Optimal Trading with Linear Costs

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CFM-Imperial Workshop on Quantitative Finance
20th october 2014

What we study at CFM

- **Predictors** of an asset's future price return
- **Risk control**
- **Costs of trading** (spread costs, impact costs)
- How to **trade optimally** under these constraints

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  Markovian mono-frequency predictor p_t

- **Risk control**

  Maximal position $|\pi_t| \leq M$

- **Costs of trading** (spread costs, impact costs)

  Linear Costs $\Gamma|\dot{\pi}_t|$

- How to **trade optimally** under these constraints

  That's the subject of this talk . . .

Outline

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Statement of the problem

If we note $r_t = \mathrm{Price}_{t+1} - \mathrm{Price}_t$, we want to maximize

$$\mathbb{L} = \int \left(E[r_t \mid p_t] \cdot \pi_t \; - \; \Gamma |\dot{\pi}_t| \right) \mathrm{d}t \qquad \text{with} \quad |\pi_t| \leq M$$

For the sake of simplicity, we consider the predictor to be **equal to its instantaneous prediction**:

$$E[r_t \mid p_t] = p_t$$

Hence, our objective is to maximize:

$$\mathbb{L} = \int \left(p_t \cdot \pi_t \; - \; \Gamma |\dot{\pi}_t| \right) \mathrm{d}t \qquad \text{with} \quad |\pi_t| \leq M$$

Conditions on the predictor

Moreover, the predictor p_t is required to be:

(i) **positively auto-correlated**:

$$\forall q,\ P(p_{t+1} > q \mid p_t) \text{ increases with } p_t$$

(ii) **Markovian**:

$$\forall \omega_{t+1},\ P(\omega_{t+1} \mid p_t, p_{t-1}, \ldots) = P(\omega_{t+1} \mid p_t)$$

(iii) **symmetric**:

$$\forall q,\ P(p_{t+1} > q \mid p_t = p)\ =\ P(p_{t+1} < -q \mid p_t = -p)$$

(iv) **unbounded**:

$$\forall q,\ \exists \epsilon_q > 0 \text{ s.t. } P(p_{t+1} > q | p_t = 0)\ > \epsilon_q$$

The naive strategy

Integrated predictability at $t = \infty$:

$$p_\infty(p_t) = \sum_{n=0}^{\infty} E[p_{t+n} \mid p_t] = E[\,\mathrm{Price}_\infty - \mathrm{Price}_t \mid p_t]$$

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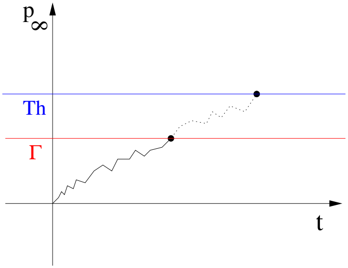

Using Bellman's dynamic programming, one can prove that the optimal strategy is a **threshold system**, i.e. there exists a threshold q such that:

$$\pi_t = \begin{cases} M & \text{if } p_t > q \\ -M & \text{if } p_t < -q \\ \pi_{t-1} & \text{if } |p_t| \leq q \end{cases}$$

Optimal behaviour

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One also proves that there exists a function $g(p_t)$ such that $g(q) = \Gamma$, which satisfies:

$$g(p_t) = p_t + \Gamma\left[P_{>q}(p_t) - P_{<-q}(p_t)\right] + \int_{-q}^{q} g(p')\, P(p'|p_t)\, \mathrm{d}p'$$

where

$$\begin{cases} P(p'|p_t) = P(p_{t+1} = p' \mid p_t) \\ P_{>q}(p_t) = P(p_{t+1} > q \mid p_t) \\ P_{<q}(p_t) = P(p_{t+1} < q \mid p_t) \end{cases}$$

The self-coherent equation

Hence, *"all we need to do"* to find q is to solve the following **self-coherent equation**:

$$i)\quad \forall p,\quad g(p) = p + \Gamma\left[P_{>q}(p) - P_{<-q}(p)\right] + \int_{-q}^{q} g(p')\, P(p'|p)\,\mathrm{d}p'$$

$$ii)\quad g(q) = \Gamma$$

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But it happened to be much trickier than we originally thought !

