively $\langle\!\langle \langle\!\langle f_0, 0 \rangle\!\rangle, \langle\!\langle g_0, g_1 \rangle\!\rangle \rangle\!\rangle$ exists). This shows that associativity follows from the following axiom.

Definition 9. The Left Pre-Summability Structure *admits witnesses* if for any $f, g \in \mathcal{C}(Y, \mathsf{D}X)$, if $\sigma \circ f \boxplus \sigma \circ g$ then $f \boxplus g$. We call this property (D-witness).

Theorem 1. *The properties* (D-*zero*)*,* (D-*com*) *and* (D-*witness*) *give to* $\mathcal{C}(X,Y)$ *the structure of a* partial commutative monoid *for any objects* X, Y*. That is, for any* $f, f_0, f_1, f_2 \in \mathcal{C}(X,Y)$*:*

- $f \boxplus 0$*,* $0 \boxplus f$ *and* $0 + f = f + 0 = f$*;*
- *If* $f_0 \boxplus f_1$ *then* $f_1 \boxplus f_0$ *and* $f_0 + f_1 = f_1 + f_0$*;*
- *If* $(f_0 + f_1) + f_2$ *or* $f_0 + (f_1 + f_2)$ *is defined, then both are defined and* $(f_0 + f_1) + f_2 = f_0 + (f_1 + f_2)$*.*

One can define inductively from this binary sum a notion of arbitrary finite sum. The empty family is always summable of sum 0. The family $(f_i)_{i \in I}$ for $I \neq \emptyset$ is summable if $\exists i_0 \in I$ such that $(f_i)_{i \in I / \{i_0\}}$ is summable and if $(\sum_{i \in I/\{i_0\}} f_i) \boxplus f_{i_0}$. Then we set $\sum_{i \in I} f_i := \sum_{i \in I/\{i_0\}} f_i + f_{i_0}$. It is shown in [2] that the choice of order for the sum is irrelevant. More precisely.

Theorem 2. *A family* $(f_i)_{i \in I}$ *is summable if and only if for all partition*[6] $I_1, \dots, I_n$ *of* I*, we have that for all* $j \in [\![1, n]\!] := \{1, \dots, n\}$*,* $(f_i)_{i \in I_j}$ *is summable and* $(\sum_{i \in I_j} f_i)_{j \in [\![1,n]\!]}$ *is summable. Moreover,* $\sum_{i \in I} f_i = \sum_{j \in [\![1,n]\!]} \sum_{i \in I_j} f_i$*.*

Definition 10. A Left Pre-Summability Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ is called a *Left Summability Structure* if (D-zero), (D-com), (D-witness) hold. A *Left Summable Category* is a category equipped with a Left Summability Structure.

1.3 Comparison with Summability Structures

In the LL setting of [2], the first author introduced a notion of Pre-Summability Structure as a Summable Pairing Structure $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ (recall Definition 1) where S is a functor for which π_0, π_1, σ are natural transformations (his S is our D).

Theorem 3. *The following are equivalent*

- $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ *is a Left Pre-Summability Structure and every morphism is additive;*
- $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ *is a Pre-Summability Structure [2].*

Proof. Let $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ be a Left Pre-Summability Structure in which every morphism is additive. By Proposition 3, for any $f \in \mathcal{C}(X,Y)$ we can define $\mathsf{S}f := \langle\!\langle f \circ \pi_0, f \circ \pi_1 \rangle\!\rangle$ and the following equations hold: $\pi_i \circ \mathsf{S}f = f \circ \pi_i$, $\sigma \circ \mathsf{S}f = f \circ \sigma$. Furthermore, S is a functor: $\pi_i \circ \mathsf{S}\mathsf{id} = \mathsf{id} \circ \pi_i = \pi_i \circ \mathsf{id}$ and $\pi_i \circ \mathsf{S}f \circ \mathsf{S}g = f \circ \pi_i \circ \mathsf{S}g = f \circ g \circ \pi_i = \pi_i \circ \mathsf{S}(f \circ g)$. Thus, by joint monicity of the π_i, $\mathsf{S}\mathsf{id} = \mathsf{id}$ and $\mathsf{S}(f \circ g) = \mathsf{S}f \circ \mathsf{S}g$. Then the equations $\pi_i \circ \mathsf{S}f = f \circ \pi_i$ and $\sigma \circ \mathsf{S}f = f \circ \sigma$ introduced above correspond to the naturality of π_0, π_1 and σ.

Conversely, let $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ be a Pre-Summability Structure in the sense of [2]. The naturality of π_0 and π_1 ensures that for any f, $f \circ \pi_0 \boxplus f \circ \pi_1$ (of witness $\mathsf{S}f$). The naturality of σ ensures that the sum of those two morphisms is $\sigma \circ \mathsf{S}f = f \circ \sigma$. Finally, $f \circ 0 = 0$ by assumption. So every morphism is additive by Proposition 3. In particular, π_0, π_1 and σ are additive, so $(\mathsf{S}, \pi_0, \pi_1, \sigma)$ is a Left Pre-Summability Structure. □

Corollary 1. *The Summability Structures introduced in [2] are exactly the Left Summability Structures for which all morphisms are additive.*

[6] Where we admit that some I_js can be empty.

2 Differential

2.1 Differential Structure

Recall from Eq. Equation (1) the main idea of the differential calculus. We generalize it to a partial additive setting: f is differentiable at x if for any u, if $x \boxplus u$ then $f'(x) \cdot u$ is defined, $f(x) \boxplus f'(x) \cdot u$ and, intuitively, $f(x+u) \simeq f(x) + f'(x) \cdot u$. Hence the differential of f can be seen as a function $\mathsf{D}f$ that maps a pair of two summable elements $\langle\!\langle x, u \rangle\!\rangle$ to a pair of two summable elements $\mathsf{D}f(x,u) = \langle\!\langle f(x), f'(x) \cdot u \rangle\!\rangle$.

Definition 11. A *Pre-Differential Structure* is a Left Summability Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ together with, for each $X, Y \in \mathsf{Obj}(\mathcal{C})$, an operator $\mathcal{C}(X,Y) \to \mathcal{C}(\mathsf{D}X, \mathsf{D}Y)$, also denoted as D, and such that $\pi_0 \circ \mathsf{D}f = f \circ \pi_0$. We define the *differential* of f as $\mathbf{d}^{\mathsf{D}}(f) := \pi_1 \circ \mathsf{D}f \in \mathcal{C}(\mathsf{D}X, Y)$. By our assumptions $\mathsf{D}f = \langle\!\langle f \circ \pi_0, \mathbf{d}^{\mathsf{D}}(f) \rangle\!\rangle$.

At this point we do not assume D to be a functor, this will be a consequence of the Chain Rule and then the equation $\pi_0 \circ \mathsf{D}f = f \circ \pi_0$ will be the naturality of π_0. We can already introduce three families of morphisms θ, $\mathbf{l}$ and $\mathbf{c}$ whose naturality will correspond to some axioms of differentiation.

Definition 12. For any object X, $\pi_0 \circ \pi_1 \boxplus \pi_1 \circ \pi_0$ and $\pi_0 \circ \pi_0 \boxplus \pi_0 \circ \pi_1 + \pi_1 \circ \pi_0$. Thus, we can define $\theta \in \mathcal{C}(\mathsf{D}^2 X, \mathsf{D}X)$ as $\theta := \langle\!\langle \pi_0 \circ \pi_0, \pi_1 \circ \pi_0 + \pi_0 \circ \pi_1 \rangle\!\rangle$.

Proof. The additivity of σ ensures that $\sigma \circ \pi_0 \boxplus \sigma \circ \pi_1$. That is, $(\pi_0 \circ \pi_0 + \pi_1 \circ \pi_0) \boxplus (\pi_0 \circ \pi_1 + \pi_1 \circ \pi_1)$. By associativity, this implies that $((\pi_0 \circ \pi_0 + \pi_1 \circ \pi_0) + \pi_0 \circ \pi_1) + \pi_1 \circ \pi_1$ is well defined, so $(\pi_0 \circ \pi_0 + \pi_1 \circ \pi_0) + \pi_0 \circ \pi_1$ is well defined. By associativity again, $\pi_0 \circ \pi_0 + (\pi_1 \circ \pi_0 + \pi_0 \circ \pi_1)$ is well defined, that is, θ exists. ☐

Definition 13. For any object X, $\langle\!\langle \pi_0, 0 \rangle\!\rangle \boxplus \langle\!\langle 0, \pi_1 \rangle\!\rangle$. We define $\mathbf{l} \in \mathcal{C}(\mathsf{D}X, \mathsf{D}^2 X)$ as $\mathbf{l} := \langle\!\langle \langle\!\langle \pi_0, 0 \rangle\!\rangle, \langle\!\langle 0, \pi_1 \rangle\!\rangle \rangle\!\rangle$.

Proof. By (D-witness), it suffices to prove that $\sigma \circ \langle\!\langle \pi_0, 0 \rangle\!\rangle = \pi_0 + 0 = \pi_0$ and $\sigma \circ \langle\!\langle 0, \pi_1 \rangle\!\rangle = 0 + \pi_1 = \pi_1$ are summable. This is the case by Proposition 1 ☐

Definition 14. For any object X, we can define $\mathbf{c} \in \mathcal{C}(\mathsf{D}^2 X, \mathsf{D}^2 X)$ as $\mathbf{c} := \langle\!\langle \langle\!\langle \pi_0 \circ \pi_0, \pi_0 \circ \pi_1 \rangle\!\rangle, \langle\!\langle \pi_1 \circ \pi_0, \pi_1 \circ \pi_1 \rangle\!\rangle \rangle\!\rangle$.

Proof. By (D-witness), it suffices to show that $\pi_0 \circ \pi_0 + \pi_0 \circ \pi_1 \boxplus \pi_1 \circ \pi_0 + \pi_1 \circ \pi_1$. But $\pi_0 \circ \pi_0 + \pi_0 \circ \pi_1 = \pi_0 \circ (\pi_0 + \pi_1)$ and $\pi_1 \circ \pi_0 + \pi_1 \circ \pi_1 = \pi_1 \circ (\pi_0 + \pi_1)$ by additivity of π_0 and π_1. Since $\pi_0 \boxplus \pi_1$, by left compatibility of addition (Proposition 2) $\pi_0 \circ (\pi_0 + \pi_1) \boxplus \pi_1 \circ (\pi_0 + \pi_1)$ which concludes the proof. ☐

It is probably easier to understand those morphisms by how they operate on witnesses. This corresponds to Proposition 8 below. The proof is a straightforward computation.

Proposition 8. *For any $x, u, v, w \in \mathcal{C}(A, X)$ such that $\langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle$ is defined,*

$$\begin{aligned} \theta \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle &= \langle\!\langle x, u + v \rangle\!\rangle \\ \mathbf{c} \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle &= \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle \\ \mathbf{l} \circ \langle\!\langle x, u \rangle\!\rangle &= \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle, \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle \end{aligned}$$

Definition 15. A *Differential Structure* consists in a Pre-Differential Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ such that the following axioms hold:

(1) (Dproj-lin) $\mathbf{d}^{\mathsf{D}}(\pi_0) = \pi_0 \circ \pi_1$, $\mathbf{d}^{\mathsf{D}}(\pi_1) = \pi_1 \circ \pi_1$;

(2) (Dsum-lin) $\mathbf{d}^{\mathsf{D}}(\sigma) = \sigma \circ \pi_1$, $\mathbf{d}^{\mathsf{D}}(0) = 0$ (additivity of the derivative operator);

(3) (D-chain) D is a functor (Chain Rule);

(4) (D-add) ι_0, θ are natural transformations (additivity of the derivative);

(5) (D-lin) $\mathbf{l}$ is a natural transformation (linearity of the derivatives);

(6) (D-Schwarz) $\mathbf{c}$ is a natural transformation (Schwarz Rule).

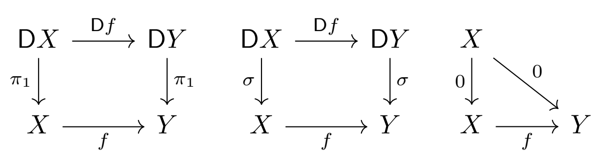
A *Coherent Differential Category* is a category $\mathcal{C}$ equipped with a Differential Structure.

The axiom (Dproj-lin) corresponds to an important structural property of D with regard to the witnesses $\langle\langle \_, \_ \rangle\rangle$. The axiom (Dsum-lin) corresponds to the additivity of the derivative, that is, $(f+g)' = f' + g'$. The axiom (D-chain) corresponds to the Chain Rule of the differential calculus. The axiom (D-add) says that $u \mapsto f'(x) \cdot u$ is additive. The axiom (D-lin) corresponds to the idea that $u \mapsto f'(x) \cdot u$ is not only additive in its second argument, but also linear. This idea is developed in [4] in the left-additive setting of Cartesian Differential Categories (see Section 6 for a comparison between Cartesian Differential Categories and our setting). The same idea can be generalized to our setting, but it would require too much technical development to be covered in this paper. Finally, the axiom (D-Schwarz) corresponds to the Schwarz rule of differential calculus, that is, the second derivative $f''(x)$ (a bilinear map) is symmetric. An account of these axioms as properties of $\mathbf{d}^{\mathsf{D}}$ can be found in Section 3.

2.2 Linearity

For the rest of this section, $\mathcal{C}$ is only assumed to be a category equipped with a Pre-Differential Structure. Any use of an axiom of Coherent Differential Categories will be made explicit.

Definition 16 (D-linearity)**.** A morphism $f \in \mathcal{C}(X, Y)$ is D-*linear* if the following diagrams commute.



Remark 2. The first diagram can also be written as $\mathbf{d}^{\mathsf{D}}(f) = f \circ \pi_1$ and means that $\mathsf{D}f = \langle\langle f \circ \pi_0, f \circ \pi_1 \rangle\rangle$.

Proposition 9. *A morphism f is D-linear if and only if it is additive and $\mathbf{d}^{\mathsf{D}}(f) = f \circ \pi_1$ (that is, $\mathsf{D}f = \langle\langle f \circ \pi_0, f \circ \pi_1 \rangle\rangle$).*

Proof. Assume that f is D-linear. Then $f \circ 0 = 0$ and, by Remark 2, $f \circ \pi_0 \boxplus f \circ \pi_1$ of witness $\mathsf{D}f$. Thus $f \circ \pi_0 + f \circ \pi_1 := \sigma \circ \mathsf{D}f = f \circ \sigma$ by the second diagram. By Proposition 3, f is additive, which concludes the proof since $\mathbf{d}^{\mathsf{D}}(f) = f \circ \pi_1$ is part of the assumptions. Conversely,

$$\begin{aligned}
\sigma \circ \mathsf{D}f &= (\pi_0 + \pi_1) \circ \mathsf{D}f \\
&= \pi_0 \circ \mathsf{D}f + \pi_1 \circ \mathsf{D}f \quad \text{by Proposition 2} \\
&= f \circ \pi_0 + f \circ \pi_1 \quad \text{by assumption} \\
&= f \circ (\pi_0 + \pi_1) = f \circ \sigma \quad \text{by additivity of } f
\end{aligned}$$

Moreover, the first and the third diagrams are in the assumption so f is D-linear. $\square$

Thus D-linear morphisms are in particular additive. As we will see, our notion of additive and D-linear morphisms ultimately coincides with that of [4], so this distinction between additivity and linearity is as relevant as it is in their setting.

Corollary 2. *(Dproj-lin) is equivalent to the linearity of π_0, π_1. (Dsum-lin) is equivalent to the linearity of σ and 0.*

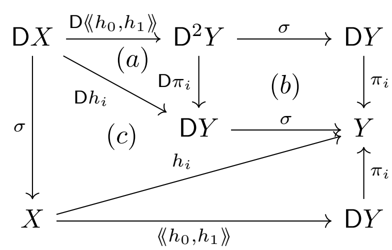
Thanks to (D-chain), (Dproj-lin) and (Dsum-lin), we can show that linear morphisms are closed under composition, witnesses and sum.

Proposition 10. *Assuming (D-chain), the composition of two linear morphisms is linear and the inverse of a linear isomorphism is linear.*

Proof. Easy verification using the functoriality of D. $\square$

Proposition 11 (D-linearity and pairing)**.** *Assume (D-chain) and (Dproj-lin). Assume that $h_0, h_1 \in \mathcal{C}(X, Y)$ are summable and both D-linear. Then $\langle\!\langle h_0, h_1 \rangle\!\rangle$ is D-linear.*

Proof. Let us do the diagram involving σ, the other two being very similar. By joint monicity of the π_i's, it suffices to solve the diagram chase below for $i = 0, 1$.



(a) is a consequence of (D-chain), (b) is a consequence of (Dproj-lin) and (c) is the D-linearity of h_i. $\square$

Proposition 12. *Assuming (D-chain) and (Dproj-lin), σ is D-linear if and only if for all $h_0, h_1 : X \to Y$ summable and both D-linear, $h_0 + h_1$ is D-linear.*

Proof. Assume that h_0, h_1 are D-linear. By Proposition 11, $\langle\!\langle h_0, h_1 \rangle\!\rangle$ is D-linear so $h_0 + h_1 = \sigma \circ \langle\!\langle h_0, h_1 \rangle\!\rangle$ is D-linear (D-linearity is closed under composition). Conversely, $\sigma = \pi_0 + \pi_1$ and π_0, π_1 are D-linear so σ is D-linear. $\square$

Corollary 3. *Assuming (Dproj-lin), (Dsum-lin) and (D-chain), $\iota_0, \iota_1, \mathbf{c}, \mathbf{l}, \theta$ are all D-linear.*

Proof. All these morphisms are obtained through pairing, sums and composition. $\square$

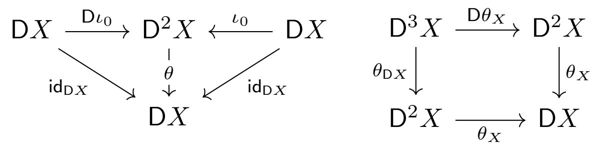
On a side note, by Remark 2 the D-linearity of π_i means that $\mathsf{D}\pi_i = \langle\!\langle \pi_i \circ \pi_0, \pi_i \circ \pi_1 \rangle\!\rangle$. In particular, it implies that $\mathbf{c} = \langle\!\langle \mathsf{D}\pi_0, \mathsf{D}\pi_1 \rangle\!\rangle$. This is very useful because the differential of a pair can then be obtained from the pair of the differentials.

Proposition 13. *Assume (Dproj-lin), (D-chain). Let $f_0, f_1 \in \mathcal{C}(X, Y)$ such that $f_0 \boxplus f_1$. Then $\mathsf{D}f_0 \boxplus \mathsf{D}f_1$ and $\langle\!\langle \mathsf{D}f_0, \mathsf{D}f_1 \rangle\!\rangle = \mathbf{c} \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle$.*

Proof. $\pi_i \circ \mathbf{c} \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle = \mathsf{D}\pi_i \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle = \mathsf{D}f_i$. $\square$

2.3 The Differentiation Monad

Proposition 14. *Assuming (D-chain), (Dproj-lin) and (Dsum-lin), the following diagrams commute.*



Proof. By Corollary 3, ι_0 is D-linear. Thus by Remark 2, $\mathsf{D}\iota_0 = \langle\!\langle \iota_0 \circ \pi_0, \iota_0 \circ \pi_1 \rangle\!\rangle = \langle\!\langle \langle\!\langle \pi_0, 0 \rangle\!\rangle, \langle\!\langle \pi_1, 0 \rangle\!\rangle \rangle\!\rangle$. Hence $\theta \circ \mathsf{D}\iota_0 = \langle\!\langle \pi_0, 0 + \pi_1 \rangle\!\rangle = \langle\!\langle \pi_0, \pi_1 \rangle\!\rangle = \mathsf{id}_{\mathsf{D}X}$ by Proposition 8. Next $\iota_0^{\mathsf{D}X} = \langle\!\langle \langle\!\langle \pi_0, \pi_1 \rangle\!\rangle, \langle\!\langle 0, 0 \rangle\!\rangle \rangle\!\rangle$ since $\langle\!\langle \pi_0, \pi_1 \rangle\!\rangle = \mathsf{id}$ and $\langle\!\langle 0^{X,X}, 0^{X,X} \rangle\!\rangle = 0^{\mathsf{D}X,\mathsf{D}X}$. By Proposition 8 again, $\theta \circ \iota_0 = \langle\!\langle \pi_0, \pi_1 + 0 \rangle\!\rangle = \langle\!\langle \pi_0, \pi_1 \rangle\!\rangle = \mathsf{id}_{\mathsf{D}X}$ so the triangles commute.

