From Gödel to Curry-Howard

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PPS/πr^2

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Once upon a time...

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- Justifying arithmetic differently
- ... Intuitionistic logic!
  - Double-negation translation (1933)
  - **Dialectica** (30's, published in 1958)

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- A translation from HA into HA^ω
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- A translation from HA into HA^ω
- That preserves intuitionistic content
- But offers two semi-classical principles:

$$\text{MP}\;\frac{\neg(\forall n \in \mathbb{N}.\, \neg P\, n)}{\exists n \in \mathbb{N}.\, P\, n} \qquad \frac{(\forall n \in \mathbb{N}.\, P\, n) \to \exists m \in \mathbb{N}.\, Q\, m}{\exists m \in \mathbb{N}.\, (\forall n \in \mathbb{N}.\, P\, n) \to Q\, m}\;\text{IP}$$

Parental advisory required

For the sake of exhaustivity, we'll take a glimpse at the historical presentation of Dialectica.

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Warning! Dusty logic inside

- Translation acting on formulæ
- Prevalence of negative connectives
- First-order logic
- Lots of arithmetic encoding
- Does not preserve β-reduction

Dialectica, Dawn of Curry-Howard:

$$\boxed{\vdash A \qquad \mapsto \qquad \vdash A^D \equiv \exists \vec{u}.\, \forall \vec{x}.\, A_D[\vec{u}, \vec{x}]}$$

$$\begin{array}{c|lll}
A \wedge B & \exists \vec{u}\, \vec{v}. & \forall \vec{x}\, \vec{y}. & A_D[\vec{u}, \vec{x}] \wedge B_D[\vec{v}, \vec{y}] \\
A \vee B & \exists \vec{u}\, \vec{v}\, b. & \forall \vec{x}\, \vec{y}. & (b = 0 \wedge A_D[\vec{u}, \vec{x}]) \vee (b = 1 \wedge B_D[\vec{v}, \vec{y}]) \\
A \to B & \exists \vec{\varphi}\, \vec{\psi}. & \forall \vec{u}\, \vec{y}. & A_D[\vec{u}, \vec{\psi}(\vec{u}, \vec{y})] \to B_D[\vec{\varphi}(\vec{u}), \vec{y}] \\
 & & & \\
\forall n.\, A[n] & \exists \vec{\varphi}. & \forall \vec{x}\, n. & A_D[\vec{\varphi}(n), \vec{x}, n] \\
\exists n.\, A[n] & \exists \vec{u}\, n. & \forall \vec{x}. & A_D[\vec{u}, n, \vec{x}]
\end{array}$$

Sound translation, blah blah blah.

A step into modernity

Let us forget the 50's, and rather jump directly to the 90's.

- Take seriously the **computational** content
- Dialectica as a **typed** object
- Works of De Paiva, Hyland, etc.

Types, types, types!

$$\exists \vec{u}.\, \forall \vec{x}.\, A_D[\vec{u}, \vec{x}]$$

$\mathbb{W}(A)$	$\mathbb{C}(A)$	$\perp_A$
Type	Type	$\mathbb{W}(A) \to \mathbb{C}(A) \to$ Prop
Witness	Counter	Orthogonality

A proof $\vdash u : A$ is a term $\vdash u : \mathbb{W}(A)$ such that $\forall x : \mathbb{C}(A).\, u \perp_A x$

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Linearized Dialectica

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- But it would be *ad-hoc*, inheriting from the encodings of Dialectica
- Let us use our our favorite tool: **Linear Logic**!

- Call-by-name decomposition of the arrow:

$$A \to B \quad \equiv \quad !A \multimap B$$

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Now we will be translating LL formulæ into LJ ones.

