A Kleene realizability semantics for the Minimalist Foundation

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Abstract

The Minimalist Foundation was ideated by M. E. Maietti and G. Sambin in [MS05] and then completed in [Mai09] by M. E. Maietti. It is intended to constitute a common core among the most relevant constructive and classical foundations. One of its novelties is that it consists of two levels: an intensional level (mTT) which should make evident the constructive contents of mathematical proofs in terms of programs, and an extensional level (emTT) formulated in a language close as much as possible to that of ordinary mathematics. Both the intensional level and the extensional level of the Minimalist Foundation consist of type systems based on versions of Martin-Lof's type theory with the addition of a primitive notion of propositions: the intensional one is based on [NPS90] and the extensional one on [Mar84].

In this talk we show how to build a predicative realizability model of the Minimalist Foundation, and in particular of its extensional level emTT, validating the Extended Church thesis (ECT).

To reach this goal it is enough to build a realizability model for the intensional level mTT validating ECT. Indeed a realizability interpretation for the extensional level emTT can be then obtained from an interpretation of mTT by composing this with the interpretation of emTT in a suitable setoid model of mTT as in [Mai09] and analyzed in [MR13].

We build the realizability model for mTT+ECT in the theory $\widehat{ID_1}$([Fef82]). This theory is formulated in the language of second-order arithmetics and it consists of PA (Peano Arithmetic) plus the existence of some (not necessary the least) fixed point for positive parameter-free arithmetical operators. Our realizability model is obtained by suitably modifying the realizability semantics in $\widehat{ID_1}$ described in [Bee85] for the extensional version of first-order Martin-Löf's type theory with one universe, which is based on Kleene realizability semantics of intuitionistic arithmetics.

In essence we interpret mTT-sets as Beeson interpreted Martin-Löf's sets following Kleene realizability in $\widehat{ID_1}$, propositions are interpreted as proof-irrelevant quotients of their Kleene realizability interpretation and the universe of mTT-small propositions as a suitable quotient of some fix-point including all the codes of small propositions.

It is worth to recall that our modifications to Beeson's model are essential, because Beeson's model for the extensional version of Martin-Löf's type theory can not validate ECT due to the inconsistency of the full axiom of choice and function extensionality with it. If we drop function extensionality and we take the intensional version of Martin-Löf's type theory then this version might be consistent with ECT, and from it we can easily derive the consistency of mTT+ECT+ full axiom of choice, but this is still an open problem.

References

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