Towards an Internalization of the Groupoid Interpretation of Type Theory

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Abstract

Homotopical interpretations of Martin-Löf type theory lead toward an interpretation of equality as a richer, more extensional notion. Extensional or axiomatic presentations of the theory with principles based on such models do not yet fully benefit from the power of dependent type theory, that is its computational character. Reconciling intensional type theory with this richer notion of equality requires to move to higher-dimensional structures where equality reasoning is explicit. In this paper, we follow this idea and develop an internalization of a groupoid interpretation of Martin-Löf type theory with one universe respecting the invariance by isomorphism principle. Our formal development relies crucially on ad-hoc polymorphism to overload notions of equality and on a conservative extension of Coq's universe mechanism with polymorphism.

1 Introduction

Our work here concentrates on the internalization in Coq of Hofmann and Streicher's groupoid model where we can have a self-contained definition of the structures involved.

Our first motivation to implement this translation is to explore the interpretation of type theory in groupoids in a completely intensional setting and in the type theoretic language, leaving no space for imprecision on the notions of equality and coherence involved. We also hope to give with this translation a basic exposition of the possible type theoretic implications of the groupoid/homotopy models, bridging a gap in the literature. On the technical side, the definition of the groupoid model actually requires to reason at a 2-dimensional level. This is due to the way we interpret the strictness in the definition of groupoids. Indeed, interpreting strictness by the fact that the internal equality coincides with the identity type requires the functional extensionlaty axiom when it comes to define for instance the groupoid on the function space. Our interpretation of strictness is closer to the idea that a groupoid is a weak ω-groupoid for which all equalities at dimension 2 are the same. That is, we only use identity types to express triviality of higher dimension, not coherences themselves. As the model that we use does not have the uniqueness of identity proof principle, the two ways of formalizing groupoids mentionned above are different. Our presentation requires less properties on identity types, but we still need the axiom of functional extensionality. Also, this indicates that if we scale to ω-groupoids, the presence of identity types in the core type theory will not be necessary anymore and so the core type theory will be axiom free. Thus, our work can be seen as a proof of concept that it is possible to interpret homotopy type theory into type theory without identity types.

We use an extension of the Coq proof assistant [1] with universe polymorphism to formally define our translation [2]. We studied a restricted source theory resembling a cut-down version of the core language of the Coq system, with only one `Type` universe (see [3] for an in-depth study of this system). This is basically Martin-Löf Type Theory (without `Type` : `Type`), with Π, Σ, `Id` types and a single universe $\mathcal{U}$.

Universe and Type Equivalence. The universe $\mathcal{U}$ is closed under Σ, Π, $\mathbb{0}$, $\mathbb{1}$, $\mathbb{2}$ and `Id` in elements of $\mathcal{U}$, *not* type equivalences. For T and U in $\mathcal{U}$, the type of (Set)-isomorphisms $T \equiv U$, is definable directly using the other type constructors. The new, proof-relevant type equivalence in the source theory for which we want to give a computational model appears in the rule below, extending the formation rules of identity types on $\mathcal{U}$. The `J` rule for type equivalences witnesses the invariance under isomorphism principle of the source type theory.

$$\frac{\Gamma \vdash i : \texttt{Elt}(A) \equiv \texttt{Elt}(B)}{\Gamma \vdash \texttt{equiv}\ i : \texttt{Id}_{\mathcal{U}}\ A\ B}$$

2 Formalization of groupoids

We formalize groupoids using type classes. Contrarily to what is done in the setoid translation, the basic notion of a morphism is an inhabitant of a relation on a type T in Type (i.e., a proof-relevant relation).

In our definition of the type Type_1 of groupoids, we do not ask that the internal equality coincides with the identity type but we model explicitly coherence laws with an equality at dimension 2, which is assumed to be irrelevant. This irrelevance is defined using a notion of contractibility expressed with identity types. One can then define groupoid morphisms (functors) preserving homs which form a pre-category with natural transformations and modifications. Groupoid equivalence itself is formalized using adjoint equivalences. We can define the pre-groupoid Type_1^1 of groupoids and homotopy equivalences. However, groupoids together with homotopy equivalences do not form a groupoid but rather a 2-groupoid. As we only have a formalization of groupoids, this can not be expressed in our setting. Nevertheless, we can state that setoids (inhabitants of Type_0, which are the targets in the interpretation of the types of our universe $\mathcal{U}$) form a groupoid. The proof that it is indeed a groupoid makes use of functional extensionality to prove contractibility of higher cells. As Type_1 appears both in the term and in the type, the use of polymorphic universe is crucial here to avoid an inconsistency.

3 Interpretation of the source type theory

Our formalization of groupoids can be organized into a model of dependent type theory. The interpretation is based on the notion of categories with families introduced by Dybjer [4] later used in [5]. This interpretation can also be seen as an extension of the Takeuti-Gandy interpretation of simple type theory, recently generalized to dependent type theory by Coquand et al. using Kan semisimplicial sets or cubical sets [6]. In our interpretation, we take advantage of universe polymorphism to interpret dependent types directly as functors into Type_0^1. We interpret contexts as groupoids. The empty context being the groupoid with exactly one element at each dimension. Types in a context Γ are (context) functors from Γ to the groupoid of setoids Type_0^1. Thus, a judgment $\Gamma \vdash A : \mathsf{Type}$ is represented as a term A of type $\mathsf{Typ}\ \Gamma$. Context extension (Rule Decl) is given by dependent sums, i.e., the judgment $\Gamma, x : A \vdash$ is represented as $\Sigma\ A$. Substitution and all typing rules can be interpreted this way. For conversion, we just have (trivial) metatheoretical result that we preserve conversion in the interpretation, as the intepretation of types just adds compatibility terms to a type, so two convertible types in the source language just get interpreted as two pairs with convertible first projections in the shallow embedding.

4 Related Work

The groupoid interpretation is due to Hofmann and Streicher [5]. This interpretation is based on the notion of categories with families introduced by Dybjer [4]. This framework has recently been used by Coquand et al. to give an interpretation in semi-simplicial sets and cubical sets [6, 7]. Although very promising, the interpretation based on cubical sets has not yet been mechanically checked, only an Haskell implementation exists. Observational Type Theory (OTT) [8], an intentional type theory where functional extensionality is native, but equality in the universe is structural.

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