Towards a Quantum Model of Linear Logic

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A "quantum" model of LL ?

- Linear logic introduced the idea that logic is in some way *actions* on *ressources*. We would like to go a bit further : interpret logic as **quantum** actions on **quantum** ressources.
- The linear implication "consumes" its hypotheses, similarly in quantum information, one can't duplicate objects (non-cloning property).
- The quantum coherence spaces model is related to a denotational model of P. Selinger for quantum computation. It uses the langage of quantum information.

1. Quantum Information
   - Density operators and superpositive applications
   - A categorical point of view
   - The fundamental adjunction
2. Quantum Coherence Spaces
3. Connectives
4. Additives and η-expansion
5. Conclusion : physics

From principles to properties

- Superposition principle → operators,
- Measures $O = \sum \alpha P_\alpha$, $p_\alpha = Tr(\rho P_\alpha)$ → positivity,
- Union of two systems → existence of a tensor product.

Objects

To represent a system state : let H be a Hilbert space

Definition

A density matrix or operator is a positive matrix (or operator) on a Hilbert space H that satisfies :

$$Tr(\rho) \leq 1.$$

Why ≤ 1 instead of $= 1$? A density operator can both represent a state or the result of an algorithm. In the latter case, the algorithm terminates with probability $Tr(\rho)$.

Morphisms

Definition

Let A be a linear application from the operators on $\mathscr{L}(H_1)$ to $\mathscr{L}(H_2)$.
A is said to be **superpositive** if, for any Hilbert space H, the linear application

$$\mathrm{Id}_{\mathscr{L}(H)} \otimes A : \quad \mathscr{L}(H \otimes H_1) \longrightarrow \mathscr{L}(H \otimes H_2)$$

satisfies that for all ρ a positive operator on $H \otimes H_1$, its image through $\mathrm{Id}_{\mathscr{L}(H)} \otimes A$ is a positive operator on $H \otimes H_2$.

It means that not only A has to transform positive operators into positive operators but also that whichever space we choose to couple to the first one, the corresponding application must conserve positiveness too.

To begin with

The main idea is to build a category whose objects are sets of positive operators and morphisms are superpositive applications.

One sees that we really need here superpositiveness since we'll use the tensor product as the product of the category.

The sets will need to satisfy certain properties, and we'll call them later quantum coherent spaces, on account of their closeness to J.-Y. Girard's classical coherent spaces.

Connectives

There are several connectives that will have to be represented, the first one being polarity, or to phrase it in a counter-intuitive way, negation :

- $\sim$ is the afore mentioned negation, considering we use linear logics it has to be involutive (for each object A, we must have $\sim\sim A = A$),
- the multiplicative connector $\otimes$ and its dual $⅋$, which is merely represented by the tensor product,
- the additive connector $\oplus$ and its dual $\&$, which will turn out to be a bit more tricky,
- ideally, the modality !, about which we may have some prospects that we will discuss in the conclusion.

Representing $\multimap$

We see that we'll also need to represent the implication connector $\multimap$ not only as $A \multimap B = \sim A ⅋ B$, but also as a set of morphisms which transform A into B.

It means that we will what is called an adjunction, that is to say a way to transform the set of homomorphisms

$$\mathrm{Hom}(A \otimes B, C)$$

into the set

$$\mathrm{Hom}(A, B \multimap C).$$

More generally, we'll give a transformation from $\mathrm{Hom}(A, B)$ to $A \multimap B$ defined as $\sim A ⅋ B$.

Goal

Let A be a set of positive operators on a Hilbert space H_1 and B on H_2. We are looking for a genaral transformation from $A \multimap B$ set of positive operators on the Hilbert space $H_1^\star \otimes H_2$ and $\mathrm{Hom}(A, B)$ set of superpositive applications from $\mathscr{L}(H_1)$ to $\mathscr{L}(H_2)$.

What we'll exhibit will be a general reversible tranformation χ from $\mathscr{L}(H_1) \to \mathscr{L}(H_2)$ to $\mathscr{L}(H_1 \otimes H_2)$ that will statisfy :

For all $F \in \mathscr{L}(H_1) \to \mathscr{L}(H_2)$, χ_F is positive $\Leftrightarrow F$ is superpositive.

