Intersection Types and Computational Complexity

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PLC and me: collaborations

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The general setting

- **ICC**: Implicit Computational Complexity

- The problem: the design of programming languages with certified computational complexity.

- The line: a **ML-like** approach
  - $\lambda$-calculus as paradigmatic programming language.
  - types as semantic properties of terms.
  - **Type assignment for $\lambda$-calculus** such that:
    - types certify the correctness of terms, in particular their complexity bound.

- We are interested, in particular, in characterizing **PTIME**.
Logical approach

- Light Logics (LLL, Girard and SLL, Lafont) where the normalization procedure is intrinsically polynomial.

Property

\[ \Pi \triangleright \Gamma \vdash A \text{ implies } \Pi \text{ normalizes to } \Pi' \text{ in } n \text{ steps} \]

where \( n \in O(|\Pi|^{l(d)}) \) (*)&

(*) \( l(d) \) is a linear function on \( d \)

(\( d = \) the maximum nesting of applications of the rule introducing the !-modality)

\[ \frac{\Gamma \vdash A}{!\Gamma \vdash !A} \] ( !)

* (**)
Decoration of the logics with $\lambda$-terms

(carefully: the obvious decoration does not enjoy subject reduction)

Property

$\Pi \triangleright \Gamma \vdash M : A$ implies $M$ normalizes to $M'$ in a number $n$ of $\beta$-reduction steps

where $n \in O(|M|^{l(d)})$ (*)&

(*) $d$ is the nesting of $!$-rules in $\Pi$

A suitable representation of data types allows for a characterization of $\text{PTIME}$.

Examples:

- **DLAL** (Baillot, Terui), derived from Light Linear Logic
- **STA** (Gaboardi, RDR), derived from Soft Linear Logic
could we do the same job with intersection types?

Possible advantages:

- to start directly from a type assignment system, skipping the difficult decoration step
- to gain in typability power

First attempt:

- to try to mimic exactly the behaviour of light logics
measuring normalization time in light logics

- the normalization of a proof can duplicate some subproofs (ending by a (!)-rule)
- the dimension of the normalized proof can be statically computed

Property

\[ \Pi \triangleright \Gamma \vdash A \text{ and } \Pi \text{ normalizes to } \Pi' \text{ implies } \]

\[ |\Pi'| = |\Pi|^l(d) \]

- the time complexity of the normalization is computed starting from its space complexity
"Classical" intersection types

- introduced by Coppo and Dezani in the 70's of last century, with the aim of enforcing the typability power of simple types.
- defined as a new logical connective, so equipped with introduction and elimination rule.
- the basic intersection type assignment system characterizes the strong normalization of terms.
- when equipped with a universal type and a partial order relation the intersection type assignment systems are the syntactical counterparts of Scott $\lambda$-models.
Classical intersection type assignment system (IC)

\[
\begin{align*}
\Gamma \vdash x : A & \quad (Ax) \\
\Gamma, x : A \vdash M : B & \quad (\rightarrow I) \\
\end{align*}
\]

\[
\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A \\
\Gamma \vdash MN : B \quad (\rightarrow E)
\]

\[
\begin{align*}
\Gamma \vdash M : A & \quad \Gamma \vdash M : B \\
\Gamma \vdash M : A \land B \quad (\land I) \\
\Gamma \vdash M : A \land B & \quad (\land E) \\
\end{align*}
\]

Property

\(\land\) enjoys:

- **Idempotency** (*Id*): \(A \land A = A\)
- **Commutativity** (*Comm*): \(A \land B = B \land A\)
- **Associativity** (*Ass*): \(A \land (B \land C) = (A \land B) \land C\)
IC and quantitative reasoning

Semantical properties

- $\Gamma \vdash M : A$ and $M \rightarrow^\beta N$ imply $\Gamma \vdash N : A$
- $M$ is typable iff it is strongly normalizing.

What about structural properties?

size of a proof?

\[
\frac{y : A \rightarrow B \vdash y : A \rightarrow B \vdash M : A}{\vdash y : A \rightarrow B \vdash yM : B}
\]
\[
\frac{\vdash M : A \quad \vdash M : A}{\vdash M : A \rightarrow B \vdash y : A \rightarrow B \vdash yx : B}
\]
\[
\frac{\vdash M : A \quad \vdash M : B}{\vdash M : A \rightarrow B \vdash y : A \rightarrow B \vdash yx : B}
\]
\[
\frac{\vdash M : A \quad \vdash M : B}{\vdash M : A \rightarrow B \vdash y : A \rightarrow B \vdash yx : B}
\]
IM: a multiplicative relevant version of IC.

