Some Varieties of Constructive Finitenes

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Introduction

In this note we consider two related notions of *constructive finiteness*. First we look at how we can define that the set of **True** positions in a stream over bool is finite. This extends the work done by Bezem, Nakata and Uustalu [1] by identifying a new formalisation which lies between two known ones, and identifying that reductions "upwards" in the hierarchy is equivalent to *Markov's Principle* and the *Weak Limited Principle of Omniscience*, respectively.

In the second part we look at *streamless sets*, recently investigated by Coquand and Spiwack [2]. A set is *streamless* if, intuitively, every stream over that set must contain duplicates. It is an open question whether the product of two streamless sets is a streamless set. Here we show that this indeed holds if we assume Markov's Principle and decidable equality.

Definitions

We start by providing the basic definitions. A stream over A is a function of type $\mathbb{N} \to A$. For a stream $f : \mathbb{N} \to \text{bool}$, $\texttt{NrOfTrue}_f : \mathbb{N} \to \mathbb{N}$ is the function which on input k returns the number of natural numbers i such that $i \leq k$ and $f(i) = \textbf{True}$, that is, $\texttt{NrOfTrue}_f\ k = |\{i \mid i \leq k \wedge f(i) = \textbf{True}\}|$.

Now we provide three different properties that can hold of Boolean streams. A function $f : \mathbb{N} \to \text{bool}$ is *eventually always false*, written $\text{eaf}(f)$ when

$$\exists n : \mathbb{N}, \forall m : \mathbb{N}, m \geq n \to f(m) = \textbf{False}\,.$$

Bezem et al. [1] refers to this as both Equation (1) and "$\mathcal{F}(\mathcal{G}\ \text{blue})s$".

Next, we say that a stream $f : \mathbb{N} \to \text{bool}$ is *bounded*, written $\text{bounded}(f)$ when there is a bound to the number of **True** positions:

$$\exists n : \mathbb{N}, \forall k : \mathbb{N}, \texttt{NrOfTrue}_f\ k \leq n.$$

Bezem et al. [1] refers to this as both Equation (2) and "$\exists n.\, \text{le}_n\, s$".

The new notion is a natural strengthening of bounded. We say that a stream $f : \mathbb{N} \to \text{bool}$ is *strictly bounded*, written $\text{sb}(f)$ when there exists a strict bound on the number of **True** positions:

$$\exists n : \mathbb{N}, (\forall k : \mathbb{N}, \texttt{NrOfTrue}_f\ k \leq n \wedge \neg\forall k : \mathbb{N}, \texttt{NrOfTrue}_f\ k \leq n-1).$$

Finally we have three axioms which are all constructively consistent, but not provable. Markov's Principle (MP) states that for any decidable predicate over natural numbers (i.e., streams over bool): $\neg\neg(\exists n : \mathbb{N}, Pn) \to \exists n : \mathbb{N}, Pn$. Intuitively MP is realized by an unbounded search, which we know (from outside the system) will terminate because of the antecedent.

Weak Limited Principle of Omniscience (WLPO) states that for any decidable predicate P, we have that $(\forall n : \mathbb{N}, P(n)) \vee (\neg\forall n : \mathbb{N}, P(n))$, while the stronger Limited Principle of Omniscience (LPO) states that $(\forall n : \mathbb{N}, P(n)) \vee (\exists n : \mathbb{N}, \neg P(n))$. It is rather easy to see that $(MP \wedge WLPO) \Rightarrow LPO$, and in fact we have the stronger $(MP \wedge WLPO) \iff LPO$.

Relations between the formulae

Bezem et al. [1] showed that for any stream f we have $\text{eaf}(f) \Rightarrow \text{bounded}(f)$. It is obvious that $\text{sb}(f) \Rightarrow \text{bounded}(f)$, and $\text{eaf}(f) \Rightarrow \text{sb}(f)$ is also clear, as the bound on the index in eaf gives us a bound where all the **True** values must reside, letting us find the exact amount of **True**'s. They furthermore show that $(\forall f, \text{bounded}(f) \Rightarrow \text{eaf}(f)) \iff LPO$.

