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THE GEOMETRY OF CUBICAL AND REGULAR TRANSITION SYSTEMS

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Résumé. Il existe des systèmes de transitions cubiques contenant des cubes ayant un nombre arbitrairement grand de faces. Un système de transition régulier est un système de transitions cubique tel que tout cube a le bon nombre de faces. Les propriétés catégoriques et homotopiques des systèmes de transitions réguliers sont similaires à celles des cubiques. On donne une description combinatoire complète des objets fibrants dans les cas cubiques et réguliers. Un des deux appendices contient un lemme indépendant sur la restriction d'une adjonction à une sous-catégorie réflective pleine.
Abstract. There exist cubical transition systems containing cubes having an arbitrarily large number of faces. A regular transition system is a cubical transition system such that each cube has the good number of faces. The categorical and homotopical results of regular transition systems are very similar to the ones of cubical ones. A complete combinatorial description of fibrant cubical and regular transition systems is given. One of the two appendices contains a general lemma of independant interest about the restriction of an adjunction to a full reflective subcategory.
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1. Introduction

Presentation

The purpose of Cattani-Sassone's notion of higher dimensional transition system introduced in [4] is to model the concurrent execution of n actions by a transition between two states labelled by a multiset $\{u_1, \dots, u_n\}$ of actions. A multiset is a set with a possible repetition of its elements: e.g. $\{u\}$ is not equal to $\{u, u\}$. A higher dimensional transition system for Cattani

$$\begin{tikzcd}
& \beta \arrow[dr, "\{b\}"] & \\
\alpha \arrow[ur, "\{a\}"] \arrow[dr, "\{b\}"'] & \{a,b\} & \delta \\
& \gamma \arrow[ur, "\{a\}"'] &
\end{tikzcd}$$

Figure 1: $a||b$: Concurrent execution of a and b

and Sassone consists of a set of states S, a set of actions L, a set of labels Σ together with a labelling map $\mu : L \to \Sigma$, and a set of tuples (α, T, β) of transitions where α and β are two states and T is a multiset of actions. All these data have to satisfy several axioms which are detailed in the original paper [4]. The higher dimensional transition $a||b$ depicted by Figure 1 consists of the transitions $(\alpha, \{a\}, \beta)$, $(\beta, \{b\}, \delta)$, $(\alpha, \{b\}, \gamma)$, $(\gamma, \{a\}, \delta)$ and $(\alpha, \{a, b\}, \delta)$. The labelling map is the identity map. Note that with $a = b$, we would get the 2-dimensional transition $(\alpha, \{a, a\}, \delta)$ which is not equal to the 1-dimensional transition $(\alpha, \{a\}, \delta)$. The latter actually does not exist in Figure 1. Indeed, the only 1-dimensional transitions labelled by the multiset $\{a\}$ are $(\alpha, \{a\}, \beta)$ and $(\gamma, \{a\}, \delta)$.

In [7], Cattani-Sassone's notion is reworded in a more convenient mathematical setting by introducing the notion of weak transition system. The transition $(\alpha, \{a, b\}, \delta)$ is then represented by the tuple (α, a, b, δ). The set of transitions has therefore to satisfy the Multiset axiom (here: if the tuple (α, a, b, δ) is a transition, then the tuple (α, b, a, δ) has to be a transition as well) and the Composition axiom which is a topological version (in the sense of topological functors) of Cattani-Sassone's interleaving axioms. The Composition axiom is called the Coherence axiom in [7]. The terminology is changed in the next paper [8] because this axiom behaves a little bit like a partial 5-ary composition in the proofs [1]. For example, the Composition axiom is the key axiom for interpreting the higher dimensional transition system modeling the n-cube as the free object generated by a "pure"

[1]In the nLab page devoted to higher dimensional transition systems, T. Porter uses the terminology "patching axiom", which is quite a good idea too.

n-dimensional transition (this weak transition system consists of two states and a n-dimensional transition going from one state to the other one) [7, Theorem 5.6]. Indeed, the free compositions generated by the Composition axiom generate all transitions of lower dimension between the intermediate states (i.e. with a source different from the initial state and a target different from the final state) . Weak transition systems assemble into a locally finitely presentable category $\mathcal{WTS}$ such that the forgetful functor forgetting the transitions, and keeping the states and the actions, is topological in the sense of [1, Definition 21.1].

The full coreflective subcategory $\mathcal{CTS}$ of cubical transition systems was then introduced in [8]. They consist of the weak transition systems which are equal to the union of their subcubes. It was preferred to the full coreflective category of $\mathcal{WTS}$ of colimits of cubes because the latter does not contain the boundary of a 2-cube. The category $\mathcal{CTS}$ is sufficient to describe the path spaces of all process algebras for any synchronization algebra because their path spaces are colimits of cubes and because all colimits of cubes are unions of cubes. Indeed, the weak transition system associated with a process algebra is obtained by starting from a labelled precubical set using the method described in [5], and by taking the free symmetric labelled precubical set generated by it [6], and then by applying the colimit-preserving realization functor from labelled symmetric precubical sets to weak transition systems constructed in [7].

However, the notion of cubical transition system is still too general. Indeed, a n-dimensional transition in a cubical transition system may have an arbitrarily large number of faces in each dimension. Here is a simple example of a 2-transition X with $2n+2$ edges for an arbitrarily large integer $n \geq 1$:

- the set of states is $\{I, F, a, b_1, \dots, b_n\}$
- the set of actions is $\{u, v\}$ with $\mu(u) \neq \mu(v)$ (μ denotes the labelling map)
- the transitions are the tuples

$$\{(I,u,v,F),(I,v,u,F),(I,u,a),(a,v,F),\\ (I,v,b_i),(b_i,u,F) \mid i \leq 1 \leq n.\}$$

The weak transition system above is cubical because it is the union, for $1 \leq i \leq n$, of the 2-cubes Z_i having the set of vertices $\{I, F, a, b_i\}$, the set of actions $\{u, v\}$ and the set of six transitions

$$\{(I, u, v, F), (I, v, u, F), (I, u, a), (a, v, F), (I, v, b_i), (b_i, u, F)\}.$$

To avoid such a behavior, it suffices to replace the Intermediate state axiom by the Unique intermediate state axiom, also called CSA2 (see Definition 2.2). The latter axiom is already introduced in [7] to formalize Cattani-Sassone's notion of higher dimensional transition systems in the setting of weak transition systems. We obtain a full reflective subcategory $\mathcal{RTS}$ of that of cubical transition systems whose objects are called the *regular transition systems*. Roughly speaking, a regular transition system is a Cattani-Sassone transition system which does not necessarily satisfy CSA1 (see Definition 2.4). There is the chain of functors

$$\mathcal{RTS} \subset_{\text{reflective}} \mathcal{CTS} \subset_{\text{coreflective}} \mathcal{WTS} \xrightarrow{\omega}_{\text{topological}} \mathbf{Set}^{\{s\}\cup\Sigma}$$

where ω is the topological functor towards a power of the category of sets forgetting the transitions: s denotes the sort of states and each element x of the set of labels Σ denotes the sort of actions labelled by x. With the notations above, one has

$$\omega(a||b) = (\{\alpha, \beta, \gamma, \delta\}, \{a\}, \{b\})$$

since the labelling map is the identity map. One has

$$\omega(X) = (\{I, F, a, b_1, \ldots, b_n\}, \{u\}, \{v\})$$

since $\mu(u) \neq \mu(v)$.

Note that none of the categories of colimits of cubes and of regular transition systems is included in the other one: see the final comment of Section 2.

This paper is devoted to the geometric properties of regular transition systems and to their relationship with cubical ones. Their study requires the use of the whole chain of functors above which is the composite of a right adjoint followed by a left adjoint followed by a topological functor. Despite the fact that colimits are different in $\mathcal{RTS}$ and in $\mathcal{CTS}$, the main results are very similar to the ones obtained for cubical transition systems in [8]. We will therefore follow the plan of [8]. The left determined model structure with respect to the cofibrations of cubical transition systems between regular ones is proved to exist. It is proved that the Bousfield localization by the cubification functor is the model structure having the same class of cofibrations and the fibrant objects are the regular transitions systems such that for any transition $(\alpha, u_1, \dots, u_n, \beta)$, the tuple $(\alpha, v_1, \dots, v_n, \beta)$ is a transition if $\mu(u_i) = \mu(v_i)$ for $1 \leq i \leq n$. The homotopical structure of this Bousfield localization will be completely elucidated. Roughly speaking, after identifying each action of a regular transition system with its label and after removing all non-discernable higher dimensional transitions, two regular transition systems are weakly equivalent if and only if they are isomorphic.

Outline of the paper

Section 2 introduces all definitions of higher dimensional transition systems used in this paper: weak, cubical, regular. It starts with the notion of *regular transition system* (Definition 2.2), and then by removing some axioms, the notions of cubical transition system and of weak transition system are recalled. This section does not contain anything new, except the notion of regular transition system. Section 3 is a technical section which provides a sufficient condition for an ω-final lift of cubical transition systems to be cubical (Theorem 3.3). This result is used in the construction of several cubical transition systems. Section 4 deals with the most elementary properties of regular transition systems. The reflection $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$ is studied. The definition of the cubification functor is recalled and its relationship with regular transition systems is explained. Section 5 establishes the existence of the left determined model structure of regular transition systems. The weak equivalences of this model structure are completely characterized. The Bousfield localization of the left determined model category of regular transition systems by the cubification functor is studied and completely elucidated in Section 6. The comparison with cubical transition systems is discussed there. The proof of Theorem 6.12 is postponed to Section A to not overload Section 6. Finally, Section 7 completely characterizes the fibrant cubical and regular transition systems in the Bousfield localizations by the cubification functor. Section B is a categorical lemma of independant interest providing a easy way to restrict an adjunction to a full reflective subcategory.