The square is a direct computation (we use simple juxtaposition for the composition of projections for the sake of readability). The bottom path can be reduced using left compatibility of addition (Proposition 2) and additivity of the projections:

$$\begin{aligned}\theta \circ \theta &= \langle\!\langle \pi_0\pi_0 \circ \theta, \pi_1\pi_0 \circ \theta + \pi_0\pi_1 \circ \theta \rangle\!\rangle \\ &= \langle\!\langle \pi_0\pi_0\pi_0, \pi_1\pi_0\pi_0 + \pi_0 \circ (\pi_1\pi_0 + \pi_0\pi_1) \rangle\!\rangle \\ &= \langle\!\langle \pi_0\pi_0\pi_0, \pi_1\pi_0\pi_0 + (\pi_0\pi_1\pi_0 + \pi_0\pi_0\pi_1) \rangle\!\rangle \,.\end{aligned}$$

The upper path can be reduced by D-linearity of θ and left compatibility of sum (Proposition 2):

$$\begin{aligned}\theta \circ \mathsf{D}\theta &= \langle\!\langle \pi_0\pi_0 \circ \mathsf{D}\theta, \pi_1\pi_0 \circ \mathsf{D}\theta + \pi_0\pi_1 \circ \mathsf{D}\theta \rangle\!\rangle \\ &= \langle\!\langle \pi_0 \circ \theta \circ \pi_0, \pi_1 \circ \theta \circ \pi_0 + \pi_0 \circ \theta \circ \pi_1 \rangle\!\rangle \\ &= \langle\!\langle \pi_0\pi_0\pi_0, (\pi_1\pi_0 + \pi_0\pi_1) \circ \pi_0 + \pi_0\pi_0\pi_1) \rangle\!\rangle \\ &= \langle\!\langle \pi_0\pi_0\pi_0, (\pi_1\pi_0\pi_0 + \pi_0\pi_1\pi_0) + \pi_0\pi_0\pi_1) \rangle\!\rangle \,.\end{aligned}$$

We conclude that those two morphisms are equal, using the associativity of the partial sum. □

Corollary 4. *(Dproj-lin), (Dsum-lin), (D-chain), (D-add) imply that* $(\mathsf{D}, \iota_0, \theta)$ *is a monad.*

3 Interpreting the axioms as properties of the derivative

In this section, $\mathcal{C}$ is only assumed to be a category equipped with a Pre-Differential Structure (Definition 11). We show that the various axioms of a Coherent Differential Category correspond to standard rules of the differential calculus, written as properties about $\mathbf{d}^{\mathsf{D}}(f)$. The results of this section are only necessary for Section 6 but they also provide some intuitions on the axioms of Coherent Differentiation.

All of the proofs are similar, and consist in using the joint monicity of π_0 and π_1 to reduce the axioms to a set of equations, then show that only one of those equations is non trivial. In what follows, "linear" always means D-linear.

Proposition 15. D *is a functor if and only if* $\mathbf{d}^{\mathsf{D}}(\mathsf{id}) = \pi_1$ *and* $\mathbf{d}^{\mathsf{D}}(g \circ f) = \mathbf{d}^{\mathsf{D}}(g) \circ \langle\!\langle f \circ \pi_0, \mathbf{d}^{\mathsf{D}}(f) \rangle\!\rangle$.

Proof. D is a functor if and only if $\mathsf{D}\mathsf{id}_X = \mathsf{id}_{\mathsf{D}X}$ and for any g, f, $\mathsf{D}(g \circ f) = \mathsf{D}g \circ \mathsf{D}f$. By joint monicity of the π_i, $\mathsf{D}\mathsf{id} = \mathsf{id}$ if and only if $\pi_i \circ \mathsf{D}\mathsf{id} = \pi_i \circ \mathsf{id} = \pi_i$. But $\pi_0 \circ \mathsf{D}\mathsf{id} = \mathsf{id} \circ \pi_0 = \pi_0$ by assumptions on Pre-Differential Structures. So $\mathsf{D}\mathsf{id} = \mathsf{id}$ if and only if $\pi_1 \circ \mathsf{D}\mathsf{id} = \pi_1$, that is, if and only if $\mathbf{d}^{\mathsf{D}}(\mathsf{id}) = \pi_1$. Similarly, $\pi_0 \circ \mathsf{D}g \circ \mathsf{D}f = g \circ \pi_0 \circ \mathsf{D}f = g \circ f \circ \pi_0 = \pi_0 \circ \mathsf{D}(g \circ f)$ by assumption on Pre-Differential-Structures. So by joint monicity of the π_i, $\mathsf{D}(g \circ f) = \mathsf{D}g \circ \mathsf{D}f$ if and only if $\pi_1 \circ \mathsf{D}(g \circ f) = \pi_1 \circ \mathsf{D}g \circ \mathsf{D}f$. By definition of $\mathbf{d}^{\mathsf{D}}$, this corresponds exactly to the equation $\mathbf{d}^{\mathsf{D}}(g \circ f) = \mathbf{d}^{\mathsf{D}}(g) \circ \mathsf{D}f = \mathbf{d}^{\mathsf{D}}(g) \circ \langle\!\langle f \circ \pi_0, \mathbf{d}^{\mathsf{D}}(f) \rangle\!\rangle$ ☐

Proposition 16. *Assuming (Dproj-lin), σ is linear if and only if $\mathsf{D}\sigma = \mathsf{D}\pi_0 + \mathsf{D}\pi_1$. Assuming (Dproj-lin) and (D-chain), σ is linear if and only if for any f_0, f_1 that are summable, $D(f_0 + f_1) = Df_0 + Df_1$ (recall that $\mathsf{D}f_0 \boxplus \mathsf{D}f_1$ by Proposition 13).*

Proof. By linearity of π_i, $\mathsf{D}\pi_i = \langle\!\langle \pi_i \circ \pi_0, \pi_i \circ \pi_1 \rangle\!\rangle$ so by Proposition 4, $\mathsf{D}\pi_0 + \mathsf{D}\pi_1 = \langle\!\langle \pi_0 \circ \pi_0 + \pi_1 \circ \pi_0, \pi_0 \circ \pi_1 + \pi_1 \circ \pi_1 \rangle\!\rangle = \langle\!\langle (\pi_0 + \pi_1) \circ \pi_0, (\pi_0 + \pi_1) \circ \pi_1 \rangle\!\rangle = \langle\!\langle \sigma \circ \pi_0, \sigma \circ \pi_1 \rangle\!\rangle$. But σ is linear if and only if $\mathsf{D}\sigma = \langle\!\langle \sigma \circ \pi_0, \sigma \circ \pi_1 \rangle\!\rangle$ by Proposition 9, that is, if and only if $\mathsf{D}\sigma = \mathsf{D}\pi_0 + \mathsf{D}\pi_1$.

For the second part of the lemma, notice that the right statement for $f_0 = \pi_0$ and $f_1 = \pi_1$ is exactly $\mathsf{D}\sigma = \mathsf{D}\pi_0 + \mathsf{D}\pi_1$, so the converse direction holds. For the forward direction, notice that

$$\begin{aligned}
&\mathsf{D}(f_0 + f_1) \\
&= \mathsf{D}(\sigma \circ \langle\!\langle f_0, f_1 \rangle\!\rangle) \\
&= \mathsf{D}\sigma \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle && \text{by (D-chain)} \\
&= (\mathsf{D}\pi_0 + \mathsf{D}\pi_1) \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle && \text{by assumptions} \\
&= \mathsf{D}\pi_0 \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle + \mathsf{D}\pi_1 \circ \mathsf{D}\langle\!\langle f_0, f_1 \rangle\!\rangle \\
&= \mathsf{D}f_0 + \mathsf{D}f_1 && \text{by (D-chain)}
\end{aligned}$$

☐

Corollary 5. *Assuming (Dproj-lin) and (D-chain), σ is linear if and only if for any f_0, f_1 that are summable, $\mathbf{d}^{\mathsf{D}}(f_0 + f_1) = \mathbf{d}^{\mathsf{D}}(f_0) + \mathbf{d}^{\mathsf{D}}(f_1)$*

Proof. The linearity of σ is equivalent to $\mathsf{D}(f_0 + f_1) = \mathsf{D}f_0 + \mathsf{D}f_1$ for any f_0, f_1 summable. By Proposition 4, this is equivalent to $\langle\!\langle (f_0 + f_1) \circ \pi_0, \mathbf{d}^{\mathsf{D}}(f_0 + f_1) \rangle\!\rangle = \langle\!\langle f_0 \circ \pi_0 + f_1 \circ \pi_0, \mathbf{d}^{\mathsf{D}}(f_0) + \mathbf{d}^{\mathsf{D}}(f_1) \rangle\!\rangle$. The left compatibility of addition (Proposition 2) ensures that the first coordinates are always equal. So σ is linear if and only if for all $f_0 \boxplus f_1$, $\mathbf{d}^{\mathsf{D}}(f_0 + f_1) = \mathbf{d}^{\mathsf{D}}(f_0) + \mathbf{d}^{\mathsf{D}}(f_1)$. ☐

Proposition 17. *The following assertions are equivalent:*

(1) ι_0 is natural;

(2) For any $f \in \mathcal{C}(X, Y)$, $\mathbf{d}^{\mathsf{D}} f \circ \iota_0 = 0$;

(3) For any $f \in \mathcal{C}(X, Y)$, any object A and $x \in \mathcal{C}(A, X)$, $\mathbf{d}^{\mathsf{D}} f \circ \langle\!\langle x, 0 \rangle\!\rangle = 0$.

Proof. (1) ⇔ (2). By joint monicity of the π_i, for any $f \in \mathcal{C}(X, Y)$, $\mathsf{D}f \circ \iota_0 = \iota_0 \circ f$ if and only if $\pi_0 \circ \mathsf{D}f \circ \iota_0 = \pi_0 \circ \iota_0 \circ f = f$ and $\pi_1 \circ \mathsf{D}f \circ \iota_0 = \pi_1 \circ \iota_0 \circ f = 0$. The first condition always hold by naturality of π_0 and definition of ι_0. So ι_0 is natural if and only if the second identity holds. This equation is precisely (2).

(2) ⇔ (3). The forward direction is directly obtained by composing the identity of (2) by x on the right. The reverse is directly obtained by taking applying the equation of (3) to $x = \mathsf{id}_X$. ☐

Proposition 18. *Assuming (Dproj-lin) and (D-chain), the following assertions are equivalent:*

(1) θ *is natural;*

(2) for any $f \in \mathcal{C}(X,Y)$, $\mathbf{d}^{\mathsf{D}} f \circ \mathsf{D}\pi_0 \boxplus \mathbf{d}^{\mathsf{D}} f \circ \pi_0$ *and* $\mathbf{d}^{\mathsf{D}} f \circ \theta = \mathbf{d}^{\mathsf{D}} f \circ \mathsf{D}\pi_0 + \mathbf{d}^{\mathsf{D}} f \circ \pi_0$*;*

(3) for any $f \in \mathcal{C}(X,Y)$*, any object* A *and any* $x, u, v \in \mathcal{C}(A,X)$ *that are summable,* $\mathbf{d}^{\mathsf{D}} f \circ \langle\langle x, u\rangle\rangle \boxplus \mathbf{d}^{\mathsf{D}} f \circ \langle\langle x, v\rangle\rangle$ *and*

$$\mathbf{d}^{\mathsf{D}} f \circ \langle\langle x, u+v\rangle\rangle = \mathbf{d}^{\mathsf{D}} f \circ \langle\langle x, u\rangle\rangle + \mathbf{d}^{\mathsf{D}} f \circ \langle\langle x, v\rangle\rangle\,.$$

Proof. (1) $\Leftrightarrow$ (2). By joint monicity of the π_i, for any $f \in \mathcal{C}(X,Y)$, $\mathsf{D}f \circ \theta = \theta \circ \mathsf{D}^2 f$ if and only if $\pi_0 \circ \mathsf{D}f \circ \theta = \pi_0 \circ \theta \circ \mathsf{D}^2 f$ and $\pi_1 \circ \mathsf{D}f \circ \theta = \pi_1 \circ \theta \circ \mathsf{D}^2 f$. The equation $\pi_0 \circ \mathsf{D}f \circ \theta = \pi_0 \circ \theta \circ \mathsf{D}^2 f$ always holds. Indeed

$$\begin{aligned}
\pi_0 \circ \mathsf{D}f \circ \theta &= f \circ \pi_0 \circ \theta && \text{by naturality of } \pi_0\\
&= f \circ \pi_0 \circ \pi_0 && \text{by definition of } \theta\\
\pi_0 \circ \theta \circ \mathsf{D}^2 f &= \pi_0 \circ \pi_0 \circ \mathsf{D}^2 f && \text{by definition of } \theta\\
&= f \circ \pi_0 \circ \pi_0 && \text{by naturality of } \pi_0
\end{aligned}$$

The left hand side of the equation $\pi_1 \circ \mathsf{D}f \circ \theta = \pi_1 \circ \theta \circ \mathsf{D}^2 f$ is $\mathbf{d}^{\mathsf{D}}(f) \circ \theta$ by definition. The right hand side rewrites as follow.

$$\begin{aligned}
&\pi_1 \circ \theta \circ \mathsf{D}^2 f\\
&= (\pi_0 \circ \pi_1 + \pi_1 \circ \pi_0) \circ \mathsf{D}^2 f\\
&= \pi_0 \circ \pi_1 \circ \mathsf{D}^2 f + \pi_1 \circ \pi_0 \circ \mathsf{D}^2 f && \text{by Proposition 2}\\
&= \pi_1 \circ \mathsf{D}\pi_0 \circ \mathsf{D}^2 f + \pi_1 \circ \pi_0 \circ \mathsf{D}^2 f && \text{by D-linearity of } \pi_0\\
&= \pi_1 \circ \mathsf{D}(\pi_0 \circ \mathsf{D}f) + \pi_1 \circ \pi_0 \circ \mathsf{D}^2 f && \text{by functoriality of D}\\
&= \pi_1 \circ \mathsf{D}(f \circ \pi_0) + \pi_1 \circ \mathsf{D}f \circ \pi_0 && \text{by naturality of } \pi_0\\
&= \pi_1 \circ \mathsf{D}f \circ \mathsf{D}\pi_0 + \pi_1 \circ \mathsf{D}f \circ \pi_0 && \text{by functoriality of D}
\end{aligned}$$

So this second equation under consideration is equivalent to the equation of (2).

(2) $\Leftrightarrow$ (3). The forward direction is directly obtained by composing the equation of (2) with $\langle\langle\langle\langle x, v\rangle\rangle, \langle\langle u, 0\rangle\rangle\rangle\rangle$ on the right. The converse is directly obtained by applying the equation of (3) to $x = \pi_0 \circ \pi_0$, $u = \pi_1 \circ \pi_0$ and $v = \pi_0 \circ \pi_1$. Indeed, $\pi_0 = \langle\langle \pi_0, \pi_1\rangle\rangle \circ \pi_0 = \langle\langle \pi_0 \circ \pi_0, \pi_1 \circ \pi_0\rangle\rangle$ and, by D-linearity of π_0, we have $\mathsf{D}\pi_0 = \langle\langle \pi_0 \circ \pi_0, \pi_0 \circ \pi_1\rangle\rangle$. ☐

Remark 3. Notice that $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) = \pi_1 \circ \mathsf{D}(\pi_1 \circ \mathsf{D}f) = \pi_1 \circ \mathsf{D}\pi_1 \circ \mathsf{D}^2 f = \pi_1 \circ \pi_1 \circ \mathsf{D}^2 f$ assuming (D-chain) and (Dproj-lin). Thus, $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f))$ is nothing more than the last coordinate of $\mathsf{D}^2 f$.

Proposition 19. *Assuming (Dproj-lin), (D-chain) and the naturality of* ι_0*, the following assertions are equivalent:*

(1) $\mathbf{l}$ *is natural;*

(2) for all morphism $f \in \mathcal{C}(Y,Z)$ $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \mathbf{l} = \mathbf{d}^{\mathsf{D}}(f)$*;*

(3) for all morphism $f \in \mathcal{C}(Y,Z)$*, for all morphisms* $x, u \in \mathcal{C}(X,Y)$*,*

$$\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \langle\langle\langle\langle x, 0\rangle\rangle, \langle\langle 0, u\rangle\rangle\rangle\rangle = \mathbf{d}^{\mathsf{D}}(f) \circ \langle\langle x, u\rangle\rangle\,.$$

Proof. By joint monicity of the π_i, $\mathbf{l}$ is natural if and only if for all f and for all $i,j \in \{0,1\}, \pi_i \circ \pi_j \circ \mathsf{D}^2 f \circ \mathbf{l} = \pi_i \circ \pi_j \circ \mathbf{l} \circ \mathsf{D} f$. By Remark 3 (and because $\pi_1 \circ \pi_1 \circ \mathbf{l} = \pi_1$), the equation for $i = j = 1$ corresponds exactly to the equation $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \mathbf{l} = \mathbf{d}^{\mathsf{D}}(f)$. Thus, it suffices to show that $\pi_i \circ \pi_j \circ \mathsf{D}^2 f \circ \mathbf{l} = \pi_i \circ \pi_j \circ \mathbf{l} \circ \mathsf{D} f$ always holds when $(i,j) \neq (1,1)$ to conclude that (1) is equivalent to (2).

- Case $i = 0, j = 0$: $\pi_0 \circ \pi_0 \circ \mathbf{l} \circ \mathsf{D} f = \pi_0 \circ \mathsf{D} f = f \circ \pi_0$ and $\pi_0 \circ \pi_0 \circ \mathsf{D}^2 f \circ \mathbf{l} = f \circ \pi_0 \circ \pi_0 \circ \mathbf{l} = f \circ \pi_0$;
- Case $i = 1, j = 0$: $\pi_0 \circ \pi_1 \circ \mathbf{l} \circ \mathsf{D} f = 0 \circ \mathsf{D} f = 0$ and $\pi_1 \circ \pi_0 \circ \mathsf{D}^2 f \circ \mathbf{l} = \pi_1 \circ \mathsf{D} f \circ \pi_0 \circ \mathbf{l} = \pi_1 \circ \mathsf{D} f \circ \iota_0 \circ \pi_0 = \pi_1 \circ \iota_0 \circ f \circ \pi_0 = 0$ thanks to the naturality of ι_0;
- Case $i = 0, j = 1$: $\pi_0 \circ \pi_1 \circ \mathbf{l} \circ \mathsf{D} f = 0 \circ \mathsf{D} f = 0$ and $\pi_0 \circ \pi_1 \circ \mathsf{D}^2 f \circ \mathbf{l} = \pi_1 \circ \mathsf{D}\pi_0 \circ \mathsf{D}^2 f \circ \mathbf{l} = \pi_1 \circ \mathsf{D} f \circ \mathsf{D}\pi_0 \circ \mathbf{l} = \pi_1 \circ \mathsf{D} f \circ \iota_0 \circ \pi_0 = \pi_1 \circ \iota_0 \circ f \circ \pi_0 = 0$ thanks to the naturality of ι_0.

