Normalisation-by-evaluation for $\lambda$-calculi

Danko Ilik

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Idea of NbE

- computational behaviour of $\lambda$-calculi traditionally studied through reduction seen as a rewrite system
- alternative: *evaluate* a $\lambda$-term in a constructive meta-language and *reify*-back the result into the object-language

$$M \xrightarrow{\parallel M \parallel} M'$$

where $M =_{\beta\eta} M'$
Advantages of NbE

- proof of Church-Rosser not necessary
- easier to deal with $\eta$-equality when not considering it as reduction
NbE for Simply Typed $\lambda$-Calculus

(Sketch on whiteboard)
Connections to Concepts from Intuitionistic Logic

Making the previous construction more precise:

- **Evaluation is Soundness**: given $\Gamma \vdash t : A$, in every world $w$ of every Kripke model, $w \models \Gamma$ implies $w \models A$

- **Reification is Completeness**: if in every world $w$ of every Kripke model, $w \models \Gamma$ implies $w \models A$, then there exists a derivation $\Gamma \vdash t : A$

$\text{NbE} = \text{Completeness} \circ \text{Soundness}$
Kripke Semantics (Possible Worlds Semantics)

A possible world is determined by atomic formulas known to hold so far.

At any later world we enrich our knowledge.
Kripke Semantics: Formal Definition

A Kripke model is given by:

- a partial order \((K, \leq)\) of worlds
- a domain of constants \(D : K \rightarrow \text{Set}\), monotone
- a relation "\(\models\)" between worlds and atomic formulas, called forcing
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Forcing is extended to composite formulas inductively:

\(w \models\)
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3. a relation “\(\models\)” between worlds and *atomic* formulas, called *forcing*

Forcing is extended to composite formulas inductively:

\[
\begin{align*}
&w \models \\
A \land B & w \models A \text{ and } w \models B
\end{align*}
\]
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\[ w \models \]

\[ A \land B \models A \text{ and } w \models B \]
\[ A \lor B \models A \text{ or } w \models B \]
\[ A \rightarrow B \text{ for any } w' \geq w, \text{ if } w' \models A \text{ then } w' \models B \]
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Forcing is extended to composite formulas inductively:

\[ w \models A \land B \quad \text{if} \quad w \models A \text{ and } w \models B \]
\[ w \models A \lor B \quad \text{if} \quad w \models A \text{ or } w \models B \]
\[ w \models A \rightarrow B \quad \text{for any } w' \geq w, \text{ if } w' \models A \text{ then } w' \models B \]
\[ w \models \forall x P(x) \quad \text{for any } w' \geq w \text{ and any } a \in D(w'), \ w' \models P(a) \]
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\[ \bot \text{ is never forced} \]
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\end{align*}
\]

Almost Tarski’s truth definition, \(\rightarrow\) (and \(\forall\)) standing out.
Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

1. \( p \lor \neg p \)
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1. $p \lor \neg p$
2. $\neg \neg p \rightarrow p$
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1. \( p \lor \neg p \)
2. \( \neg \neg p \to p \)
3. 2 implies 1
Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

1. \( p \lor \neg p \)
2. \( \neg
\neg p \rightarrow p \)
3. 2 implies 1
4. \( (p \rightarrow q) \rightarrow (\neg p \lor q) \)
Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

1. \( p \lor \neg p \)
2. \( \neg \neg p \rightarrow p \)
3. 2 implies 1
4. \( (p \rightarrow q) \rightarrow (\neg p \lor q) \)
5. \( (p \rightarrow (q \lor r)) \rightarrow ((p \rightarrow q) \lor (p \rightarrow r)) \)
Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

1. $p \lor \neg p$
2. $\neg \neg p \to p$
3. 2 implies 1
4. $(p \to q) \to (\neg p \lor q)$
5. $(p \to (q \lor r)) \to ((p \to q) \lor (p \to r))$
6. $\neg \forall x \neg P(x) \to \exists x P(x)$
Instances of NbE