Prerequisites and notations

All categories are locally small. The set of maps in a category $\mathcal{K}$ from X to Y is denoted by $\mathcal{K}(X, Y)$. The initial (final resp.) object, if it exists, is always denoted by $\varnothing$ ($\mathbf{1}$ resp.). The identity of an object X is denoted by Id_X. A subcategory is always isomorphism-closed. We refer to [2] for locally presentable categories, to [19] for combinatorial model categories, and to [1] for topological categories, i.e. categories equipped with a topological functor towards a power of the category of sets. We refer to [12] and to [11] for model categories. For general facts about weak factorization systems, see also [13]. The reading of the first part of [16], published in [15], is recommended for any reference about good, cartesian, and very good cylinders.

2. Regular higher dimensional transition systems

This section does not contain anything new, except the notion of *regular transition system*. It collects definitions and facts about the various notions of transition systems which were expounded in the previous papers of this series [7] and [8]. To keep this section concise, the definition of a regular transition system is given first, and then by removing some axioms, the definitions of a cubical transition system and of a weak transition system are recalled. It is necessary to recall all these definitions because most of the proofs of this paper make use of the whole chain of functors

$$\mathcal{RTS} \subset_{\text{reflective}} \mathcal{CTS} \subset_{\text{coreflective}} \mathcal{WTS} \xrightarrow{\omega}_{\text{topological}} \mathbf{Set}^{\{s\}\cup\Sigma}$$

where $\mathbf{Set}$ is the category of sets.

Notation 2.1. A nonempty set of labels Σ is fixed.

Definition 2.2. *A* regular higher dimensional transition system *consists of a triple*

$$X = (S, \mu : L \to \Sigma, T = \bigcup_{n\geq 1} T_n)$$

where S is a set of states, *where L is a set of* actions, *where $\mu : L \to \Sigma$ is a set map called the* labelling map, *and finally where $T_n \subset S \times L^n \times S$*

*for $n \geq 1$ is a set of n-*transitions *or n-*dimensional transitions *such that one has:*

- *(All actions are used) For every $u \in L$, there is a 1-transition (α, u, β).*
- *(Multiset axiom) For every permutation σ of $\{1, \ldots, n\}$ with $n \geq 2$, if the tuple $(\alpha, u_1, \ldots, u_n, \beta)$ is a transition, then the tuple*

  $$(\alpha, u_{\sigma(1)}, \ldots, u_{\sigma(n)}, \beta)$$

  is a transition as well.
- *(Composition axiom [2]) For every $(n+2)$-tuple $(\alpha, u_1, \ldots, u_n, \beta)$ with $n \geq 3$, for every $p, q \geq 1$ with $p + q < n$, if the five tuples*

  $$(\alpha, u_1, \ldots, u_n, \beta), (\alpha, u_1, \ldots, u_p, \nu_1), (\nu_1, u_{p+1}, \ldots, u_n, \beta),$$
  $$(\alpha, u_1, \ldots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \ldots, u_n, \beta)$$

  are transitions, then the $(q+2)$-tuple $(\nu_1, u_{p+1}, \ldots, u_{p+q}, \nu_2)$ is a transition as well.
- *(Unique intermediate state axiom or CSA2) [3]. For every $n \geq 2$, every p with $1 \leq p < n$ and every transition $(\alpha, u_1, \ldots, u_n, \beta)$ of X, there exists a unique state ν such that both $(\alpha, u_1, \ldots, u_p, \nu)$ and $(\nu, u_{p+1}, \ldots, u_n, \beta)$ are transitions.*

A map of regular transition systems

$$f : (S, \mu : L \to \Sigma, (T_n)_{n\geq 1}) \to (S', \mu' : L' \to \Sigma, (T'_n)_{n\geq 1})$$

consists of a set map $f_0 : S \to S'$, a commutative square

$$\begin{tikzcd} L \arrow[r, "\mu"] \arrow[d, "\widetilde{f}"'] & \Sigma \arrow[d, equal] \\ L' \arrow[r, "\mu'"'] & \Sigma \end{tikzcd}$$

[2] This axiom is called the Coherence axiom in [7] and [8].

[3] This axiom is also called CSA2 in [7]

such that if the tuple $(\alpha, u_1, \dots, u_n, \beta)$ *is a transition, then the tuple*

$$(f_0(\alpha), \widetilde{f}(u_1), \dots, \widetilde{f}(u_n), f_0(\beta))$$

is a transition. The corresponding category is denoted by $\mathcal{RTS}$. *The* n-*transition* $(\alpha, u_1, \dots, u_n, \beta)$ *is also called a* transition from α to β. *The maps* f_0 *and* $\widetilde{f}$ *will be also denoted by* f.

Notation 2.3. The labelling map from the set of actions to the set of labels will be very often denoted by μ. The set of states of a regular transition system X is denoted by X^0.

The category $\mathcal{RTS}$ of regular higher dimensional transition systems is a full subcategory of the category of *cubical transition systems* $\mathcal{CTS}$ introduced in [8]. By definition, a cubical transition system satisfies all axioms of higher dimensional transition system but one: the Unique intermediate state axiom is replaced by the Intermediate state axiom, the state ν is not necessarily unique anymore. The category $\mathcal{CTS}$ is a full subcategory of the category of *weak transition systems* $\mathcal{WTS}$ introduced in [7]. By definition, a weak transition system satisfies all axioms of regular transition systems but two: the Unique intermediate state axiom is removed and an action is not necessarily used. Weak transition system is the "minimal" definition: the multiset axiom is indeed required to ensure that the concurrent execution of n actions does not depend on the order of the labelling, and the composition axiom is required (even if its use is often hidden) e.g. to ensure that labelled n-cubes are free objects (e.g. see the proof of [7, Theorem 5.6]). One has the inclusions of full subcategories $\mathcal{RTS} \subset \mathcal{CTS} \subset \mathcal{WTS}$. The inclusion $\mathcal{RTS} \subset \mathcal{CTS}$ is strict since the introduction gives an example of cubical transition system which is not regular. The situation is summarized in Table 1. Let us recall now the definition of CSA1 for this sequence of definitions to be complete:

Definition 2.4. *[7, Definition 4.1 (2)] and [8, Definition 7.1] A cubical transition system satisfies the* First Cattani-Sassone axiom *(CSA1) if for every transition* (α, u, β) *and* (α, u', β) *such that the actions* u *and* u' *have the same label in* Σ, *one has* $u = u'$.

The category $\mathcal{WTS}$ is locally finitely presentable and the functor

$$\omega : \mathcal{WTS} \longrightarrow \mathbf{Set}^{\{s\}\cup\Sigma}$$

	C-S	Regular	Cubical	Weak
Multiset axiom	yes	yes	yes	yes
Composition axiom	yes	yes	yes	yes
All actions used	yes	yes	yes	no
Intermediate state axiom	yes	yes	yes	no
Unique Intermediate state axiom	yes	yes	no	no
CSA1	yes	no	no	no

Table 1: Summary of all variants of transition systems (C-S meaning Cattani-Sassone).

taking the weak higher dimensional transition system

$$(S, \mu : L \to \Sigma, (T_n)_{n\geq 1})$$

to the $(\{s\} \cup \Sigma)$-tuple of sets $(S, (\mu^{-1}(x))_{x\in\Sigma}) \in \mathbf{Set}^{\{s\}\cup\Sigma}$ is topological by [7, Theorem 3.4].

Let us recall that the paradigm of *topological functor* is the underlying set functor from the category of general topological spaces to that of sets. The notion of topological functor is a generalization of the notions of initial and final topologies [1]. More precisely, a functor $\omega : \mathcal{C} \to \mathcal{D}$ is *topological* if each cone $(f_i : X \to \omega A_i)_{i\in I}$ where I is a class has a unique ω-initial lift (the *initial structure*) $(\overline{f}_i : A \to A_i)_{i\in I}$, i.e.: 1) $\omega A = X$ and $\omega\overline{f}_i = f_i$ for each $i \in I$; 2) given $h : \omega B \to X$ with $f_i h = \omega\overline{h}_i$, $\overline{h}_i : B \to A_i$ for each $i \in I$, then $h = \omega\overline{h}$ for a unique $\overline{h} : B \to A$. Topological functors can be characterized as functors such that each cocone $(f_i : \omega A_i \to X)_{i\in I}$ where I is a class has a unique ω-final lift (the *final structure*) $\overline{f}_i : A_i \to A$, i.e.: 1) $\omega A = X$ and $\omega\overline{f}_i = f_i$ for each $i \in I$; 2) given $h : X \to \omega B$ with $hf_i = \omega\overline{h}_i$, $\overline{h}_i : A_i \to B$ for each $i \in I$, then $h = \omega\overline{h}$ for a unique $\overline{h} : A \to B$. A limit (resp. colimit) in $\mathcal{C}$ is calculated by taking the limit (resp. colimit) in $\mathcal{D}$, and by endowing it with the initial (resp. final) structure. In particular, a topological functor is faithful and it creates all limits and colimits.

The category $\mathcal{CTS}$ is a full coreflective locally finitely presentable subcategory of $\mathcal{WTS}$ by [8, Corollary 3.15]. The composite functor

$$\mathcal{CTS} \subset \mathcal{WTS} \xrightarrow{\omega} \mathbf{Set}^{\{s\}\cup\Sigma}$$

is faithful and colimit-preserving.

The inclusion $\mathcal{CTS} \subset \mathcal{WTS}$ is strict. Here are two families of examples of weak transition systems which are not cubical:

1. The weak transition system $\underline{x} = (\varnothing, \{x\} \subset \Sigma, \varnothing)$ for $x \in \Sigma$ is not cubical because the action x is not used.

2. Let $n \geq 0$. Let $x_1, \dots, x_n \in \Sigma$. The *pure n-transition*
$$C_n[x_1, \dots, x_n]^{ext}$$
is the weak transition system with the set of states $\{0_n, 1_n\}$, with the set of actions
$$\{(x_1, 1), \dots, (x_n, n)\}$$
and with the transitions all $(n+2)$-tuples
$$(0_n, (x_{\sigma(1)}, \sigma(1)), \dots, (x_{\sigma(n)}, \sigma(n)), 1_n)$$
for σ running over the set of permutations of the set $\{1, \dots, n\}$. It is not cubical for $n > 1$ because it does not contain any 1-transition. Intuitively, the pure transition is a cube without faces of lower dimension.