Thus (1) and (2) are equivalent.

Next (2) is a particular case of (3) for $x = \pi_0$ and $u = \pi_1$. Conversely, assuming (2) we have that $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \langle\!\langle \langle\!\langle x, 0 \rangle\!\rangle, \langle\!\langle 0, u \rangle\!\rangle \rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \mathbf{l} \circ \langle\!\langle x, u \rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(f) \circ \langle\!\langle x, u \rangle\!\rangle$. □

Proposition 20. *Assuming (Dproj-lin) and (D-chain), the following assertions are equivalent:*

(1) $\mathbf{c}$ *is natural;*

(2) for all morphism $f \in \mathcal{C}(Y,Z)$*,* $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \mathbf{c} = \mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f))$*;*

(3) for all morphism $f \in \mathcal{C}(Y,Z)$ *and* $x, u, v, w \in \mathcal{C}(X,Y)$ *that are summable,*

$$\begin{aligned}&\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle \\ &= \mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle\end{aligned}$$

Proof. By joint monicity of the π_i, $\mathbf{c}$ is natural if and only if for all f and for all $i,j \in \{0,1\}, \pi_i \circ \pi_j \circ \mathsf{D}^2 f \circ \mathbf{c} = \pi_i \circ \pi_j \circ \mathbf{c} \circ \mathsf{D}^2 f$. But $\pi_i \circ \pi_j \circ \mathbf{c} \circ \mathsf{D}^2 f = \pi_j \circ \pi_i \circ \mathsf{D}^2 f$. Then, by Remark 3, the equation for $i = j = 1$ corresponds exactly to the equation $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \mathbf{c} = \mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f))$. Thus, it suffices to show that $\pi_i \circ \pi_j \circ \mathsf{D}^2 f \circ \mathbf{c} = \pi_i \circ \pi_j \circ \mathbf{c} \circ \mathsf{D}^2 f$ when $(i,j) \neq (1,1)$ to conclude that (1) is equivalent to (2).

- $i = 0, j = 0$: $\pi_0 \circ \pi_0 \circ \mathbf{c} \circ \mathsf{D}^2 f = \pi_0 \circ \pi_0 \circ \mathsf{D}^2 f = f \circ \pi_0 \circ \pi_0$ and $\pi_0 \circ \pi_0 \circ \mathsf{D}^2 f \circ \mathbf{c} = f \circ \pi_0 \circ \pi_0 \circ \mathbf{c} = f \circ \pi_0 \circ \pi_0$
- $i = 1, j = 0$: $\pi_1 \circ \pi_0 \circ \mathbf{c} \circ \mathsf{D}^2 f = \pi_0 \circ \pi_1 \circ \mathsf{D}^2 f = \pi_1 \circ \mathsf{D}\pi_0 \circ \mathsf{D}^2 f = \pi_1 \circ \mathsf{D} f \circ \mathsf{D}\pi_0$ and $\pi_1 \circ \pi_0 \circ \mathsf{D}^2 f \circ \mathbf{c} = \pi_1 \circ \mathsf{D} f \circ \pi_0 \circ \mathbf{c} = \pi_1 \circ \mathsf{D} f \circ \mathsf{D}\pi_0$
- $i = 0, j = 1$: $\pi_0 \circ \pi_1 \circ \mathbf{c} \circ \mathsf{D}^2 f = \pi_1 \circ \pi_0 \circ \mathsf{D}^2 f = \pi_1 \circ \mathsf{D} f \circ \pi_0$ and $\pi_0 \circ \pi_1 \circ \mathsf{D}^2 f \circ \mathbf{c} = \pi_1 \circ \mathsf{D}\pi_0 \circ \mathsf{D}^2 f \circ \mathbf{c} = \pi_1 \circ \mathsf{D} f \circ \mathsf{D}\pi_0 \circ \mathbf{c} = \pi_1 \circ \mathsf{D} f \circ \pi_0$.

Thus (1) and (2) are equivalent.

Next, (2) is a particular case of (3) for $x = \pi_0 \circ \pi_0$, $u = \pi_0 \circ \pi_1$, $v = \pi_1 \circ \pi_0$ and $w = \pi_1 \circ \pi_1$. Conversely, if (3) holds then $\mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \mathbf{c} \circ \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(\mathbf{d}^{\mathsf{D}}(f)) \circ \langle\!\langle \langle\!\langle x, v \rangle\!\rangle, \langle\!\langle u, w \rangle\!\rangle \rangle\!\rangle$. □

4 Compatibility with the cartesian product

We assume in this section that $\mathcal{C}$ is cartesian. We also assume that $\mathcal{C}$ is equipped with a Left Summability Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$.

Notations 2. We use & for the cartesian product, following the notations of LL. For any objects Y_0, Y_1, the projection will be written as $\mathbf{p}_i \in \mathcal{C}(Y_0 \,\&\, Y_1, Y_i)$ and the pairing of $f_0 \in \mathcal{C}(X, Y_0)$ and $f_1 \in \mathcal{C}(X, Y_1)$ as $\langle f_0, f_1 \rangle$[7]. Finally, the terminal object will be written $\top$ and we write $\mathbf{t}_X$ the unique morphism of $\mathcal{C}(X, \top)$. Note that the uniqueness of the pairing in the universal property of the cartesian product can be understood as the joint monicity of the $\mathbf{p}_i$.

4.1 Cartesian product and Summability Structure

Definition 17. The Summability Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ is *compatible with the cartesian product* if $\langle 0, 0 \rangle = 0$ and if for all $f_0, g_0 \in \mathcal{C}(X, Y_0)$ and $f_1, g_1 \in \mathcal{C}(X, Y_1)$, $\langle f_0, f_1 \rangle \boxplus \langle g_0, g_1 \rangle$ if and only if $f_0 \boxplus g_0$ and $f_1 \boxplus g_1$ and

$$\langle f_0, f_1 \rangle + \langle g_0, g_1 \rangle = \langle f_0 + f_1, g_0 + g_1 \rangle \,.$$

That is, if the sum on pairs is the coordinate-wise sum. Let us break down this definition in more details.

Proposition 21. *The following are equivalent*

- $\mathbf{p}_0, \mathbf{p}_1$ *are additive;*
- $\langle 0, 0 \rangle = 0$ *and for all* $f_0, g_0 \in \mathcal{C}(X, Y_0)$ *and* $f_1, g_1 \in \mathcal{C}(X, Y_1)$*, if* $\langle f_0, f_1 \rangle \boxplus \langle g_0, g_1 \rangle$ *then* $f_0 \boxplus g_0$*,* $f_1 \boxplus g_1$ *and* $\langle f_0, f_1 \rangle + \langle g_0, g_1 \rangle = \langle f_0 + f_1, g_0 + g_1 \rangle$*.*

Proof. Assume that $\mathbf{p}_0, \mathbf{p}_1$ are additive. Then $\mathbf{p}_i \circ 0 = 0 = \mathbf{p}_i \circ \langle 0, 0 \rangle$. Thus by joint monicity, $0 = \langle 0, 0 \rangle$. Furthermore, assume that $\langle f_0, f_1 \rangle \boxplus \langle g_0, g_1 \rangle$. Then by additivity of $\mathbf{p}_i$, $\mathbf{p}_i \circ \langle f_0, f_1 \rangle = f_i$ and $\mathbf{p}_i \circ \langle g_0, g_1 \rangle = g_i$ are summable and $f_i + g_i = \mathbf{p}_i \circ (\langle f_0, f_1 \rangle + \langle g_0, g_1 \rangle)$. So the joint monicity of the $\mathbf{p}_i$ implies that $\langle f_0, f_1 \rangle + \langle g_0, g_1 \rangle = \langle f_0 + f_1, g_0 + g_1 \rangle$.

Conversely, since $\langle 0, 0 \rangle = 0$ we have $\mathbf{p}_i \circ 0 = \mathbf{p}_i \circ \langle 0, 0 \rangle = 0$. Let $f, g \in \mathcal{C}(X, Y_0 \,\&\, Y_1)$ that are summable. One can write $f = \langle \mathbf{p}_0 \circ f, \mathbf{p}_1 \circ f \rangle$ and $g = \langle \mathbf{p}_0 \circ g, \mathbf{p}_1 \circ g \rangle$. Since $f \boxplus g$ we have $\mathbf{p}_i \circ f \boxplus \mathbf{p}_i \circ g$ and $f + g = \langle \mathbf{p}_0 \circ f + \mathbf{p}_0 \circ g, \mathbf{p}_1 \circ f + \mathbf{p}_1 \circ g \rangle$. Applying $\mathbf{p}_i$ on this equation yields that $\mathbf{p}_i \circ (f + g) = \mathbf{p}_i \circ f + \mathbf{p}_i \circ g$ so $\mathbf{p}_i$ is additive. □

Corollary 6. *If* $\mathbf{p}_0$ *and* $\mathbf{p}_1$ *are additive, then for all* $f_0, g_0 \in \mathcal{C}(X_0, Y_0)$ *and* $f_1, g_1 \in \mathcal{C}(X_1, Y_1)$*, if* $f_0 \& f_1 \boxplus g_0 \& g_1$ *then* $f_0 \boxplus g_0$*,* $f_1 \boxplus g_1$ *and* $f_0 \& f_1 + g_0 \& g_1 = (f_0 + g_0) \& (f_1 + g_1)$*.*

Proof. We simply use the fact that $f \,\&\, g = \langle f \circ \mathbf{p}_0, g \circ \mathbf{p}_1 \rangle$ and Proposition 21 together with the left compatibility of sum with regard to composition (Proposition 2). □

We now assume that the projections $\mathbf{p}_0$ and $\mathbf{p}_1$ are additive. This allows to define a morphism $\mathbf{c}_\& \in \mathcal{C}(\mathsf{D}(X_0 \,\&\, X_1), \mathsf{D}X_0 \,\&\, \mathsf{D}X_1)$ for any objects X_0, X_1 as $\mathbf{c}_\& := \langle \langle\!\langle \mathbf{p}_0 \circ \pi_0, \mathbf{p}_0 \circ \pi_1 \rangle\!\rangle, \langle\!\langle \mathbf{p}_1 \circ \pi_0, \mathbf{p}_1 \circ \pi_1 \rangle\!\rangle \rangle$. In other words, $\pi_i \circ \mathbf{p}_j \circ \mathbf{c}_\& = \mathbf{p}_j \circ \pi_i$ and

$$\mathbf{c}_\& \circ \langle\!\langle \langle f_0, f_1 \rangle, \langle g_0, g_1 \rangle \rangle\!\rangle = \langle \langle\!\langle f_0, g_0 \rangle\!\rangle, \langle\!\langle f_1, g_1 \rangle\!\rangle \rangle \,.$$

This is very reminiscent of the flip $\mathbf{c}$ (it swaps the two middle coordinates), except that there are no summability conditions associated with the $\langle \_, \_ \rangle$ pairing.

[7] We start the indices by 0 in order to stay consistant with the notations of π_0 and π_1

Theorem 4. *The following assertions are equivalent*

(1) $\mathbf{c}_\&$ *is an isomorphism;*

(2) $\pi_0 \mathbin{\&} \pi_0 \boxplus \pi_1 \mathbin{\&} \pi_1$*;*

(3) for any $f_0, g_0 \in \mathcal{C}(X, Y_0)$*,* $f_1, g_1 \in \mathcal{C}(X, Y_1)$*, if* $f_0 \boxplus g_0$ *and* $f_1 \boxplus g_1$ *then* $f_0 \mathbin{\&} f_1 \boxplus g_0 \mathbin{\&} g_1$*;*

(4) for any $f_0, g_0 \in \mathcal{C}(X, Y_0)$*,* $f_1, g_1 \in \mathcal{C}(X, Y_1)$*, if* $f_0 \boxplus g_0$ *and* $f_1 \boxplus g_1$ *then* $\langle f_0, f_1\rangle \boxplus \langle g_0, g_1\rangle$

and then $\langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} \pi_1 \rangle\!\rangle = \mathbf{c}_\&^{-1}$.

Proof. (1) ⇒ (2): Assume that $\mathbf{c}_\&$ is an isomorphism, of inverse w. Then $\pi_i \circ \mathbf{p}_j = \pi_i \circ \mathbf{p}_j \circ \mathbf{c}_\& \circ w = \mathbf{p}_j \circ \pi_i \circ w$. But $\mathbf{p}_j \circ (\pi_i \mathbin{\&} \pi_i) = \pi_i \circ \mathbf{p}_j$ by naturality of $\mathbf{p}_j$ so $\mathbf{p}_j \circ \pi_i \circ w = \mathbf{p}_j \circ (\pi_i \mathbin{\&} \pi_i)$. By joint monicity of the $\mathbf{p}_j$'s we have $\pi_i \circ w = (\pi_i \mathbin{\&} \pi_i)$. That is $w = \langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} \pi_1 \rangle\!\rangle$.

(2) ⇒ (1): Assume that $\pi_0 \mathbin{\&} \pi_0 \boxplus \pi_1 \mathbin{\&} \pi_1$, of witness w. Then, $\mathbf{p}_j \circ \pi_i \circ w = \mathbf{p}_j \circ (\pi_i \mathbin{\&} \pi_i) = \pi_i \circ \mathbf{p}_j$. Hence

$$\mathbf{p}_j \circ \pi_i \circ w \circ \mathbf{c}_\& = \pi_i \circ \mathbf{p}_j \circ \mathbf{c}_\& = \mathbf{p}_j \circ \pi_i$$
$$\pi_i \circ \mathbf{p}_j \circ \mathbf{c}_\& \circ w = \mathbf{p}_j \circ \pi_i \circ w = \pi_i \circ \mathbf{p}_j$$

By joint monicity of the $\mathbf{p}_j$'s and of the π_i's we get $w \circ \mathbf{c}_\& = \mathsf{id}_{\mathsf{D}(X_0 \& X_1)}$ and $\mathbf{c}_\& \circ w = \mathsf{id}_{\mathsf{D}X_0 \& \mathsf{D}X_1}$.

(2) ⇒ (3): We have $\langle\!\langle f_0, g_0 \rangle\!\rangle \in \mathcal{C}(X, \mathsf{D}Y_0)$ and $\langle\!\langle f_1, g_1 \rangle\!\rangle \in \mathcal{C}(X, \mathsf{D}Y_1)$. Let $w = \langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} \pi_1 \rangle\!\rangle \circ (\langle\!\langle f_0, g_0 \rangle\!\rangle \mathbin{\&} \langle\!\langle f_1, g_1 \rangle\!\rangle)$. We have $\pi_0 \circ w = f_0 \mathbin{\&} f_1$ and $\pi_1 \circ w = g_0 \mathbin{\&} g_1$ so that $f_0 \mathbin{\&} f_1 \boxplus g_0 \mathbin{\&} g_1$.

(3) ⇒ (2): (2) is a particular case of case (3).

(3) ⇒ (4): Assume that $f_0 \boxplus g_0$ and $f_1 \boxplus g_1$. Then by assumption, $f_0 \mathbin{\&} f_1 \boxplus g_0 \mathbin{\&} g_1$. Let $w = \langle\!\langle f_0 \mathbin{\&} f_1, g_0 \mathbin{\&} g_1 \rangle\!\rangle \circ \langle \mathsf{id}, \mathsf{id} \rangle$. Then $\pi_0 \circ w = \langle f_0, f_1 \rangle$ and $\pi_1 \circ w = \langle g_0, g_1 \rangle$ so that $\langle f_0, f_1 \rangle \boxplus \langle g_0, g_1 \rangle$.

(4) ⇒ (3): Assume that $f_0 \boxplus g_0$ and $f_1 \boxplus g_1$. Then $f_0 \circ \mathbf{p}_0 \boxplus g_0 \circ \mathbf{p}_0$ and $f_1 \circ \mathbf{p}_1 \boxplus g_1 \circ \mathbf{p}_1$ by left compatibility wrt. composition (Proposition 2). Hence, by assumption, $\langle f_0 \circ \mathbf{p}_0, f_1 \circ \mathbf{p}_1 \rangle \boxplus \langle g_0 \circ \mathbf{p}_0, g_1 \circ \mathbf{p}_1 \rangle$. That is $f_0 \mathbin{\&} f_1 \boxplus g_0 \mathbin{\&} g_1$. □

Corollary 7. *A Summability Structure is compatible with the cartesian product if and only if* $\mathbf{p}_0, \mathbf{p}_1$ *are additive and* $\mathbf{c}_\&$ *is an isomorphism.*

4.2 Cartesian product and Differential Structure

We now assume that $\mathcal{C}$ is a cartesian category with a Pre-Differential Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$.

Definition 18. The (Pre-)Differential Structure $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ is *compatible with the cartesian product* if the underlying Summability Structure is compatible with the cartesian product, and if $\mathbf{p}_0, \mathbf{p}_1$ are D-linear. A *Cartesian Coherent Differential Category* (CCDC) is a Coherent Differential Category with a cartesian product which is compatible with the Differential Structure.

We assume that $\mathcal{C}$ is a CCDC. By D-linearity of $\mathbf{p}_0$ and $\mathbf{p}_1$, all constructions involving the cartesian product are D-linear.

Proposition 22. *If $h_0 \in \mathcal{C}(X, Y_0)$ and $h_1 \in \mathcal{C}(X, Y_1)$ are D-linear, then $\langle h_0, h_1\rangle$ is D-linear. If $f_0 \in \mathcal{C}(X_0, Y_0)$ and $f_1 \in \mathcal{C}(X_1, Y_1)$ are D-linear, then $f_0 \mathbin{\&} f_1$ is D-linear.*

Proof. For the first statement we proceed as for Proposition 11 except that the paring as a summable pair is replaced by the pairing of the cartesian product. The second statement follows from the first one, because $f_0 \mathbin{\&} f_1 = \langle f_0 \circ \mathbf{p}_0, f_1 \circ \mathbf{p}_1\rangle$, the projections are D-linear, and D-linearity is closed under composition. □

For any objects X_0, X_1, there is a natural transformation $\langle \mathsf{D}\mathbf{p}_0, \mathsf{D}\mathbf{p}_1\rangle \in \mathcal{C}(\mathsf{D}(X_0 \mathbin{\&} X_1), \mathsf{D}X_0 \mathbin{\&} \mathsf{D}X_1)$. By D-linearity of $\mathbf{p}_0$ and $\mathbf{p}_1$ this natural transformation is equal to $\langle\langle\!\langle \mathbf{p}_0 \circ \pi_0, \mathbf{p}_0 \circ \pi_1\rangle\!\rangle, \langle\!\langle \mathbf{p}_1 \circ \pi_0, \mathbf{p}_1 \circ \pi_1\rangle\!\rangle\rangle = \mathbf{c}_\&$. It yields a result similar to Proposition 13.

Proposition 23. *For any $f_0 \in \mathcal{C}(X, Y_0)$ and $f_1 \in \mathcal{C}(X, Y_1)$, $\langle \mathsf{D}f_0, \mathsf{D}f_1\rangle = \mathbf{c}_\& \circ \mathsf{D}\langle f_0, f_1\rangle$*

Proof. $\mathbf{p}_i \circ \mathbf{c}_\& \circ \mathsf{D}\langle f_0, f_1\rangle = \mathsf{D}\mathbf{p}_i \circ \mathsf{D}\langle f_0, f_1\rangle = \mathsf{D}f_i$. □

4.3 Partial derivatives

One can define from $\mathbf{c}_\&^{-1}$ two natural transformations

$$\begin{aligned}\Phi^0 &= (\mathbf{c}_\&)^{-1} \circ (\mathsf{id}_{\mathsf{D}X_0} \mathbin{\&} \iota_0) \in \mathcal{C}(\mathsf{D}X_0 \mathbin{\&} X_1, \mathsf{D}(X_0 \mathbin{\&} X_1))\\ \Phi^1 &= (\mathbf{c}_\&)^{-1} \circ (\iota_0 \mathbin{\&} \mathsf{id}_{\mathsf{D}X_1}) \in \mathcal{C}(X_0 \mathbin{\&} \mathsf{D}X_1, \mathsf{D}(X_0 \mathbin{\&} X_1))\end{aligned}$$

Note that $\mathbf{c}_\&$, $(\mathbf{c}_\&)^{-1}$, Φ^0 and Φ^1 are all D-linear, thanks to Propositions 10 and 22 and Corollary 3.