Then, we'll define $A \multimap B$ as

$$\{\chi_F \geq 0 \mid F(A) \subseteq B\}.$$

1. Quantum Information
2. Quantum Coherence Spaces
   - Basics
   - An important exemple
3. Connectives
4. Additives and η-expansion
5. Conclusion : physics

From classic to quantic (leaving the diagonal) #1

Writing classical coherence spaces in terms of diagonal square matrix :

$X \subseteq \{a_1, a_2, ..., a_n\}$ can be seen as

$$M_X := \begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \ddots & \\ & & & \varepsilon_n \end{pmatrix}$$ where ε_i is 1 if $a_i \in X$ and 0 otherwise.

From that point of view, $X \overset{\frown}{\smile} Y$ (that is, $\#|X \cap Y| \leq 1$) can be rewritten :

$$Tr(M_X M_Y) \leq 1$$

From classic to quantic (leaving the diagonal) #2

Writing probabilistic coherence spaces in terms of diagonal square matrix :

X, a "probabilistic subset" of $\{a_1, a_2, ..., a_n\}$ can be seen as

$$M_x := \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad \text{where } \lambda_i = p(a_i \in X)$$

Again, we can rewrite polarity as :

$$Tr(M_x M_y) \leq 1$$

Polarity #1

We will use the following isomorphism between $\mathscr{L}(E)$ and $\mathscr{L}(E^\star)$, called *transposition* :

Definition

If E is a Hilbert space and $f \in \mathscr{L}(E)$, we define the *transposed* of f (notation ${}^t\!f$) by :

$$\text{For all } \varphi \in E^\star : \quad {}^t\!f(\varphi) = \varphi \circ f$$

Transposition enjoys all the properties we need : preservation of positivity, of the trace, of the norm...

Polarity #2

We now consider positive hermitians, a natural generalisation of the diagonal matrix presentation we saw.

Definition

Two positive operators $f \in \mathscr{H}^+(E)$ and $g \in \mathscr{H}^+(E^\star)$ are **polar** (notation $f \downarrow\!\!\sim g$) whenever :

$$Tr({}^t f.g) \leq 1$$

The polar of $X \subseteq \mathscr{H}^+(E)$: $\sim X := \{\ y \mid\ y \downarrow\!\!\sim x\text{, for all } x \in X\ \}$

Quantum Coherence Space

Definition

We call a Quantum Coherence Space (QCS), a subset $X \subseteq \mathscr{H}^+(E)$ such that X is equal* to its bipolar, *i.e.* :

$$X = \sim\sim X$$

* : As we use Hilbert spaces, we identify E and its bidual, $E^{\star\star}$

Definition

If $X \subseteq \mathscr{H}^+(E)$ we will call E the *support* of X, notation $|X|$.

Adjunction #1

We have an isomorphism between $\mathscr{L}(E_1^\star) \otimes \mathscr{L}(E_2)$ and $\mathscr{L}(E_1) \to \mathscr{L}(E_2)$. It is defined by :

Definition

If $F \in \mathscr{L}(E_1) \to \mathscr{L}(E_2)$, χ_F is the only operator on $E_1^\star \otimes E_1$ satisfying :

For all $x \in \mathscr{L}(E_1)$ and $y \in \mathscr{L}(E_2)$: $Tr(F(x).y) = Tr(\chi_F.{}^t x \otimes y)$

Remark : the polarity is compatible with this adjuction, i.e.,

$$A \multimap \underset{\sim}{\downarrow} \simeq \sim A$$

Adjunction #2

Theorem

F is superpositive if and only if χ_F is positive.