\[
\text{(Ax)} \quad \frac{\Gamma \vdash M : A}{x : A \vdash x : A}
\]

\[
(\land I) \quad \frac{\Gamma \vdash M : A \quad \Delta \vdash M : B}{\Gamma \land \Delta \vdash M : A \land B}
\]

\[
(\Gamma \vdash x : A \vdash M : B)
\]

\[
\lor
\]

\[
(\Gamma \vdash M : B \quad x \notin \text{dom}(\Gamma))
\]

\[
(\rightarrow I) \quad \frac{\Gamma \vdash M : A \rightarrow B}{\Gamma \vdash \lambda x. M : A \rightarrow B}
\]

\[
(\rightarrow E) \quad \frac{\Gamma \vdash M : A \rightarrow B \quad \Delta \vdash N : A}{\Gamma \land \Delta \vdash MN : B}
\]

Note:

- \(\land\) is both a logical connective (by rule \((\land I)\), and a structural operator, acting on the contexts (by rules \((\rightarrow E)\) and \((\land I)\))

- the properties \(Id, Comm, Ass\) for \(\land\) are no more derived, but they depend on the definition of \(\land\) on the left
Define:

\[
\Gamma \land \Delta(x) = \begin{cases} 
  \Gamma(x) & \text{if } x \in \text{dom}(\Gamma) \text{ and } x \notin \text{dom}(\Delta) \\
  \Delta(x) & \text{if } x \in \text{dom}(\Delta) \text{ and } x \notin \text{dom}(\Gamma) \\
  \Gamma(x) \land \Delta(x) & \text{if } x \in \text{dom}(\Delta) \cup \text{dom}(\Gamma) \\
  \text{undefined} & \text{otherwise}
\end{cases}
\]

- **IM** coincide with the system **I** of Kfoury and Wells
- **IM** enjoys subject reduction iff \( \land \) is considered modulo \( Id, Comm, Ass \) (Mairson and Neergaard)
Properties of IM

Semantical properties

**IM** gives type to all and only the strongly normalizing terms.

What about structural properties? No changes with respect to **IC**.
Intersection without idempotency

- \((n\text{-ary})\) intersection as structural rule, so types are restricted to proper types (no intersection on the right of the arrow):

\[
A ::= a \mid \sigma \rightarrow A
\]

\[
\sigma ::= A \mid A \land A
\]

- \(\land\) is considered modulo \(Comm\) and \(Ass\) (no \(Id\)).

The system \(IM\{Comm, Ass\}\) is:

\[
\frac{x : A \vdash x : A}{(Ax)}
\]

\[
\frac{\Gamma, x : \sigma \vdash M : B}{\Gamma \vdash \lambda x. M : \sigma \rightarrow B} \quad (\rightarrow I) \quad \frac{\Gamma \vdash M : B \quad x \notin \text{dom}(\Gamma)}{\Gamma \vdash \lambda x. M : A \rightarrow B} \quad (\rightarrow I_0)
\]

\[
\frac{\Gamma_0 \vdash M : A_1 \land ... \land A_n \rightarrow B \quad \left(\Gamma_i \vdash N : A_i\right)_{1 \leq i \leq n}}{\land_{0 \leq i \leq n} \Gamma_i \vdash MN : B} \quad (\rightarrow E)
\]
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Introduction

Intersection Types

Without idempotency

Without associativity

Complexity

Conclusions

Semantical properties

- IM\{Comm, Ass\} enjoys subject reduction
- IM\{Comm, Ass\} characterizes strong normalization of terms

Structural properties

- Let \( M \) be strongly normalizing. Then \( M \) has infinite types, but every derivation \( \Pi \triangleright \Gamma \vdash M : A \) has the same shape.
- \( \Pi \triangleright \Gamma \vdash M : A \) and \( M \xrightarrow{\beta} M' \) implies \( \Pi' \triangleright \Gamma \vdash M' : A \) where \( |\Pi'| = |\Pi| - n \).

\[
\begin{align*}
(x : A_i & \vdash x : A_i)_{1 \leq i \leq n} \\
\cdots & \\
\Gamma_0, x : A_1 \land \ldots \land A_n & \vdash M : B \\
\Gamma_0 & \vdash \lambda x.M : A_1 \land \ldots \land A_n \rightarrow B \\
\land_{0 \leq i \leq n} \Gamma_i & \vdash \lambda x.MN : B \\
(\rightarrow I) & \\
(\Gamma_i \vdash N : A_i)_{1 \leq i \leq n} & \vdash (\rightarrow E) \\
\sim & \\
\Gamma_0 \land_{0 \leq i \leq n} \Gamma_i & \vdash M[N/x] : B
\end{align*}
\]
Complexity of reduction

Let $\Pi \triangleright \Gamma \vdash M : A$, for some $\Gamma$ and $A$. Then $M \xrightarrow{\beta}^* nf(M)$ in a number of $\beta$-reduction steps $\leq |\Pi|$.

