Higher Inductive Types as Homotopy-Initial Algebras

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Introduction

In Extensional Type Theory we have a well-known correspondence (Dybjer 1996) between

1. Inductive types: finite types 0, 1, 2, ..., natural numbers $\mathbb{N}$, lists $\mathsf{List}[A]$, well-founded trees $\mathsf{W}_{x:A}B(x)$, etc.
2. Initial algebras of a certain form

$(\mathbb{N}, 0, \mathsf{suc})$ is initial among algebras of the form (C, z, c), where $z : C$ and $c : C \longrightarrow C$.

Initial: there is a unique function $h : \mathbb{N} \longrightarrow C$ which preserves the constructors (a *homomorphism*).

Introduction

In Intensional Type Theory this correspondence breaks down: we cannot prove (definitional) uniqueness.

In Homotopy Type Theory, we can prove *propositional* uniqueness, and more: we have a correspondence (Awodey et al, 2012) between

1. Inductive types: 0, 1, 2, $\mathbb{N}$, List$[A]$, $\mathsf{W}_{x:A}B(x)$, etc. with *propositional computation rules*
2. *Homotopy-initial* algebras of a certain form

$(\mathbb{N}, 0, \mathsf{suc})$ is homotopy-initial among algebras of the form (C, z, c).

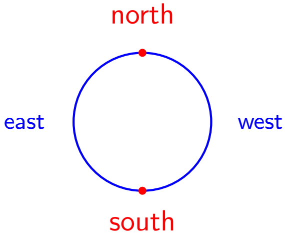
Homotopy-initial: the type of homomorphisms from $(\mathbb{N}, 0, \mathsf{suc})$ to any other algebra (C, z, c) is contractible.

Higher Inductive Types

A powerful tool in HoTT are *Higher-Inductive Types* (HITs):

1. HITs extend ordinary inductive types by allowing constructors involving *path spaces* of X (e.g., $c : a =_X b$) rather than just points of X (e.g., c : X).

   E,g., the circle $\mathbf{S}^1$ is a HIT generated by four constructors:



north : $\mathbf{S}^1$

south : $\mathbf{S}^1$

east : north $=_{\mathbf{S}^1}$ south

west : north $=_{\mathbf{S}^1}$ south

Higher Inductive Types

2. Many interesting constructions arise as HITs: spheres $\mathbf{S}^n$, interval, torus T, quotients, pushouts, suspensions, integers $\mathbb{Z}$, truncations $||A||$ (aka squash types), ...
3. *Open question*: Which computation rules should be propositional vs. definitional? *Here we assume the former.*
3. *Open problem*: finding a unifying schema for HITs (**not** a subject of this talk).

The subject of this talk: *Can a manageable class of HITs be characterized by a universal property - as homotopy-initial algebras?*

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The subject of this talk: *Can a manageable class of HITs be characterized by a universal property - as homotopy-initial algebras? Yes!*

W-suspensions

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"Generalized suspensions": vacuous induction on point constructors; arbitrary number of path constructors between any two point constructors.

Induction and higher-dimensionality remain orthogonal, which gives W-supsensions a well-behaved elimination principle.

W-suspensions: point constructors

The W-suspension type W is a HIT generated by

$$\begin{aligned} \mathsf{point} &: \Pi_{a:A}(B(a) \longrightarrow \mathsf{W}) \longrightarrow \mathsf{W} \\ \mathsf{path} &: \ldots \end{aligned}$$

where, just like for well-founded trees,

- A is the type of *point constructors*
- $B : A \longrightarrow \texttt{type}$ gives the arity of each point constructor

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Example: The type $\mathbb{N}$ has two point constructors: one for zero and one for successor. Thus, $\mathbb{N}$ is a W-suspension with $A :\equiv 2$ and B given by $\top \mapsto 0, \bot \mapsto 1$.

W-suspensions: path constructors

The W-suspension type W is a HIT generated by

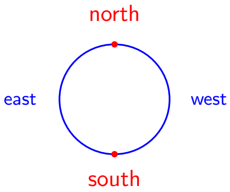
$$\begin{aligned}\mathsf{point} &: \Pi_{a:A}(B(a) \longrightarrow \mathsf{W}) \longrightarrow \mathsf{W}\\ \mathsf{path} &: \Pi_{c:C}\Pi_{b_F:B(F(c))\longrightarrow \mathsf{W}}\Pi_{b_G:B(G(c))\longrightarrow \mathsf{W}}\\ &\qquad \mathsf{point}\Big(F(c), b_F\Big) =_{\mathsf{W}} \mathsf{point}\Big(G(c), b_G\Big)\end{aligned}$$

where

- C is the type of *path constructors*
- $F : C \longrightarrow A$ and $G : C \longrightarrow A$ give the left and right endpoints of each path constructor