With the matrix representation in some base, there is an easier way to compute χ_F :

$$\chi_F := \begin{pmatrix} F(E_{1,1}) & \dots & F(E_{1,n}) \\ \vdots & \ddots & \vdots \\ F(E_{n,1}) & \dots & F(E_{n,n}) \end{pmatrix}$$

where $E_{i,j}$ is the square matrix with 1 in position (i,j) and 0 elsewhere

Bipolar theorem

This theorem caracterises the geometry of QCS :

Theorem

A set of positive hermitian $X \subseteq \mathscr{H}^+(E)$, is equal to its bipolar $\sim\sim X$ if and only if :

- *0 is in X*
- *X is convex and closed*
- *if $x \in X$ and $y \leq x$, then $y \in X$*

Proof uses convex projection theorem and linear algebra.

Quantum Booleans and density operators

The set of density operators on a Hilbert E (wich are used to represent the state of a quantum system) is a QCS :

$$Dens_E := \{\ h \in \mathscr{H}^+(E) \mid Tr(h) \leq 1\ \}$$

contains 0*, is convex and closed, and downward closed for* $\leq$ *(order on positives)*

We will call this set "quantum booleans" or "canonical positive" over E, notation P_E .

Its polar is : $\sim P = N_{E^\star} := \{\ h \in \mathscr{H}^+(E) \mid |||h||| \leq 1\ \}$ called "canonical negative" over $E^\star$.

$|||h|||$ *is the operator norm of* h *:* $|||h||| := Max_{v \in E} \frac{||h(v)||}{||v||}$

Definitions

Definition

The QCS corresponding to linear implication, $X \multimap Y$, is defined on the support $|X|^\star \otimes |Y|$, by :

$$X \multimap Y := \{\ \chi_F \mid F \in X \to^+ Y\ \}$$

That is, the set of the χ_F corresponding to the F superpositive such that $F(X) \subseteq Y$.

From this, we derive definitions of $\otimes$ and $\parr$, by duality :

$$X \parr Y = \sim X \multimap Y \quad \text{and} \quad X \otimes Y = \sim(\sim X \parr \sim Y)$$

Proposition

$$X \otimes Y = \sim\sim\{\ x \otimes y \mid x \in X, y \in Y\ \}$$

Another point of view for ⊗

Proposition

Let A, B be two coherent spaces that we suppose bounded. We denote A_m (resp. B_m) the set of maximal elements of A (resp. B), and C_m the convex hull of $\{a \otimes b \mid a \in A_m \text{ and } b \in B_m\}$. The QCS $A \otimes B$ is equal to the set :

$$\{\rho \mid \exists \rho_m \in C_m \text{ such that } \rho_m \geq \rho\}.$$

Hence, the set of maxima in $A \otimes B$ is equal to C_m.

Since the traces of states in quantum physics are supposed to be less than 1, the quantum coherent spaces formed by physically meaningful operators are bounded.

Separability #1

Suppose now that A_m contains the operators that represents admissible states for a first system S_A and B_m for a second one S_B.

Then, $(A \otimes B)_m$ is the convex hull of A_m and B_m. That is to say, $(A \otimes B)_m$ is the set of the separable states one can obtain by combining S_A and S_B.

Being separable for two states means you can act on one without touching to the other one. States that are not separable are called entangled.
Entanglement is the property of quantum states that gives a special interest to quantum computation.

Separability #2

Consequently, even though we can not represent separable states with the simple operation of $\otimes$, we have this property onto the set of maxima.

What's more, one can deduce from the precedent property that $A \otimes B$ is the QCS generated by the separable states obtained by combining the systems S_A and S_B.

Regarding the canonical exemple of QCS P_H (which represents all possible states on a Hilbert space H, we have that the tensor connective $P_{H_1} \otimes P_{H_2}$ is the QCS generated by all possible separable states on $H_1 \otimes H_2$, that is to say :

$$P_{H_1} \otimes P_{H_2} = \sim\sim\{\text{separable states over} H_1 \otimes H_2\}.$$

Coupling two systems

Remember that $P_H = \{\rho \geq 0 \mid \mathrm{Tr}(\rho) \leq 1\}$.