Proposition 24. $\Phi^0 = \langle\!\langle \pi_0 \mathbin{\&} \mathsf{id}_{X_1}, \pi_1 \mathbin{\&} 0\rangle\!\rangle$ *and* $\Phi^1 = \langle\!\langle \mathsf{id}_{X_0} \mathbin{\&} \pi_0, 0 \mathbin{\&} \pi_1\rangle\!\rangle$

Proof. By Theorem 4, $(\mathbf{c}_\&)^{-1} = \langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} \pi_1\rangle\!\rangle$. and the result follows by a straightforward computation. □

Definition 19 (Partial derivative)**.** If $f \in \mathcal{C}(X_0 \mathbin{\&} X_1, Y)$ one can define $\mathsf{D}_0 f := \mathsf{D}f \circ \Phi^0 \in \mathcal{C}(\mathsf{D}X_0 \mathbin{\&} X_1, \mathsf{D}Y)$ and $\mathsf{D}_1 f := \mathsf{D}f \circ \Phi^1 \in \mathcal{C}(X_0 \mathbin{\&} \mathsf{D}X_1, Y)$, the *partial derivatives* of f.

Proposition 25. *For any $f \in \mathcal{C}(X_0 \mathbin{\&} X_1, Y)$, $\pi_0 \circ \mathsf{D}_0 f = f \circ (\pi_0 \mathbin{\&} \mathsf{id})$ and $\pi_0 \circ \mathsf{D}_1 f = f \circ (\mathsf{id} \mathbin{\&} \pi_0)$.*

Proof. $\pi_0 \circ \mathsf{D}_0 f = \pi_0 \circ \mathsf{D}f \circ \Phi^0 = f \circ \pi_0 \circ \Phi^0 = f \circ (\pi_0 \mathbin{\&} \mathsf{id})$ by Proposition 24. The proof for Φ^1 is similar. □

Proposition 26. *The following diagram commutes.*

$$\begin{array}{ccccc}\mathsf{D}(X_0 \mathbin{\&} \mathsf{D}X_1) & \xleftarrow{\Phi^0} & \mathsf{D}X_0 \mathbin{\&} \mathsf{D}X_1 & \xrightarrow{\Phi^1} & \mathsf{D}(\mathsf{D}X_0 \mathbin{\&} X_1)\\ {\scriptstyle \mathsf{D}\Phi^1}\downarrow & & & & \downarrow{\scriptstyle \mathsf{D}\Phi^0}\\ \mathsf{D}^2(X_0 \mathbin{\&} X_1) & & \xrightarrow{\quad\quad\mathbf{c}\quad\quad} & & \mathsf{D}^2(X_0 \mathbin{\&} X_1)\end{array}$$

Proof. We use Proposition 24 to compute $\mathsf{D}\Phi^1 \circ \Phi^0$ and $\mathsf{D}\Phi^0 \circ \Phi^1$. Since Φ^0 is D-linear, $\mathsf{D}\Phi^0 = \langle\!\langle \Phi^0 \circ \pi_0, \Phi^0 \circ \pi_1\rangle\!\rangle$ by Remark 2. Thus

$$\begin{aligned}\mathsf{D}\Phi^0 \circ \Phi^1 &= \mathsf{D}\Phi^0 \circ \langle\!\langle \mathsf{id}_{X_0} \mathbin{\&} \pi_0, 0 \mathbin{\&} \pi_1\rangle\!\rangle\\ &= \langle\!\langle \Phi^0 \circ (\mathsf{id}_{X_0} \mathbin{\&} \pi_0), \Phi^0 \circ (0 \mathbin{\&} \pi_1)\rangle\!\rangle\\ &= \langle\!\langle \langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} 0\rangle\!\rangle, \langle\!\langle 0 \mathbin{\&} \pi_1, 0 \mathbin{\&} 0\rangle\!\rangle\rangle\!\rangle\end{aligned}$$

Similarly, $\mathsf{D}\Phi^1 \circ \Phi^0 = \langle\!\langle \langle\!\langle \pi_0 \mathbin{\&} \pi_0, 0 \mathbin{\&} \pi_1\rangle\!\rangle, \langle\!\langle \pi_1 \mathbin{\&} 0, 0 \mathbin{\&} 0\rangle\!\rangle\rangle\!\rangle$. The commutation results from Proposition 8. □

Proposition 27. *The following diagram commutes*

$$\begin{array}{ccccc}
\mathsf{D}(X_0 \mathbin{\&} \mathsf{D}X_1) & \xleftarrow{\Phi^0} & \mathsf{D}X_0 \mathbin{\&} \mathsf{D}X_1 & \xrightarrow{\Phi^1} & \mathsf{D}(\mathsf{D}X_0 \mathbin{\&} X_1) \\
{\scriptstyle \mathsf{D}\Phi^1}\downarrow & & \downarrow{\scriptstyle \mathbf{c}_{\&}^{-1}} & & \downarrow{\scriptstyle \mathsf{D}\Phi^0} \\
\mathsf{D}^2(X_0 \mathbin{\&} X_1) & \xrightarrow{\theta} & \mathsf{D}(X_0 \mathbin{\&} X_1) & \xleftarrow{\theta} & \mathsf{D}^2(X_0 \mathbin{\&} X_1)
\end{array}$$

Proof. Thanks to the computation of $\mathsf{D}\Phi^0 \circ \Phi^1$ in the proof of Proposition 26, we know that $\theta \circ \mathsf{D}\Phi^0 \circ \Phi^1 = \langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} 0 + 0 \mathbin{\&} \pi_1 \rangle\!\rangle = \langle\!\langle \pi_0 \mathbin{\&} \pi_0, \pi_1 \mathbin{\&} \pi_1 \rangle\!\rangle$ by Corollary 6. So $\theta \circ \mathsf{D}\Phi^0 \circ \Phi^1 = (\mathbf{c}_{\&})^{-1}$ by Theorem 4. A similar computation yields the result for $\theta \circ \mathsf{D}\Phi^1 \circ \Phi^0$. ☐

Remark 4. We can check that the natural morphisms Φ^0, Φ^1 are a *strength* [8, 9] for the monad $(\mathsf{D}, \iota_0, \theta)$. Then the diagram of Proposition 27 means that this monad is a *commutative monad*. The diagrams can be checked by hand, but are also a consequence of very generic properties about strong monads on cartesian categories.

As mentioned in [10] in paragraph 2.3, any monad (M, η, μ) on a cartesian category can be endowed with the Structure of *a colax symmetric monoidal monad*[8] taking

- $\mathsf{n}^0 := \mathbf{t}_{\mathsf{M}\top} \in \mathcal{C}(\mathsf{M}\top, \top)$
- $\mathsf{n}^2_{X_1,X_2} := \langle \mathsf{M}\mathbf{p}_1, \mathsf{M}\mathbf{p}_2 \rangle \in \mathcal{C}(\mathsf{M}(X_1 \mathbin{\&} X_2), \mathsf{M}X_1 \mathbin{\&} \mathsf{M}X_2)$

When $\mathsf{M} = \mathsf{D}$, $\mathsf{n}^2 = \mathbf{c}_{\&}$. If n^2 and n^0 are isos, M becomes a *(strong) symmetric monoidal monad*. This is what happens here, as $\mathbf{c}_{\&}$ is a natural isomorphism and we can show that n^0 is an isomorphism of inverse ι_0 using the join monicity of the π_i. But symmetric monoidal monad are the same as commutative monads as shown in [8, 12], and it turns out that the strengths induced from the symmetric monoidal structure are exactly Φ^0 and Φ^1.

The axioms (D-Schwarz) and (D-add) carry to the setting of partial derivatives very naturally thanks to Propositions 26 and 27 respectively, giving the full fledged Schwarz and Leibniz rules. The fact that the Leibniz rule is a consequence of the additivity of the derivative is not surprising, as it is also the case in the usual differential calculus: $f'(x,y) \cdot (u,v) = f'(x,y) \cdot (u,0) + f'(x,y) \cdot (0,v) = \frac{\partial f}{\partial x}(x,y) \cdot u + \frac{\partial f}{\partial y}(x,y) \cdot v$.

Proposition 28 (Leibniz rule)**.** $\mathsf{D}f \circ \mathbf{c}_{\&}^{-1} = \theta \circ \mathsf{D}_0\mathsf{D}_1 f = \theta \circ \mathsf{D}_1\mathsf{D}_0 f$

Proof. Let us prove that $\mathsf{D}f \circ \mathbf{c}_{\&}^{-1} = \theta \circ \mathsf{D}_0\mathsf{D}_1 f$.

$$\begin{aligned}
\theta \circ \mathsf{D}_0\mathsf{D}_1 f &= \theta \circ \mathsf{D}(\mathsf{D}f \circ \Phi^1) \circ \Phi^0 \quad \text{by definition} \\
&= \theta \circ \mathsf{D}^2 f \circ \mathsf{D}\Phi^1 \circ \Phi^0 \quad \text{by (D-chain)} \\
&= \mathsf{D}f \circ \theta \circ \mathsf{D}\Phi^0 \circ \Phi^0 \quad \text{by (D-add)} \\
&= \mathsf{D}f \circ \mathbf{c}_{\&}^{-1} \quad \text{by Proposition 27}
\end{aligned}$$

☐

The proof of $\mathsf{D}f \circ \mathbf{c}_{\&}^{-1} = \theta \circ \mathsf{D}_1\mathsf{D}_0 f$ is similar.

Proposition 29 (Schwarz rule)**.** $\mathsf{D}_0\mathsf{D}_1 f = \mathbf{c} \circ \mathsf{D}_1\mathsf{D}_0 f$

Proof. very similar to that of Proposition 28, except that it uses the naturality of $\mathbf{c}$ included in (D-Schwarz) instead of the naturality of θ. ☐

[8] Also called oplax symmetric monoidal monad, or symmetric comonoidal monad, or Hopf monad, see [11]

4.4 Generalization to arbitrary finite products

Notations 3. Recall that the existence of arbitrary finite products is equivalent to the existence of a binary product and a terminal object. In order to stay consistant with the current notations, we write the finite products starting from 0: $X_0 \& \cdots \& X_n$. We allow empty products, with the convention that taking $n = -1$ yields a product $X_0 \& \cdots \& X_{-1} := \top$.

The constructions above can be extended to arbitrary finite products. On can indeed define a (symmetric monoidal) natural transformation $\mathbf{c}^n_\& \in \mathcal{C}(\mathsf{D}(X_0 \& \cdots \& X_n), \mathsf{D}X_0 \& \cdots \& \mathsf{D}X_n)$ inductively by $(\mathbf{c}_\&)_X := \mathbf{t}_{\mathsf{D}\top} \in \mathcal{C}(\mathsf{D}\top, \top)$, $(\mathbf{c}^0_\&)_X := \mathsf{id}_{\mathsf{D}X} \in \mathcal{C}(\mathsf{D}X, \mathsf{D}X)$ and $\mathbf{c}^{n+1}_\& := \mathbf{c}_\& \circ \langle \mathbf{c}^n_\&, \mathsf{id}_{\mathsf{D}X_{n+1}} \rangle$. By associativity of the cartesian product, this definition does not depend on the actual parenthesizing of $X_0 \& \cdots \& X_n$.

Notations 4. Let $X_0, Y_0, \ldots, X_n, Y_n \in \mathsf{Obj}(\mathcal{C})$. Let $i \in [\![0, n]\!]$ and let $f_k \in \mathcal{C}(X_k, Y_k)$ for each $k \neq i$. Let $g \in \mathcal{C}(X_i, Y_i)$. Let $(g; f_{-i}) := f_0 \& \cdots \& f_{i-1} \& g \& f_{i+1} \& \cdots \& f_n$ in which we use f_i everywhere except at position i where we use g.

Similarly to the binary case, one can then define a strength $\Phi^i \in \mathcal{C}(X_0 \& \cdots \& \mathsf{D}X_i \& \cdots \& X_n, \mathsf{D}(X_0 \& \cdots \& X_n))$ as

$$\Phi^i := (\mathbf{c}^n_\&)^{-1} \circ (\mathsf{id}_{\mathsf{D}X_i}; (\iota_0)_{-i})$$

Proposition 30. *$\mathbf{c}^n_\&$ is an isomorphism and $(\mathbf{c}^n_\&)^{-1} = \langle\!\langle \pi_0 \& \cdots \& \pi_0, \pi_1 \& \cdots \& \pi_1 \rangle\!\rangle$. Hence, $\Phi^i = \langle\!\langle (\pi_0; \mathsf{id}_{-i}), (\pi_1; 0_{-i}) \rangle\!\rangle$.*

Proof. The equation on $\mathbf{c}^n_\&$ is obtained by unfolding the inductive definition and using Theorem 4. The equations on the Φ^i's follow from this, as in Proposition 24. □

Definition 20. For any $f \in \mathcal{C}(X_0 \& \cdots \& X_n, Y)$ one can define the *i-th partial derivative of f* as $\mathsf{D}_i f := \mathsf{D}f \circ \Phi^i \in \mathcal{C}(X_0 \& \cdots \& \mathsf{D}X_i \& \cdots \& X_n, \mathsf{D}Y)$.

Proposition 31. $\pi_0 \circ \mathsf{D}_i f = f \circ (\pi_0; \mathsf{id}_{-i})$.

Proof. Same as Proposition 25 □

Definition 21. For any $X \in \mathsf{Obj}(\mathcal{C})$ and $n \geq 0$, we define $\theta^k_X \in \mathcal{C}(\mathsf{D}^{n+1}X, \mathsf{D}X)$ as the composition of k copies of θ: $\theta^0_X = \mathsf{id}_{\mathsf{D}X}$ and $\theta^{k+1}_X = \theta^k_X \circ \theta_{\mathsf{D}^k X}$. We define similarly $\pi^k_i \in \mathcal{C}(\mathsf{D}^k X, X)$.

Note that $\theta^k = \langle\!\langle \pi_0^{k+1}, \sum_{j=0}^k \pi_0^j \circ \pi_1 \circ \pi_0^{k-j} \rangle\!\rangle$. In other words, the right component of θ^k sums over all of the possible combinations of k left projections and one right projection. One can prove a generalization of Proposition 27 for $n \geq 1$,

$$(\mathbf{c}^n_\&)^{-1} = \theta^n \circ \mathsf{D}^n \Phi^{\alpha(n)} \circ \cdots \circ \mathsf{D}\Phi^{\alpha(1)} \circ \Phi^{\alpha(0)}$$

for any α permutation of $[\![0, n]\!]$. As in Proposition 28, this generalizes the Leibniz Rule to the n-ary case.

Proposition 32 (Leibniz, generalized)**.** *For any $n \geq 1$ and for any α permutation of $[\![0, n]\!]$,*

$$\mathsf{D}f \circ (\mathbf{c}^n_\&)^{-1} = \theta^n \circ \mathsf{D}_{\alpha(n)} \ldots \mathsf{D}_{\alpha(0)} f\,.$$

4.5 Multilinear morphism

Definition 22. A morphism $f \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n, Y)$ is *multilinear* (and more precisely, $(n+1)$*-linear*) if for any $i \in [\![0, n]\!]$, $\pi_1 \circ \mathsf{D}_i f = f \circ (\pi_1; \mathsf{id}_{-i})$. Note that the 1-linear morphisms are exactly the D-linear ones.

As a sanity check of the notion, we can use the result below together with the Leibniz rule to show a result similar to the fact that in differential calculus, if Φ is a bilinear map, then $\Phi'(x, y) \cdot (u, v) = \Phi(x, v) + \Phi(u, y)$.

Lemma 1. *For any $f \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n, Y)$ and $i, j \in [\![0, n]\!]$ such that $i \neq j$,*

$$\mathsf{D}\pi_0 \circ \mathsf{D}_i \mathsf{D}_j f = \mathsf{D}_i f \circ (\pi_0; \mathsf{id}_{-j})$$

Proof. This is a direct computation

$$\begin{aligned}
\mathsf{D}\pi_0 \circ \mathsf{D}_i \mathsf{D}_j f &= \mathsf{D}\pi_0 \circ \mathsf{D}(\mathsf{D}_j f) \circ \Phi^i \\
&= \mathsf{D}(\pi_0 \circ \mathsf{D}_j f) \circ \Phi^i \quad \text{by (D-chain)} \\
&= \mathsf{D}(f \circ (\pi_0; \mathsf{id}_{-j})) \circ \Phi^i \quad \text{by Proposition 31} \\
&= \mathsf{D}f \circ \mathsf{D}(\pi_0; \mathsf{id}_{-j}) \circ \Phi^i \quad \text{by (D-chain)} \\
&= \mathsf{D}f \circ \Phi^i \circ (\pi_0; \mathsf{id}_{-j}) \quad \Phi^i \text{ natural and } i \neq j \\
&= \mathsf{D}_i f \circ (\pi_0; \mathsf{id}_{-j})
\end{aligned}$$

□

Theorem 5. *For any $f \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n, Y)$ $(n+1)$-linear,*

$$\begin{aligned}
\pi_0 \circ \mathsf{D}f \circ (\mathbf{c}_\&)^{-1} &= f \circ (\pi_0 \,\&\, \cdots \,\&\, \pi_0) \\
\pi_1 \circ \mathsf{D}f \circ (\mathbf{c}_\&)^{-1} &= f \circ (\pi_1 \,\&\, \pi_0 \,\&\, \cdots \,\&\, \pi_0) + \cdots + \ f \circ (\pi_0 \,\&\, \cdots \,\&\, \pi_0 \,\&\, \pi_1)
\end{aligned}$$

Proof. We will write the proof for $n = 1$. The general case relies on the same arguments. The first equation is just a direct consequence of the naturality of π_0 and Proposition 24. For the second equation, Proposition 28 ensures that $\pi_1 \circ \mathsf{D}f \circ \mathbf{c}_\&^{-1} = \pi_1 \circ \theta \circ \mathsf{D}_0\mathsf{D}_1 f = \pi_1 \circ \pi_0 \circ \mathsf{D}_0\mathsf{D}_1 f + \pi_0 \circ \pi_1 \circ \mathsf{D}_0\mathsf{D}_1 f$. We can compute those two summands separately.

$$\begin{aligned}
\pi_1 \circ \pi_0 \circ \mathsf{D}_0\mathsf{D}_1 f &= \pi_1 \circ \mathsf{D}_1 f \circ (\pi_0 \,\&\, \mathsf{id}) \quad \text{by Proposition 25} \\
&= f \circ (\mathsf{id} \,\&\, \pi_1) \circ (\pi_0 \,\&\, \pi_1) \quad \text{by bilinarity of } f \\
&= f \circ (\pi_0 \,\&\, \pi_1)
\end{aligned}$$

$$\begin{aligned}
\pi_0 \circ \pi_1 \circ \mathsf{D}_0\mathsf{D}_1 f &= \pi_1 \circ \mathsf{D}\pi_0 \circ \mathsf{D}_0\mathsf{D}_1 f \quad \text{by linearity of } \pi_0 \\
&= \pi_1 \circ \mathsf{D}_0 f \circ (\mathsf{id} \,\&\, \pi_0) \quad \text{by Lemma 1} \\
&= f \circ (\pi_1 \,\&\, \mathsf{id}) \circ (\mathsf{id} \,\&\, \pi_0) \quad \text{by bilinarity of } f \\
&= f \circ (\pi_1 \,\&\, \pi_0)
\end{aligned}$$

Which concludes the proof. □

We can expand on the ideas of the proof Lemma 1 to show the following result. This result is crucial, as it explains how to project on a series of partial derivatives.