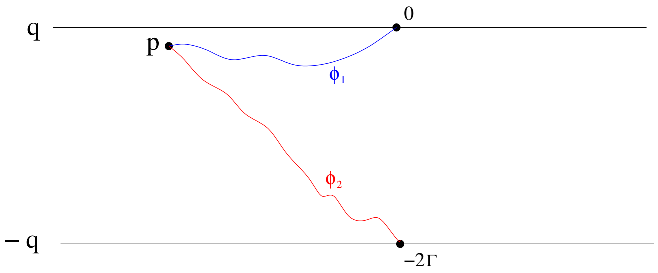
Outline

A new method

The intuitive idea is to do the following:

1. Suppose that the threshold q is known **in the future**.
2. Calculate in which cases it is worth switching your position at **present time**.
3. The threshold will be the **break-even value**.

Starting at $p \leq q$ with $\pi = -M$, is it worth trading $\Delta\pi$?



Suppose that we do trade $\Delta\pi$, then:

Path ϕ_1

$$\begin{aligned} \text{Proba} &= P(\phi_1|p) \\ \text{Gain} &= \Delta\pi \cdot \int_z \phi_1(z)\mathrm{d}z \\ \text{Cost} &= 0 \end{aligned}$$

Path ϕ_2

$$\begin{aligned} \text{Proba} &= P(\phi_2|p) \\ \text{Gain} &= \Delta\pi \cdot \int_z \phi_2(z)\mathrm{d}z \\ \text{Cost} &= 2\Gamma \cdot \Delta\pi \end{aligned}$$

Path integrals

Hence, it is worth trading if, and only if:

$$\Delta\pi \;\cdot \int_{\substack{\phi_b=p \\ -q<\phi(z)<q,\; z\in]0,T_\phi[}}^{|\phi_e|\geq q} \left[\int_z \phi(z)\,\mathrm{d}z - 2\Gamma \cdot \mathbf{1}_{\{\phi_e \leq -q\}}(\phi) \right] P(\phi|p)\,\mathcal{D}\phi \;\geq\; 0$$

where ϕ is any path of length T_ϕ, starting at ϕ_b and ending at ϕ_e.

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As q determines the exact frontier where it is worth trading, we have:

$$\boxed{\int_{\substack{\phi_b = q \\ -q < \phi(z) < q,\ z \in]0, T_\phi[}}^{|\phi_e| \geq q} \left[\int_z \phi(z)\,\mathrm{d}z - 2\Gamma \cdot \mathbf{1}_{\{\phi_e \leq -q\}}(\phi) \right] P(\phi|q)\,\mathcal{D}\phi \ = \ 0}$$

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Although more sophisticated, this equation will be simpler to solve !

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Ornstein-Uhlenbeck predictor

Consider now an **Ornstein-Uhlenbeck** predictor:

$$\mathrm{d}p = -\varepsilon p \,\mathrm{d}t + \beta \,\mathrm{d}X_t$$

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If we note

$$\mathcal{L}(p) = \int_{\substack{\phi_b = p \\ -q<\phi(z)<q,\ z\in]0,T_\phi[}}^{|\phi_e|\geq q} \left[\int_z \phi(z)\,\mathrm{d}z \right] P(\phi|p)\,\mathcal{D}\phi$$

$$\mathcal{P}(p) = \int_{\substack{\phi_b = p \\ -q<\phi(z)<q,\ z\in]0,T_\phi[}}^{\phi_e<-q} P(\phi|p)\,\mathcal{D}\phi$$

then the equation becomes:

$$\mathcal{L}(q) = 2\Gamma \cdot \mathcal{P}(q)$$

The Kolmogorov backward equations

With Itō's lemma one can prove that $\mathcal{L}$ and $\mathcal{P}$ satisfy :

$$\frac{1}{2}\beta^2 \frac{\partial^2 \mathcal{L}}{\partial p^2} - \varepsilon p \frac{\partial \mathcal{L}}{\partial p} = -p$$

$$\frac{1}{2}\beta^2 \frac{\partial^2 \mathcal{P}}{\partial p^2} - \varepsilon p \frac{\partial \mathcal{P}}{\partial p} = 0$$

with initial conditions

$$\begin{cases} \mathcal{L}(q) = 0 \\ \mathcal{L}(-q) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{P}(q) = 0 \\ \mathcal{P}(-q) = 1 \end{cases}$$

