On the Complexity of Negative Quantification*

Aleksy Schubert, Paweł Urzyczyn,[1] and Konrad Zdanowski[2]

[1] University of Warsaw
[alx,urzy]@mimuw.edu.pl
[2] Cardinal Stefan Wyszyński University in Warsaw
k.zdanowski@uksw.edu.pl

Abstract

Universal quantifiers in first-order formulas are classified as positive (co-variant) and negative (contra-variant) depending on their position. We define the negative fragment of a logic so that it consists of formulas where all quantifiers are negative. We prove that the decision problem for the negative forall-arrow fragment of intuitionistic predicate logic becomes *co*-NEXPTIME complete under the restriction that all predicates are of fixed arity. This contrasts with an earlier result that the full negative logic is EXPSPACE complete.

The usual way of classifying the quantifier complexity of a formula is by counting the number of quantifier alternations. This concept can be defined even if formulas of a given logic have no equivalent prenex normal form. One simply counts alternations between positive and negative occurrences of quantifiers. In the context of intuitionistic logic this gives the, so called, Mints hierarchy [1]. Here, we are interested in the first level of this hierarchy consisting of formulas with only negative occurrences of quantifiers. For the logic with universal quantification and implication only this class may be defined by the following grammar, where quantifier free formulas are represented by Δ:

- $\Sigma_1 ::= \Delta \mid \Pi_1 \to \Sigma_1$;
- $\Pi_1 ::= \Delta \mid \Sigma_1 \to \Pi_1 \mid \forall x\, \Pi_1$.

We establish the complexity of the set of Σ_1 theorems of intuitionistic predicate logic in a vocabulary with predicates of bounded arity and with no function symbols. We show that this fragment is *co*-NEXPTIME complete. This may be compared with the earlier work which shows that Σ_1 fragment with no restriction on the arity of predicates is EXPSPACE complete [3], as well as with Π_1 fragment which is 2-*co*-NEXPTIME hard [2]. The higher classes in the hierarchy, Σ_2 and Π_2, become undecidable even in the monadic vocabulary [3].

To obtain a lower bound we define a tiling problem of covering the space $\mathbb{N} \times \{0,1\}^*$ with a finite set of tiles $\mathcal{T}$, according to a given set of deterministic rules $\mathcal{G} : \mathcal{T}^4 \to \mathcal{T}^2$. The set of tiles $\mathcal{T}$ is assumed to include two distinguished elements, E and OK. Rules in $\mathcal{G}$ define a tiling function $T_\mathcal{G} : \mathbb{N} \times \{0,1\}^* \to \mathcal{T}$, given by:

- $T_\mathcal{G}(n, w) = \mathrm{E}$, when $n = 0$ or $w = \varepsilon$.
- $T_\mathcal{G}(m{+}1, wi) = \pi_i(\mathcal{G}(\mathrm{K}, \mathrm{L}, \mathrm{M}, \mathrm{N}))$, for $i = 0, 1$, where π_i stands for the i-th projection, and $\mathrm{K} = T_\mathcal{G}(m, wi)$, $\mathrm{L} = T_\mathcal{G}(m, w)$, $\mathrm{M} = T_\mathcal{G}(m{+}1, w)$, and $\mathrm{N} = T_\mathcal{G}(m{+}2, w)$;

One can imagine a tiling of $\mathbb{N} \times \{0,1\}^*$ as a full binary tree labeled by rows of tiles (the label of a node $w \in \{0,1\}^*$ is the sequence of tiles $T_\mathcal{G}(n, w)$, for all n).

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Let $s \in \mathbb{N}$. The problem $\mathcal{G}$ is *s-solvable* iff, for every w with $|w| = s$, there is a prefix w' of w and a number $m \leq s$ such that $T_{\mathcal{G}}(m, w') = \text{OK}$. That is, an OK tile must be reached at every branch of the tree of length s and it must be at most the s-th tile in the row.

Since s is presented in binary, one can show by a routine argument that the problem whether a given $\mathcal{G}$ is s-solvable belongs to the class of *co*-NEXPTIME complete problems. We show how to code this problem by a Σ_1 formula φ such that s–solvability of $\mathcal{G}$ corresponds to provability of φ. Moreover, φ can be written as $\forall\bar{y}_1\psi_1 \to \cdots \to \forall\bar{y}_k\psi_k \to \psi_{k+1}$, where all ψ_i are quantifier free. After encoding s-solvability of $\mathcal{G}$ using predicates of arity 2, we use a syntactic translation to replace them by monadic predicates.

For the upper bound we define a refutation system for the Σ_1 fragment of intuitionistic logic. We show that the size of a refutation may be bounded by 2^{n^k}, where n is the length of a disproved formula φ and k corresponds, roughly, to the maximal arity of predicates in φ.

The main ingredient here is the observation that while proving a Σ_1 formula φ from the set of Π_1 assumptions Γ there is no need of introducing new variables besides those which are free in φ and Γ (assuming that there is at least one). The above restricts the number of possible sequents that may occur in an alleged proof of $\Gamma \vdash \varphi$. After constructing a refutation tree one can show that some of its branches can be cut off so that the existence of the full refutation tree follows from the existence of its fragment of exponential size.

References

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