Proposition 33. *Let $n \geq 0$, $f \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n)$, $d \geq 0$ and $i, i_1, \ldots, i_d \in [\![0, n]\!]$. Then,*

$$\mathsf{D}^d \pi_0 \circ \mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f = \mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} f \circ (\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i})$$

where $h_d(i) = \#\{k \in [\![1, d]\!] \mid i_k = i\}$. Furthermore, if f is $(n+1)$-linear, then

$$\mathsf{D}^d \pi_1 \circ \mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f = \mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} f \circ (\mathsf{D}^{h_d(i)} \pi_1; \mathsf{id}_{-i})$$

Proof. By induction on d. The case $d = 0$ is Proposition 31 for π_0, and the definition of n-linearity for π_1. We deal with the inductive step for π_0. The inductive step for π_1 is dealt with similarly.

$$\begin{aligned}
&\mathsf{D}^{d+1} \pi_0 \circ \mathsf{D}_{i_{d+1}} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f \\
&\qquad = \mathsf{D}(\mathsf{D}^d \pi_0) \circ \mathsf{D}(\mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f) \circ \Phi^{i_{d+1}} \quad \text{by definition} \\
&\qquad = \mathsf{D}(\mathsf{D}^d \pi_0 \circ \mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f) \circ \Phi^{i_{d+1}} \quad \text{by (D-chain)} \\
&\qquad = \mathsf{D}(\mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f \circ (\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i})) \circ \Phi^{i_{d+1}} \quad \text{induction hypothesis} \\
&\qquad = \mathsf{D}\mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f \circ \mathsf{D}(\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i}) \circ \Phi^{i_{d+1}} \quad \text{by (D-chain)}
\end{aligned}$$

The next step is to use the naturality of $\Phi^{i_{d+1}}$:

$$\mathsf{D}(f_0 \,\&\, \cdots \,\&\, f_n) \circ \Phi^{i_{d+1}} = (\mathsf{D} f_{i_{d+1}}; f_{-i_{d+1}})$$

If $i_{d+1} = i$, then

$$\mathsf{D}(\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i}) \circ \Phi^{i_{d+1}} = \Phi^{i_{d+1}} \circ (\mathsf{D}^{h_d(i)+1} \pi_0; \mathsf{id}_{-i})$$

If $i_{d+1} \neq i$ then

$$\mathsf{D}(\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i}) \circ \Phi^{i_{d+1}} = \Phi^{i_{d+1}} \circ (\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i})$$

In both case,

$$\mathsf{D}(\mathsf{D}^{h_d(i)} \pi_0; \mathsf{id}_{-i}) \circ \Phi^{i_{d+1}} = \Phi^{i_{d+1}} \circ (\mathsf{D}^{h_{d+1}(i)} \pi_0; \mathsf{id}_{-i})$$

Consequently:

$$\begin{aligned}
\mathsf{D}^{d+1} \pi_0 \circ \mathsf{D}_{i_{d+1}} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f &= \mathsf{D}\mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f \circ \Phi^{i_{d+1}} \circ (\mathsf{D}^{h_{d+1}(i)} \pi_0; \mathsf{id}_{-i}) \\
&= \mathsf{D}_{i_{d+1}(i)} \mathsf{D}_{i_d} \ldots \mathsf{D}_{i_1} \mathsf{D}_i f \circ (\mathsf{D}^{h_{d+1}} \pi_0; \mathsf{id}_{-i})
\end{aligned}$$

which concludes the proof. ◻

This property instantiated in $d = 1$ gives a generalization of Lemma 1.

Corollary 8. *If $f \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n)$ is $(n+1)$-linear, then for any $i, j \in [\![0, n]\!]$ such that $i \neq j$ and for any $k \in \{0, 1\}$,*

$$\mathsf{D}\pi_k \circ \mathsf{D}_i \mathsf{D}_j f = \mathsf{D}_i f \circ (\pi_k; \mathsf{id}_{-j})$$

$$\mathsf{D}\pi_k \circ \mathsf{D}_i \mathsf{D}_i f = \mathsf{D}_i f \circ (\mathsf{D}\pi_k; \mathsf{id}_{-i})$$

We can use this corollary to show that the partial derivative of a $(n+1)$-linear morphism is also $(n+1)$-linear.

Theorem 6. *If $f \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, X_n)$ is $(n+1)$-linear, then for any $i \in [\![0, n]\!]$, $\mathsf{D}_i f$ is $(n+1)$-linear.*

Proof. Let $j \in [\![0, n]\!]$. The goal is to prove that $\pi_1 \circ \mathsf{D}_j \mathsf{D}_i f = \mathsf{D}_i f \circ (\pi_1; \mathsf{id}_{-j})$. By joint monicity of the π_k, it suffices to prove that $\pi_k \circ \pi_1 \circ \mathsf{D}_j \mathsf{D}_i f = \pi_k \circ \mathsf{D}_i f \circ (\pi_1; \mathsf{id}_{-j})$ for any $k \in \{0, 1\}$. If $i \neq j$,

$$\begin{aligned}
\pi_k \circ \pi_1 \circ \mathsf{D}_j \mathsf{D}_i f &= \pi_1 \circ \mathsf{D}\pi_k \circ \mathsf{D}_j \mathsf{D}_i f \quad \text{by D-linearity of } \pi_1 \\
&= \pi_1 \circ \mathsf{D}_j f \circ (\pi_k; \mathsf{id}_{-i}) \quad \text{by Corollary 8} \\
&= f \circ (\pi_1; \mathsf{id}_{-j}) \circ (\pi_k; \mathsf{id}_{-i}) \quad \text{since } f \text{ is } (n+1)\text{-linear} \\
&= f \circ (\pi_k; \mathsf{id}_{-i}) \circ (\pi_1; \mathsf{id}_{-j}) \quad \text{since } i \neq j \\
&= \pi_k \circ \mathsf{D}_i f \circ (\pi_1; \mathsf{id}_{-j}) \quad \text{since } f \text{ is } (n+1)\text{-linear}
\end{aligned}$$

The case $i = j$ is very similar

$$\begin{aligned}
\pi_k \circ \pi_1 \circ \mathsf{D}_i \mathsf{D}_i f &= \pi_1 \circ \mathsf{D}\pi_k \circ \mathsf{D}_i \mathsf{D}_i f \quad \text{by D-linearity of } \pi_1 \\
&= \pi_1 \circ \mathsf{D}_i f \circ (\mathsf{D}\pi_k; \mathsf{id}_{-i}) \quad \text{by Corollary 8} \\
&= f \circ (\pi_1; \mathsf{id}_{-i}) \circ (\mathsf{D}\pi_k; \mathsf{id}_{-i}) \quad \text{since } f \text{ is } (n+1)\text{-linear} \\
&= f \circ (\pi_k; \mathsf{id}_{-i}) \circ (\pi_1; \mathsf{id}_{-i}) \quad \text{since } \pi_k \text{ is D-linear} \\
&= \pi_k \circ \mathsf{D}_i f \circ (\pi_1; \mathsf{id}_{-i}) \quad \text{since } f \text{ is } (n+1)\text{-linear.}
\end{aligned}$$

□

Composition with a linear morphism preserves multilinearity. Thus, the Leibniz rule ensures that if f is multilinear then $\mathsf{D}f$ is also multilinear.

Proposition 34. *If $f \in \mathcal{C}(X_0 \& \cdots \& X_n, Y)$ is $(n+1)$-linear and $h \in \mathcal{C}(Y, Z)$ is linear, then $h \circ f$ is $(n+1)$-linear.*

Proof. This follows from a straightforward computation $\pi_1 \circ \mathsf{D}_i(h \circ f) = \pi_1 \circ \mathsf{D}(h \circ f) \circ \Phi^i = \pi_1 \circ \mathsf{D}h \circ \mathsf{D}f \circ \Phi^i = h \circ \pi_1 \circ \mathsf{D}_i f = h \circ f \circ (\pi_1; \mathsf{id}_{-i})$. □

Theorem 7. *If $f \in \mathcal{C}(X_0 \& \cdots \& X_n, Y)$ is $(n+1)$-linear, then $\mathsf{D}f \circ (\mathbf{c}^n_\&)^{-1} \in \mathcal{C}(\mathsf{D}X_0 \& \cdots \& \mathsf{D}X_n, \mathsf{D}Y)$ is also $(n+1)$-linear.*

Proof. By Leibniz (Proposition 32), $\mathsf{D}f \circ (\mathbf{c}^n_\&)^{-1} = \theta^n \circ \mathsf{D}_{\alpha(n)} \ldots \mathsf{D}_{\alpha(0)} f$. But the partial derivatives preserves multilinearity by Theorem 6 and composition by θ^n on the left preserves multilinearity by Proposition 34. □

5 Models arising as the Kleisli category of the exponential comonad of a model of LL

The development of section is not necessary for the rest of the paper, but shows that many examples of Cartesian Coherent Differential Categories arise from models of linear logic.

5.1 Coherent Differentiation in a linear setting

Let $\mathcal{L}$ be a category. We write the composition of $f \in \mathcal{L}(X, Y)$ with $g \in \mathcal{L}(Y, Z)$ as $g\,f$ to stress the intuition that the morphisms of $\mathcal{L}$ are linear. We assume that $\mathcal{L}$ is a model of LL, and more precisely a Seely Category in the sense of [13]. The axioms of Seely Category include the existence of a cartesian product & and a comonad $(!, \mathsf{der}, \mathsf{dig})$ on $\mathcal{L}$, with $\mathsf{der}_X \in \mathcal{L}(!X, X)$ and $\mathsf{dig}_X \in \mathcal{L}(!X, !!X)$. The Kleisli category $\mathcal{L}_!$ of this comonad is defined by taking the same objects as $\mathcal{L}$ and taking $\mathcal{L}_!(X,Y) = \mathcal{L}(!X,Y)$. Composition is defined in this category as $g \circ f = g\,!f\,\mathsf{dig}$ and the identity at X is der_X. It is well known that $\mathcal{L}_!$ is a cartesian (closed) category, with the same cartesian product & as $\mathcal{L}$.

The goal of this section is to show that Coherent Differentiation as introduced in [2] in the setting of LL gives $\mathcal{L}_!$ a CCDC structure.

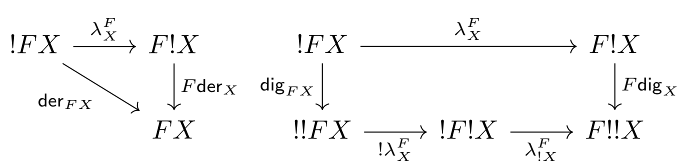
Theorem 8. *Any Differential Structure on a Summable Category $\mathcal{L}$ (in the sense of [2]) induces a CCDC structure on $\mathcal{L}_!$.*

Let us detail first what the assumption means. The category $\mathcal{L}$ is said to be *summable* [2] if it has a Summability Structure $(\mathsf{S},\pi_0,\pi_1,\sigma)$ in the sense of the first author. By Theorem 3, this means that $(\mathsf{S},\pi_0,\pi_1,\sigma)$ is a Left Summability Structure in the sense of Definition 10 where every morphism is additive and $\mathsf{S}f := \langle\!\langle f \circ \pi_0, f \circ \pi_1 \rangle\!\rangle$. Then, we can define ι_i, θ, $\mathbf{l}$ and $\mathbf{c}$ as usual[9]. The difference is that the additivity of every morphism ensures that those families are natural transformations for the functor S. In particular, $(\mathsf{S},\iota_0,\theta)$ is *de facto* a monad. The category $\mathcal{L}$ is said to be *summable as a cartesian category* if $\langle \mathsf{S}\mathbf{p}_0, \mathsf{S}\mathbf{p}_1 \rangle = \mathbf{c}_\&$ is an isomorphism[10]. Because every morphism of $\mathcal{L}$ is additive, this corresponds by Corollary 7 to the fact that the cartesian product is compatible with the Left Summability Structure.

It is well known that there is a faithful functor $\mathsf{Der} : \mathcal{L} \to \mathcal{L}_!$ which maps X to X and $f \in \mathcal{L}(X,Y)$ to $f\,\mathsf{der}_X \in \mathcal{L}(!X,Y)$. This functor induces a Left Summability Structure $(\mathsf{D}, \mathsf{Der}(\pi_0), \mathsf{Der}(\pi_1), \mathsf{Der}(\sigma))$ on $\mathcal{L}_!$ (where $\mathsf{D}X := \mathsf{S}X$) compatible with the cartesian product & of $\mathcal{L}_!$. The reason is that Der preserves monicity and additivity, thanks to the well known fact that $\mathsf{Der}(h) \circ f = h\,f$. Finally, the definition of Der ensures that $\mathsf{Der}(\langle\!\langle f_0, f_1 \rangle\!\rangle) = \langle\!\langle \mathsf{Der}(f_0), \mathsf{Der}(f_1) \rangle\!\rangle$. In particular, the families of morphism generated by the Left Summability Structure $(\mathsf{D}, \mathsf{Der}(\pi_0), \mathsf{Der}(\pi_1), \mathsf{Der}(\sigma))$ in Definitions 6 and 12 to 14 are $\mathsf{Der}(\iota_i)$, $\mathsf{Der}(\theta)$, $\mathsf{Der}(\mathbf{l})$ and $\mathsf{Der}(\mathbf{c})$ respectively.

Then a *Differential Structure* on a summable category $\mathcal{L}$ consists of a natural transformation $\partial_X \in \mathcal{L}(!\mathsf{S}X, \mathsf{S}!X)$ following some axioms called (∂-chain), (∂-local), (∂-lin), (∂-&) and (∂-Schwarz) (see [2] for the diagrams). The first axiom, (∂-lin), is a compatibility condition of ∂ with regard to dig and der, making ∂ a *distributive law* between the functor S and the comonad $!\_$.

Definition 23. A *distributive law* between a functor $F : \mathcal{L} \to \mathcal{L}$ and the comonad $!\_$ on $\mathcal{L}$ is a natural transformation $\lambda^F \in \mathcal{L}(!FX, F!X)$ such that the two following diagrams commute.



A definition of distributive laws can be found in [14], together with the two following observations (corollary 5.11)[11].

Proposition 35. *Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. There is a bijection between distributive laws $\lambda_F \in \mathcal{L}(!FX, \mathsf{S}!X)$ and* liftings *$\hat{F}$ of F on $\mathcal{L}_!$. A lifting $\hat{F}$ of F is a functor $\hat{F} : \mathcal{L}_! \to \mathcal{L}_!$ such that $\hat{F}X = FX$ and $\hat{F}(\mathsf{Der}(h)) = \mathsf{Der}(Fh)$.*

[9] Note that in [2], θ is called τ

[10] We can show that the condition required in [2] that $0 \in \mathcal{L}(\mathsf{S}\top, \top)$ is an isomorphism always hold, using the joint monicity of the π_i

[11] These observations are made in the more general setting of 2-categories

Proof. Given a distributive law $\lambda^F \in \mathcal{L}(!FX, F!X)$, one can define an extension mapping X to FX and $f \in \mathcal{L}_!(X,Y)$ to $Ff\,\lambda^F_X \in \mathcal{L}_!(FX, FY)$. We can check that it is a functor using the diagrams of distributive laws, and a lifting[12] of F using the naturality of λ^F. Conversely, any lifting $\hat{F}$ of F induces a family $\lambda^F_X = \hat{F}\mathsf{id}_{!X} \in \mathcal{L}_!(!FX, F!X)$. The two diagrams of distributive law comes from the functoriality of $\hat{F}$ and the naturality comes from the fact that $\hat{F}$ is an extension of F. □

Remark 5. Let $F, G : \mathcal{L} \to \mathcal{L}$ be two functors, with respective lifting $\hat{F}$ and $\hat{G}$ associated to the distributive laws $\lambda^F \in \mathcal{L}(!FX, F!X)$ and $\lambda^G \in \mathcal{L}(!GX, G!X)$. Then $\hat{G}\hat{F}$ is a lifting of GF and the distributive law associated with $\hat{G}\hat{F}$ is the following natural transformation: $\lambda^{GF}_X = G\lambda^F_X\,\lambda^G_{FX} \in \mathcal{L}(!GFX, GF!X)$.

Proposition 36. *Let $F, G : \mathcal{L} \to \mathcal{L}$ be two endofunctors. Assume that $\hat{F}$ and $\hat{G}$ are lifting of F and G respectively, and let λ^F and λ^G be their respective associated distributive law. Let $\alpha_X \in \mathcal{L}(FX, GX)$ be a natural transformation. Then $\mathsf{Der}(\alpha_X) \in \mathcal{L}_!(\hat{F}X, \hat{G}X)$ is natural if and only if the following diagram commutes.*

$$\begin{array}{ccc} !FX & \xrightarrow{\lambda^F} & F!X \\ {\scriptstyle !\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ !GX & \xrightarrow[\lambda^G]{} & G!X \end{array} \tag{3}$$

Proof. straightforward computation. □

In the case of differentiation, the axiom (∂-chain) implies that $\partial \in \mathcal{C}(!\mathsf{S}X, \mathsf{S}!X)$ is a distributive law between the comonad ! and the functor S. This means that S can be lifted to an endofunctor D on $\mathcal{L}_!$. Besides, there is a trivial distributive law $\mathsf{id}_{!X} \in \mathcal{L}(!X, !X)$ associated to the lifting of the identity functor on $\mathcal{L}$ to the identity functor on $\mathcal{L}_!$. Finally, as we saw in Remark 5, there is a distributive law $\mathsf{S}\partial_X\,\partial_{\mathsf{S}X} \in \mathcal{C}(!\mathsf{S}^2X, \mathsf{S}^2!X)$ associated to D^2, the lifting of S^2 to $\mathcal{L}_!$.

Then (∂-local) consists in an instance of Equation (3) in which $F = \mathsf{S}$, $G = \mathsf{Id}$ and $\alpha = \pi_0$. This means that $\mathsf{Der}(\pi_0) \in \mathcal{L}_!(\mathsf{D}X, X)$ is a natural transformation. Thus, $(\mathsf{D}, \mathsf{Der}(\pi_0), \mathsf{Der}(\pi_1), \mathsf{Der}(\sigma))$ is a Pre-Differential Structure on $\mathcal{L}_!$ (in the sense of Definition 11) and (D-chain) holds. Moreover, since D is a lifting of S, for any $h \in \mathcal{L}(X,Y)$, the morphism $\mathsf{Der}(h) \in \mathcal{L}_!(X,Y)$ is D-linear. Indeed, $\mathsf{Der}(\pi_0) \circ \mathsf{D}(\mathsf{Der}(h)) = \mathsf{Der}(\pi_0) \circ \mathsf{Der}(\mathsf{S}h) = \mathsf{Der}(\pi_0\,\mathsf{S}h) = \mathsf{Der}(h\,\pi_0) = \mathsf{Der}(h) \circ \mathsf{Der}(\pi_0)$. As a result, $\mathsf{Der}(\pi_i), \mathsf{Der}(\sigma), \mathsf{Der}(\mathbf{p}_i)$ are all linear so (Dproj-lin), (Dsum-lin) hold and the Pre-Differential Structure is compatible with the cartesian product.

