Higher Inductive Types as Homotopy-Initial Algebras

Kristina Sojakova

Carnegie Mellon University, Pittsburgh, USA
kristinas@cmu.edu

Homotopy Type Theory (HoTT, [9]) is a new field of mathematics based on the recently-discovered correspondence between Martin-Löfs constructive type theory and abstract homotopy theory. Under this new interpretation, types are topological spaces, terms are points in spaces, and proofs of identity are paths between points. Since proofs of identity are themselves terms (of an identity type), we can also talk about *higher paths*, for instance, proofs that two proofs of identity are equal, and so on.

Type-theoretically, HoTT is an extension of the intensional Martin-Löf dependent type theory, with two new features motivated by abstract homotopy theory: Voevodsky's *univalence axiom* [3] and *higher inductive types* ([5, 6]). The theory remains intensional in the sense that no form of Streicher's K-rule [8] or the identity reflection rule are admissible (as the groupoid model constructed in [2] shows). In fact, the former is incompatible with univalence.

Higher inductive types are important because they allow us to represent a variety of mathematical objects - such as spheres, tori, pushouts, and quotients - within the type theory. Ordinarily, an inductive type X can be understood as being freely generated by a collection of constructors for X. A higher inductive types X also permits constructors involving *path spaces* of X: for example, the circle S^1 can be represented as the higher inductive type generated by a single point constructor $\mathsf{base} : \mathsf{S}^1$ and a single path constructor $\mathsf{loop} : \mathsf{base} =_{\mathsf{S}^1} \mathsf{base}$, where we use $=$ to denote propositional equality. The induction principle associated with this simple definition is powerful enough to show, e.g., that the fundamental group of S^1 is the group of integers (see [4]).

We investigate a variant of higher inductive types whose computational behavior is determined up to a higher path. We show that in this setting, higher inductive types are characterized by the universal property of being a homotopy-initial algebra. In the case of the circle S^1, the data $(\mathbf{S}^1, \mathsf{base}, \mathsf{loop})$ together can be thought of as defining an $\mathbf{S}^1$*-algebra*. The recursion principle for the circle then says that given any other $\mathbf{S}^1$-algebra (X, x, s), where X is a type, $x : X$ is a point, and s is a loop based at x, there exist an $\mathbf{S}^1$*-homomorphism* (f, β, θ) from $(\mathbf{S}^1, \mathsf{base}, \mathsf{loop})$ to (X, x, s).

An $\mathbf{S}^1$-homomorphism (f, β, θ) between $\mathbf{S}^1$-algebras (X, x, s) and (Y, y, r) consists of a map $f : X \to Y$, a path $\beta : f(x) = y$, and a higher path $\theta : f(s) \cdot \beta = \beta \cdot r$, where $f(s)$ records the effect of the map f on the path s. An $\mathbf{S}^1$-algebra is called *homotopy-initial* [1] if the type of homomorphisms to any other algebra is contractible, meaning it consists of an element which is unique up to a higher path, which is itself unique up to a higher path, and so on.

Theorem 1. *An* $\mathbf{S}^1$*-algebra satisfies the formation, introduction, elimination, and computation rules for a circle if and only if it is homotopy-initial.*

This theorem follows from an analogous result for a more general class of higher inductive types we call *W-suspensions*. For the full account we refer to [7].

References

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