Requirements

- We will be interpreting the formulæ of linear logic:

$$A, B ::= A \otimes B \mid A \parr B \mid !A \mid ?A \mid A \oplus B \mid A \,\&\, B$$

- Sufficient to define $\mathbb{W}(A)$, $\mathbb{C}(A)$ and $\perp_A$
- Duality for free:
  - $\mathbb{W}(A^\perp) \equiv \mathbb{C}(A)$ and conversely
  - Orthogonal by complementation:

$$\frac{u \not\perp_A x}{x \perp_{A^\perp} u}$$

Linear decomposition

	$\mathbb{W}$	$\mathbb{C}$
$A \to B$	$\begin{cases} \mathbb{W}(A) \to \mathbb{W}(B) \\ \mathbb{C}(B) \to \mathbb{W}(A) \to \mathbb{C}(A) \end{cases}$	$\mathbb{W}(A) \times \mathbb{C}(B)$
$A \multimap B$	$\begin{cases} \mathbb{W}(A) \to \mathbb{W}(B) \\ \mathbb{C}(B) \to \mathbb{C}(A) \end{cases}$	$\mathbb{W}(A) \times \mathbb{C}(B)$
$!A$	$\mathbb{W}(A)$	$\mathbb{W}(A) \to \mathbb{C}(A)$

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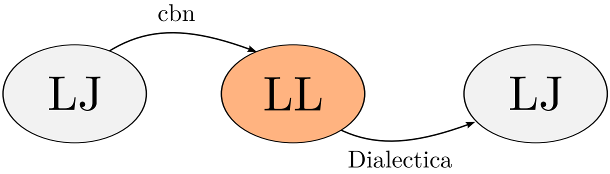
$!A$	$\mathbb{W}(A)$	$\mathbb{W}(A) \to \mathbb{C}(A)$

$$\frac{u \perp_A \psi\, y \quad \to \quad \varphi\, u \perp_B y}{(\varphi, \psi) \perp_{A \multimap B} (u, y)} \qquad \frac{u \perp_A z\, u}{u \perp_{!A} z}$$

Intepretation of the call-by-name λ-calculus

We're now trying to translate the λ-calculus through Dialectica.



- First through the call-by-name linear decomposition into LL;
- Then into LJ with the linear Dialectica.

Brief reminder

We recall here the call-by-name translation of the λ-calculus into LL:

$$[\![A \to B]\!] \equiv ![\![A]\!] \multimap [\![B]\!]$$

$$[\![A \times B]\!] \equiv ![\![A]\!] \otimes ![\![B]\!]$$

$$[\![A + B]\!] \equiv ![\![A]\!] \oplus ![\![B]\!]$$

$$[\![\Gamma \vdash A]\!] \equiv \bigotimes ![\![\Gamma]\!] \vdash [\![A]\!]$$

Prolegomena

In order to interpret the λ-calculus, we need the following:

Dummy term

For all type A, there exists $\vdash \varnothing_A : \mathbb{W}(A)$.

Decidability of the orthogonality

The $\perp_A$ relation is decidable. In particular, there must exist some λ-term

$$@^A : \mathbb{W}(A) \to \mathbb{W}(A) \to \mathbb{C}(A) \to \mathbb{W}(A)$$

with the following behaviour:

$$u_1 @_x^A u_2 \cong \texttt{if } u_1 \perp_A x \texttt{ then } u_2 \texttt{ else } u_1$$

Did you solve the organization issue?

If we were to use the translation as is, we would bump up into an unbearable bureaucracy. Instead, we are going to use the following isomorphism.

$$[\![x_1 : \Gamma_1, \ldots x_n : \Gamma_n \vdash t : A]\!] \cong \mathbb{W}(\Gamma) \to \left\{ \begin{array}{l} \mathbb{W}(A) \\ \mathbb{C}(A) \to \mathbb{C}(\Gamma_1) \\ \vdots \\ \mathbb{C}(A) \to \mathbb{C}(\Gamma_n) \end{array} \right.$$

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Which results in the following translations:

$$[\![\vec{x} : \Gamma \vdash t : A]\!] \equiv \begin{cases} \vec{x} : \mathbb{W}(\Gamma) \vdash t^{\bullet} : \mathbb{W}(A) \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_1} : \mathbb{C}(A) \rightarrow \mathbb{C}(\Gamma_1) \\ \vdots \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_n} : \mathbb{C}(A) \rightarrow \mathbb{C}(\Gamma_n) \end{cases}$$

Translation

For $(-)^\bullet$:

$$
\begin{array}{lcl}
x^\bullet & \equiv & x \\
(\lambda x.\, t)^\bullet & \equiv & \left\{ \begin{array}{l} \lambda x.\, t^\bullet \\ \lambda \pi x.\, t_x\, \pi \end{array} \right. \\
(t\, u)^\bullet & \equiv & (\texttt{fst}\ t^\bullet)\, u^\bullet
\end{array}
$$