The result is not yet satisfying! we would like to express the bound on the reduction length as function of the size of the term.

But the system is interesting in itself.

- **IM{{\textit{Comm}, \textit{Ass}}}** is the core of system R of De Carvalho
- **IM{{\textit{Comm}, \textit{Ass}}},** when equipped with a universal type and an equivalence relation on types, is the syntactical account of the class of linear relational $\lambda$-models (Paolini, Piccolo, RDR)
Intersection without $Id$ and $Ass$

The idea:

- use the lack of $Ass$ to stratify the intersection
- use the intersection on the left for contracting different premises
  (inspired to Soft Linear Logic of Lafont)

The system $IM\{Comm\}$:

$$
\begin{align*}
\Gamma, x : A &\vdash x : A \quad (Ax) \\
\Gamma &\vdash M : A \\
\Gamma &\vdash \lambda x.M : B \rightarrow A \quad (\rightarrow I_0)
\end{align*}
$$

$$
\begin{align*}
\Gamma, x : \sigma &\vdash M : A \\
\Gamma &\vdash \lambda x.M : \sigma \rightarrow A \quad (\rightarrow I) \\
\Gamma &\vdash M : \sigma \rightarrow A \\
\Delta &\vdash N : \sigma \quad \Gamma \# \Delta \\
\Gamma, \Delta &\vdash MN : A \quad (\rightarrow E)
\end{align*}
$$

$$
\begin{align*}
\bigwedge_{i=1}^{n} \Gamma_i &\vdash M : \sigma_1 \quad ... \\
\Gamma_1 &\vdash M : \sigma_1 \\
\Gamma_n &\vdash M : \sigma_n \\
n &> 1
\end{align*}
$$

$$
\begin{align*}
\bigwedge_{i=1}^{n} \Gamma_i &\vdash M : \sigma_1 \land ... \land \sigma_n \quad (\land_n)
\end{align*}
$$

$$
\begin{align*}
\Gamma, x_1 : \sigma_1, ..., x_n : \sigma_n &\vdash M : \tau \\
\Gamma, x : \sigma_1 \land ... \land \sigma_n &\vdash M[x/x_1, ..., x/x_n] : \tau \quad (m)
\end{align*}
$$
Properties of IM\{Comm\}

Semantical properties

- IM\{Comm\} enjoys subject reduction
- IM\{Comm\} characterizes strong normalization of terms

Structural properties

Let \( \Pi \vdash \Gamma \vdash M : A \), and \( M \xrightarrow{\beta} N \) in \( i \) steps.

- \( i \leq |M|^{d(\Pi)+1} \)
- \( |N| \leq |M|^{d(\Pi)+1} \)

where \( d(\Pi) \) is the maximal nesting of rule \((\land_n)\) in \( \Pi \).

- \( d(\Pi) \) is an upper bound on the number of duplications in \( M \), it does not represent duplication in \( \Pi \)
- \( i \leq |M|^{d(\Pi)+1} \) does not mean that the reduction time is polynomial, since \( d(\Pi) \) can depend on \( \Pi \)
Problems of IM$\{Comm\}$

In IM$\{Comm\}$ data types cannot be uniformly typed:

- **Numerals:** $n = \lambda xy.x^n y$
  $\vdash n : ((A \to A) \land \ldots \land (A \to A)) \to A \to A$

- **Binary words:**
  - $101 = \lambda xyz.x(y(xz))$:
    $\vdash 101 : ((A \to A) \land (A \to A)) \to (A \to A) \to A \to A$
  - $111 = \lambda xyz.x(xz)$:
    $\vdash 111 : ((A \to A) \land (A \to A) \land (A \to A)) \to (A \to A) \to A \to A$
Allowing \( Id \) and \( Comm \)

- types enjoy idempotency and commutativity
- the use of multiple contraction is limited:
  - linear premises can be contracted only if they are identical
  - intersection is replaced by stratification, to allow multiplexor of degree 1
  - technically, the previous conditions reflect in having two different contexts, for linear and not linear premises

- Types:
  \[
  A ::= a \mid \sigma \rightarrow A \\
  \sigma ::= A \mid \{\sigma, \ldots, \sigma\}^1_{\geq 1} \quad n \geq 1
  \]