- Simply typed lambda calculus (Berger-Schwichtenberg 1991)
- Untyped lambda calculus (Filinski-Rohde, 2004)
- Martin-Löf type theory (Abel-Coquand-Dybjer, 2007)
- ...
Part II: $\check{\lambda}\mu\check{\mu}$ and its Kripke Semantics

joint work with Hugo Herbelin and Gyesik Lee (ROSAEC, Korea)
Proof Systems For Classical Logic

▶ call/cc and Pierce’s Law (Griffin, 1990)
Proof Systems For Classical Logic

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- $\lambda\mu$-calculus (Parigot, 1992)
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- $\lambda\mu$-calculus (Parigot, 1992)
- $\overline{\lambda}\mu$-calculus (Herbelin, 1995)
Proof Systems For Classical Logic

- call/cc and Pierce’s Law (Griffin, 1990)
- $\lambda\mu$-calculus (Parigot, 1992)
- $\check{\lambda}\mu$-calculus (Herbelin, 1995)
- $\check{\lambda}\mu\check{\mu}$ — calculus (Curien-Herbelin, 2000)
\(\bar{\lambda}\mu\tilde{\mu}:\) Syntax and Reduction Rules

**Syntax** 3 categories: **commands**, **terms** and **evaluation contexts**

\[
\begin{align*}
c & ::= \langle t \parallel e \rangle \\
t & ::= x \mid \mu\alpha.c \mid \lambda x.t \\
e & ::= \alpha \mid \tilde{\mu}x.c \mid t \cdot e
\end{align*}
\]

**Reduction**

(\(\mu\)) \(\langle \mu\alpha.c \parallel e \rangle \rightarrow c[e/\alpha]\)

(\(\tilde{\mu}\)) \(\langle t \parallel \tilde{\mu}x.c \rangle \rightarrow c[t/x]\)

(\(\beta\)) \(\langle \lambda x.t \parallel t' \cdot e \rangle \rightarrow \langle t' \parallel \tilde{\mu}x.\langle t \parallel e \rangle \rangle\)

**Critical pair** \(\langle \mu\alpha.c \parallel \tilde{\mu}x.c' \rangle\): CBN, CBV strategies
The Sequent Calculus $\text{LK}_{\mu\tilde{\mu}}$

$$\frac{\Gamma \vdash v : A | \Delta \quad \Gamma | e : A \vdash \Delta}{\langle v \parallel e \rangle : (\Gamma \vdash \Delta)} \text{(Cut)}$$

$$\frac{(x : A), \Gamma \vdash x : A | \Delta}{(Ax_R)}$$

$$\frac{\Gamma | \alpha : A \vdash (\alpha : A), \Delta}{(Ax_L)}$$

$$\frac{c : (\Gamma \vdash (\alpha : A), \Delta)}{\Gamma \vdash \mu \alpha . c : A | \Delta} \text{ (\mu)}$$

$$\frac{c : ((x : A), \Gamma \vdash \Delta)}{\Gamma | \tilde{\mu} x . c : A \vdash \Delta} \text{ (\tilde{\mu})}$$

$$\frac{(x : A), \Gamma \vdash (t : B) | \Delta}{\Gamma \vdash \lambda x . t : A \to B | \Delta} \text{ (\to_R)}$$

$$\frac{\Gamma \vdash t : A | \Delta \quad \Gamma | e : B \vdash \Delta}{\Gamma | t \cdot e : A \to B \vdash \Delta} \text{ (\to_L)}$$

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Normalisation-by-evaluation for $\lambda$-calculi
Kripke Semantics for \( \text{LK}_{\mu\tilde{\mu}} \): Motivation

2 “new” ingredients for the model cake:
Kripke Semantics for $\text{LK}_{\mu\tilde{\mu}}$: Motivation

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exploding nodes
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- Intuitionistic completeness for Kripke semantics has seen only a **classical** proof
Kripke Semantics for $LK_{\mu\tilde{\mu}}$: Motivation

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- Actually, the proof cannot be intuitionistic (Gödel, Kreisel, 1962)
Kripke Semantics for $\text{LK}_{\mu\tilde{\mu}}$: Motivation

2 “new” ingredients for the model cake:
exploding nodes

- Intuitionistic completeness for Kripke semantics has seen only a \textit{classical} proof
- Actually, the proof \textit{cannot be} intuitionistic (Gödel, Kreisel, 1962)
- Yet, an \textit{intuitionistic} proof by Veldman (1976), introducing exploding nodes
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2 “new” ingredients for the model cake: exploding nodes

