All derivations of groupoid laws are propositionally equal.

Marc Lasson[1]

INRIA
PPS, Université Paris Diderot
marc.lasson@inria.fr

Garner, van den Berg [6] and Lumsdsaine [3] independently showed that in type theory, each type can be equipped with a structure of weak ω-groupoids. For this, they show that a minimal fragment MLID of Martin-Löf type theory, where identity types are the only allowed type constructors, bears a weak ω-category structure. Informally, these results state the possibility to express groupoid laws of weak ω-groupoids as types and in each case to find a canonical inhabitant of these types reflecting the fact that the law holds. Identities, inversion and concatenation of paths, associativities, idempotency of inversion, horizontal and vertical compositions of 2-paths, are all examples of groupoid laws.

In this work[1], we follow a syntactic approach proposed by Brunerie [2] to formalize the notion of groupoid laws. We call *groupoid law* any closed type $\forall\Gamma.c$ such that the sequent $\Gamma \vdash c : \mathsf{Type}$ is derivable in MLID and the context Γ is contractible. A *contractible context* is a context of the following shape: $X : \mathsf{Type}, x : X, x_1 : C_1, y_1 : M_1 = x_1, \ldots, x_n : C_n, y_n : M_n = x_n$ where x_i does not occur in M_i. A canonical inhabitant of a groupoid law may always be obtained by successive path inductions. Some examples of groupoid laws are given in Figure 1.

Moreover, one can prove that any term M such that $\Gamma \vdash M : c$ is derivable in MLID is extensionally equal to the canonical one. The natural question we answer positively here is: does this uniqueness property of groupoid laws holds in the whole Martin-Löf type theory (MLTT) (with function spaces, universes, sigma types, and inductive families) ? We prove that if $\Gamma \vdash M : c$ is derivable in MLTT then M is equal to the canonical derivation of the groupoid law.

The main idea of the proof is to use successive path inductions to reduce the problem of the uniqueness of inhabitants of a given groupoid law to the uniqueness of the canonical point inhabiting a parametric loop space. Given a base type X and a point $x : X$, the n-th *loop space* and its *canonical point* are inductively defined by:

$$\Omega_0(A,a) := A \qquad \omega_0(A,a) := a$$
$$\Omega_{n+1}(A,a) := \Omega_n(a = a, 1_a) \qquad \omega_{n+1}(A,a) := \omega_n(a = a, 1_a)$$

where $1_a : a = a$ denotes the reflexivity. Thus for any integer n, $\forall X : \mathsf{Type}, x : X.\Omega_n(X,x)$ is a groupoid law inhabited by $\lambda X : \mathsf{Type}, x : X.\omega_n(X,x)$ (note that using one universe, it is possible to internalize the quantification over n; everything that we state here will be true whether or not this is used). We call this groupoid law the n-th *parametric loop space*.

The 0-th parametric loop space, is the polymorphic type $\forall X : \mathsf{Type}.X \to X$ of identity functions, and its canonical inhabitant is $\lambda X : \mathsf{Type}, x : X.x$, ie. the identity function. This term is the only one in MLTT up to function extensionality inhabiting its type. The standard tool to prove this kind of property is by using Reynold's parametricity theory [4] which was introduced to study the behavior of type quantifications within polymorphic λ-calculus (a.k.a. System F). It refers to the concept that well-typed programs cannot inspect types; they must

[1]We believe that Lumsdaine's construction of a contractible globular operad may be described in our framework, but we have not checked it. More generally, a precise study of how models of MLID restricted to contractible contexts compare to definitions of ω-groupoids is out of the scope of the present work.

$$
\begin{array}{rcl}
\texttt{id} & : & \forall X : \mathsf{Type}. X \to X \\
\texttt{sym} & : & \forall X : \mathsf{Type}, x : X. x = y \to y = x \\
\texttt{concat} & : & \forall X : \mathsf{Type}, x : X, y : X. x = y \to \forall z : X. y = z \to x = z \\
\texttt{assoc} & : & \forall X : \mathsf{Type}, x : X, y : X, p : x = y, z : X, q : y = z, t : X, r : z = t. \\
& & \quad \mathsf{concat}\, X\, x\, z\, (\mathsf{concat}\, X\, x\, y\, p\, z\, q)\, t\, r = \mathsf{concat}\, X\, x\, y\, p\, t\, (\mathsf{concat}\, X\, y\, z\, q\, t\, r) \\
\texttt{horizontal} & : & \forall X : \mathsf{Type}, x : X, y : X, p : x = y, p' : x = y. p = p' \to \\
& & \quad \forall z : X, q : x = z, q' : x = z. q = q' \to \mathsf{concat}\, X\, x\, y\, p\, z\, q = \mathsf{concat}\, X\, x\, y\, p'\, z\, q'
\end{array}
$$

Figure 1: Examples of groupoid laws with aliases for canonical inhabitants

behave uniformly with respect to abstract types. Reynolds formalizes this notion by showing that polymorphic programs satisfy the so-called logical relations defined by induction on the structure of types. This tool has been extended by Bernardy et al. [1] to dependent type systems. It provides a uniform translation of terms, types and contexts preserving typing (the so-called *abstraction theorem*). In its unary version (the only needed for this work), logical relations are defined by associating to any well-formed type $A : \mathsf{Type}$ a predicate $[\![A]\!] : A \to \mathsf{Type}$ and to any inhabitant $M : A$ a witness $[\![M]\!] : [\![A]\!]\, M$ that the M satisfies the predicate. This translation may be extended to cope with identity types by taking $[\![a = b]\!] : a = b \to \mathsf{Type}$ to be the predicate defined by $\lambda p : a = b. p_*([\![a]\!]) = [\![b]\!]$ where p_* is the transport along p of the predicate generated by the common type of a and b. Then, it is easy -although quite verbose- to find a translation of introduction and elimination rules of identity types as well as checking that these translations preserve computation rules. This allows to extend the Bernardy's abstraction theorem to identity types. Using this framework, we are able to generalize the uniqueness property of the polymorphic identity type to any parametric loop space. The proof proceed by induction on the dimension of the loop space and uses algebraic properties of transport.

This work shows that parametricity theory may be used to deduce properties about the algebraic structure of identity types. The most interesting question that remains open is whether or not we can extend the translation and the uniqueness property of groupoid laws to deal with Voevodsky's univalence axiom.

References

[1] Jean-Philippe Bernardy, Patrik Jansson, and Ross Paterson. Proofs for free - parametricity for dependent types. *J. Funct. Program.*, 22(2):107–152, 2012.

[2] Guillaume Brunerie. Syntactic grothendieck weak ∞-groupoids. Available as `http://uf-ias-2012.wikispaces.com/file/view/SyntacticInfinityGroupoidsRawDefinition.pdf`.

[3] Peter LeFanu Lumsdaine. Weak omega-categories from intensional type theory. In Pierre-Louis Curien, editor, *TLCA*, volume 5608 of *LNCS*, pages 172–187. Springer, 2009.

[4] John C. Reynolds. Types, Abstraction and Parametric Polymorphism. In *IFIP Congress*, pages 513–523, 1983.

[5] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.

[6] Benno van den Berg and Richard Garner. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society*, 102(2):370–394, 2011.