We give now some important examples of regular transition systems. In each of the following examples, the axioms of regular higher dimensional transition systems are satisfied for trivial reasons.

Notation 2.5. For $n \geq 1$, let $0_n = (0, \dots, 0)$ (n-times) and $1_n = (1, \dots, 1)$ (n-times). By convention, let $0_0 = 1_0 = ()$.

1. Every set X may be identified with the cubical transition system having the set of states X, with no actions and no transitions.

2. For every $x \in \Sigma$, let us denote by $\uparrow\! x\! \uparrow$ the cubical transition system with four states $\{1, 2, 3, 4\}$, one action x and two transitions $(1, x, 2)$ and $(3, x, 4)$. The cubical transition system $\uparrow\! x\! \uparrow$ is called the *double transition (labelled by x)* where $x \in \Sigma$.

Let us introduce now the cubical transition system corresponding to the labelled n-cube.

Proposition 2.6. *[7, Proposition 5.2] Let $n \geq 0$ and $x_1, \ldots, x_n \in \Sigma$. Let $T_d \subset \{0,1\}^n \times \{(x_1, 1), \ldots, (x_n, n)\}^d \times \{0,1\}^n$ (with $d \geq 1$) be the subset of $(d+2)$-tuples*

$$((\epsilon_1, \ldots, \epsilon_n), (x_{i_1}, i_1), \ldots, (x_{i_d}, i_d), (\epsilon'_1, \ldots, \epsilon'_n))$$

such that

- *$i_m = i_n$ implies $m = n$, i.e. there are no repetitions in the list $(x_{i_1}, i_1), \ldots, (x_{i_d}, i_d)$*
- *for all i, $\epsilon_i \leq \epsilon'_i$*
- *$\epsilon_i \neq \epsilon'_i$ if and only if $i \in \{i_1, \ldots, i_d\}$.*

Let $\mu : \{(x_1, 1), \ldots, (x_n, n)\} \to \Sigma$ be the set map defined by $\mu(x_i, i) = x_i$. Then

$$C_n[x_1, \ldots, x_n] = (\{0,1\}^n, \mu : \{(x_1, 1), \ldots, (x_n, n)\} \to \Sigma, (T_d)_{d\geq 1})$$

is a well-defined cubical transition system called the n-cube.

The n-cubes $C_n[x_1, \ldots, x_n]$ for all $n \geq 0$ and all $x_1, \ldots, x_n \in \Sigma$ are regular by [7, Proposition 5.2] and [7, Proposition 4.6]. For $n = 0$, $C_0[]$, also denoted by C_0, is nothing else but the one-state higher dimensional transition system $(\{()\}, \mu : \varnothing \to \Sigma, \varnothing)$.

By [8, Theorem 3.6], the category $\mathcal{CTS}$ is a small-injectivity class of $\mathcal{WTS}$. More precisely being cubical is equivalent to being injective with respect to the set of inclusions $C_n[x_1, \ldots, x_n]^{ext} \subset C_n[x_1, \ldots, x_n]$ and $\underline{x_1} \subset C_1[x_1]$ for all $n \geq 0$ and all $x_1, \ldots, x_n \in \Sigma$. Note that the composition axiom plays a central role in this result.

Finally, let us notice that there is the isomorphism of weak transition systems

$$\uparrow x \uparrow \cong \varinjlim (C_1[x] \leftarrow \underline{x} \to C_1[x])$$

for any label x of Σ, the colimit being taken in $\mathcal{WTS}$. The double transition $\uparrow x \uparrow$ is an example of cubical transition system, and even of regular transition system, which is not a colimit of cubes. Another example of regular transition system which is not a colimit of cubes is the boundary of a labelled 2-cube (see [8]). This was the main motivation for introducing cubical transition systems. Conversely, by [7, Proposition 9.7], there exists a labelled precubical set K such that its realization $\mathbb{T}(K)$ as weak transition system does not satisfy CSA2. Every labelled precubical set is a colimit of cubes, therefore $\mathbb{T}(K)$ is a colimit of cubes since the realization functor from labelled symmetric precubical sets to weak transition systems is colimit-preserving. Hence the weak transition system $\mathbb{T}(K)$ is an example of a colimit of cubes which is not regular (but it is cubical as any colimit of cubes).

3. Intermediate state axiom and ω-final lifts

Let S be a set of objects of a locally presentable category $\mathcal{K}$. For each object X of $\mathcal{K}$, the colimit of the natural forgetful functor $\widehat{S}{\downarrow}X \to \mathcal{K}$, where $\widehat{S}$ is the full small category of $\mathcal{K}$ generated by S, is denoted by ($s \in S$ may be omitted)

$$\varinjlim_{\substack{s \to X \\ s \in \mathcal{S}}} s.$$

By [17, Proposition 3.1(i)], the full subcategory of colimits of objects of $\mathcal{S}$ is a coreflective subcategory $\mathcal{K}_{\mathcal{S}}$ of $\mathcal{K}$. The right adjoint to the inclusion functor $\mathcal{K}_{\mathcal{S}} \subset \mathcal{K}$ is precisely given by the functorial mapping

$$X \mapsto \varinjlim_{\substack{s \to X \\ s \in \mathcal{S}}} s.$$

By [8, Theorem 3.11], a weak transition system is cubical if and only if it is canonically a colimit of cubes and double transitions. In other terms, a weak transition system X is cubical if and only if the canonical map

$$q_X : \varinjlim_{\substack{f : C_n[x_1, \ldots, x_n] \to X \\ f : \uparrow x \uparrow \to X}} \mathrm{dom}(f) \to X$$

is an isomorphism. The functorial mapping $X \mapsto \mathrm{dom}(q_X)$ is the coreflection of the inclusion $\mathcal{CTS} \subset \mathcal{WTS}$. The image of $\underline{x}$ for any $x \in \Sigma$ by the coreflection $\mathcal{WTS} \to \mathcal{CTS}$ is therefore the initial cubical transition system $\varnothing$. This implies that the category $\mathcal{CTS}$ is not a concretely coreflective subcategory of $\mathcal{WTS}$ over ω because the set of actions is not preserved. Hence there is no reason for an ω-final lift of cubical transition systems to be cubical. This holds anyway in the situation of Theorem 3.3 which will be used several times in the paper.

Proposition 3.1. *Let $X = \varinjlim X_i$ be a colimit of weak transition systems. If all X_i satisfy the Intermediate state axiom, then so does X.*

Proof. Let T_i be the image by the canonical map $X_i \to X$ of the set of transitions of X_i. Let $G_0 = \bigcup_i T_i$. Let us define G_α by induction on the transfinite ordinal $\alpha \geq 0$. If α is a limit ordinal, then let $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. If the set of tuples G_α is defined, then let $G_{\alpha+1}$ be obtained from G_α by adding the set of all $(q+2)$-tuples $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$ such that there exist five tuples $(\alpha, u_1, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_p, \nu_1)$, $(\nu_1, u_{p+1}, \dots, u_n, \beta)$, $(\alpha, u_1, \dots, u_{p+q}, \nu_2)$ and $(\nu_2, u_{p+q+1}, \dots, u_n, \beta)$ of the set G_α for some $p \geq 1$ and $q \geq 1$. For cardinality reason, the transfinite sequence stabilizes and by [7, Proposition 3.5], there exists an ordinal α_0 such that G_{α_0} is the set of transitions of X. Every transition of G_0 satisfies the Intermediate state axiom since it is satisfies by all X_i. Suppose that all transitions of G_α satisfies the Intermediate state axiom. Take a tuple $(\nu_1, u_{p+1}, \dots, u_{p+q}, \nu_2)$ of $G_{\alpha+1}$ like above. Suppose that $q \geq 2$ and let $q > r \geq 1$. There exists a state ν_3 of X such that the tuples $(\alpha, u_1, \dots, u_{p+r}, \nu_3)$ $(\nu_3, u_{p+r+1}, \dots, u_n, \beta)$ are two transitions of G_α since all transitions of G_α satisfy the Intermediate state axiom by induction hypothesis. From the five tuples

$$(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_{p+r}, \nu_3), (\nu_3, u_{p+r+1}, \dots, u_n, \beta)$$
$$(\alpha, u_1, \dots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \dots, u_n, \beta)$$

of G_α, one deduces that the tuple $(\nu_3, u_{p+r+1}, \dots, u_{p+q}, \nu_2)$ belongs to $G_{\alpha+1}$. From the five tuples

$$(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \nu_1), (\nu_1, u_{p+1}, \dots, u_n, \beta),$$
$$(\alpha, u_1, \dots, u_{p+r}, \nu_3), (\nu_3, u_{p+r+1}, \dots, u_n, \beta),$$

one deduces that the tuple $(\nu_1, u_{p+1}, \dots, u_{p+r}, \nu_3)$ belongs to $G_{\alpha+1}$. Hence $G_{\alpha+1}$ satisfies the Intermediate state axiom. One deduces that X satisfies the Intermediate state axiom. □

Proposition 3.2. *Consider the following map, functorial with respect to the weak transition system X:*

$$r_X : \varinjlim_{\substack{f : C_n[x_1,\dots,x_n] \to X \\ f : \underline{x} \to X}} \mathrm{dom}(f) \to X.$$

The map r_X is always bijective on states and actions and one-to-one on transitions. The map r_X is an isomorphism if and only if X satisfies the Intermediate state axiom.