\({\sigma, \ldots, \sigma}\) is a set!
The system IM{Comm, I d}

\[
\begin{align*}
(Ax) & \quad \frac{\Gamma, x : A \vdash M : B \quad x \notin \text{dom}(\Gamma, \Delta)}{\Gamma \vdash \lambda x. M : A \to B} \quad (\rightarrow I_w) \\
& \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \to B} \quad (\rightarrow I_l) \\
& \quad \frac{\Gamma \vdash \lambda x. M : \sigma \to B}{\Gamma \vdash \lambda x. M : \sigma \to B} \quad (\rightarrow I_s) \\
& \quad \frac{\Gamma_1 \vdash M : \sigma \to A \quad \Gamma_2 \vdash N : \sigma \quad \Gamma_1, \Delta_1 \# \Gamma_2, \Delta_2}{\Gamma_1, \Gamma_2 \vdash MN : A} \quad (\rightarrow E) \\
& \quad \frac{\Gamma, x_1 : A, \ldots, x_n : A \vdash M : \tau}{\Gamma \vdash \lambda x : \{A\} \vdash M[x/x_1, \ldots, x_n] : \tau} \quad (m_l) \\
& \quad \frac{\Gamma \vdash \lambda x : \{\sigma_1, \ldots, \sigma_n\} \vdash M[x/x_1, \ldots, x_n] : \tau}{\Gamma \vdash \lambda x : \{\sigma_1, \ldots, \sigma_n\} \vdash M[x/x_1, \ldots, x_n] : \tau} \quad (m_s) \\
& \quad \frac{\Gamma_i \vdash M : \sigma_i \quad 1 \leq i \leq n \geq 1}{\bot \vdash \{\Gamma_i\} \cup \{\Delta_i\} \vdash M : \{\sigma_1, \ldots, \sigma_n\}} \quad (\text{strat})
\end{align*}
\]

where \((\Gamma \cup \Delta)(x) = \Gamma(x) \cup \Delta(x)\) and \(\{\Gamma\}(x) = \{\Gamma(x)\}\)
Properties of IM{ Comm, Id }

Semantical properties

- IM{ Comm, Id } enjoys subject reduction
- IM{ Comm, Id } characterizes strong normalization of terms

Structural properties

Let $\Pi \triangleright \Gamma | \Delta \vdash M : A$, and $M \xrightarrow{\beta}^* nf(M)$ in $n$ steps.

- $n \leq | M |^{d(\Pi)+1}$
- $| nf(M) | \leq | M |^{d(\Pi)+1}$

where $d(\Pi)$ is the maximal nesting of rule ( strat ) in $\Pi$. 
Data Types in $\text{IM}\{\text{Comm}, \text{I}d\}$

In $\text{IM}\{\text{Comm}, \text{I}d\}$ data types can be uniformly typed:

- **Numerals:** $n = \lambda xy.x^n y$
  
  $\vdash n : \{A \rightarrow A\} \rightarrow A \rightarrow A$

  $\vdash n : \{nA \rightarrow A\}^n \rightarrow A \rightarrow A$

- **Binary words:**
  
  - $\underline{101} = \lambda xyz.x(y(xz))$:
    
    $\vdash \underline{101} : \{A \rightarrow A\} \rightarrow \{A \rightarrow A\} \rightarrow A \rightarrow A$

    $\vdash \underline{101} : \{\{A \rightarrow A\}\} \rightarrow \{A \rightarrow A\} \rightarrow A \rightarrow A$

  - $\underline{111} = \lambda xyz.x(x(xz))$:
    
    $\vdash \underline{111} : \{A \rightarrow A\} \rightarrow \{A \rightarrow A\} \rightarrow A \rightarrow A$

    $\vdash \underline{111} : \{\{\{A \rightarrow A\}\}\} \rightarrow \{A \rightarrow A\} \rightarrow A \rightarrow A$

**Property**

All data types can be typed by derivations of depth 0.
The complete system

\[ IM\{Comm, Id\} \]

\[ \frac{\Gamma \vdash M : A \quad a \notin \text{FV}(\Gamma, \Delta)}{\Gamma \vdash M : \forall a.A} \quad (\forall I) \]

\[ \frac{\Gamma \vdash M : \forall a.B}{\Gamma \vdash M : B[A/a]} \quad (\forall E) \]
Polynomial soundness (De Benedetti, RDR)
Let $M$ be the representation in $IM\{Comm, Id\}$ of the function $\phi : N^p \rightarrow N$. Then, for every sequence $w_1, ..., w_p$ of binary words, $Mw_1...w_p$ reduces to normal form in a number of steps which is polynomial in $|w_1| + ... + |w_p|$.

Polynomial completeness (De Benedetti, RDR)
For every polynomial function $\phi : N^p \rightarrow N$, there is a term $M$ representing it in $IM\{Comm, Id\}$.
Conclusions

- Intersection can be seen both as a logical connective on types and as structural rule on the premises.

- The lack of the properties $Id, Comm, Ass$ reflects in a structural behaviour:
  - The lack of $Id$ gives an account of duplication.
  - The lack of $Ass$ gives an account of stratification (nesting of duplications).
  - The lack of $Comm$? A system without $Id, Ass, Comm$ has been used by Di Gianantonio, Honsell, Lenisa to model game semantics. So I guess it give an account of sequentiality.