Solution

One obtains :

$$\mathcal{L}(p) = \frac{1}{\varepsilon}\left(p - \frac{q}{\mathrm{I}}\int_0^p e^{av^2}\,\mathrm{d}v\right)$$
$$\mathcal{P}(p) = \frac{1}{2}\left(1 - \frac{1}{\mathrm{I}}\int_0^p e^{av^2}\,\mathrm{d}v\right)$$

with

$$\mathrm{I} = \int_0^q e^{av^2}\,\mathrm{d}v \qquad \text{and} \qquad a = \frac{\varepsilon}{\beta^2}.$$

Hence:

$$\boxed{q = \frac{\beta}{\sqrt{\varepsilon}}\, F^{-1}\left(\frac{\Gamma\varepsilon^{3/2}}{\beta}\right) \qquad \text{with} \qquad F(x) = x - e^{-x^2}\int_0^x e^{v^2}\,\mathrm{d}v}$$

Properties of the solution

- The solution can also be expressed as a **threshold on the integrated predictability** p_∞:

$$q_\infty = \Gamma \cdot H\left(\frac{\sigma_\infty \sqrt{2}}{\Gamma}\right) \qquad \text{with} \qquad H(x) = x\, F^{-1}\left(\frac{1}{x}\right).$$

where $\sigma_\infty = \frac{\beta}{\sqrt{2\varepsilon^3}}$ is the standard deviation of the integrated predictability.

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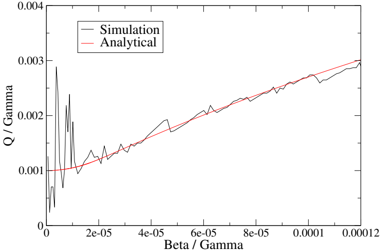
- Comparison with simulation results:



Outline

1. Preliminary results
2. The path-integral technique
3. Solution for an Ornstein-Uhlenbeck predictor
4. Extension to a band system

Connection with Richard Martin's results

By pure chance, Jean-Philippe encountered **Richard Martin** while we were writing our article, and discovered that he had a very similar formula in an apparently different context[1] :

$$\left(\frac{3\sigma^2\epsilon}{2b}\right)^{1/3} \qquad \text{with} \qquad \begin{cases} \epsilon & \to \Gamma \\ \sigma & \to \beta\sqrt{\varepsilon} \\ b & \to \varepsilon \end{cases}$$

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This result was in the case of a **quadratic risk control**, i.e. where we optimize the following:

$$\mathbb{L} = \int \left(p_t \cdot \pi_t \; - \; \Gamma|\dot{\pi}| - \lambda\pi^2 \right) \mathrm{d}t$$

Can we generalize our **path-integral technique** to also solve this problem ?

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Band system

It is a well-known fact[2] that, for a quadratic risk control, the optimal system is a **band system**, also known as a **DT-NT-DT system**.

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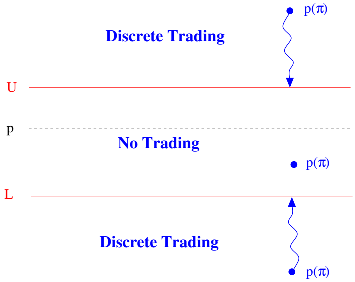

where $p(\pi) = 2\lambda\pi$

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The idea

To a predictor value p_1 we associate the value of the predictor **whose upper bound is the lower bound of** p_1, i.e. $u(p_2) = \ell(p_1) = \ell$.

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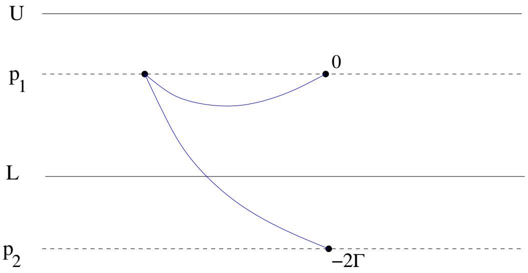

Being at $\pi = \ell$ with a predictor $p = p_1$, we wonder if it is worth buying an **infinitesimal amount** $\delta\pi$

$\Rightarrow$ this gives a **first equation** relating p_1, ℓ and p_2.