Proof. Let α be a state of X. Then there exists a map $C_0 \to X$ sending the unique state of C_0 to α. Hence r_X is onto on states. Let α and β be two states of $\mathrm{dom}(r_X)$ sent to the same state γ of X. Then the diagram $\{\alpha\} \leftarrow \{\gamma\} \to \{\beta\}$ is a subdiagram in the colimit calculating $\mathrm{dom}(r_X)$. Hence $\alpha = \beta$ in $\mathrm{dom}(r_X)$. So r_X is bijective on states. Let u be an action of X. Then there exists a map $\underline{\mu(u)} \to X$ sending the action $\mu(u)$ to u. This implies that r_X is onto on actions. Let u and v be two actions of $\mathrm{dom}(r_X)$ sent to the same action w of X. Then the diagram $\{\mu(u)\} \leftarrow \{\mu(w)\} \to \{\mu(v)\}$ is a subdiagram in the colimit calculating $\mathrm{dom}(r_X)$. Hence $u = v$ in $\mathrm{dom}(r_X)$ and r_X is bijective on actions. Hence by [10, Proposition 4.4], r_X is always one-to-one on transitions.

By Proposition 3.1, the weak transition system $\mathrm{dom}(r_X)$ satisfies the Intermediate state axiom. Therefore, if r_X is an isomorphism, then X satisfies the Intermediate state axiom. Conversely, let us suppose that X satisfies the Intermediate state axiom. Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition of X. This transition gives rise to a map of weak transition systems $\phi : C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \to X$. Since X satisfies the Intermediate state axiom, it is injective with respect to the inclusion $C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \subset C_n[\mu(u_1), \dots, \mu(u_n)]$ (see the proof of [8, Theorem 3.6]) [4]. Hence ϕ factors as a composite $C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \to C_n[\mu(u_1), \dots, \mu(u_n)] \to X$. By definition of $\mathrm{dom}(r_X)$, ϕ factors as a composite

$$C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \longrightarrow C_n[\mu(u_1), \dots, \mu(u_n)] \longrightarrow \mathrm{dom}(r_X) \xrightarrow{r_X} X.$$

Hence r_X is onto on transitions. □

[4]Note that the composition axiom of weak transition systems is used here. It is worth noting that its use is often hidden.

Theorem 3.3. *Let $(f_i : \omega(A_i) \to W)_{i\in I}$ be a cocone of $\mathbf{Set}^{\{s\}\cup\Sigma}$ such that the weak transition systems A_i are cubical for all $i \in I$. Then the ω-final lift $\overline{W}$ satisfies the Intermediate state axiom. Assume moreover that every action u of W is the image of an action of A_{i_u} for some $i_u \in I$. Then the ω-final lift $\overline{W}$ is cubical.*

Proof. Let $\mathcal{C}$ be the full subcategory of weak transition systems satisfying the Intermediate axiom. By Proposition 3.2 and [17, Proposition 3.1(i)], the category is a full coreflective subcategory of $\mathcal{WTS}$, the right adjoint being given by the kelleyfication-like functor $X \mapsto \mathrm{dom}(r_X)$. Unlike the coreflection from $\mathcal{WTS}$ to $\mathcal{CTS}$, the new coreflection preserves the set of actions (and also the set of states). This means that the category $\mathcal{C}$ is concretely coreflective over ω. Hence $\overline{W}$ satisfies the Intermediate state axiom by the dual of [1, Proposition 21.31]. Let u be an action of $\overline{W}$. Then, by hypothesis, there exists an action v of some A_{i_u} such that the map $f_{i_u} : A_{i_u} \to W$ takes v to u. Since A_{i_u} is cubical by hypothesis, there exists a transition (α, v, β) of A_{i_u}. Hence the triple $(f_{i_u}(\alpha), u, f_{i_u}(\beta))$ is a transition of $\overline{W}$. This means that all actions of $\overline{W}$ are used. In other terms, $\overline{W}$ is cubical. □

Note that we have also proved that the forgetful functor $\mathcal{C} \subset \mathcal{WTS} \xrightarrow{\omega} \mathbf{Set}^{\{s\}\cup\Sigma}$ is topological by [1, Theorem 21.33]. We give the first application of this result. It states that the image of a cubical transition system is cubical.

Corollary 3.4. *Let $f : X \to Y$ be a map of weak transition systems. Let L_X (L_Y resp.) be the set of actions of X (Y resp.). Then f factors as a composite $X \to f(X) \to Y$ such that the map $f(X) \to Y$ is the inclusion $f(X^0) \subset Y^0$ on states and the inclusion $f(L_X) \subset L_Y$ on actions. If X is cubical, then $f(X)$ is cubical.*

Proof. Consider the ω-final lift $f(X)$ of the map of $\mathbf{Set}^{\{s\}\cup\Sigma}$

$$\omega(X) \longrightarrow (f(X^0), f(L_X))$$

induced by f. Then $f(X)$ is a solution. Assume now that X is cubical. By Theorem 3.3, the weak transition system $f(X)$ is cubical and the proof is complete. □

4. Most elementary properties of regular transition systems

A weak transition system satisfies the Unique intermediate state axiom or CSA2 if and only if it is orthogonal to the set of inclusions

$$C_n[x_1, \dots, x_n]^{ext} \subset C_n[x_1, \dots, x_n]$$

for all $n \geq 0$ and all $x_1, \dots, x_n \in \Sigma$ by [7, Theorem 5.6]. By [2, Theorem 1.39], there exists a functor

$$\mathrm{CSA}_2 : \mathcal{WTS} \to \mathcal{WTS}$$

such that for any weak transition system Y satisfying CSA2 and any weak transition system X, the weak transition system $\mathrm{CSA}_2(X)$ satisfies CSA2 and there is a natural bijection $\mathcal{WTS}(X, Y) \cong \mathcal{WTS}(\mathrm{CSA}_2(X), Y)$. Write

$$\eta_X : X \to \mathrm{CSA}_2(X)$$

for the unit of this adjunction. The following proposition provides an easy way to check that a cubical transition system is regular.

Proposition 4.1. *Let X be a cubical transition system. Let Y be a weak transition system satisfying CSA2. Let $f : X \to Y$ be a map of weak transition systems which is one-to-one on states. Then X is regular.*

Note that the hypothesis that X is cubical cannot be removed. Indeed, the inclusion

$$C_n[x_1, \dots, x_n]^{ext} \subset C_n[x_1, \dots, x_n]$$

for $x_1, \dots, x_n \in \Sigma$ is one-to-one on states because it is the inclusion

$$\{0_n, 1_n\} \subset \{0, 1\}^n.$$

The target $C_n[x_1, \dots, x_n]$ satisfies CSA2. But the pure n-transition

$$C_n[x_1, \dots, x_n]^{ext}$$

does not satisfy CSA2 for $n \geq 2$ because it does not even satisfy the Intermediate state axiom.

Proof. Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition of X with $n \geq 2$. Let $1 \leq p \leq n-1$. Since X is cubical, there exist two states ν_1 and ν_2 such that $(\alpha, u_1, \dots, u_p, \nu_i)$ and $(\nu_i, u_{p+1}, \dots, u_n, \beta)$ are transitions of X for $i = 1, 2$. Then the five tuples

$$\begin{gathered}(f(\alpha), f(u_1), \dots, f(u_n), f(\beta)),\\ (f(\alpha), f(u_1), \dots, f(u_p), f(\nu_1)), (f(\nu_1), f(u_{p+1}), \dots, f(u_n), f(\beta))\\ (f(\alpha), f(u_1), \dots, f(u_p), f(\nu_2)), (f(\nu_2), f(u_{p+1}), \dots, f(u_n), f(\beta))\end{gathered}$$

are transitions of Y. Since Y satisfies CSA2 by hypothesis, one has $f(\nu_1) = f(\nu_2)$. Since f is one-to-one on states by hypothesis, one obtains $\nu_1 = \nu_2$. Therefore X satisfies CSA2. □

Proposition 4.2. *Let X be a cubical transition system. There exists a push-out diagram of cubical transition systems*

$$\begin{tikzcd} X^0 \arrow[r, "\subset"] \arrow[d, "(\eta_X)^0"'] & X \arrow[d, "\eta_X"] \\ \mathrm{CSA}_2(X)^0 \arrow[r, "\subset"] & \mathrm{CSA}_2(X) \end{tikzcd}$$

where the horizontal maps are the inclusion of the set of states into the corresponding cubical transition system. For all cubical transition systems X, the unit map $\eta_X : X \to \mathrm{CSA}_2(X)$ is onto on states and the identity on actions.

Once again, the hypothesis that X is cubical cannot be removed. Indeed, let us consider again the case of a pure n-transition $X = C_n[x_1, \dots, x_n]^{ext}$ with $x_1, \dots, x_n \in \Sigma$. Then $\mathrm{CSA}_2(X) = C_n[x_1, \dots, x_n]$ by [7, Theorem 5.6]: in plain English, the n-cube is the free regular transition system generated the pure transition consisting of its $n!$ n-dimensional transitions. The commutative square

$$\begin{tikzcd} \{0_n, 1_n\} \arrow[r, "\subset"] \arrow[d, "(\eta_{C_n[x_1,\dots,x_n]^{ext}})^0"'] & C_n[x_1, \dots, x_n]^{ext} \arrow[d, "\eta_{C_n[x_1,\dots,x_n]^{ext}}"] \\ \{0,1\}^n \arrow[r, "\subset"] & C_n[x_1, \dots, x_n] \end{tikzcd}$$

is not a pushout diagram. The unit map $\eta_{C_n[x_1,\dots,x_n]^{ext}}$ is not onto on states. However, it is still bijective on actions.

We could actually prove that the map $\eta_X : X \to \mathrm{CSA}_2(X)$ is always bijective on actions for any weak transition system X. We leave the proof of this fact to the interested reader because it will not be used in this paper.