Proposition

Let H_1 and H_2 be two Hilbet spaces. The QCS $P_{H_1} \parr P_{H_2}$ is merely $P_{H_1 \otimes H_2}$. That is to say, $\parr$ combines the sets of all possible states on H_1 and H_2 into the set of all possible states on $H_1 \otimes H_2$.

i.e.

$$P_{H_1 \otimes H_2} = P_{H_1} \parr P_{H_2}$$

Definitions

We define the $\oplus$ and $\&$ connectives, on the support $|X| \oplus |Y|$, by :

Definition

$X \oplus Y := \sim\sim(\{x \oplus 0 | x \in X\} \cup \{0 \oplus y | y \in Y\})$

Definition

$X \& Y := \{ \ h \mid p_{|X|} h p_{|X|} \in X \ \text{ and } \ p_{|Y|} h p_{|Y|} \in Y \ \}$

By $p_{|X|}$, we mean the orthogonal projection on $|X|$ in the hilbert $|X| \oplus |Y|$.

These two definitions are dual :

Proposition

$$X \oplus Y = \sim(\sim X \& \sim Y)$$

Distributivity

As expected, $\otimes$ distributes over $\oplus$:

(And dually, $⅋$ distributes over $\&$)

Proposition

For all A, B, C $\quad A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$

The proof only consist in lifting the isomorphism
$|A| \otimes (|B| \oplus |C|) \simeq (|A| \otimes |B|) \oplus (|A| \otimes |C|)$ *on supports.*

Some ideal experiment

Let Ω_A and Ω_B be two regions of $\mathbb{R}^3$ such that $\Omega_A \cap \Omega_B = \phi$ and V a potential such that

$$V \begin{cases} = \infty & \text{outside } \; \Omega_A \cup \Omega_B \\ \text{is bounded} & \text{in } \; \Omega_A \cup \Omega_B \end{cases}$$

Let $V_A = \infty$ on Ω_B, $V_A = V$ outside.

Some ideal experiment II

One can see that a particule submited to V_A (resp. V_B) stays in Ω_A.

Let's say that initally we have a particule in Ω_A or Ω_B and submited to V. Then, we cut the potential (we suppose we can do it in no time, abruptedly and perfectly). The particule delocalizes itself and occupies the whole space $\mathbb{R}^3$, meaning also both Ω_A and Ω_B.

After what, we plug the potential again and we found out that the particule is restricted to $\Omega_A \cup \Omega_B$ and that the information contained in Ω_A is independent from the one in Ω_B, since the two regions are separated from infinite potential (they evolve independently).

Interpretation of $\oplus$

Then imagine that you allow a system to be represented by a certain set of states A or B, linearly independant from each other. The states resulting in a stochastic combination of states of A and of B are the convex hull of $A \cup B$.

Proposition

Let A and B two bounded QCS and A_m, B_m the sets of their maxima, and C_m the convex hull of $A_m \oplus 0_{|B|} \cup 0_{|A|} \oplus B_m$. Then $A \oplus B$ can be rewritten as :

$$A \oplus B = \{\rho \mid \exists \rho_m \in C_n \text{ such that } \rho \leq \rho_m\}.$$

Therefore, the sets of stochastic combination of A_m and B_m is the set of maxima of $A \oplus B$. Once more, there's no direct result on $A \oplus B$ but on its maxima.

Interpreting LL additive rules

The proof :

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathtt{Ax})}{A \vdash A \oplus B}\,(\oplus\text{-r}) \qquad \dfrac{\dfrac{}{B \vdash B}\,(\mathtt{Ax})}{B \vdash A \oplus B}\,(\oplus\text{-l})}{A \oplus B \vdash A \oplus B}\,(\&)$$

is interpreted by : $\begin{pmatrix} U & W \\ W^\dagger & V \end{pmatrix} \longrightarrow \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ *i.e.* $Id_{\mathscr{L}(|A|)} \oplus Id_{\mathscr{L}(|B|)}$

But the proof :

$$\dfrac{}{A \oplus B \vdash A \oplus B}\,(\mathtt{Ax})$$

is interpreted by the identity, $Id_{\mathscr{L}(|A| \oplus |B|)}$

Combining superpositive operators

Using the properties of superpositive operators, one can define an operation, noted $\dot{+}$, such that :