Furthermore, (∂-lin) consists in two instances of Equation (3) in which $\alpha = \iota_0 \in \mathcal{L}(\mathsf{S}X, X)$ and $\alpha = \theta \in \mathcal{L}(\mathsf{S}^2X, X)$ respectively. So (∂-lin) ensures that $\mathsf{Der}(\iota_0)$ and $\mathsf{Der}(\theta)$ are natural, this is exactly (D-add). Finally, (∂-Schwarz) consists in an instance of Equation (3) in which $\alpha = \mathbf{c}$, meaning that $\mathsf{Der}(\mathbf{c}) \in \mathcal{L}_!(\mathsf{S}^2X, \mathsf{S}^2X)$ is natural. So (∂-Schwarz) ensures (D-Schwarz). The only lacking axiom is (D-lin) that corresponds to the naturality of $\mathsf{Der}(\mathbf{l})$. Thanks to Proposition 36, it would hold if and only if the diagram below commutes.

$$\begin{array}{ccccc} !\mathsf{S}X & & \xrightarrow{\quad\partial_X\quad} & & \mathsf{S}!X \\ {\scriptstyle !\mathbf{l}}\downarrow & & & & \downarrow{\scriptstyle \mathbf{l}_{!X}} \\ !\mathsf{S}^2X & \xrightarrow[\partial_{\mathsf{S}X}]{} & \mathsf{S}!\mathsf{S}X & \xrightarrow[\mathsf{S}\partial_X]{} & \mathsf{S}^2!X \end{array}$$

[12]The terminology extension is also used. We use the term lifting in order to stick to the terminology of [14]

This diagram is not mentioned in [2] but makes perfectly sense in the setting of Coherent Differentiation in LL and holds in all known LL models of coherent differentiation. The study of the consequences of this diagram is left for further work.

To conclude, any Differential Structure on $\mathcal{L}$ (in the sense of [2]) induces the structure of a CCDC on $\mathcal{L}_!$.

Remark 6. The only remaining axiom is (∂-&) that deals with the Seely isomorphisms $\mathsf{m}^n \in \mathcal{L}(!X_0 \otimes \ldots \otimes !X_n, !(X_0 \mathbin{\&} \ldots \mathbin{\&} X_n))$ of the Seely category $\mathcal{L}$. It is possible to define in LL a notion of multilinearity: given any $l \in \mathcal{L}(X_0 \otimes \ldots \otimes X_n, Y)$, one can define $\widetilde{l} \in \mathcal{L}_!(X_0 \& \ldots \& X_n, Y)$ as $\widetilde{l} = l\,(\mathsf{der} \otimes \ldots \otimes \mathsf{der})\,(\mathsf{m}^n)^{-1}$. Then a morphism in $\mathcal{L}_!(X_0 \& \ldots \& X_n, Y)$ is $(n+1)$-*linear* (in the sense of LL) if it can be written as $\widetilde{h}$ for some h. The axiom (∂-&) allows to show that any $(n+1)$-linear morphism in the sense of LL is also $(n+1)$-linear in the sense of Definition 22. A proof of this fact can be implicitly found in Theorem 4.26 of [15]. This is a crucial fact, because it shows that what really matters is the $(n+1)$-linearity in terms of CCDC rather than the $(n+1)$-linearity in terms of LL.

Many models of LL have a Differential Structure, such as Coherence Spaces, Non Uniform Coherence Spaces and Probabilistic Coherence Spaces. Thus, their Kleisli categories are all CCDCs. This provides a rich variety of examples. We present here the example of Probabilistic Coherence Spaces.

5.2 The example of Probabilistic Coherence Spaces

A *Probabilistic Coherence Space* (PCS) is a pair $X = (|X|, \mathsf{P}X)$ where $|X|$ is a set and $\mathsf{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ satisfies $\mathsf{P}X = \{x \in (\mathbb{R}_{\geq 0})^{|X|} \mid \forall x' \in \mathcal{P}'\ \langle x, x' \rangle := \sum_{a \in |X|} x_a x'_a \leq 1\}$ for some $\mathcal{P}' \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ called a *predual* of X. To avoid ∞ coefficients it is also assumed that $\forall a \in |X|\ 0 < \sup_{x' \in \mathcal{P}'} x'_a < \infty$ and then it is easily checked that for all $\forall a \in |X|\ 0 < \sup_{x \in \mathsf{P}X} x_a < \infty$.

A multiset of elements of a set I is a function $m : I \to \mathbb{N}$ such that the set $\mathsf{supp}(m) = \{i \in I \mid m(i) \neq 0\}$ is finite. The set $\mathcal{M}_{\mathrm{fin}}(I)$ of these multisets is the free commutative monoid generated by I. We use $[i_1, \ldots, i_k]$ for the $m \in \mathcal{M}_{\mathrm{fin}}(I)$ such that $m(i) = \#\{j \mid i_j = i\}$, for $i_1, \ldots, i_k \in I$.

Given PCSs X and Y, a function $f : \mathsf{P}X \to \mathsf{P}Y$ is *analytic*[13] if there is a *matrix* $t \in (\mathbb{R}_{\geq 0})^{\mathcal{M}_{\mathrm{fin}}(|X|) \times |Y|}$ such that, for all $x \in \mathsf{P}X$ and $b \in |Y|$, one has $f(x)_b = \sum_{(m,b) \in \mathcal{M}_{\mathrm{fin}}(|X|) \times |Y|} t_{m,b} x^m$ where $x^m = \prod_{a \in |X|} x_a^{m(a)}$. Thanks to the fact that all the coefficients in t are finite, it is not difficult to see that they can be recovered from the function f itself by means of iterated differentiation, see [17]. So an analytic function has *exactly one* associated matrix.

The identity function $\mathsf{P}X \to \mathsf{P}X$ is analytic (of matrix t given by $t_{m,a} = \delta_{m,[a]}$) and the composition of two analytic functions is still analytic. We use **APcoh** for the category whose objects are PCSs and morphisms are analytic functions. For instance, if 1 is the PCS $(\{*\}, [0,1])$ then $f_1, f_2 : [0,1] \to [0,1]$ given by $f_1(x) = 1 - \sqrt{1 - x^2}$ and $f_2(x) = e^{x-1}$ are in $\mathbf{APcoh}(1,1)$, but $f_3(x) = 2x - x^2$ is not because of the negative coefficient. The (pointwise) sum of two analytic functions $\mathsf{P}X \to \mathsf{P}Y$ is always well defined $\mathsf{P}X \to \mathbb{R}_{\geq 0}^{|Y|}$, but is not necessarily in $\mathbf{APcoh}(X,Y)$ so **APcoh** is not left-additive[14].

If X is a PCS then $\mathsf{D}X = (\{0,1\} \times |X|, \mathsf{P}(\mathsf{D}X) = \{z \in (\mathbb{R}_{\geq 0})^{\{0,1\} \times |X|} \mid \pi_0(z) + \pi_1(z) \in \mathsf{P}X\})$, where $\pi_i(z)_a = z_{i,a}$, is a PCS. Then $\pi_0, \pi_1 \in \mathbf{APcoh}(\mathsf{D}X, X)$ and we have also

[13] There is also a purely functional characterization of these functions as those which are totally monotone and Scott continuous, see [16]

[14] At least for this most natural addition.

$\sigma \in \mathbf{APcoh}(\mathsf{D}X, X)$ given by $\sigma(z) = \pi_0(z) + \pi_1(z)$. In other words $\mathsf{D}X$ is the PCS whose elements are the pairs $(x, u) \in \mathsf{P}X^2$ such that $x + u \in \mathsf{P}X$. In that way we have equipped **APcoh** with a Left Pre-Summability Structure and the associated notion of summability is the obvious one: $f_0, f_1 \in \mathbf{APcoh}(X, Y)$ are summable if their pointwise sum $f_0 + f_1$ is in $\mathbf{APcoh}(X, Y)$. It is easily checked that this Left Pre-Summability Structure is a Left Summability Structure (see Definition 10).

As explained in Section 2.1, differentiation boils down to extending the operation D to morphisms in such a way that the conditions of Definition 15 is satisfied. Given $f \in \mathbf{APcoh}(X, Y)$ of matrix t and $(x, u) \in \mathsf{P}(\mathsf{D}X)$ we have

$$
\begin{aligned}
f(x+u) &= \sum_{(m,b)\in\mathcal{M}_{\mathrm{fin}}(|X|)\times|Y|} t_{m,b}(x+u)^m \\
&= \sum_{(m,b)\in\mathcal{M}_{\mathrm{fin}}(|X|)\times|Y|} t_{m,b} \sum_{p\leq m} \binom{m}{p} x^{m-p} u^p \\
&= f(x) + \sum_{a\in\mathsf{supp}(m)} \binom{m}{[a]} x^{m-[a]} u_a + r(x,u) \\
&= f(x) + \sum_{a\in\mathsf{supp}(m)} m(a) x^{m-[a]} u_a + r(x,u)
\end{aligned}
$$

where we use $\mathsf{supp}(m) = \{a \in |X| \mid m(a) > 0\}$ and $\binom{m}{p} = \prod_{a\in|X|} \binom{m(a)}{p(a)} \in \mathbb{N}$ when $p \leq m$ for the pointwise order. In these expressions the remainder $r(x, u)$ is a power series in x and u whose all monomials have total degree > 1 in u (such as $x_a u_b u_c$ if $a, b, c \in |X|$). In particular $\|r(x, u)\| \in o(\|u\|)$ where $\|x\| = \sup\{\langle x, x'\rangle \mid x' \in \mathcal{P}'\} \in [0, 1]$ for any predual $\mathcal{P}'$ of X (this norm does not depend on the choice of $\mathcal{P}'$). Using Definition 11 we set $\mathbf{d}^{\mathsf{D}}(f)(x, u) = \sum_{a\in\mathsf{supp}(m)} m(a) x^{m-[a]} u_a$ and since all coefficients of t are ≥ 0 we have $f(x)+\mathbf{d}^{\mathsf{D}}(f)(x, u) \leq f(x+u)$ for the pointwise order so that $\mathsf{D}f(x, u) = (f(x), \mathbf{d}^{\mathsf{D}}(f)(x, u)) \in \mathsf{P}(\mathsf{D}Y)$. We have defined in that way an analytic function $\mathsf{D}f \in \mathbf{APcoh}(\mathsf{D}X, \mathsf{D}Y)$ and it is easily checked that **APcoh** is a coherent differential category in the sense of Definition 15. For the two examples above we get $\mathbf{d}^{\mathsf{D}}(f_2)(x, u) = e^{x-1}u$ and $\mathbf{d}^{\mathsf{D}}(f_1)(x, u) = xu/\sqrt{1-x^2}$ which seems to be undefined when $x = 1$ but is not because then *we must have* $u = 0$ and so $\mathbf{d}^{\mathsf{D}}(f_2)(1, 0) = 0$.

An analytic $f \in \mathbf{APcoh}(X, Y)$ is *linear* if its matrix t satisfies that whenever $t_{m,b} \neq 0$, one has $m = [a]$ for some $a \in |X|$. This notion of linearity[15] coincides with both additivity Definition 3 and D-linearity Definition 16.

The category **APcoh** is Cartesian, with $\top = (\emptyset, \{0\})$ and $X \,\&\, Y = (\{0\} \times |X| \cup \{1\} \times |Y|), \{z \in (\mathbb{R}_{\geq 0})^{\{0\}\times|X|\cup\{1\}} \mid \mathbf{p}_0(z) \in \mathsf{P}X \text{ and } \mathbf{p}_1(z) \in \mathsf{P}Y\}$ which is easily seen to be a PCS ($\mathbf{p}_i$ is defined exactly as π_i) such that $\mathsf{P}(X \,\&\, Y) = \mathsf{P}X \times \mathsf{P}Y$ up to a trivial bijection. The projections $\mathbf{p}_i$ are additive, and $\mathbf{c}_\&$ (see Section 4.1) is an iso: if $((x, u), (y, v)) \in \mathsf{P}(\mathsf{D}X \,\&\, \mathsf{D}Y)$ then $((x, y), (u, v)) \in \mathsf{P}(\mathsf{D}(X \,\&\, Y))$ since $(x, y) + (u, v) = (x + u, y + v)$ so the summability structure is compatible with the Cartesian product by Corollary 7.

An $f \in \mathbf{APcoh}(X_0 \,\&\, X_1, Y)$ is *bilinear* in X_0, X_1 if it is linear (or additive) separately in both inputs, which is equivalent to saying that its matrix t satisfies that if $t_{m,b} \neq 0$ then $m = [(0, a_0), (1, a_1)]$ with $a_i \in |X_i|$ for $i = 0, 1$. Let $\mathsf{N} = (\mathbb{N}, \{x \in (\mathbb{R}_{\geq 0})^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} x_n \leq 1\})$ which represents the type of integers in **APcoh**, then the function $h : \mathbf{APcoh}(\mathsf{N} \,\&\, \mathsf{N} \,\&\, \mathsf{N}, \mathsf{N})$ given by $h(u, x, y) = u_0 x + (\sum_{n=1}^{\infty} u_n) y$ is bilinear in N, $\mathsf{N} \,\&\, \mathsf{N}$ and can be understood as

[15] Which arises from the fact that **APcoh** is the Kleisli category of the comonad "!" on the PCS model of LL of [17].

an `ifzero` operator. The function $k \in \mathbf{APcoh}(\mathsf{N}, \mathsf{N})$ such that $k(x)_n = x_{n+1}$ is linear and represents the successor operation.

6 Link with Cartesian Differential Categories

We show in this section that CCDCs are a generalization of Cartesian Differential Categories [4].

6.1 Cartesian Left Additive Categories

We rely on the presentation of [18] for Left Additive Categories, as it corresponds to a minimal set of assumptions.

Definition 24. A Left Additive Category is a category such that each hom-set is a commutative monoid, with addition $+$ and zero 0 left compatible with composition, that is $(f + g) \circ h = f \circ h + g \circ h$ and $0 \circ f = 0$

Definition 25. A map h is additive if addition is compatible with composition with h on the left, that is $h \circ (f + g) = h \circ f + h \circ g$ and $h \circ 0 = 0$. Note that the identity is additive, and additive morphisms are closed under addition and composition.

Definition 26. A Cartesian Left Additive Category is a Left Additive Category such that the projections are additive.

Given any Cartesian Left Additive category $\mathcal{C}$, one can define a Summable Pairing Structure (Definition 1) $(\mathsf{D}_{\&}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_0 + \mathbf{p}_1)$ with $\mathsf{D}_{\&} X = X \mathbin{\&} X$. Then one can check that all morphisms are summable (the witness of $f \boxplus g$ is $\langle f, g \rangle$). Moreover, the fact that the category is left additive ensures that the notion of sum induced by $(\mathsf{D}_{\&}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_0 + \mathbf{p}_1)$ coincides with the native structure of monoid on the homs-et. In particular, a morphism is additive in the sense of Definition 3 if and only if it is additive in the sense of Definition 26. Consequently, $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{p}_0 + \mathbf{p}_1$ are additive. Thus, $(\mathsf{D}_{\&}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_0 + \mathbf{p}_1)$ is a Left Pre-Summability Structure. Finally, it is *de facto* a Left Summability Structure because (D-witness) trivially holds (everything is summable), and (D-zero), (D-com) hold thanks to the fact that everything is summable and that $(\mathcal{C}(X, Y), +, 0)$ is a commutative monoid.

Conversely, any Left Summability Structure on $\mathcal{C}$ of shape $(\mathsf{D}_{\&}, \mathbf{p}_0, \mathbf{p}_1, \sigma)$ with $\mathsf{D}_{\&} X = X \mathbin{\&} X$ endow each hom-set with a commutative monoid structure and Proposition 2 ensures that the category is left additive. Then, as above, a morphism is additive in the sense of Definition 3 if and only if it is additive in the sense of Definition 26. Thus $\mathbf{p}_0, \mathbf{p}_1$ are additive so the category is Cartesian left additive. Moreover $\sigma = \mathbf{p}_0 + \mathbf{p}_1$ by Proposition 1 so the Left Summability Structure induced by the monoid on the hom-set coincides with the Left Summability Structure we started from. To summarize.

Theorem 9. *Let $\mathcal{C}$ be a cartesian category. Define $\mathsf{D}_{\&} X = X \mathbin{\&} X$. There is a bijection between the monoid structures on the hom-set that makes $\mathcal{C}$ a Cartesian Left Additive category and the Left Summability Structures $(\mathsf{D}, \pi_0, \pi_1, \sigma)$ on $\mathcal{C}$ such that $\mathsf{D} = \mathsf{D}_{\&}$, $\pi_0 = \mathbf{p}_0$ and $\pi_1 = \mathbf{p}_1$.*

Remark 7. Any Left Summability Structure on $\mathcal{C}$ of shape $(\mathsf{D}_{\&}, \mathbf{p}_0, \mathbf{p}_1, \sigma)$ with $\mathsf{D}_{\&} X = X \& X$ is *de facto* compatible with the cartesian product. The additivity of $\mathbf{p}_0$ and $\mathbf{p}_1$ is part of the axioms of summability, and $\mathbf{c}_{\&}$ turns out to be equal to $\mathbf{c}$ so it is an isomorphism because it is involutive.

6.2 Cartesian Differential Categories

We give the axioms of a Cartesian Differential Category following the alternative formulation of [5] for convenience. Note that the axiom $\mathbf{d}(\mathsf{id}) = \mathbf{p}_1$ seems to be missing from the axioms given in [5], although it can be found in the original formulation in [4].

Definition 27. A Cartesian Differential Category is a cartesian Left Additive Category $\mathcal{C}$ equipped with a differential combinator $\mathbf{d}$ that maps each morphism $f \in \mathcal{C}(X, Y)$ to a morphism $\mathbf{d}(f) \in \mathcal{C}(X \,\&\, X, Y)$ such that

(1) $\mathbf{d}(\mathbf{p}_0) = \mathbf{p}_0 \circ \mathbf{p}_1$, $\mathbf{d}(\mathbf{p}_1) = \mathbf{p}_1 \circ \mathbf{p}_1$;

(2) $\mathbf{d}(0) = 0$ and $\mathbf{d}(f + g) = \mathbf{d}(f) + \mathbf{d}(g)$;

(3) $\mathbf{d}(\mathsf{id}) = \pi_1$ and $\mathbf{d}(g \circ f) = \mathbf{d}(g) \circ \langle f \circ \pi_0, \mathbf{d}(f)\rangle$;

(4) $\mathbf{d}(f) \circ \langle x, 0\rangle = 0$ and $\mathbf{d}(f) \circ \langle x, u + v\rangle = \mathbf{d}(f) \circ \langle x, u\rangle + \mathbf{d}(f) \circ \langle x, v\rangle$;

(5) $\mathbf{d}(\mathbf{d}(f)) \circ \langle\langle x, 0\rangle, \langle 0, u\rangle\rangle = \mathbf{d}(f) \circ \langle x, u\rangle$;

(6) $\mathbf{d}(\mathbf{d}(f)) \circ \langle\langle x, u\rangle, \langle v, w\rangle\rangle = \mathbf{d}(\mathbf{d}f) \circ \langle\langle x, v\rangle, \langle u, w\rangle\rangle$.

There is usually another axiom, that states that $\mathbf{d}(\langle f, g\rangle) = \langle \mathbf{d}(f), \mathbf{d}(g)\rangle$. But as observed in [18], this axiom is a consequence of the linearity of the projections and of the chain rule so we discard it. Besides, we chose a different ordering of the axioms to stick closer to the presentation of Cartesian Coherent Differential Categories.

Let $\mathcal{C}$ be a left additive category. As stated in Theorem 9, $\mathcal{C}$ admits a Summability Structure $(\mathsf{D}_\&, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_0 + \mathbf{p}_1)$ compatible with the cartesian product whose induced notion of sum coincides with the sum in the hom-set. Then, there is a bijection between Pre-Differential Structures on top of this Summability Structure and differential combinators in the sense of Definition 27: we can define D from $\mathbf{d}$ as $\mathsf{D}(f) := \langle f \circ \mathbf{p}_0, \mathbf{d}(f)\rangle$, and we can define $\mathbf{d}$ from D as $\mathbf{d} := \mathbf{d}^{\mathsf{D}}$.