Proof. The natural transformation from the state set functor $(-)^0 : \mathcal{CTS} \to \mathbf{Set} \subset \mathcal{CTS}$ to the identity functor of $\mathcal{CTS}$ gives rise to a commutative diagram of cubical transition systems:

$$\begin{tikzcd} X^0 \arrow[r, "\subset"] \arrow[d, "(\eta_X)^0"'] & X \arrow[d, "\eta_X"] \\ \mathrm{CSA}_2(X)^0 \arrow[r, "\subset"] & \mathrm{CSA}_2(X). \end{tikzcd}$$

Consider the pushout diagram of cubical transition systems

$$\begin{tikzcd} X^0 \arrow[r, "\subset"] \arrow[d, "(\eta_X)^0"'] & X \arrow[d, "\eta_X"] \\ \mathrm{CSA}_2(X)^0 \arrow[r] & Z. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By the universal property of the pushout, the unit map $\eta_X : X \to \mathrm{CSA}_2(X)$ factors uniquely as a composite

$$X \longrightarrow Z \longrightarrow \mathrm{CSA}_2(X).$$

Since the forgetful functor $\omega : \mathcal{WTS} \to \mathbf{Set}^{\{s\}\cup\Sigma}$ forgetting the transitions is topological, and since the inclusion $\mathcal{CTS} \subset \mathcal{WTS}$ is colimit-preserving, the state set functor $X \mapsto X^0$ from $\mathcal{CTS}$ to $\mathbf{Set}$ is colimit-preserving. Hence the set map $Z^0 \to \mathrm{CSA}_2(X)^0$ is bijective. Therefore, by Proposition 4.1, the cubical transition system Z satisfies CSA2. Hence we obtain $Z = \mathrm{CSA}_2(X)$

by the universal property of the adjunction. The functor taking a cubical transition system to its set of actions is the composite functor

$$\mathcal{CTS} \subset \mathcal{WTS} \xrightarrow{\omega} \mathbf{Set}^{\{s\}\cup\Sigma} \longrightarrow \mathbf{Set}^{\Sigma} \xrightarrow{L\mapsto \coprod_{x\in\Sigma} L_x} \mathbf{Set}$$

which is colimit-preserving as well. Therefore, one obtains the pushout diagram of sets

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & \text{set of actions of } X \arrow[d] \\
\varnothing \arrow[r] & \text{set of actions of } \mathrm{CSA}_2(X). \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

This means that $X \to \mathrm{CSA}_2(X)$ is the identity on actions. By Corollary 3.4, there exists a cubical transition system $\eta_X(X)$ such that $\eta_X : X \to \mathrm{CSA}_2(X)$ factors as a composite $X \to \eta_X(X) \to \mathrm{CSA}_2(X)$ such that the map $\eta_X(X) \to \mathrm{CSA}_2(X)$ is the inclusion $\eta_X(X^0) \subset \mathrm{CSA}_2(X)^0$ on states and an inclusion on actions. By Proposition 4.1, $\eta_X(X)$ satisfies CSA2. Therefore $\eta_X(X) = \mathrm{CSA}_2(X)$ by the universal property of the adjunction. Hence the map $\eta_X : X \to \mathrm{CSA}_2(X)$ is onto on states. □

Proposition 4.3. *If X is cubical, then $\mathrm{CSA}_2(X)$ is regular. In particular, if X is regular, then $\mathrm{CSA}_2(X)$ is regular.*

Proof. By definition, $\mathrm{CSA}_2(X)$ satisfies the Unique Intermediate State axiom. By Proposition 4.2, the unit $X \to \mathrm{CSA}_2(X)$ is the identity on actions. Therefore all actions of $\mathrm{CSA}_2(X)$ are used since they are used in X which is cubical. □

Proposition 4.4. *The category $\mathcal{RTS}$ is a full reflective subcategory of $\mathcal{CTS}$ and the reflection is the functor $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$ which is the restriction of $\mathrm{CSA}_2 : \mathcal{WTS} \to \mathcal{WTS}$ to cubical transition systems.*

Proof. Let X be a cubical transition system and Y a regular transition system. By Proposition 4.3, one has the bijection of sets

$$\mathcal{CTS}(X,Y) \cong \mathcal{RTS}(\mathrm{CSA}_2(X), Y).$$

It is therefore the left adjoint of the inclusion $\mathcal{RTS} \subset \mathcal{CTS}$. □

Proposition 4.5. *The category $\mathcal{RTS}$ is locally finitely presentable.*

Proof. We already know that the cubes together with the double transitions are a dense generator of $\mathcal{CTS}$ by [8, Theorem 3.11 and Corollary 3.12]. But they are regular. So $\mathcal{RTS}$ has a dense and hence strong generator because colimits in $\mathcal{RTS}$ are calculated, first, by taking the colimits in $\mathcal{CTS}$ and, then, the image by the reflection $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$. The category $\mathcal{RTS}$ is also cocomplete for the same reason. The proof is complete with [2, Theorem 1.20]. $\square$

We can now introduce the cubification functor.

Definition 4.6. *[7] [8, Definition 3.13] Let $X \in \mathcal{WTS}$. The* cubification functor *is the functor $\underline{\mathrm{Cub}} : \mathcal{WTS} \longrightarrow \mathcal{WTS}$ defined by*

$$\underline{\mathrm{Cub}}(X) = \varinjlim_{C_n[x_1,\dots,x_n]\to X} C_n[x_1, \dots, x_n],$$

the colimit being taken in $\mathcal{WTS}$.

For any $X \in \mathcal{WTS}$, the weak transition system $\underline{\mathrm{Cub}}(X)$ is cubical and the colimit can be taken in $\mathcal{CTS}$ since the latter is coreflective in $\mathcal{WTS}$.

Proposition 4.7. *Let X be a weak transition system. Then the canonical map*

$$\pi_X : \underline{\mathrm{Cub}}(X) \longrightarrow X$$

is bijective on states.

Proof. The argument is given in the proof of [8, Theorem 3.11]. $\square$

Proposition 4.8. *Let X be a regular transition system. Then the cubical transition system $\underline{\mathrm{Cub}}(X)$ is regular and the colimit*

$$\varinjlim_{C_n[x_1,\dots,x_n]\to X} C_n[x_1, \dots, x_n]$$

is the same in $\mathcal{RTS}$, in $\mathcal{CTS}$ and in $\mathcal{WTS}$.

Proof. The weak transition system $\underline{\mathrm{Cub}}(X)$ is cubical because it is a colimit of cubes. The canonical map $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ is bijective on states by Proposition 4.7. Therefore $\underline{\mathrm{Cub}}(X)$ is regular by Proposition 4.1. We already know that the colimit is the same in $\mathcal{CTS}$ and in $\mathcal{WTS}$ since $\mathcal{CTS}$ is a full coreflective subcategory of $\mathcal{WTS}$. The functor $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$ is a left adjoint to the inclusion $\mathcal{RTS} \subset \mathcal{CTS}$ by Proposition 4.4. So it is colimit-preserving and one obtains, because the cubes are regular, the isomorphism:

$$\mathrm{CSA}_2\left(\varinjlim^{\mathcal{CTS}} C_n[x_1,\dots,x_n]\right) \cong \varinjlim^{\mathcal{RTS}} C_n[x_1,\dots,x_n].$$

The left-hand term is $\mathrm{CSA}_2(\underline{\mathrm{Cub}}(X))$ which is isomorphic to $\underline{\mathrm{Cub}}(X)$ since $\underline{\mathrm{Cub}}(X)$ is regular. $\square$

5. The left determined model category of regular transition systems

Let us start this section with a few remarks about the terminology.

Notation 5.1. For every map $f : X \to Y$ and every natural transformation $\alpha : F \to F'$ between two endofunctors of $\mathcal{K}$, the map $f \star \alpha$ is defined by the diagram:

$$\begin{tikzcd}
FX \arrow[r, "f"] \arrow[d, "\alpha_X"'] & FY \arrow[d] \arrow[ddr, bend left, "\alpha_Y"] & \\
F'X \arrow[r] \arrow[drr, bend right, "F'f"'] & \bullet \arrow[dr, "f\star\alpha"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & F'Y.
\end{tikzcd}$$

For a set of morphisms $\mathcal{A}$, let $\mathcal{A} \star \alpha = \{f \star \alpha, f \in \mathcal{A}\}$.

Let $(\mathcal{C},\mathcal{W},\mathcal{F})$ be a model structure on a locally presentable category $\mathcal{K}$ where $\mathcal{C}$ is the class of cofibrations, $\mathcal{W}$ the class of weak equivalences and $\mathcal{F}$ the class of fibrations. A cylinder for $(\mathcal{C},\mathcal{W},\mathcal{F})$ is a triple $(\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}, \gamma^0 \oplus \gamma^1 : \mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Cyl}, \sigma : \mathrm{Cyl} \Rightarrow \mathrm{Id})$ consisting of a functor $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ and two natural transformations $\gamma^0 \oplus \gamma^1 : \mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Cyl}$ and

$\sigma : \mathrm{Cyl} \Rightarrow \mathrm{Id}$ such that the composite $\sigma \circ (\gamma^0 \oplus \gamma^1)$ is the codiagonal functor $\mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Id}$ and such that the functorial map $\sigma_X : \mathrm{Cyl}(X) \to X$ belongs to $\mathcal{W}$ for every object X. We will often use the notation $\gamma = \gamma^0 \oplus \gamma^1$. The cylinder is *good* if the functorial map $\gamma_X : X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration for every object X. It is *very good* if, moreover, the map $\sigma_X : \mathrm{Cyl}(X) \to X$ is a trivial fibration for every object X. A good cylinder is *cartesian* if

- The functor $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ has a right adjoint $\mathrm{Path} : \mathcal{K} \to \mathcal{K}$ called the path functor.

- There are the inclusions $\mathcal{C} \star \gamma^\epsilon \subset \mathcal{C}$ for $\epsilon = 0, 1$ and $\mathcal{C} \star \gamma \subset \mathcal{C}$.

The notions above can be adapted to a cofibrantly generated weak factorization system $(\mathcal{L}, \mathcal{R})$ by considering the combinatorial model structure

$$(\mathcal{L}, \mathrm{Mor}(\mathcal{K}), \mathcal{R}).$$

They can be also extended to any set of maps I by considering the associated cofibrantly generated weak factorization system in the sense of [3, Proposition 1.3].