Translation II

For t_x :

$$
\begin{array}{lcl}
x_x & \equiv & \lambda\pi.\,\pi \\
 & : & \mathbb{C}(A) \to \mathbb{C}(A) \\
y_x & \equiv & \lambda\pi.\,\varnothing \\
 & : & \mathbb{C}(A) \to \mathbb{C}(\Gamma_i) \\
(\lambda y.\,t)_x & \equiv & \lambda(y,\pi).\,t_x\,\pi \\
 & : & \mathbb{W}(A) \times \mathbb{C}(B) \to \mathbb{C}(\Gamma_i) \\
(t\,u)_x & \equiv & \lambda\pi.\,u_x\,((\mathtt{snd}\ t^\bullet)\,\pi\,u^\bullet)\ @_\pi\ t_x\,(u^\bullet,\pi) \\
 & : & \mathbb{C}(B) \to \mathbb{C}(\Gamma_i)
\end{array}
$$

It just works... Does it?

Soundness

If $\vdash t : A$, then $\vdash t^\bullet : \mathbb{W}(A)$, and in addition, for all $\pi : \mathbb{C}(A)$, $t^\bullet \perp_A \pi$.

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Soundness

If $\vdash t : A$, then $\vdash t^\bullet : \mathbb{W}(A)$, and in addition, for all $\pi : \mathbb{C}(A)$, $t^\bullet \perp_A \pi$.

Sadness

The translation is still not stable by β-reduction.

Almost there

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- We want multisets $\mathfrak{M}$ (think of lists)!
- We just change:

$$\begin{array}{rcl} \mathbb{C}(!A) & \equiv & \mathbb{W}(A) \to \mathbb{C}(A) \\ \mathbb{C}(!A) & \equiv & \mathbb{W}(A) \to \mathfrak{M}\ \mathbb{C}(A) \end{array}$$

- Term interpretation is almost unchanged:
  - $\varnothing$ becomes the empty set;
  - $@$ becomes union
  - ... plus a bit of monadic boilerplate
- We do not need orthogonality anymore...

What about the computational content?

This gives us the following types for the translation:

$$
[\![\vec{x} : \Gamma \vdash t : A]\!] \equiv \begin{cases} \vec{x} : \mathbb{W}(\Gamma) \vdash t^\bullet : \mathbb{W}(A) \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_1} : \mathbb{C}(A) \to \mathfrak{M}\ \mathbb{C}(\Gamma_1) \\ \vdots \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_n} : \mathbb{C}(A) \to \mathfrak{M}\ \mathbb{C}(\Gamma_n) \end{cases}
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- What on earth is t_{x_i}?

An unbearable suspense

A small interlude of ~~advertisement~~ **definitions** to introduce you to the KAM.

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Closures	c	$::=$	(t, σ)
Environments	σ	$::=$	$\emptyset \mid \sigma + (x := c)$
Stacks	π	$::=$	$\varepsilon \mid c \cdot \pi$
Processes	p	$::=$	$\langle c \mid \pi \rangle$

Push	$\langle (t\,u, \sigma) \mid \pi \rangle$	$\rightarrow$	$\langle (t, \sigma) \mid (u, \sigma) \cdot \pi \rangle$
Pop	$\langle (\lambda x.\, t, \sigma) \mid c \cdot \pi \rangle$	$\rightarrow$	$\langle (t, \sigma + (x := c)) \mid \pi \rangle$
Grab	$\langle (x, \sigma + (x := c)) \mid \pi \rangle$	$\rightarrow$	$\langle c \mid \pi \rangle$
Garbage	$\langle (x, \sigma + (y := c)) \mid \pi \rangle$	$\rightarrow$	$\langle (x, \sigma) \mid \pi \rangle$

The Krivine Machine™

Closures all the way down

Let:

- a term $\vec{x} : \Gamma \vdash t : A$
- a closure $\sigma \vdash \Gamma$
- a stack $\vdash \pi : A^{\perp}$ (i.e. $\pi^{\bullet} : \mathbb{C}(A)$)

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- a stack $\vdash \pi : A^{\perp}$ (i.e. $\pi^{\bullet} : \mathbb{C}(A)$)

Then $t_{x_i}\,\pi^{\bullet}$ is the multiset made of **the stacks encountered by** x_i while evaluating $\langle (t, \sigma) \mid \pi \rangle$, i.e.

$$(t_{x_i}\{\vec{x} := \sigma\})\,\pi^{\bullet} = [\rho_1^{\bullet}; \ldots ; \rho_m^{\bullet}]$$

$$\begin{array}{ccc} \langle (t, \sigma) \mid \pi \rangle & \longrightarrow^* & \langle (x_i, \sigma_1) \mid \rho_1 \rangle \\ & \vdots & \vdots \\ & \longrightarrow^* & \langle (x_i, \sigma_m) \mid \rho_m \rangle \end{array}$$

Otherwise said, Dialectica tracks the Grab rule.