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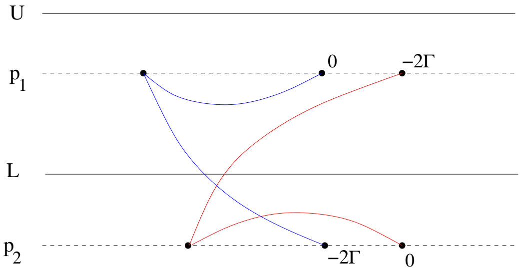

Being at $\pi = \ell$ with a predictor $p = p_2$, we wonder if it is worth selling an **infinitesimal amount** $\delta\pi$

$\Rightarrow$ this gives a **second equation** relating p_1, ℓ and p_2.

The band equations

$$\int\limits_{\substack{\phi_e \geq p_1 \,\vee\, \phi_e \leq p_2 \\ \phi_b = p_1 \\ p_1 < \phi(z) < p_2,\ z \in]0, T_\phi[}} \left[\int_z (\phi(z) - \ell)\, \mathrm{d}z - 2\Gamma \cdot \mathbf{1}_{\{\phi_e \leq p_2\}}(\phi) \right] P(\phi|p_1)\, \mathcal{D}\phi \ = 0$$

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The band equations

If we note

$$\mathcal{L}(p) = \int_{\substack{\phi_b = p \\ p_1 < \phi(z) < p_2,\ z \in]0, T_\phi[}}^{\phi_e \geq p_1 \ \vee \ \phi_e \leq p_2} \left[\int_z \phi(z)\,\mathrm{d}z \right] P(\phi|p)\,\mathcal{D}\phi$$

$$\mathcal{T}(p) = \int_{\substack{\phi_b = p \\ p_1 < \phi(z) < p_2,\ z \in]0, T_\phi[}}^{\phi_e \geq p_1 \ \vee \ \phi_e \leq p_2} \left[\int_z \mathrm{d}z \right] P(\phi|p)\,\mathcal{D}\phi$$

$$\mathcal{P}(p) = \int_{\substack{\phi_b = p \\ p_1 < \phi(z) < p_2,\ z \in]0, T_\phi[}}^{\phi_e \leq p_2} P(\phi|p)\,\mathcal{D}\phi$$

then the equations become:

$$\mathcal{L}(p_1) - \ell \cdot \mathcal{T}(p_1) - 2\Gamma \cdot \mathcal{P}(p_1) = 0$$
$$\mathcal{L}(p_2) - \ell \cdot \mathcal{T}(p_2) - 2\Gamma \cdot \mathcal{P}(p_2) = 0$$

Kolmogorov backward equations

$$\frac{1}{2}\beta^2 \frac{\partial^2 \mathcal{L}}{\partial p^2} - \varepsilon p \frac{\partial \mathcal{L}}{\partial p} = -p$$

with initial conditions $\mathcal{L}(p_1) = \mathcal{L}(p_2) = 0$

This gives :

$$\mathcal{L}(p) = \frac{1}{\varepsilon}\left(p - \frac{p_2 - p_1}{\mathrm{I}} \int_{p_1}^{p} e^{ax^2}\,\mathrm{d}x\right)$$

with

$$a = \frac{\varepsilon}{\beta^2} \quad \text{and} \quad \mathrm{I} = \int_{p_1}^{p_2} e^{ax^2}\,\mathrm{d}x$$

Kolmogorov backward equations

$$\frac{1}{2}\beta^2 \, \frac{\partial^2 \mathcal{T}}{\partial p^2} \; - \; \varepsilon p \, \frac{\partial \mathcal{T}}{\partial p} \; = \; -1$$

with initial conditions $\mathcal{T}(p_1) = \mathcal{T}(p_2) = 0$

This gives :

$$\mathcal{T}(p) = \frac{2aJ}{\varepsilon}\left(\frac{1}{\mathrm{I}}\int_{p_1}^{p} e^{ax^2}\,\mathrm{d}x - \frac{1}{J}\int_{p_1}^{p} e^{ax^2}\left[\int_{p_1}^{x} e^{-ay^2}\,\mathrm{d}y\right]\mathrm{d}x\right)$$

with

$$J = \iint\limits_{p_2 \leqslant x \leqslant y \leqslant p_1} e^{a(x^2 - y^2)} \, \mathrm{d}x \, \mathrm{d}y$$