Definition 5.2. *Let $n \geq 1$ and $x_1, \dots, x_n \in \Sigma$. Let $\partial C_n[x_1, \dots, x_n]$ be the regular transition system defined by removing from the n-cube $C_n[x_1, \dots, x_n]$ all its n-transitions. It is called the* boundary *of $C_n[x_1, \dots, x_n]$.*

Notation 5.3. Denote by $\mathcal{I}$ the set of maps of cubical transition systems:

$$\begin{aligned}\mathcal{I} = \{C : \varnothing \to \{0\}, R : \{0,1\} \to \{0\}\} \\ \cup \{\partial C_n[x_1, \dots, x_n] \to C_n[x_1, \dots, x_n] \mid n \geq 1 \text{ and } x_1, \dots, x_n \in \Sigma\} \\ \cup \{C_1[x] \to \uparrow x \uparrow \mid x \in \Sigma\}.\end{aligned}$$

By [8, Corollary 6.8] and [10, Theorem 4.6], there exists a (necessarily unique) left determined model category structure on $\mathcal{CTS}$ (denoted by $\mathcal{CTS}$ as well) with the set of generating cofibrations $\mathcal{I}$. A map of cubical transition systems is a cofibration of this model structure if and only if it is one-to-one on actions. By [8, Proposition 5.5], this model category has a cartesian and very good cylinder $\mathrm{Cyl} : \mathcal{CTS} \to \mathcal{CTS}$ defined on objects as follows: for a cubical transition system $X = (S, \mu : L \to \Sigma, T)$, $\mathrm{Cyl}(X)$ has the same set of states S, the set of actions $L \times \{0,1\}$ with the labelling map $L \times \{0,1\} \to L \to \Sigma$ and a tuple $(\alpha, (u_1, \epsilon_1), \dots, (u_n, \epsilon_n), \beta)$ is a transition of $\mathrm{Cyl}(X)$ if and only if $(\alpha, u_1, \dots, u_n, \beta)$ is a transition of X. The map $\gamma_X^\epsilon : X \to \mathrm{Cyl}(X)$ for $\epsilon = 0, 1$ is induced by the identity on states and by the mapping $u \mapsto (u, \epsilon)$ on actions. The map $\sigma_X : \mathrm{Cyl}(X) \to X$ is induced by the identity on states and by the projection $(u, \epsilon) \mapsto u$ on actions.

Proposition 5.4. *One has the natural isomorphism of cubical transition systems*

$$\mathrm{CSA}_2(\mathrm{Cyl}(X)) \cong \mathrm{Cyl}(\mathrm{CSA}_2(X))$$

for every cubical transition system X.

Proof. We have just recalled that the canonical map $\sigma_X : \mathrm{Cyl}(X) \to X$ is bijective on states. Therefore, by Proposition 4.1, one has $\mathrm{Cyl}(\mathcal{RTS}) \subset \mathcal{RTS}$. By Proposition 4.2, for every cubical transition system X, one has the pushout diagram of weak transition systems (and of cubical transition systems since colimits are the same):

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{CSA}_2(X)^0 \arrow[r] & \mathrm{CSA}_2(X). \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Since $\mathrm{Cyl} : \mathcal{CTS} \to \mathcal{CTS}$ is a left adjoint, one obtains the pushout diagram of cubical transition systems:

$$\begin{tikzcd} \mathrm{Cyl}(X^0) \arrow[r] \arrow[d] & \mathrm{Cyl}(X) \arrow[d] \\ \mathrm{Cyl}(\mathrm{CSA}_2(X)^0) \arrow[r] & \mathrm{Cyl}(\mathrm{CSA}_2(X)). \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

For any set E viewed as a cubical transition system, one has $\mathrm{Cyl}(E) = E$.

Therefore one obtains the pushout diagram of cubical transition systems:

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & \mathrm{Cyl}(X) \arrow[d] \\
\mathrm{CSA}_2(X)^0 \arrow[r] & \mathrm{Cyl}(\mathrm{CSA}_2(X)).
\end{tikzcd}$$

Since $\mathrm{CSA}_2(X)$ is regular, the cubical transition system $\mathrm{Cyl}(\mathrm{CSA}_2(X))$ is regular. Therefore, by Proposition 4.2, the cubical transition systems

$$\mathrm{Cyl}(\mathrm{CSA}_2(X))$$

and

$$\mathrm{CSA}_2(\mathrm{Cyl}(X))$$

satisfy the same universal property. Hence we obtain the natural isomorphism

$$\mathrm{CSA}_2(\mathrm{Cyl}(X)) \cong \mathrm{Cyl}(\mathrm{CSA}_2(X)).$$

□

Theorem 5.5. *There exists a (necessarily unique) left determined model category structure on $\mathcal{RTS}$ (denoted by $\mathcal{RTS}$) such that the set of generating cofibrations is $\mathrm{CSA}_2(\mathcal{I}) = \mathcal{I}$ and such that the fibrant objects are the fibrant cubical transition systems which are regular. The cartesian cylinder is the restriction to $\mathcal{RTS}$ of the cylinder of $\mathcal{CTS}$ defined above. The restricted cylinder is very good. The reflection $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$ is a left Quillen homotopically surjective functor. The inclusion $\mathcal{RTS} \subset \mathcal{CTS}$ reflects weak equivalences.*

Proof. Thanks to Proposition B.1 applied with Proposition 5.4, we see that $\mathrm{Cyl} : \mathcal{CTS} \to \mathcal{CTS}$ and its right adjoint $\mathrm{Path} : \mathcal{CTS} \to \mathcal{CTS}$ restrict to endofunctors of $\mathcal{RTS}$. We then apply [15, Lemma 5.2] which is reexplained also in [10, Theorem 9.3]. The only thing which remains to be proved is that the restriction $\mathrm{Cyl} : \mathcal{RTS} \to \mathcal{RTS}$ is a very good cylinder. Consider the following commutative square of solid arrows of $\mathcal{RTS}$:

$$\begin{tikzcd}
\mathrm{CSA}_2(A) \arrow[r] \arrow[d, "\mathrm{CSA}_2(f)"'] & \mathrm{Cyl}(X) \arrow[d, "\sigma_X"] \\
\mathrm{CSA}_2(B) \arrow[r] \arrow[ur, dashed, "k"] & X
\end{tikzcd}$$

where $f \in \mathcal{I}$ and $X \in \mathcal{RTS}$. Because of the adjunction, the existence of a lift k is equivalent to the existence of a lift in the following commutative square of solid arrows of $\mathcal{CTS}$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & \mathrm{Cyl}(X) \arrow[d, "\sigma_X"] \\
B \arrow[r] \arrow[ur, dashed, "\ell"] & X.
\end{tikzcd}$$

So the restriction of Cyl to $\mathcal{RTS}$ is very good as well. $\square$

The end of the section is devoted to a characterization of the weak equivalences of the left-determined model structure $\mathcal{RTS}$.

Proposition 5.6. *(Compare with [8, Proposition 7.8]) Every regular transition system satisfying CSA1 is fibrant in $\mathcal{RTS}$. The category of regular transition systems satisfying CSA1 is a small-orthogonality class of $\mathcal{RTS}$.*

Proof. Every regular transition system satisfying CSA1 is fibrant in $\mathcal{CTS}$ by [8, Proposition 7.8], and therefore fibrant in $\mathcal{RTS}$ by Corollary 5.5. A regular transition system is CSA1 if and only if it is orthogonal to the maps $\sigma_{C_1[x]} : \mathrm{Cyl}(C_1[x]) \to C_1[x]$ for all $x \in \Sigma$. $\square$

The full subcategory of regular transition systems satisfying CSA1 is therefore a full reflective subcategory by [2, Theorem 1.39]. Write $\mathrm{CSA}_1^{\mathcal{RTS}} : \mathcal{RTS} \longrightarrow \mathcal{RTS}$ for the reflection. The full subcategory of cubical transition systems satisfying CSA1 is also a small-orthogonality class and a full reflective subcategory of $\mathcal{CTS}$ by [8, Proposition 7.2]. Write $\mathrm{CSA}_1^{\mathcal{CTS}}$:

$\mathcal{CTS} \longrightarrow \mathcal{CTS}$ for the reflection. The functor $\mathrm{CSA}_1^{\mathcal{RTS}} : \mathcal{RTS} \to \mathcal{RTS}$ ($\mathrm{CSA}_1^{\mathcal{CTS}} : \mathcal{CTS} \to \mathcal{CTS}$ resp.) can be defined as follows. Let $X_0 = X$. We construct by transfinite induction a sequence of regular (cubical resp.) transition systems as follows: if for $\alpha \geq 0$, there exist two transitions (α, u, β) and (α, u', β) with $u \neq u'$ and $\mu(u) = \mu(u')$, consider the pushout diagram in $\mathcal{RTS}$ (in $\mathcal{CTS}$ resp.)

$$\begin{tikzcd} \mathrm{Cyl}(C_1[\mu(u)]) \arrow[r, "\substack{(\mu(u),1,0) \mapsto u \\ (\mu(u),1,1) \mapsto u'}"] \arrow[d, "\sigma_{C_1[\mu(u)]}"'] & X_\alpha \arrow[d] \\ C_1[\mu(u)] \arrow[r] & X_{\alpha+1}, \end{tikzcd}$$

otherwise let $X_{\alpha+1} = X_\alpha$. If α is a limit ordinal, then let $X_\alpha = \varinjlim_{\beta<\alpha} X_\beta$, the colimit being calculated $\mathcal{RTS}$ (in $\mathcal{CTS}$ resp.). By a cardinality argument (all maps $X_\alpha \to X_{\alpha+1}$ are onto on actions), the sequence stabilizes. The colimit is $\mathrm{CSA}_1^{\mathcal{RTS}}(X)$ ($\mathrm{CSA}_1^{\mathcal{CTS}}(X)$ resp.).

