Proving and Computing with the Harthong-Reeb line using Ω-integers

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The Harthong-Reeb line offers an alternative model to describe the continuum. It uses only integers to represent real numbers (viewing them at different scales). This is appealing, especially when considering fields of applications such as discrete geometry or geometric computations.

In TYPES 2011, we presented a formal description of the Harthong-Reeb line based on axiomatic non-standard integers (intuitively they correspond to $\mathbb{Z}$ where we introduce a - new - infinitely large number ω). We formally proved using Coq [5] that this description of the Harthong-Reeb line does satisfy all the axioms for constructive real numbers proposed by D. Bridges [1].

However, having an axiomatic description of the underlying non-standard integers prevents us from computing in this formalism. Thus it has been investigated how to implement these non-standard integers using Laugwitz-Schmieden integers [2].

Laugwitz-Schmieden integers (also known as Ω-integers) are sequences of elements of $\mathbb{Z}$. This representation of non-standard integers allows to build a constructive description of the Harthong-Reeb line. Thus, we can implement the Ω-arithmetization scheme (adapted from Euler's one) proposed in [3]. Using extraction in Ocaml and the graphical interface of Ocaml, we obtain discrete representations of *continuous* functions at different scales as shown in Fig. 1.

One of our goals is to establish the correction of this Ω-arithmetization scheme for continuous functions as Fleuriot does in [4] using hyperreals. To do so, we must first establish that the Harthong-Reeb line based on Laugwitz-Schmieden integers verifies Bridges' axioms.

As showed by Chollet et al. in [2], most properties of the constructive real line hold for this model based on Laugwitz-Schmieden (except three of them: two properties about the order and the least upper bound principle). These restrictions come from the fact that Ω-integers are a very rich structure where many Ω-integers have no interpretation as naive (or standard) integers. Chollet et al. thus propose an alternative axiom system which is very close to Bridges' one, but defines an alternative continuum.

We formally proved using Coq that the description of this continuum based on Ω-integers actually verifies all of these (new) axioms. Work in progress consists in finding a way to characterize a subset of the Harthong-Reeb line based on Ω-integers which corresponds to constructive real numbers (such as those available in CoRN). We then expect to prove using Coq that this subset is actually isomorphic to the constructive real numbers of CoRN.

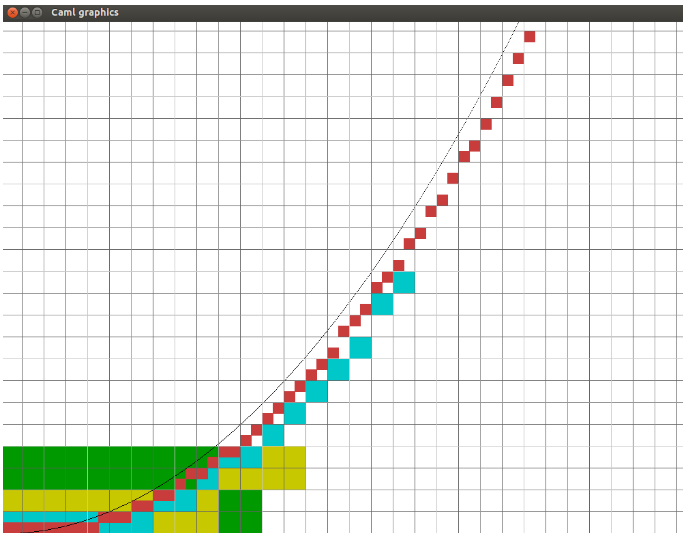

Figure 1: The arithmetization of the function $t \mapsto \frac{t^2}{6}$. Graphs of the function $\widetilde{t} \mapsto \widetilde{x}(\widetilde{t})$ are drawn at different ranks of the involved infinite integer sequences.

References

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