Kolmogorov backward equations

$$\frac{1}{2}\beta^2 \, \frac{\partial^2 \mathcal{P}}{\partial p^2} \; - \; \varepsilon p \, \frac{\partial \mathcal{P}}{\partial p} \; = \; 0$$

with initial conditions $\mathcal{P}(p_1) = 0$ and $\mathcal{P}(p_2) = 1$

This gives :

$$\mathcal{P}(p) = \frac{1}{\mathrm{I}} \int_{p_1}^{p} e^{ax^2} \mathrm{d}x$$

The lower bound value is

$$\ell = \frac{e^{-ap_1^2} - e^{-ap_2^2}}{2a\int_{p_1}^{p_2} e^{-ax^2}\mathrm{d}x}$$

where p_2 is given by:

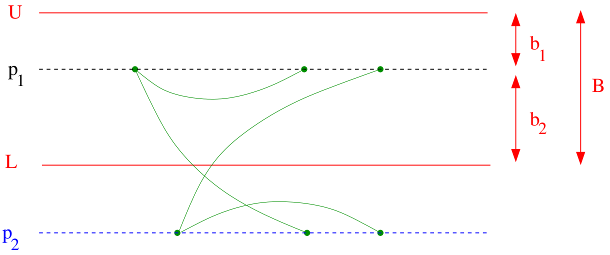
$$\boxed{p_1 - p_2 \ = \ 2\Gamma\varepsilon - \mathrm{I}e^{-ap_1^2} + \frac{J\cdot(e^{-ap_1^2} - e^{-ap_2^2})}{\mathrm{I}'}}$$

with :

$$a = \frac{\varepsilon}{\beta^2}\ , \quad \mathrm{I} = \int_{p_1}^{p_2} e^{ax^2}\mathrm{d}x\ , \quad \mathrm{I}' = \int_{p_1}^{p_2} e^{-ax^2}\mathrm{d}x\ , \quad J = \iint_{p_2\leqslant x\leqslant y\leqslant p_1} e^{a(x^2-y^2)}\,\mathrm{d}x\,\mathrm{d}y$$

And similarly for the upper bound.

Asymptotic behaviour



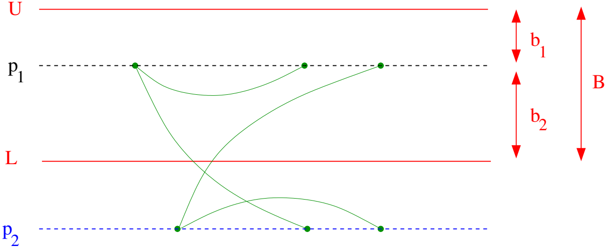
If $p_1, p_2 \to 0$ then

$$b_1 - b_2 \to 0 \qquad \textbf{(symmetric band)}$$

If $p_1, p_2 \to +\infty$ then

$$b_1 \to 0 \qquad \textbf{(totally asymmetric band)}$$

Asymptotic behaviour



If $p_1, p_2 \to 0$ then

$$B \to 2\sqrt[3]{\frac{3}{2}\Gamma\beta^2}$$ **(Martin and Schöneborn's result)**

If $p_1, p_2 \to +\infty$ then

$$B \to \sqrt{2\Gamma\varepsilon\, p_1}$$ **(infinite band size)**

Final remark on risk control

Quadratic risk control could actually be replaced by any risk control of the form

$$\mathcal{R}(\pi) = \lambda|\pi|^z \qquad \text{with} \qquad z > 0$$

Indeed, only the relation between the predictors space and the positions space changes.

More precisely, the positions space is a **topological deformation** of the predictors space given by the function $\mathcal{R}'(\pi)$.

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More precisely, the positions space is a **topological deformation** of the predictors space given by the function $\mathcal{R}'(\pi)$.

The MaxPos constraint can itself be seen as a risk control of the form

$$\mathcal{R}(\pi) = \lambda \left(\frac{\pi}{M}\right)^z \qquad \text{with} \qquad z \to +\infty$$

Thus, the threshold system can be seen as a **degenerate form of a band system**, with a strong deformation of the positions space.

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