- Intuitionistic completeness for Kripke semantics has seen only a **classical** proof
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- The same relaxation happens for the intuitionistic proof for classical logic (Krivine, 1996) and for Beth models (Friedman, 60’s or 70’s)
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“refinement” of \(\models\)

- identify “strongly refutes” as primitive, define “forcing” and “refutation” by orthogonality
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- a relation "\(\models\)" between worlds and *atomic* formulas, called **forcing**

Forcing is extended to composite formulas inductively:

- \(w \models \top\)
- \(A \land B \models w \models A\) and \(w \models B\)
- \(A \lor B \models w \models A\) or \(w \models B\)
- \(A \rightarrow B\) for any \(w' \geq w\), if \(w' \models A\) then \(w' \models B\)
- \(\forall x P(x)\) for any \(w' \geq w\) and any \(a \in D(w')\), \(w' \models P(a)\)
- \(\exists x P(x)\) there is \(a \in D(w)\) such that \(w \models P(a)\)
- \(\bot\) is never forced
Kripke Semantics for Classical Logic

A Kripke model is given by:

- a partial order \((K, \leq)\) of worlds
- a domain of constants \(D : K \to \text{Set}\), monotone
- a relation “\((\cdot) : (\cdot) \models_s\)” between worlds and atomic formulas, called forcing strong refutation
- a marker of exploding worlds \(\models_\bot\)
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- a relation “\((\cdot) : (\cdot) \models_s\)” between worlds and *atomic* formulas, called **forcing strong refutation**
- a marker of *exploding* worlds \(\models_{\bot}\)

Than we define by orthogonality:

**forcing** \(w : \models A\) iff for all \(w' \geq w\), \(w' : A \models_s\) implies \(w' : \models_{\bot}\)

**refutation** \(w : A \models\) iff for all \(w' \geq w\), \(w' : \models A\) implies \(w' : \models_{\bot}\)
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A Kripke model is given by:

- a partial order \((K, \leq)\) of worlds
- a domain of constants \(D : K \rightarrow \text{Set}\), monotone
- a relation “\((\cdot) : (\cdot) \vdash_s\)” between worlds and atomic formulas, called forcing strong refutation
- a marker of exploding worlds \(\vdash_{\perp}\)

Then we define by orthogonality:

- forcing \(w : \vdash A\) iff for all \(w' \geq w\), \(w' : A \vdash_s\) implies \(w' : \vdash_{\perp}\)
- refutation \(w : A \vdash\) iff for all \(w' \geq w\), \(w' : \vdash A\) implies \(w' : \vdash_{\perp}\)

mutually with the extension of strong refutation to composite formulas: \ldots
Kripke Semantics for Classical Logic

... mutually with the extension of strong refutation to composite formulas:

\[ w : \square \not\vdash \]
Kripke Semantics for Classical Logic

... mutually with the extension of strong refutation to composite formulas:

\[ w : \square \vdash \]

\[ A \lor B \text{ w refutes } A \text{ and } w \text{ refutes } B \]
Kripke Semantics for Classical Logic

... mutually with the extension of strong refutation to composite formulas:

\[ w : \models \Box \vdash \]

- \( A \lor B \) \( w \) refutes \( A \) and \( w \) refutes \( B \)
- \( A \land B \) \( w \) refutes \( A \) or \( w \) refutes \( B \)
Kripke Semantics for Classical Logic

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  \forall x P(x) & \text{ there is } a \in D(w) \text{ such that } w \text{ refutes } P(a)
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- \( A \lor B \) \( w \) refutes \( A \) and \( w \) refutes \( B \)
- \( A \land B \) \( w \) refutes \( A \) or \( w \) refutes \( B \)
- \( A \rightarrow B \) \( w \) forces \( A \) and \( w \) refutes \( B \)
- \( \forall x P(x) \) there is \( a \in D(w) \) such that \( w \) refutes \( P(a) \)
- \( \exists x P(x) \) for all \( w' \geq w \) and all \( a \in D(w') \), \( w \) refutes \( P(a) \)
Kripke Semantics for Classical Logic

\[ \cdots \text{mutually with the extension of } \textit{strong} \text{ refutation to composite formulas:} \]

\[ \mathbf{w} : \models \square \vdash \]