Besides, we show in Section 3 that there is a one to one correspondence between the axioms of Coherent Differential Categories concerning D and the axioms of Cartesian Differential categories concerning $\mathbf{d}^{\mathsf{D}} = \mathbf{d}$. The axioms (1) are the same. The correspondence of (2) is exactly Corollary 5. The correspondence of (3) is exactly Proposition 15. The correspondence of (4) is exactly Proposition 17 and Proposition 18. The correspondence of (5) is exactly Proposition 19. Finally, the correspondence of (6) is exactly Proposition 20.

The Differential Structures on top of the Left Summability Structure $(\mathsf{D}_\&, \mathbf{p}_0, \mathbf{p}_1, \sigma)$ are *de facto* compatible with the cartesian product, because the linearity of $\mathbf{p}_0$ and $\mathbf{p}_1$ is included in (1). To conclude.

Theorem 10. *The Cartesian Differential Categories are exactly the Cartesian Coherent Differential Categories in which* $\mathsf{D}X = X \,\&\, X$, $\pi_0 = \mathbf{p}_0$, $\pi_1 = \mathbf{p}_1$.

Remark 8. In [4], h is said to be linear if $\mathbf{d}(h) = h \circ \mathbf{p}_1$. At first glance, it seems to be a weaker condition than the one of Definition 16 as h has no reason to be additive. But in any CCDC, (D-add) ensures that $\mathbf{d}^{\mathsf{D}}(h) = h \circ \pi_1$ is a necessary and sufficient condition to ensure h is D-linear. Indeed, assume that $\mathbf{d}^{\mathsf{D}}(h) = h \circ \pi_1$. Then $h \circ 0 = h \circ \mathbf{p}_1 \circ \langle\!\langle 0, 0\rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(h) \circ \langle\!\langle 0, 0\rangle\!\rangle = 0$ by Proposition 17 and $h \circ (f_0 + f_1) = h \circ \pi_1 \circ \langle\!\langle 0, f_0 + f_1\rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(h) \circ \langle\!\langle 0, f_0 + f_1\rangle\!\rangle = \mathbf{d}^{\mathsf{D}}(h) \circ \langle\!\langle 0, f_0\rangle\!\rangle + \mathbf{d}^{\mathsf{D}}(h) \circ \langle\!\langle 0, f_1\rangle\!\rangle$ by Proposition 18. Hence, $h \circ 0 = \mathbf{d}^{\mathsf{D}}(h) \circ \langle\!\langle 0, f_0\rangle\!\rangle + \mathbf{d}^{\mathsf{D}}(h) \circ \langle\!\langle 0, f_1\rangle\!\rangle = h \circ \pi_1 \circ \langle\!\langle 0, f_0\rangle\!\rangle + h \circ \pi_1 \circ \langle\!\langle 0, f_1\rangle\!\rangle = h \circ f_0 + h \circ f_1$. Thus, h is additive and $\mathbf{d}^{\mathsf{D}}(h) = h \circ \pi_1$ so h is D-linear.

7 A first order Coherent Differential language

Let us briefly introduce a first order language associated to these models. Note that a development of a whole Coherent Differential PCF can already be found in [15], with a semantics based on LL Coherent Differentiation, of which our language can be roughly considered as a fragment. Our main contribution here is that CCDCs provide the tools for a much more principled and synthetic treatment of the semantics. This tighter connection between syntax and semantics allows for the development of new ideas, such as a more systematic treatment of multilinearity.

7.1 Terms

Definition 28. Le $\mathcal{B}$ be a set of ground type symbols, ranged over by $\alpha, \beta, \ldots$ For any $\alpha \in \mathcal{B}$ and $h \in \mathbb{N}$, $\mathsf{D}^h\alpha$ is a ground type. General types are inductively defined by

$$A, B, C := \mathsf{D}^h\alpha \mid A \,\&\, B\,.$$

where h is an arbitrary element of $\mathbb{N}$.

For any type A, we define the type $\mathsf{D}A$ inductively on A by $\mathsf{D}\mathsf{D}^h\alpha = \mathsf{D}^{h+1}\alpha$ and $\mathsf{D}(A \,\&\, B) = \mathsf{D}A \,\&\, \mathsf{D}B$.

Definition 29. Let $\phi, \psi, \ldots$ be function symbols. Each function symbol ϕ is uniquely assigned a *function type* of the form $A_0, \ldots, A_n \to B$ where A_i and B are types. Then, $n+1$ is called the arity of ϕ, denoted as $\mathsf{ar}(\phi)$.

A function symbol ϕ of type $A_0, \ldots, A_n \to B$ will be interpreted in section Section 7.2 as a $(n+1)$-linear morphisms $[\![\phi]\!] \in \mathcal{C}([\![A_0]\!] \,\&\, \cdots \,\&\, [\![A_n]\!], [\![B]\!])$ (recall Definition 22). Note that the types A_i can themselves be products and need not to be ground types. For example, a 2-linear map in $\mathcal{C}((A\&B)\&C, D)$ can by no means be seen as a 3-linear map in $\mathcal{C}(A\&B\&C, D)$.

Definition 30. Define *functions* as

$$f, g, \ldots := \phi \mid \pi_i^A \mid \mathbf{p}_i^{A,B} \mid \iota_i^A \mid \theta_n^A$$

where $i \in \{0, 1\}$, $n \geq 0$, ϕ are function symbols and A, B are types. Each function f has a function type: π_0^A, π_1^A have type $\mathsf{D}A \to A$, ι_0^A, ι_1^A have type $A \to \mathsf{D}A$, the θ_n^A have type $\mathsf{D}^{n+1}A \to \mathsf{D}A$ and $\mathbf{p}_0^{A,B}, \mathbf{p}_1^{A,B}$ have types $A \,\&\, B \to A$ and $A \,\&\, B \to B$ respectively. Notice that projections have arity 1 and not 2. The type attached to the constructors π_i, $\mathbf{p}_i$, ι_i and θ_n will always be kept implicit in what follows.

Remark 9. Taking $n = -1$ allows to write constants.

Definition 31. Let $\mathcal{V}$ be a set of variable symbols. The set Λ_1 of terms is defined inductively as follows

$$t, u, \ldots := \langle t_0, t_1 \rangle \mid f^\zeta(t_0, \ldots, t_n) \mid x$$

where $x \in \mathcal{V}$, f are function symbols of arity $n+1$ and $\zeta \in [\![0, n]\!]^*$, the set of finite words[16] of elements of $[\![0, n]\!]$.

[16] Such a word represents a successive application of partial derivatives on the multilinear symbol f, more on this in the semantics section.

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \text{ (Var)} \qquad \frac{\Gamma \vdash t_0 : A \qquad \Gamma \vdash t_1 : B}{\Gamma \vdash \langle t_0, t_1 \rangle : A \,\&\, B} \text{ (Pair)}$$

$$\frac{f : A_0, \dots, A_n \to B \qquad \forall i, \Gamma \vdash t_i : \mathsf{D}^{|\zeta|_i} A_i}{\Gamma \vdash f^{\zeta}(t_0, \dots, t_n) : \mathsf{D}^{|\zeta|} B} \text{ (App)}$$

Figure 1: Typing rules

$$\partial(x, y) = \begin{cases} x & \text{if } y = x \\ \iota_0(y) & \text{otherwise} \end{cases}$$
$$\partial(x, \langle t_0, t_1 \rangle) = \langle \partial(x, t_0), \partial(x, t_1) \rangle$$
$$\partial(x, f^{\zeta}(t_0, \dots, t_n)) = \theta_n(f^{\zeta n \cdots 10}(\partial(x, t_0), \dots, \partial(x, t_n)))$$

Figure 2: Differential of a term

Remark 10. Nothing prevents us from adding to this calculus non multilinear function symbols, assuming that the formal derivatives for the function symbols are also provided. We focus on multilinear functions though, due to the nature of the basic operations of PCF. A Coherent Differential PCF would contain a base type `nat`, two function symbols `pred` and `succ` of type `nat` $\to$ `nat`, a family of function symbols $\texttt{if}^A$ of type $\texttt{nat}, A \,\&\, A \to A$ (conditional) and a family of function symbols $\texttt{let}^A$ of type $\texttt{nat}, (\texttt{nat} \to A) \to A$ (call-by-value on the type of integers). An analysis of the semantics of these symbols in Coherent Differentiation in the LL setting of [15] or in the example of Section 5.2 indeed shows that `pred` and `succ` should be interpreted as linear morphisms, and that $\texttt{if}^A$ and $\texttt{let}^A$ should be interpreted as 2-linear morphisms. Using the PCF fixpoint operator it is then possible to write terms whose interpretation is not multilinear. For instance, f_1 of Section 5.2 is the semantics of a term, see [19].

Notations 5. For any word ζ, we write $|\zeta|$ for its length, and $|\zeta|_j$ for the number of occurrences of the letter j. We will write f for f^{ϵ}, where ϵ is the empty word. Besides, when $\mathsf{ar}(f) = 0$, a word $\zeta \in [\![0, 0]\!]^*$ can be uniquely seen as an integer (its length $|\zeta|$). We will then write $f^{(d)}$ for f^{ζ}, where $d = |\zeta|$.

We introduce the typing rules in Figure 1. The systematic treatment of multilinear morphisms allows for a great factorization of the rules. We write $f : A_0, \dots, A_n \to B$ if f has type $A_0, \dots, A_n \to B$.

Given any term t, one can define a term $\partial(x, t)$ by induction on t. The inductive steps are given in Figure 2.

Proposition 37. *If* $\Gamma, x : A \vdash t : B$ *then* $\Gamma, x : \mathsf{D}A \vdash \partial(x, t) : \mathsf{D}B$

Proof. By induction on the typing derivation.

- If the last rule applied is (Var) then the first possibility is that $t = x$ and $\Gamma, x : A \vdash x : A$. But then, $\partial(x, x) = x$ and $\Gamma, x : \mathsf{D}A \vdash x : \mathsf{D}A$. The second possibility is that $t = y$ with $y \neq x$ and $\Gamma \vdash y : B$. But then, $\partial(x, y) = \iota_0(y)$ and $\Gamma \vdash \iota_0(y) : \mathsf{D}B$. Thus, $\Gamma, x : \mathsf{D}A \vdash \iota_0(y) : \mathsf{D}B$ in both cases.

- If the last rule applied is (Pair), then $t = \langle t_0, t_1\rangle$, t is of type $B_0 \& B_1$, $\Gamma, x : A \vdash t_0 : B_0$ and $\Gamma, x : A \vdash t_1 : B_1$. But $\partial(x, t) = \langle \partial(x, t_0), \partial(x, t_1)\rangle$. By induction hypothesis $\Gamma, x : \mathsf{D}A \vdash \partial(x, t_0) : \mathsf{D}B_0$ and $\Gamma, x : \mathsf{D}A \vdash \partial(x, t_1) : \mathsf{D}B_1$. Thus, by applying (Pair), $\Gamma, x : \mathsf{D}A \vdash \langle \partial(x, t_0), \partial(x, t_1)\rangle : \mathsf{D}B_0 \& \mathsf{D}B_1$. But $\mathsf{D}B_0 \& \mathsf{D}B_1 = \mathsf{D}(B_0 \& B_1)$ so $\Gamma, x : \mathsf{D}A \vdash \partial(x, \langle t_0, t_1\rangle) : \mathsf{D}(B_0 \& B_1)$.
- If the last rule applied is (App) then $t = f^{\zeta}(t_0, \dots, t_n)$, f has some type $A_0, \dots, A_n \to B$, and $\Gamma, x : A \vdash t : \mathsf{D}^{|\zeta|}B$. Besides, for any i, $\Gamma, x : A \vdash t_i : \mathsf{D}^{|\zeta|_i}A_i$. By induction hypothesis, $\Gamma, x : \mathsf{D}A \vdash \partial(x, t_i) : \mathsf{D}^{|\zeta|_i+1}A_i$. But $|\zeta n \cdots 10|_i = |\zeta|_i + 1$ so applying the (App) rule gives a derivation for $\Gamma, x : \mathsf{D}A \vdash f^{\zeta n \cdots 10}(\partial(x, t_0), \dots, \partial(x, t_n)) : \mathsf{D}^{|\zeta|+n+1}B$. Applying the (App) rule again for $f = \theta_n$ yields a derivation of $\Gamma, x : \mathsf{D}A \vdash \theta_n(f^{\zeta n \cdots 10}(\partial(x, t_0), \dots, \partial(x, t_n))) : \mathsf{D}^{|\zeta|+1}B$, which concludes the proof.

□

7.2 Semantics

Let $\mathcal{C}$ be a CCDC. For the sake of simplicity, we assume that $\mathsf{D}(X \& Y) = \mathsf{D}X \& \mathsf{D}Y$ and $\mathbf{c}_\& = \mathsf{id}$. Assume that we are given an object $[\![\alpha]\!]$ for any ground type symbol α. Then one can interpret any type as an object: $[\![\mathsf{D}^h\alpha]\!] = \mathsf{D}^h[\![\alpha]\!]$ and $[\![A \& B]\!] = [\![A]\!] \& [\![B]\!]$. It follows by a straightforward induction that $[\![\mathsf{D}A]\!] = \mathsf{D}[\![A]\!]$. This interpretation extends as usual to contexts, setting $[\![x_0 : A_0, \dots, x_n : A_n]\!] = [\![A_0]\!] \& \cdots \& [\![A_n]\!]$. The semantics of the empty context is $\top$.

Assume that we are given a $(n+1)$-linear morphism $[\![\phi]\!] \in \mathcal{C}([\![A_0]\!] \& \cdots \& [\![A_n]\!], [\![B]\!])$ for any function symbol $\phi : A_0, \dots, A_n \to B$. Then any function $f : A_0, \dots, A_n \to B$ can be interpreted as a $(n+1)$-linear morphism $[\![f]\!]$ setting $[\![\pi_i]\!] = \pi_i$, $[\![\iota_i]\!] = \iota_i$, $[\![\theta_n]\!] = \theta^n$ (as defined in Definition 21) and $[\![\mathbf{p}_i]\!] = \mathbf{p}_i$.

Remark 11. Since $\mathbf{c}_\& = \mathsf{id}$, $\mathsf{D}\mathbf{p}_i = \mathsf{D}\mathbf{p}_i \circ (\mathbf{c}_\&)^{-1} = \mathsf{D}\mathbf{p}_i \circ \langle\!\langle \pi_0 \& \pi_0, \pi_1 \& \pi_1\rangle\!\rangle = \langle\!\langle \mathbf{p}_i \circ (\pi_0 \& \pi_0), \mathbf{p}_i \circ (\pi_1 \& \pi_1)\rangle\!\rangle = \langle\!\langle \pi_0 \circ \mathbf{p}_i, \pi_1 \circ \mathbf{p}_i\rangle\!\rangle = \mathbf{p}_i$. Besides, $\langle \mathsf{D}f_0, \mathsf{D}f_1\rangle = \mathsf{D}\langle f_0, f_1\rangle$ by Proposition 23

Theorem 11. *For any term t such that $\Gamma \vdash t : A$, we can define $[\![t]\!]_\Gamma \in \mathcal{C}([\![\Gamma]\!], [\![A]\!])$. This morphism does not depends on the typing derivation of $\Gamma \vdash t : A$.*

Proof. We proceed by induction on the term.

- If $t = x$ then the last typing rule must be (Var). It implies that $\Gamma = \Gamma_0, x : A, \Gamma_1$. Define $[\![x]\!]_\Gamma = \mathbf{p}_{|\Gamma_0|} \in \mathcal{C}([\![\Gamma_0]\!] \& [\![A]\!] \& [\![\Gamma_1]\!], [\![A]\!])$.
- If $t = \langle t_0, t_1\rangle$ then the last typing rule must be (Pair), so t is of type $A \& B$, $\Gamma \vdash t_0 : A$ and $\Gamma \vdash t_1 : B$. By induction, one can define $[\![t_0]\!]_\Gamma \in \mathcal{C}([\![\Gamma]\!], [\![A]\!])$ and $[\![t_1]\!]_\Gamma \in \mathcal{C}([\![\Gamma]\!], [\![B]\!])$. Then we define $[\![\langle t_0, t_1\rangle]\!]_\Gamma = \langle [\![t_0]\!]_\Gamma, [\![t_1]\!]_\Gamma\rangle \in \mathcal{C}([\![\Gamma]\!], [\![A \& B]\!])$.
- If $t = f^{\zeta}(t_0, \dots, t_n)$ with $f : A_0, \dots, A_n \to B$ then the last typing rule must be (App). That is, t must be of type $D^{|\zeta|}B$ for some type B and for all i, we have a derivation of $\Gamma \vdash t_i : \mathsf{D}^{|\zeta|_i}A_i$. By induction hypothesis, we can define $[\![t_i]\!]_\Gamma \in \mathcal{C}([\![\Gamma]\!], [\![D^{|\zeta|_i}A_i]\!])$. But $[\![D^{|\zeta|_i}A_i]\!] = \mathsf{D}^{|\zeta|_i}[\![A_i]\!]$ and $\mathsf{D}_{\zeta_k} \dots \mathsf{D}_{\zeta_1}[\![f]\!] \in \mathcal{C}(\mathsf{D}^{|\zeta|_0}[\![A_0]\!] \& \cdots \& \mathsf{D}^{|\zeta|_n}[\![A_n]\!], \mathsf{D}^{|\zeta|}[\![B]\!])$. Thus, we can set $[\![f^{\zeta_1 \cdots \zeta_k}(t_0, \dots, t_n)]\!]_\Gamma = \mathsf{D}_{\zeta_k} \dots \mathsf{D}_{\zeta_1}[\![f]\!] \circ \langle [\![t_0]\!]_\Gamma, \dots, [\![t_n]\!]_\Gamma\rangle$.

□

Notations 6. We will write $[\![x]\!]_\Gamma = \mathbf{p}_x$ to designate the projection on $[\![\Gamma]\!]$ to the coordinate where x appears in Γ.

Remark 12. In particular, $[\![\pi_i^{(d)}(t)]\!] = \mathsf{D}^d\pi_i \circ [\![t]\!]$, $[\![\iota_i^{(d)}(t)]\!] = \mathsf{D}^d\iota_i \circ [\![t]\!]$, $[\![\theta_n^{(d)}(t)]\!] = \mathsf{D}^d(\theta^n) \circ [\![t]\!]$. More importantly, $[\![\mathbf{p}_i^{(d)}(t)]\!] = \mathsf{D}^d\mathbf{p}_i \circ [\![t]\!] = \mathbf{p}_i \circ [\![t]\!]$ because of our assumption that $\mathbf{c}_\&$ is the identity.

Notations 7. For any word $\zeta = \zeta_1 \cdots \zeta_k$ in $[\![0, n]\!]^k$, define $\mathsf{D}_\zeta := \mathsf{D}_{\zeta_k} \dots \mathsf{D}_{\zeta_1}$. Then for any $f \in \mathcal{C}(X_0 \,\&\, \dots \,\&\, X_n, Y)$, $\mathsf{D}_\zeta f \in \mathcal{C}(\mathsf{D}^{|\zeta|_0} X_0 \,\&\, \cdots \,\&\, \mathsf{D}^{|\zeta|_n} X_n, \mathsf{D}^{|\zeta|} Y)$. Note that $\mathsf{D}_{\zeta\cdot\delta} = \mathsf{D}_\delta \mathsf{D}_\zeta$. Then, Proposition 33 can be seen as the property that for any f $(n+1)$-linear, for any word δ of length d, $\mathsf{D}^d\pi_i \circ \mathsf{D}_\delta \mathsf{D}_j f = \mathsf{D}_\delta f \circ (\mathsf{D}^{|\delta|_j}\pi_i; \mathsf{id}_{-j})$

The main result of this section on the calculus consists in showing that the semantics of the syntactical derivative operation corresponds to the derivative in the model.