If $F \in \Gamma \otimes A \to^{+} C$ **and** $G \in \Gamma \otimes B \to^{+} D$

Then $F \dot{+} G \in \Gamma \otimes (A \oplus B) \to^{+} C \oplus D$

Remark : $Id_{\mathscr{L}(E)} \dot{+} Id_{\mathscr{L}(F)} = Id_{\mathscr{L}(E \oplus F)}$

This operation on morphisms would correspond to a rule of the form :

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash D}{\Gamma, A \oplus B \vdash C \oplus D}\ (\mathtt{Add})$$

or equivalently

$$\frac{\vdash \Gamma, A, C \qquad \vdash \Gamma, B, D}{\vdash \Gamma, A \& B, C \oplus D}\ (\mathtt{Add})$$

The corresponding rule

The (Add) rule can be derived in LL :

$$\dfrac{\dfrac{\Gamma, A \vdash C}{\Gamma, A \vdash C \oplus D}\ (\oplus\text{-r}) \qquad \dfrac{\Gamma, B \vdash D}{\Gamma, B \vdash C \oplus D}\ (\oplus\text{-l})}{\Gamma, A \oplus B \vdash C \oplus D}\ (\&)$$

But the interpretation differs in the QCS model, for instance :

$$\dfrac{\dfrac{}{A \vdash A}\ (\mathtt{Ax}) \qquad \dfrac{}{B \vdash B}\ (\mathtt{Ax})}{A \oplus B \vdash A \oplus B}\ (\mathtt{Add})$$

is interpreted by :

$$Id_{\mathscr{L}(|A|)} \dotplus Id_{\mathscr{L}(|B|)} \left(= Id_{\mathscr{L}(|A| \oplus |B|)}\right) \neq Id_{\mathscr{L}(|A|)} \oplus Id_{\mathscr{L}(|B|)}$$

A (Cut)-elimination case for (Add)

Let's look at a case of (Cut)-elimination for the (Add) rule :

Work in progress...

The same elimination case for the derived rule

`Work in progress...`

Quantum additives ?

- What happens if we "switch" the definitions of additives, that is to say, if we write :

Definition

$$X \oplus_q Y := \{\ h \geq 0\ |\exists x \in X, y \in Y, \lambda, \mu \in \mathbb{R}^+,\ \text{such that}\ \lambda + \mu \leq 1\ ,\ p_{|X|} h p_{|X|} = \lambda x \text{ and } p_{|Y|} h p_{|Y|} = \mu y \}$$

and

Definition

$$X \&_q Y := \sim\sim\{\ x \oplus y \mid x \in X\ ,\ y \in Y\ \}$$

- $\oplus_q$ corresponds to a natural operation on spaces in quantum mechanics.
- We obtain odd commutation relations ($\oplus_q / \parr$ and $\&_q / \otimes$).
- But this might be of some interest to interpret exponentials.

Fock spaces

- Fock spaces are based on operators of anihilation and creation :
$$a_k^\dagger \text{ et } a_k$$
- That is to say, a basis of this space is given by $\prod a_{k_j}^\dagger |0\rangle$.
- There are two kinds of Fock spaces : if the a_k commute, we have a symmetric space, if they anticommute, an anti symmetric one.

Antisymmetry

In the latter case, Fock space corresponds to the exterior algebra.

Proposition

Let H_1, H_2 be two vector spaces and $\Lambda H_1, \Lambda H_2$ their respective exterior algebra. There's a canonical isometry between $\Lambda(H_1 \oplus H_2)$ and $\Lambda H_1 \otimes \Lambda H_2$.

Which leads us to think there might be a glint of hope as to express $!A$ as a QCS with support $\Lambda|A|$.

By lifting the isometry on Hilbert spaces up to a transformation onto their operators, we expect to find an isomorphism between $!(X \& Y)$ and $(!X) \otimes (!Y)$.

Other rules

- Operators of creation and anihilation provides us intuitive transformations.
- Creation from $X \otimes !X$ to $!X$
- Anihilation from $\sim X \otimes !X$ to $!X$

Questions ?