Let X be a regular transition system. The canonical map

$$X \to \mathrm{CSA}_1^{\mathcal{CTS}}(X)$$

is then a transfinite composition of pushouts in $\mathcal{CTS}$ of maps of $\{\sigma_{C_1[x]} \mid x \in \Sigma\}$. Since a colimit is calculated in $\mathcal{RTS}$ by taking the colimit in $\mathcal{CTS}$ and by taking the image by the functor CSA_2, the map $\mathrm{CSA}_2(X) = X \to \mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X))$ is a transfinite composition of pushouts in $\mathcal{RTS}$ of maps of $\{\sigma_{C_1[x]} \mid x \in \Sigma\}$. Thus, $\mathrm{CSA}_1^{\mathcal{RTS}}(X)$ is orthogonal to $\mathrm{CSA}_2(X) = X \to \mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X))$. Hence the canonical map $X \to \mathrm{CSA}_1^{\mathcal{RTS}}(X)$ factors uniquely as a composite

$$X \longrightarrow \mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X)) \longrightarrow \mathrm{CSA}_1^{\mathcal{RTS}}(X).$$

Proposition 5.7. *There exists a regular transition system X such that the "comparison map"*

$$\mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X)) \to \mathrm{CSA}_1^{\mathcal{RTS}}(X)$$

is not an isomorphism.

Proof. A cubical transition system is completely defined by giving the list of all transitions and the actions identified by the labelling map. We consider the regular transition system X having the transitions

$$\begin{gathered}(\alpha,u_1,u_2,\beta),(\alpha,u_2,u_1,\beta),(\alpha,u_1,\chi),(\chi,u_2,\beta),(\alpha,u_2,\nu),(\nu,u_1,\beta),\\(\alpha,u_1',u_2',\beta),(\alpha,u_2',u_1',\beta),(\alpha,u_1',\chi'),(\chi',u_2',\beta),(\alpha,u_2',\nu'),(\nu',u_1',\beta),\\(\gamma,v,\chi),(\gamma,v',\chi'),(U_1,u_1,V_1),(U_1,u_1',V_1),(U_2,u_2,V_2),(U_2,u_2',V_2)\end{gathered}$$

such that all actions are labelled by some $x\in\Sigma$. By applying the functor $\mathrm{CSA}_1^{\mathcal{CTS}}:\mathcal{CTS}\to\mathcal{CTS}$ to X, the actions u_i and u_i' are identified because of the presence of the transitions

$$(U_1,u_1,V_1),(U_1,u_1',V_1),(U_2,u_2,V_2),(U_2,u_2',V_2).$$

The functor $\mathrm{CSA}_1^{\mathcal{CTS}}:\mathcal{CTS}\to\mathcal{CTS}$ does not make the identification $v=v'$ because these two actions are used in the transitions (γ,v,χ) and (γ,v',χ') and because it is assumed that $\chi\neq\chi'$. The cubical transition system

$$\mathrm{CSA}_1^{\mathcal{CTS}}(X)$$

therefore consists of the transitions [5]

$$\begin{gathered}(\alpha,u_1,u_2,\beta),(\alpha,u_2,u_1,\beta),(\alpha,u_1,\chi),(\chi,u_2,\beta),(\alpha,u_2,\nu),(\nu,u_1,\beta),\\(\alpha,u_1,u_2,\beta),(\alpha,u_2,u_1,\beta),(\alpha,u_1,\chi'),(\chi',u_2,\beta),(\alpha,u_2,\nu'),(\nu',u_1,\beta),\\(\gamma,v,\chi),(\gamma,v',\chi'),(U_1,u_1,V_1),(U_2,u_2,V_2).\end{gathered}$$

The latter cubical transition system is not regular. Indeed, in the regular transition system $\mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X))$, the identifications of states $\chi=\chi'$ and $\nu=\nu'$ are made. We obtain for $\mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X))$ the list of transitions

$$\begin{gathered}(\alpha,u_1,u_2,\beta),(\alpha,u_2,u_1,\beta),(\alpha,u_1,\chi),(\chi,u_2,\beta),(\alpha,u_2,\nu),(\nu,u_1,\beta),\\(\gamma,v,\chi),(\gamma,v',\chi),(U_1,u_1,V_1),(U_2,u_2,V_2).\end{gathered}$$

The map $\mathrm{CSA}_2(\mathrm{CSA}_1^{\mathcal{CTS}}(X))\to\mathrm{CSA}_1^{\mathcal{RTS}}(X)$ therefore identifies the actions v and v'. Hence it is not an isomorphism. □

[5]The states are preserved by $\mathrm{CSA}_1^{\mathcal{CTS}}$ since the canonical map $X\to\mathrm{CSA}_1^{\mathcal{CTS}}(X)$ is a transfinite composition of pushouts of maps of the form $\mathrm{Cyl}(C_1[z])\to C_1[z]$ for $z\in\Sigma$, because these maps are all of them state-preserving and because the state set functor from $\mathcal{CTS}$ to $\mathbf{Set}$ is colimit-preserving. Beware of the fact that the functor $\mathrm{CSA}_1^{\mathcal{RTS}}$ is not state-preserving.

Proposition 5.8. *(Compare with [8, Proposition 7.4]) Let Y be a regular transition system satisfying CSA1. Let X be a regular transition system. Then two homotopy equivalent maps $f, g : X \to Y$ are equal. In other terms, each of the two canonical maps $X \to \mathrm{Cyl}(X)$ induces a bijection $\mathcal{RTS}(\mathrm{Cyl}(X), Y) \cong \mathcal{RTS}(X, Y)$.*

Proof. By [8, Proposition 7.4], one has the bijection of sets

$$\mathcal{CTS}(\mathrm{Cyl}(X), Y) \cong \mathcal{CTS}(X, Y),$$

the binary product being calculated in $\mathcal{CTS}$. The category $\mathcal{RTS}$ is a full reflective subcategory of $\mathcal{CTS}$ by Proposition 4.4. Thus, there is the bijection $\mathcal{RTS}(\mathrm{Cyl}(X), Y) \cong \mathcal{RTS}(X, Y)$ where the binary product is calculated in $\mathcal{RTS}$. □

The following model-categorical lemma is implicitly used several times in [8] and [10] and it will be used again several times in this paper. Let us state it clearly:

Lemma 5.9. *Let $\mathcal{M}$ be a left proper combinatorial model category such that the generating cofibrations are maps between finitely presentable objects. Let $\mathcal{C}$ be a class of weak equivalences of $\mathcal{M}$ satisfying the following condition: in every pushout diagram of $\mathcal{M}$ of the form*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "g \in \mathcal{C}"'] & C \arrow[d, "f"] \\ B \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

either ϕ is a cofibration or f is an isomorphism. Then every map of $\mathbf{cell}_{\mathcal{M}}(\mathcal{C})$ is a weak equivalence of $\mathcal{M}$, where $\mathbf{cell}_{\mathcal{M}}(\mathcal{C})$ is the class of transfinite composition of pushouts of maps of $\mathcal{C}$.

Proof. Since $\mathcal{M}$ is left proper, f is always a weak equivalence of $\mathcal{M}$. By [18, Proposition 4.1], the class of weak equivalences of $\mathcal{M}$ is closed under transfinite composition. Hence the proof is complete. □

Lemma 5.10. *For all $x \in \Sigma$, the map $\sigma_{C_1[x]} : \mathrm{Cyl}(C_1[x]) \to C_1[x]$ satisfies the conditions of Lemma 5.9 for $\mathcal{M} = \mathcal{RTS}$.*

Proof. Consider a pushout diagram of $\mathcal{RTS}$

$$\begin{tikzcd} \mathrm{Cyl}(C_1[x]) \arrow[r, "\phi"] \arrow[d, "\sigma_{C_1[x]}"'] & C \arrow[d, "f"] \\ C_1[x] \arrow[r] & D. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $f : C \to D$ factors as a composite $f : C \to E \to \mathrm{CSA}_2(E) = D$ where E is the colimit in $\mathcal{CTS}$. If ϕ is not a cofibration, then ϕ is constant on actions. In this case, $C \cong E$ by the proof of [8, Theorem 7.10], therefore E is regular. One obtains $D = \mathrm{CSA}_2(E) \cong E \cong C$. Hence f is an isomorphism. $\square$

Theorem 5.11. *(Compare with [8, Theorem 7.10]) A map $f : X \to Y$ of regular transition systems is a weak equivalence for the left determined model structure of $\mathcal{RTS}$ if and only if the map $\mathrm{CSA}_1^{\mathcal{RTS}}(f) : \mathrm{CSA}_1^{\mathcal{RTS}}(X) \to \mathrm{CSA}_1^{\mathcal{RTS}}(Y)$ is an isomorphism.*

Proof. By Lemma 5.10, a map of regular transition systems $f : X \to Y$ is a weak equivalence if and only if the map $\mathrm{CSA}_1^{\mathcal{RTS}}(f) : \mathrm{CSA}_1^{\mathcal{RTS}}(X) \to \mathrm{CSA}_1^{\mathcal{RTS}}(Y)$ is a weak equivalence. Since $\mathrm{CSA}_1^{\mathcal{RTS}}(X)$ and $\mathrm{CSA}_1^{\mathcal{RTS}}(Y)$ are fibrant by Proposition 5.6, a map of regular transition systems $f : X \to Y$ is a weak equivalence if and only if the map $\mathrm{CSA}_1^{\mathcal{RTS}}(f) : \mathrm{CSA}_1^{\mathcal{RTS}}(X) \to \mathrm{CSA}_1^{\mathcal{RTS}}(Y)$ is a homotopy equivalence. The proof is complete with Proposition 5.8. $\square$

6. Bousfield localization of the regular t.s. by the cubification functor

We now deal with the Bousfield localization of $\mathcal{RTS}$ by the cubification functor $\underline{\mathrm{Cub}}$ and we compare this Bousfield localization with the one of $\mathcal{CTS}$ by the same cubification functor.

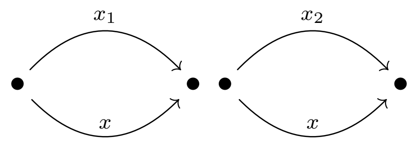

Figure 2: The cubical transition system $Z_x^{x_1,x_2}$ contains four states and three actions x_1, x_2, x with $\mu(x_1) = \mu(x_2) = x$.