Theorem 12. *If $\Gamma, x : A \vdash t : B$ then $[\![\partial(x,t)]\!]_{\Gamma,x:\mathsf{D}A} = \mathsf{D}_1[\![t]\!]_{\Gamma,x:A}$ where $[\![t]\!]_{\Gamma,x:A}$ is seen as a morphisms of $\mathcal{C}([\![\Gamma]\!] \,\&\, [\![A]\!], [\![B]\!])$.*

Proof. By induction on t.

- If $t = x$ then $[\![t]\!]_{\Gamma,x:A} = \mathbf{p}_1 \in \mathcal{C}([\![\Gamma]\!] \,\&\, [\![A]\!], [\![A]\!])$. Then $\mathsf{D}_1\mathbf{p}_1 = \mathsf{D}\mathbf{p}_1 \circ \Phi^1 = \mathsf{D}\mathbf{p}_1 \circ \langle\!\langle \mathsf{id} \,\&\, \pi_0, 0 \,\&\, \pi_1 \rangle\!\rangle = \langle\!\langle \mathbf{p}_1 \circ (\mathsf{id} \,\&\, \pi_0), \mathbf{p}_1 \circ (0 \,\&\, \pi_1) \rangle\!\rangle = \langle\!\langle \pi_0 \circ \mathbf{p}_1, \pi_1 \circ \mathbf{p}_1 \rangle\!\rangle = \mathbf{p}_1$ using Proposition 24 and the linearity of $\mathbf{p}_1$.
- If $t = y \neq x$ then $[\![t]\!]_{\Gamma,x:A} = [\![y]\!]_\Gamma \circ \mathbf{p}_0 = \mathbf{p}_y \circ \mathbf{p}_0 \in \mathcal{C}([\![\Gamma]\!] \,\&\, [\![A]\!], [\![B]\!])$. Then $\mathsf{D}_1(\mathbf{p}_y \circ \mathbf{p}_0) = \mathsf{D}\mathbf{p}_y \circ \mathsf{D}\mathbf{p}_0 \circ \Phi^1 = \mathsf{D}\mathbf{p}_y \circ \mathsf{D}\mathbf{p}_0 \circ \langle\!\langle \mathsf{id} \,\&\, \pi_0, 0 \,\&\, \pi_1 \rangle\!\rangle = \mathsf{D}\mathbf{p}_y \circ \langle\!\langle \mathbf{p}_0 \circ (\mathsf{id} \,\&\, \pi_0), \mathbf{p}_0 \circ (0 \,\&\, \pi_1) \rangle\!\rangle = \mathsf{D}\mathbf{p}_y \circ \langle\!\langle \mathbf{p}_0, 0 \rangle\!\rangle = \langle\!\langle \mathbf{p}_y \circ \mathbf{p}_0, 0 \rangle\!\rangle = [\![\iota_0(y)]\!] = [\![\partial(x,y)]\!]$.
- If $t = \langle t_0, t_1 \rangle$, then $[\![\partial(x,t)]\!] = [\![\langle \partial(x,t_0), \partial(x,t_1) \rangle]\!] = \langle [\![\partial(x,t_0)]\!], [\![\partial(x,t_1)]\!] \rangle$. By induction hypothesis, $[\![\partial(x,t)]\!] = \langle \mathsf{D}_1[\![t_0]\!], \mathsf{D}_1[\![t_1]\!] \rangle$. But $\langle \mathsf{D}_1[\![t_0]\!], \mathsf{D}_1[\![t_1]\!] \rangle = \langle \mathsf{D}[\![t_0]\!] \circ \Phi^1, \mathsf{D}[\![t_1]\!] \circ \Phi^1 \rangle = \langle \mathsf{D}[\![t_0]\!], \mathsf{D}[\![t_1]\!] \rangle \circ \Phi^1$. By Remark 11, this is equal to $\mathsf{D}\langle [\![t_0]\!], [\![t_1]\!] \rangle \circ \Phi^1 = \mathsf{D}_1 \langle [\![t_0]\!], [\![t_1]\!] \rangle = \mathsf{D}_1[\![t]\!]$.
- If $t = f^\zeta(t_0, \dots, t_n)$ then $\partial(x,t) = \theta_n(f^{\zeta n \cdots 10}(\partial(x,t_0), \dots, \partial(x,t_n)))$. Thus, $[\![\partial(x,t)]\!] = \theta^n \circ \mathsf{D}_{n\cdots 10}\mathsf{D}_\zeta f \circ \langle [\![\partial(x,t_0)]\!], \dots, [\![\partial(x,t_n)]\!] \rangle = \theta^n \circ \mathsf{D}_{n\cdots 10}\mathsf{D}_\zeta f \circ \langle \mathsf{D}_1[\![t_0]\!], \dots, \mathsf{D}_1[\![t_n]\!] \rangle$ by induction hypothesis. But then, the Leibniz rule (Proposition 32) states that $\theta^n \circ \mathsf{D}_{n\cdots 10}\mathsf{D}_\zeta f = \mathsf{D}\mathsf{D}_\zeta f$. Thus, $[\![\partial(x,t)]\!] = \mathsf{D}\mathsf{D}_\zeta f \circ \langle \mathsf{D}[\![t_0]\!] \circ \Phi^1, \dots, \mathsf{D}[\![t_n]\!] \circ \Phi^1 \rangle = \mathsf{D}\mathsf{D}_\zeta f \circ \langle \mathsf{D}[\![t_0]\!], \dots, \mathsf{D}[\![t_n]\!] \rangle \circ \Phi^1 = \mathsf{D}(\mathsf{D}_\zeta f \circ \langle [\![t_0]\!], \dots, [\![t_n]\!] \rangle) \circ \Phi^1 = \mathsf{D}[\![t]\!] \circ \Phi^1 = \mathsf{D}_1[\![t]\!]$.

□

7.3 Reduction

We introduce in this section a set of reduction rules that deals with the differential content of the terms. The set of rules is more compact than the one given in [15], but covers all of the rules concerning the fragment we are looking at.

Remark 13. We could have added a construct $t[u/x]$ for explicit substitutions, with the typing rule

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash t[u/x] : B}(\text{Cut})$$

As well as reduction rules that performs the substitution steps (for example, $x[u/x] \to_{\Lambda_1} u$). We decided not to do so, as the use of explicit substitutions introduces some subtleties that will anyway be dealt with in the higher order setting in a more unified approach.

The main difference with the differential lambda-calculus of [20] is the absence of sum, because we do not want a non deterministic typing rules such as

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash t + u : A}$$

But the reduction of a π_1 against a θ will introduce sums. Handling sum without the typing rule above is tricky, because of subject reduction. There will be no guarantee indeed that if $\Gamma \vdash t + u : A$ and $t \to_{\Lambda_1} t'$ then $\Gamma \vdash t' + u : A$. For this reason, we chose a conservative approach, by keeping sums as a formal multi-set on top of the terms.

Definition 32. A term multi-set is a finite multi-set of term.

See Section 5.2 for the notations we use on multisets. We define a reduction $\to_{\Lambda_1}$ from terms to term multi-sets. The reduction rules are given in Figure 3. Then we define $\to^?_{\Lambda_1}$ as the "reflexive" closure of $\to_{\Lambda_1}$. That is, $t \to^?_{\Lambda_1} L$ if $t \to_{\Lambda_1} L$ or if $L = [t]$. It allow to lifts $\to_{\Lambda_1}$ to a reduction from a term multi-set to a term multi-set in a monadic fashion: if for all i, $t_i \to^?_{\Lambda_1} L_i$, then

$$[t_1, \dots, t_n] \to_{\mathcal{M}(\Lambda_1)} \sum_{i=1}^{n} L_i$$

where $\sum$ is the multi-set union, that is, the pointwise sum of the functions $L_i : \Lambda_1 \to \mathbb{N}$.

$$\begin{aligned}
\mathbf{p}_i^{(d)}(\langle t_0, t_1 \rangle) &\to_{\Lambda_1} [t_i] \\
\pi_i^{(d)}(f^{\zeta j \delta}(t_0, \dots, t_n)) &\to_{\Lambda_1} [f^{\zeta\delta}(t_0, \dots, \pi_i^{(|\delta|_j)}(t_j), \dots, t_n)] \quad \text{where } |\delta| = d \\
\pi_i^{(d)}(\iota_i^{(d)}(t)) &\to_{\Lambda_1} [t] \\
\pi_i^{(d)}(\iota_{1-i}^{(d)}(t)) &\to_{\Lambda_1} [\,] \\
\pi_0^{(d)}(\theta_n^{(d)}(t)) &\to_{\Lambda_1} [(\pi_0^{(d)})^{n+1}(t)] \\
\pi_1^{(d)}(\theta_n^{(d)}(t)) &\to_{\Lambda_1} \sum_{k=0}^{n} [(\pi_0^{(d)})^k \pi_1^{(d)} (\pi_0^{(d)})^{n-k}(t)]
\end{aligned}$$

Here, $(\pi_i^{(d)})^n$ is a notation for n successive applications of $\pi_i^{(d)}$.

Figure 3: Reduction rules

Definition 33. A term multi-set $[t_1, \dots, t_n]$ of type A in context Γ is $\mathcal{C}$-summable if $[\![t_1]\!]_\Gamma, \dots, [\![t_n]\!]_\Gamma$ are summable (in the sense of Theorem 2). Then, we define $[\![[t_1, \dots, t_n]]\!]_\Gamma = [\![t_1]\!]_\Gamma + \dots + [\![t_n]\!]_\Gamma$. Note that $[\,]$ is always $\mathcal{C}$-summable, and $[\![[\,]]\!] = 0$.

The main point of Coherent Differentiation is that the reduction $\to_{\Lambda_1}$ will always introduce term multi-sets that are $\mathcal{C}$-summable, for any model $\mathcal{C}$.

Theorem 13 (Invariance of semantics under reduction)**.** *For any $\Gamma \vdash t : A$, if $t \to_{\Lambda_1} L$ then L is $\mathcal{C}$-summable and $[\![L]\!]_\Gamma = [\![t]\!]_\Gamma$.*

Proof. Let us consider every application of the rule $\to_{\Lambda_1}$. Note that when a term multi-set has one element, it is always $\mathcal{C}$-summable and $[\![[t]]\!] = [\![t]\!]$.

$$\begin{aligned}
[\![\mathbf{p}_i^{(d)}(\langle t_0, t_1 \rangle)]\!] &= \mathsf{D}^d \mathbf{p}_i \circ \langle [\![t_0]\!], [\![t_1]\!] \rangle \\
&= \mathbf{p}_i \circ \langle [\![t_0]\!], [\![t_1]\!] \rangle \\
&= [\![t_i]\!]
\end{aligned}$$

The rule below is the one where most of the differential content appears. Recall that $[\![f]\!]$ is assumed to be multilinear, for any function f. It implies that $\mathsf{D}_\zeta[\![f]\!]$ is also multilinear by Theorem 6, so it is possible to apply Proposition 33 on it.

$$\begin{aligned}
&[\![\pi_i^{(d)}(f^{\zeta j \delta}(t_0,\dots,t_n))]\!] \\
&\quad= \mathsf{D}^d\pi_i \circ \mathsf{D}_\delta \mathsf{D}_j \mathsf{D}_\zeta[\![f]\!] \circ \langle [\![t_0]\!],\dots,[\![t_n]\!]\rangle \\
&\quad= \mathsf{D}_\delta \mathsf{D}_\zeta[\![f]\!] \circ (\mathsf{D}^{|\delta|_j}\pi_i; \mathsf{id}_{-j}) \circ \langle [\![t_0]\!],\dots,[\![t_n]\!]\rangle \quad \text{by Proposition 33} \\
&\quad= \mathsf{D}_{\zeta\delta}[\![f]\!] \circ \langle [\![t_0]\!],\dots,\mathsf{D}^{|\delta|_j}\pi_i \circ [\![t_j]\!],\dots,[\![t_n]\!]\rangle \\
&\quad= [\![f^{\zeta\delta}(t_0,\dots,\pi_i^{(|\delta|_j)}(t_j),\dots,t_n)]\!]
\end{aligned}$$

The three next rules are rather standard and are consequence of the definition of π_i, ι_j and θ^n.

$$\begin{aligned}
[\![\pi_i^{(d)}(\iota_i^{(d)}(t))]\!] &= \mathsf{D}^d\pi_i \circ \mathsf{D}^d\iota_i \circ [\![t]\!] \\
&= \mathsf{D}^d(\pi_i \circ \iota_i) \circ [\![t]\!] \quad \text{by (D-chain)} \\
&= \mathsf{D}^d\mathsf{id} \circ [\![t]\!] = [\![t]\!] \quad \text{by (D-chain)}
\end{aligned}$$

$$\begin{aligned}
[\![\pi_i^{(d)}(\iota_{1-i}^{(d)}(t))]\!] &= \mathsf{D}^d\pi_i \circ \mathsf{D}^d\iota_{1-i} \circ [\![t]\!] \\
&= \mathsf{D}^d(\pi_i \circ \iota_{1-i}) \circ [\![t]\!] \quad \text{by (D-chain)} \\
&= \mathsf{D}^d 0 \circ [\![t]\!] = 0 \circ [\![t]\!] \quad \text{by (Dsum-lin)} \\
&= 0 = [\![[\,]]\!]
\end{aligned}$$

$$\begin{aligned}
[\![\pi_0^{(d)}(\theta_n^{(d)}(t))]\!] &= \mathsf{D}^d\pi_0 \circ \mathsf{D}^d\theta^n \circ [\![t]\!] \\
&= \mathsf{D}^d(\pi_0 \circ \theta^n) \circ [\![t]\!] \quad \text{by (D-chain)} \\
&= \mathsf{D}^d(\pi_0^{n+1}) \circ [\![t]\!] \\
&= (\mathsf{D}^d\pi_0)^{n+1} \circ [\![t]\!] \quad \text{by (D-chain)} \\
&= [\![(\pi_0^{(d)})^{n+1}(t)]\!]
\end{aligned}$$

The last rule is where finite multisets of size greater than 1 are introduced. Most line of equation below should be interpreted as "the sum above is well defined, so the sum below is well defined and both are equal".

$$\begin{aligned}
[\![\pi_1^{(d)}(\theta_n^{(d)}(t))]\!] &= \mathsf{D}^d\pi_1 \circ \mathsf{D}^d\theta^n \circ [\![t]\!] \\
&= \mathsf{D}^d(\pi_1 \circ \theta^n) \circ [\![t]\!] \quad \text{by (D-chain)} \\
&= \left(\mathsf{D}^d(\sum_{k=0}^{n} \pi_0^k \circ \pi_1 \circ \pi_0^{n-k})\right) \circ [\![t]\!] \\
&= \left(\sum_{k=0}^{n} \mathsf{D}^d(\pi_0^k \circ \pi_1 \circ \pi_0^{n-k})\right) \circ [\![t]\!] \quad \text{by (Dsum-lin) and Proposition 16} \\
&= \sum_{k=0}^{n} \mathsf{D}^d(\pi_0^k \circ \pi_1 \circ \pi_0^{n-k}) \circ [\![t]\!] \quad \text{by Proposition 2}
\end{aligned}$$

$$= \sum_{k=0}^{n} (\mathsf{D}^d \pi_0)^k \circ \mathsf{D}^d \pi_1 \circ (\mathsf{D}^d \pi_0)^{n-k} \circ [\![t]\!] \quad \text{by (D-chain)}$$

$$= \sum_{k=0}^{n} [\![(\pi_0^{(d)})^k \pi_1^{(d)} (\pi_0^{(d)})^{n-k}(t)]\!]$$

Thus, $\sum_{k=0}^{n}[(\pi_0^{(d)})^k \pi_1^{(d)} (\pi_0^{(d)})^{n-k}(t)]$ is $\mathcal{C}$-summable of semantics $[\![\pi_1^{(d)}(\theta_n^{(d)}(t))]\!]$ □

Corollary 9. *For any term multi-set $\Gamma \vdash L : A$ that is $\mathcal{C}$-summable, if $L \to_{\mathcal{M}(\Lambda_1)} L'$ then L' is $\mathcal{C}$-summable and $[\![L']\!]_\Gamma = [\![L]\!]_\Gamma$.*

Proof. Assume that $[t_1, \dots, t_n]$ is $\mathcal{C}$-summable and that $[t_1, \dots, t_n] \to_{\mathcal{M}(\Lambda_1)} L$. That is, for any i, $t_i \to_{\Lambda_1}^{?} [t_i^1, \dots, t_i^{k_i}]$ and $L = \sum_{i=1}^{n}[t_i^1, \dots, t_i^{k_i}]$. Then by Theorem 13, for any i, $[\![t_i^1]\!], \dots, [\![t_i^{k_i}]\!]$ are summable of sum $[\![t_i]\!]$. By assumption, $[\![t_1]\!], \dots, [\![t_n]\!]$ are summable, that is, $\sum_{j=1}^{k_1}[\![t_1^j]\!], \dots, \sum_{j=1}^{k_n}[\![t_n^j]\!]$ are summable. By Theorem 2, it means that the family $[\![t_1^1]\!], \dots, [\![t_1^{k_1}]\!], \dots, [\![t_n^1]\!], \dots, [\![t_n^{k_n}]\!]$ is summable of sum

$$\sum_{i=1}^{n} \sum_{j=1}^{k_i} [\![t_i^j]\!] = \sum_{i=1}^{n} [\![t_i]\!]$$

Thus L is $\mathcal{C}$-summable and $[\![L]\!] = [\![[t_1, \dots, t_n]]\!]$. □

The usage of such terms multi-sets seems to be somewhat non deterministic. But any multi-set generated by reductions of the calculus can be interpreted as a summable family in deterministic models such as Probabilistic Coherence Spaces[17] or Non Uniform Coherence Spaces. This determinism of the models allows to prove in [15] a result that roughly state that whenever a closed term of *type integer* reduces to a term multi-set $C + [\underline{\nu}]$ (where $\underline{\nu}$ are the usual integer variables of PCF), then $[\![C]\!] = 0$. That is, only one of the branches of the reduction rule

$$\pi_1^{(d)}(\theta_n^{(d)}(t)) \to_{\Lambda_1} \sum_{k=0}^{n}[(\pi_0^{(d)})^k \pi_1^{(d)} (\pi_0^{(d)})^{n-k}(t)]$$

produces a non empty multiset. The proof relies on the fact that any term of type integer will be interpreted in **APcoh** as a dirac distribution δ_n on $\mathbb{N}$ or as the zero distribution, because the calculus does not feature any form of probabilistic branching. Thus, a term multiset of type integer is **APcoh**-summable if and only if there is at most one term in the multiset whose semantic is not 0. In particular, $[\![\underline{\nu}]\!]^{\mathbf{APcoh}} = \delta_\nu$ and $C + [\underline{\nu}]$ is **APcoh**-summable (by Corollary 9) so $[\![C]\!]^{\mathbf{APcoh}} = 0$. Note that a similar proof can be done, using the semantics of Non Uniform Coherence Spaces instead of Probabilistic Coherence Spaces. This observation lead to the development of a completely deterministic Krivine Machine, using a form of memory on the projections path.

Conclusion

We have introduced and studied a general categorical framework for Coherent Differentiation, a new approach to the differential calculus which does not require the ambient category to be (left-)additive. We have also proposed some basic syntactical constructs accounting in a term language for these new categorical constructs. These are the foundations for a

[17]Probabilistic branching is by no mean a form of non determinism

principled and systematic approach to the denotational semantics of functional programming languages like (probabilistic) PCF extended with Coherent Differentiation. As shown in [15] such an extension can perfectly feature deterministic or probabilistic behaviors, in sharp contrast with the Differential λ-calculus [21] which is inherently non-deterministic. The next step will be to specialize the present general axiomatization to the case where the category is cartesian closed.

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