Proof-Relevant Rewriting Strategies (WIP)

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Generalized Rewriting

- **Equational reasoning** `x = y |- x + 1 ==> y + 1`
- **Logical reasoning** `x <-> y |- (x /\ y) ==> (x /\ x)`
- **Rewriting** `y ~> z |- x ~> y ==> x ~> z`
- **Abstract data types, quotients/setoids**
  `s, t : list, x =set y |- union x y =set x`
  `==> union x x =set x`

Rewriting in Type Theory

Moving from substitution to congruence.

- Built-in substitution: Leibniz equality.
  $\Pi\ A\ (P : A \rightarrow \texttt{Type})\ (x\ y : A),\ P\ x \rightarrow x = y \rightarrow P\ y.$
  - ✓ Applies to any context
  - ✗ Large proof term: repeats the context that depends on x
  - ✗ Restricted to equality, one rewrite at a time

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- Congruence.
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  - × Applies at the toplevel only
  - ✓ Small proof term: mentions the changed terms only
  - ✓ Generalizes to n-ary, parallel rewriting
  - × Still restricted to equality

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One can build a set of combinators to rewrite in depth: HOL conversions [Paulson 83].

Basin [NuPRL, 94], Sacerdoti Coen [Coq, 04]

- Generalized to any relation
  Proper (iff $\Longrightarrow$ iff) not $\triangleq \Pi\ P\ Q,\ P \leftrightarrow Q \rightarrow \neg\ P \leftrightarrow \neg\ Q$
- Multiple signatures for a given constant
  Proper (impl $\longrightarrow$ impl) not

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Requires proof search:

- Heuristic in NUPRL based on subrelations ($\mathrm{impl} \subset \mathrm{iff}$)
- Complete procedure in COQ.

Both are monolithic algorithms with a primitive notion of signature: a list of atomic relations (with variance).

A New Look at Generalized Rewriting

Sozeau [JFR 2009]

- Extensible signatures (shallow embedding)

  $\mathrm{all} : \forall\ A : \mathtt{Type}, (A \to \mathtt{Prop}) \to \mathtt{Prop}$

  $\Pi\ A, \mathsf{Proper}\ (\mathrm{pointwise\_relation}\ A\ \mathrm{iff} \Longrightarrow \mathrm{iff})\ (\mathtt{@}\mathrm{all}\ A)$

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- An algebraic presentation, supporting higher-order functions (rewriting under binders) and polymorphism:

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- Generic morphism declarations.
- Support for subrelations, quotienting the signatures.
- Rewriting on operators/functions, parallel rewrites. . .

Proper

```
Class Proper {A} (R : relation A) (m : A) : Prop :=
  proper : R m m.

Instance reflexive_proper `(Reflexive A R) (x : A) : Proper R x.
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Instance not_P : Proper (iff ++> iff) not.
```

Outline

1. Generalized Rewriting in Type Theory
2. Proof-relevant relations
3. Rewriting Strategies

Impredicativity helped

All fine with relations in $\texttt{Prop}$, how about $\texttt{Type}$-valued relations?

$$\text{Proper} : \Pi A : \texttt{Type}_i, (A \to A \to \texttt{Type}_j) \to A \to \texttt{Type}_j.$$

Need to show, under $A : \texttt{Type}_i$:

$$\begin{array}{ll} \text{Proper} & ((A \to A \to \texttt{Type}_j) \to A \to \texttt{Type}_j) \\ & (\mathrm{iso\_rel}\ A \longrightarrow \mathrm{eq}\ A \longrightarrow \mathrm{iso}) \\ & (\text{Proper}\ A) \end{array}$$

Inconsistency: $\texttt{Type}_{\mathsf{max}(i,j+1)} \not\leq \texttt{Type}_i$

Universe Polymorphism

With full universe polymorphism (Sozeau & Tabareau [ITP'14]):

$$\mathsf{Proper}_{i\,j} : \Pi A : \mathtt{Type}_i, (A \to A \to \mathtt{Type}_j) \to A \to \mathtt{Type}_j$$

We can show, under $A : \mathtt{Type}_i$:

$$\begin{array}{ll} \mathsf{Proper}_{i'\,j'} & ((A \to A \to \mathtt{Type}_j) \to A \to \mathtt{Type}_j) \\ & (\mathrm{iso\_rel}\ A \longrightarrow \mathrm{eq}\ A \longrightarrow \mathrm{iso}) \\ & (\mathsf{Proper}_{i\,j}\ A) \end{array}$$

With constraint: $\mathsf{max}(i, j+1) \leq i'$.
Actually, a non-polymorphic $\mathrm{crelation}(A : \mathtt{Type}_i) := A \to\to \mathtt{Type}_j$ is already problematic: no relation equivalence or subrelation definition possible.