Example: the circle $\mathbf{S}^1$ as a W-suspension

Revisiting the circle:



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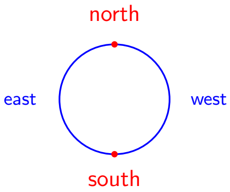

we see that $\mathbf{S}^1$ is a W-suspension with

- $A :\equiv 2$
- B is given by $\top, \bot \mapsto 0$
- $C :\equiv 2$
- F is given by $\top, \bot \mapsto \text{north}$
- G is given by $\top, \bot \mapsto \text{south}$

Main Theorem

Theorem

In HoTT, the existence of W*-suspensions is equivalent to the existence of a suitable algebra* $\left(\mathsf{W}, \mathsf{point}, \mathsf{path}\right)$ *which is homotopy-initial.*

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In HoTT, the existence of the circle $\mathbf{S}^1$ *is equivalent to the existence of a suitable algebra* $\left(\mathbf{S}^1, \mathsf{north}, \mathsf{south}, \mathsf{east}, \mathsf{west}\right)$ *which is homotopy-initial.*

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Corollary

In HoTT, the existence of the natural numbers $\mathbb{N}$ *is equivalent ...*

and so on

Proof Idea

We show that for any algebra $(\mathsf{W}, \mathsf{point}, \mathsf{path})$, the induction principle is equivalent to the simpler *recursion principle* plus a *uniqueness condition*, which are in turn equivalent to homotopy-initiality:

Induction = Recursion + Uniqueness = Homotopy-Initiality

Recursion principle: for any algebra (C, p, r), we have a homomorphism from $(\mathsf{W}, \mathsf{point}, \mathsf{path})$ to (C, p, r).

Uniqueness condition: any two homomorphisms from $(\mathsf{W}, \mathsf{point}, \mathsf{path})$ to (C, p, r) are propositionally equal.

Proof Idea

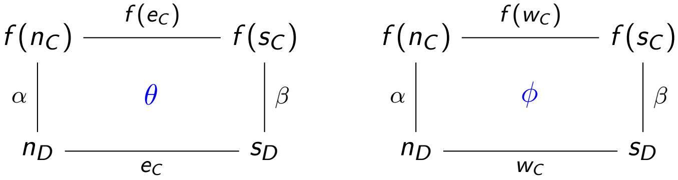
For the circle $\mathbf{S}^1$:

Definition

A homomorphism from (C, n_C, s_C, e_C, w_C) to (D, n_D, s_D, e_D, w_D) is a map $f : C \longrightarrow D$ together with paths

$$\alpha : f(n_C) = n_D$$
$$\beta : f(s_C) = s_D$$

and higher paths θ, ϕ:



Proof Idea

The uniqueness condition for $\mathbf{S}^1$ thus says that any two homomorphisms $(f, \alpha_f, \beta_f, \theta_f, \phi_f)$ and $(g, \alpha_g, \beta_g, \theta_g, \phi_g)$ from $(\mathbf{S}^1, \mathsf{north}, \mathsf{south}, \mathsf{east}, \mathsf{west})$ to (C, n_C, s_C, e_C, w_C) are equal.

This is the same as saying that

1. There is a path $p : f = g$ (a **propositional** η**-rule**).
2. The (higher) paths $\alpha_f, \beta_f, \theta_f, \phi_f$ and $\alpha_g, \beta_g, \theta_g, \phi_g$ are suitably related over p.

Conclusion

We have

- Introduced a class of higher inductive types, which is relatively simple and subsumes types like
  - well-founded trees $\mathsf{W}_{x:A}B(x)$, hence the types of natural numbers $\mathbb{N}$, lists $\mathsf{List}[A]$, ...
  - the interval I
  - all the spheres $\mathbf{S}^n$
  - ordinary suspensions $\mathsf{susp}(A)$

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    - all the spheres $\mathbf{S}^n$
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  with propositional computational rules.
- Shown that this class can be characterized as a homotopy-initial algebra of a certain form; *thus equating the proof-theoretic concept of a higher-inductive type with a particular universal property*.

Conclusion

Open questions:

- What other HITs arise naturally as W-suspensions?
- Does homotopy-initiality scale to other HITs such as set and groupoid quotients, higher-level truncations, the torus, ... ?

References:

- P. Dybjer, Representing Inductively Defined Sets by Well-orderings in Martin-Löf's Type Theory, 1996.
- S. Awodey, N. Gambino, and K. Sojakova, Inductive Types in Homotopy Type Theory, 2012.