We find that the new notion of strictly bounded falls neatly between the two previous notions, completing the picture:

Lemma 1. $(\forall f, \text{sb}(f) \Rightarrow \text{eaf}(f)) \iff MP$*.*

Lemma 2. $(\forall f, \text{bounded}(f) \Rightarrow \text{sb}(f)) \iff WLPO$*.*

As neither MP, WLPO or LPO hold constructively, we get a strict hierarchy where eaf $\Rightarrow$ sb and sb $\Rightarrow$ bounded holds constructively, but none of the other directions hold without further assumptions.

Streamless

In this section we will look at streamless sets. A set B is streamless if for all streams $g : \mathbb{N} \to B$ we have indices $i < j$ and $g(i) = g(j)$. See Coquand and Spiwack [2] for further elaborations. It is an open problem whether $A \times B$ is streamless whenever A and B are. Here we look at two special cases in which it holds.

First we observe several properties of streamless sets. If we have a stream g over streamless B we can make a new stream over $B \times \mathbb{N}$, such that for every $\langle b, i\rangle$ in the new stream, b occurs at least twice in g. We get this letting it begin with the pair $\langle g(j), j\rangle$ with j as above, and then continuing likewise on the stream we get by starting g at index j. We can iterate this process, and for every $n : \mathbb{N}$ we get a stream such that every element in the new stream gives a $b : B$ and an index such that b occurs at least n times in g before the index. Given a stream $g : \mathbb{N} \to B$ we denote by $g^x : \mathbb{N} \to B \times \mathbb{N}$ the stream which gives pairs $\langle b, i\rangle$ such that there is at least x occurrences of b before i in g.

Finally, we say that a set A is bounded by $n : \mathbb{N}$ if every list over A with more than n elements must contain duplicates.

Lemma 3. *If A is bounded by n and B is streamless then $A \times B$ is streamless.*

Proof sketch. Assuming $g : \mathbb{N} \to A \times B$ we can look at $g_2 : \mathbb{N} \to B$, its second projection. By looking at $(g_2)^{n+1}(0)$ we get a pair $\langle b, i\rangle$ such that b occurs at least $n + 1$ times before i in g_2. As this is a bounded range we can extract the list $[j_1, \dots, j_i]$ of indices where b occurs in g_2. Note that $[g_1(j_1), \dots, g_1(j_i)]$ is $n+1$ elements of A, so there must be at least two indices $j_k < j_l$ such that $g_1(j_k) = g_1(j_l)$. As $g_2(j_k) = b = g_2(j_l)$ we get $g(j_k) = g(j_l)$. □

Lemma 4. *Assuming Markov's Principle and decidable equality on A we have that $A \times B$ is streamless whenever A and B are.*

Proof sketch. Assume a stream $g : \mathbb{N} \to A \times B$. We then define the predicate $P(n) :=$"for $\langle b, i\rangle = (g_2)^n(0)$ we have duplicates in the list generated from taking the A-elements from the n pairs before i with b as their second element." Notice that $P(n)$ is decidable as long as B is streamless and A has decidable equality.

We want to prove $\neg\neg\exists n, P(n)$ when A is streamless. From $\neg\exists n, P(n)$ we get that for every n we have a list of n elements of A without duplicates. This allows us to construct a stream over A without duplicates, contradicting that A is streamless.

Using Markov's Principle we get an n such that $P(n)$, and by essentially the same argument as for Lemma 3 we get the two indices $i < j$ with $g(i) = g(j)$. $\square$

References

[1] Marc Bezem, Keiko Nakata, and Tarmo Uustalu. On streams that are finitely red. *Logical Methods in Computer Science*, 8(4), 2012.

[2] Thierry Coquand and Arnaud Spiwack. Constructively finite? In *Contribuciones científicas en honor de Mirian Andrés Gómez*, pages 217–230. Universidad de La Rioja, 2010.