Computational relations

Generalized rewriting now handles:

- The function space "relation": rewrite `x` to `y` in `C` $=$ `? : C[x]` $\rightarrow$ `C[y]`
- Isomorphism of types
- Computationally relevant relations, e.g. CoRN's appartness relation on reals.
- Hom-types of categories which are not `Prop`-based setoids, e.g. groupoids.

Outline

Automated rewriting

An efficiency concern: `autorewrite` does `repeat rewrite`.

- Crawls through the whole goal each time.
- Applies transitivity of rewriting at the top-level only, resulting in large proof-terms.

We want to allow the specification of precise rewriting strategies (e.g. bottomup, innermost, repeated...) that avoid this.

- Traversal of the goal specified by the user.
- Applies transitivity of rewriting at inner points of the term, resulting in shorter proof-terms.

Rewriting strategies

- Based on ELAN's rewriting strategies
- Implemented using the LogicT monad (failure/success continuations) for efficient backtracking and clear semantics.
- Using the existing generalized rewriting framework to produce Proper constraints and build the rewriting proofs.

Interface: `rewrite_strat` $strategy$ $(\texttt{in}\ t)?$

Rewriting strategies

s, t, u	$::=$	`(<-)?` c	(right to left?) lemma
	$\mid$	`fail` $\mid$ `id`	failure $\mid$ identity
	$\mid$	`refl`	reflexivity
	$\mid$	`progress` s	progress
	$\mid$	`try` s	failure catch
	$\mid$	s `;` u	composition
	$\mid$	$s \mid\mid t$	left-biased choice
	$\mid$	`repeat` s	iteration $(+)$
	$\mid$	`subterm(s)?` s	one or all subterms
	$\mid$	`innermost` s	innermost first
	$\mid$	`hints` $hintdb$	apply first matching hint
	$\mid$	`eval` $redexpr$	apply reduction
	$\mid$	`fold` c	fold expression
	$\mid$	`pattern` p	pattern matching

Derived strategies

$$
\begin{array}{lcl}
\texttt{try}\ s & = & s \mathbin{||} \texttt{id} \\
\texttt{any}\ s & = & \texttt{fix}\ u.\texttt{try}\ (s\ \texttt{;}\ u) \\
\texttt{repeat}\ s & = & s\ \texttt{;}\ \texttt{any}\ s \\
\texttt{bottomup}\ s & = & \texttt{fix}\ bu.((\texttt{progress}\ (\texttt{subterms}\ bu)) \mathbin{||} s)\ \texttt{;}\ \texttt{try}\ bu \\
\texttt{topdown}\ s & = & \texttt{fix}\ td.(s \mathbin{||} (\texttt{progress}\ (\texttt{subterms}\ td)))\ \texttt{;}\ \texttt{try}\ td \\
\texttt{innermost}\ s & = & \texttt{fix}\ i.((\texttt{subterm}\ i) \mathbin{||} s) \\
\texttt{outermost}\ s & = & \texttt{fix}\ o.(s \mathbin{||} (\texttt{subterm}\ o))
\end{array}
$$

An example

Suppose the theory of monoids on `T`.

A goal: `x y : T |- x` $\bullet$ `((`ϵ $\bullet$ `y)` $\bullet$ ϵ`)`.

- `autorewrite with monoids` will do two rewrites with both unit laws, the proof term will be roughtly twice the goal size.
- `rewrite_strat (topdown (repeat (hints monoids)))` will first rewrite ϵ $\bullet$ `y` to `y` and directly after, `y` $\bullet$ ϵ to `y`, resulting in a proof term of size roughly that of the initial goal, and will be twice as fast as well.

Advantages

- Improved performance by replacing `autorewrite` tactic used in `Ring` with: `topdown` (`hints` $Esimpl$)
- Avoid mixing of rewrite with `Ltac` constructs, e.g.: (`rewrite` l_1 || ... || `progress rewrite` l_n) becomes `rewrite_strat` (l_1 || ... || `progress` l_n) which traverses the term just once.
- Another common pattern:

```
match goal with
|- context [t] => rewrite l
end
                =
rewrite_strat (topdown (pattern t; term l))
```

Future work

- Debug & release
- Benchmarks

The End