Look

$$
\begin{array}{lcl}
x_x & \equiv & \lambda\pi.\,[\pi] \\
 & : & \mathbb{C}(A) \to \mathfrak{M}\ \mathbb{C}(A) \\
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(t\,u)_x & \equiv & \lambda\pi.\,(((\mathtt{snd}\ t^\bullet)\,\pi\,u^\bullet) \gg\!= u_x)\ @\ t_x\,(u^\bullet,\pi) \\
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(We can generalize this interpretation to algebraic datatypes.)

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  - $\rightsquigarrow$ First class access to those stacks with $(-)_x$
  - $\rightsquigarrow$ Or through a control operator

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- We can do the same thing with other calling conventions
  - $\rightsquigarrow$ The protohistoric Dialectica was call-by-name
  - $\rightsquigarrow$ Choose your favorite translation into LL!

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- Alas, no way to solve it without changing totally Dialectica.

The faulty one is the application case (more generally duplication).

$$(t\,u)_x \equiv \lambda\pi.\,(((\texttt{snd}\ t^\bullet)\,\pi\,u^\bullet) \gg\!= u_x)\ @\ t_x\,(u^\bullet, \pi)$$

- What about more expressive systems?
- We follow the computation intuition we presented
- ... and we apply Dialectica to dependent types
  - $\rightsquigarrow$ subsuming first-order logic;
  - $\rightsquigarrow$ a proof-relevant $\forall$;
  - $\rightsquigarrow$ towards CC^ω and further!

Main lines

- We keep the CBN λ-calculus
  - $\rightsquigarrow$ it can be lifted readily to dependent types
  - $\rightsquigarrow$ $A \to B$ becomes $\Pi x : A.\, B$
  - $\rightsquigarrow$ $A \times B$ becomes $\Sigma x : A.\, B$
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  - $\rightsquigarrow$ nothing special to do!
- Design choice: types have no computational content (effect-free):
  - $\rightsquigarrow$ a bit disappointing;
  - $\rightsquigarrow$ but it works...
  - $\rightsquigarrow$ and the usual CC presentation does not help much!

Type translation

Idea: if A is a type,

$$\begin{array}{lcl} A^\bullet \equiv & (\mathbb{W}(A), \mathbb{C}(A)) : \mathtt{Type} \times \mathtt{Type} & \\ A_x \equiv & \lambda\pi.\, [] & \text{(effect-free)} \end{array}$$

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We get:

$$\begin{array}{lcl} \mathtt{Type}^\bullet & \equiv & (\mathtt{Type} \times \mathtt{Type}, 1) \\ \mathtt{Type}_x & \equiv & \lambda\pi.\,[\,] \\ (\Pi y : A.\, B)^\bullet & \equiv & \left(\begin{array}{c} (\Pi y : \mathbb{W}(A).\, \mathbb{W}(B)) \\ \times \\ (\Pi y : \mathbb{W}(A).\, \mathbb{C}(B) \to \mathfrak{M}\ \mathbb{C}(A)) \end{array}, \Sigma y : \mathbb{W}(A).\, \mathbb{C}(B) \right) \\ (\Pi y : A.\, B)_x & \equiv & \lambda\pi.\,[\,] \end{array}$$

(We can obtain inductives + dependent destruction quite easily.)

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  - $\leadsto$ A bit of delimited control (the $(-)_x$ part)
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- But is is partially wrong:
  - $\leadsto$ it is oblivious of sequentiality
  - $\leadsto$ how can we fix it?
- The delimited control part can be lifted seamlessly to CC^ω
  - $\leadsto$ as soon as we have a little bit more than CC
  - $\leadsto$ we need a more computation-relevant presentation of CC

Thanks for your attention.