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\[ \exists x P(x) \quad \text{for all } w' \geq w \text{ and all } a \in D(w'), w \text{ refutes } P(a) \]

\[ \bot \quad \text{always} \]
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\[ \bot \quad \text{always} \]

\[ \top \quad \text{never} \]
Resemblance to Intuitionistic Forcing

\[ w : \vdash A \rightarrow B \iff \text{for all } w' \geq w, w' : \vdash A \Rightarrow w' : \vdash B \]  

(1)

\[ w : \vdash \forall x. A(x) \iff \text{for all } w' \geq w \text{ and } t \in D(w'), w' : \vdash A(t) \]  

(2)

\[ w : \vdash \bot \iff w : \vdash \bot \]  

(3)

\[ w : \vdash \top \iff \text{true} \]  

(4)

\[ w : \vdash A \land B \iff w : \vdash A \text{ and } w : \vdash B \]  

(5)

\[ w : \vdash A \lor B \iff w : \vdash A \text{ or } w : \vdash B \]  

(6)

\[ w : \vdash \exists x. A(x) \iff \text{for some } t \in D(w), w : \vdash A(t) \]  

(7)
Soundness of $\text{LK}_{\tilde{\mu}}$ for Kripke Semantics

\[ c : (\Gamma \vdash \Delta) \implies \text{for any } w, w : \models \Gamma \text{ and } w : \Delta \models \text{ implies } w : \models \bot \]
\[ \Gamma \vdash t : A|\Delta \implies \text{for any } w, w : \models \Gamma \text{ and } w : \Delta \models \text{ implies } w : \models A \]
\[ \Gamma|e : A \vdash \Delta \implies \text{for any } w, w : \models \Gamma \text{ and } w : \Delta \models \text{ implies } w : A \models \]

Proof.
By mutual induction on the derivations. \qed
We prove Completeness for a universal Kripke model. From that, Completeness for any Kripke model follows.

The Universal Kripke model $U$ is obtained by putting:

- **possible worlds** $K$ is $\{(\Gamma, \Delta) | \Gamma : \text{tvar} \rightarrow \text{typ}, \Delta : \text{evar} \rightarrow \text{typ}\}$
- **partial order** $(\Gamma, \Delta) \leq (\Gamma', \Delta')$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$
- **exploding nodes** $(\Gamma, \Delta) \vdash \bot$ iff $\Gamma \vdash_{\text{cf}} \Delta$
Strong Completeness of Kripke Semantics for LK\(_{\mu\tilde{\mu}}\)

\[(\Gamma, \Delta) : \vdash A \iff \text{there is a term } t \text{ such that } \Gamma \vdash_{cf} t : A|\Delta\]

\[(\Gamma, \Delta) : A \models \iff \text{there is an ev. context } e \text{ such that } \Gamma|e : A \vdash_{cf} \Delta\]

**Proof.**

By induction on the type \(A\).

**Remarks:**

- Only case for disjunction not straightforward
- Richer semantics – simpler completeness proof
Strong Completeness of Kripke Semantics for LK\(_{\mu\tilde{\mu}}\)

- Normalisation-by-evaluation
Strong Completeness of Kripke Semantics for $\text{LK}_{\mu\tilde{\mu}}$

- Normalisation-by-evaluation
- Experiments show: All reduction rules preserved
Strong Completeness of Kripke Semantics for LK$_{\mu \tilde{\mu}}$

- Normalisation-by-evaluation
- Experiments show: All reduction rules preserved
- Experiments show: Call-by-name discipline
Future Work

- Compare with the \textit{call-by-value} variant of Kripke semantics
Future Work

- Compare with the *call-by-value* variant of Kripke semantics
- Compare to a constructive Henkin-style proof (Krivine, Berardi-Valentini)
Future Work

- Compare with the *call-by-value* variant of Kripke semantics
- Compare to a constructive Henkin-style proof (Krivine, Berardi-Valentini)
- Formalise in Coq the quantifier part