Let $x \in \Sigma$. Consider the unique map $p_x : C_1[x] \sqcup C_1[x] \to \uparrow x \uparrow$ bijective on states and sending the actions of the source $C_1[x] \sqcup C_1[x]$ to their label. Let us factor p_x as a composite (all maps are bijective on states)

$$C_1[x] \sqcup C_1[x] \xhookrightarrow[p_x^{cof}]{\substack{x_1 \mapsto x_1 \\ x_2 \mapsto x_2}} Z_x^{x_1,x_2} \xrightarrow[\simeq]{\substack{x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x \mapsto x}} \uparrow x \uparrow$$

with $Z_x^{x_1,x_2}$ is depicted in Figure 2, and where x_1 and x_2 are the two actions of $C_1[x] \sqcup C_1[x]$ with $\mu(x_1) = \mu(x_2) = x$. The left-hand map is a cofibration because it is one-to-one on actions. One has the isomorphisms

$$\begin{aligned} \mathrm{CSA}_1^{\mathcal{RTS}}(Z_x^{x_1,x_2}) \cong \mathrm{CSA}_1^{\mathcal{RTS}}(\uparrow x \uparrow) \cong \mathrm{CSA}_1^{\mathcal{CTS}}(Z_x^{x_1,x_2}) \\ \cong \mathrm{CSA}_1^{\mathcal{CTS}}(\uparrow x \uparrow) \cong \uparrow x \uparrow, \end{aligned}$$

so the right-hand map is a weak equivalence of $\mathcal{CTS}$ by [8, Theorem 7.10] and of $\mathcal{RTS}$ by Theorem 5.11. Therefore p_x^{cof} is a cofibrant replacement of p_x both in $\mathcal{CTS}$ and in $\mathcal{RTS}$.

Notation 6.1. Let $\mathcal{S} = \{p_x \mid x \in \Sigma\}$ and $\mathcal{S}^{cof} = \{p_x^{cof} \mid x \in \Sigma\}$.

Proposition 6.2. *For a cubical transition system X, the following statements are equivalent:*

1. *The labelling map μ is one-to-one.*
2. *X is $\mathcal{S}$-injective.*
3. *X is $\mathcal{S}$-orthogonal.*

If any one of these statements is true, then X satisfies CSA1 and is $\mathcal{S}^{cof}$-orthogonal.

Proof. The equivalence $(1) \iff (2) \iff (3)$ and the fact that these three conditions imply CSA1 is [10, Proposition 8.2]. Let X be a cubical transition system satisfying (1). Consider the diagram of cubical transition systems:

$$\begin{tikzcd} C_1[x] \sqcup C_1[x] \arrow[r, "\phi"] \arrow[d, "p_x^{cof}"'] & X \\ Z_x^{x_1,x_2}, \arrow[ur, dashed, "\ell"] & \end{tikzcd}$$

where x_1 and x_2 are the two actions of $C_1[x] \sqcup C_1[x]$. Define ℓ on states by $\ell(\alpha) = \phi(\alpha)$ for all states α, and on actions by $\ell(x_i) = \phi(x_i)$ for $i = 1, 2$ and $\ell(x) = \phi(x_1)$. Since X satisfies (1), one has $\phi(x_1) = \phi(x_2)$. We deduce that ℓ is a well-defined map of cubical transition systems. The map ℓ is the only solution because p_x^{cof} is bijective on states and the image by ℓ of the actions of $Z_x^{x_1,x_2}$ is necessarily the unique action of X labelled by x. Hence X is $\mathcal{S}^{cof}$-orthogonal. □

Proposition 6.3. *(Compare with [8, Proposition 8.4]) For every regular transition system X, the canonical map $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ belongs to* $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$

Proof. The difficulty is, once again, that colimits are not calculated in the same way in $\mathcal{RTS}$ and in $\mathcal{CTS}$. Let $(u_1^i, u_2^i)_{i \in I}$ be the family of pairs of actions of X such that $\pi_X(u_1^i) = \pi_X(u_2^i)$, which implies $\mu(u_1^i) = \mu(u_2^i)$. Since X is cubical, for all $i \in I$, there exist 1-transitions $(\alpha_j^i, u_j^i, \beta_j^i)$ of X for $j = 1, 2$. Let $\phi^i : C_1[\mu(u_1^i)] \sqcup C_1[\mu(u_2^i)] \to X$ be the map of cubical transition systems sending the two 1-transitions of the source to $(\alpha_j^i, u_j^i, \beta_j^i)$ for $j = 1, 2$. Since $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ is the identity on states by Proposition 4.7, one obtains the following commutative diagram of regular transition systems:

$$\begin{tikzcd}
\coprod_{i\in I} C_1[\mu(u_1^i)] \sqcup C_1[\mu(u_2^i)] \arrow[r, "\phi^i"] \arrow[d, "\coprod_{i\in I} p_{\mu(u_1^i)}"'] & \underline{\mathrm{Cub}}(X) \arrow[d, "\pi_X"] \\
\coprod_{i\in I} {\uparrow}\mu(u_1^i){\uparrow} \arrow[r] & X.
\end{tikzcd}$$

Consider the pushout diagram of regular transition systems:

$$\begin{tikzcd}
\coprod_{i\in I} C_1[\mu(u_1^i)] \sqcup C_1[\mu(u_2^i)] \arrow[r, "\phi^i"] \arrow[d, "\coprod_{i\in I} p_{\mu(u_1^i)}"'] & \underline{\mathrm{Cub}}(X) \arrow[d] \\
\coprod_{i\in I} {\uparrow}\mu(u_1^i){\uparrow} \arrow[r] & Z.
\end{tikzcd}$$

The colimit Z is calculated in $\mathcal{RTS}$ by taking the colimit T in $\mathcal{CTS}$ and by taking the image by the reflection CSA_2. Hence the map $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ factors as a composite

$$\underline{\mathrm{Cub}}(X) \longrightarrow T \longrightarrow \mathrm{CSA}_2(T) = Z \stackrel{h}{\longrightarrow} X.$$

The map $\underline{\mathrm{Cub}}(X) \to T$ is a pushout in $\mathcal{CTS}$ of the map $\coprod_{i\in I} p_{\mu(u_1^i)}$. The latter is bijective on states, therefore the map $\underline{\mathrm{Cub}}(X) \to T$ is bijective on states as well. The map $T \to Z$ is onto on states by Proposition 4.2. Hence the map $g : \underline{\mathrm{Cub}}(X) \to Z$ is onto on states. Let α and β be two states of $\underline{\mathrm{Cub}}(X)$ mapped to the same state γ of Z. Then γ is mapped to $\pi_X(\alpha) = \pi_X(\beta)$ by $Z \to X$. Hence $\alpha = \beta$ by Proposition 4.7. Therefore $g : \underline{\mathrm{Cub}}(X) \to Z$ is bijective on states, and so is the map of cubical transition systems $h : Z \to X$. By construction, the latter map is one-to-one on actions. Therefore $h : Z \to X$ is one-to-one on transitions by [10, Proposition 4.4]. Any action u is used by a 1-transition (α, u, β) of X. Hence $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ is onto on actions. Thus, there exists an action v of

$\underline{\mathrm{Cub}}(X)$ such that $\pi_X(v) = u$. This means that $h(g(v)) = u$. Hence h is onto on actions as well. To conclude that h is an isomorphism, consider a transition $(\alpha, u_1, \dots, u_n, \beta)$ of X. It gives rise to a map of weak transition systems $C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \to X$ which factors as a composite $C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \to C_n[\mu(u_1), \dots, \mu(u_n)] \to X$ since X is cubical. One obtains the composite map of weak transition systems

$$C_n[\mu(u_1), \dots, \mu(u_n)]^{ext} \longrightarrow C_n[\mu(u_1), \dots, \mu(u_n)] \longrightarrow \underline{\mathrm{Cub}}(X) \longrightarrow Z \longrightarrow X.$$

Hence h is onto on transitions. □

Lemma 6.4. *For all $x \in \Sigma$, the map $p_x : C_1[x] \sqcup C_1[x] \to \uparrow x \uparrow$ satisfies the conditions of Lemma 5.9 for $\mathcal{M} = \underline{\mathbf{L}}_{\underline{\mathrm{Cub}}} \mathcal{RTS}$.*

Proof. Consider a pushout diagram of $\mathcal{RTS}$

$$\begin{tikzcd} C_1[x] \sqcup C_1[x] \arrow[r, "\phi"] \arrow[d, "p_x"'] & C \arrow[d, "f"] \\ \uparrow x \uparrow \arrow[r] & D. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $f : C \to D$ factors as a composite $f : C \to E \to \mathrm{CSA}_2(E) = D$ where E is the colimit in $\mathcal{CTS}$. If ϕ is not a cofibration, then ϕ is constant on actions. In this case, $C \cong E$ by the proof of [8, Proposition 8.5], therefore E is regular. One obtains $D = \mathrm{CSA}_2(E) \cong E \cong C$. Hence f is an isomorphism. □

Theorem 6.5. *(Compare with [8, Theorem 8.6]) Let $\mathcal{W}_{\underline{\mathrm{Cub}}}$ be the Grothendieck localizer generated by the class of maps $f : X \to Y$ of regular transition systems such that $\underline{\mathrm{Cub}}(f) : \underline{\mathrm{Cub}}(X) \to \underline{\mathrm{Cub}}(Y)$ is a weak equivalence of $\mathcal{RTS}$ (the left determined model structure). Let $\mathcal{W}(\mathcal{S})$ be the Grothendieck localizer generated by the set of maps $\mathcal{S}$. Then one has $\mathcal{W}_{\underline{\mathrm{Cub}}} = \mathcal{W}(\mathcal{S})$.*

Proof. The proof is *mutatis mutandis* the proof of [8, Theorem 8.6]. Let us sketch it. By Proposition 6.3, the counit $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ belongs to

$\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ for all regular transition systems. By Lemma 6.4, one deduces that $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S}) \subset \mathcal{W}(\mathcal{S})$. Hence, for all regular transition systems X, the counit $\pi_X : \underline{\mathrm{Cub}}(X) \to X$ belongs to $\mathcal{W}(\mathcal{S})$. Let $f : X \to Y$ be a map of $\mathcal{W}_{\underline{\mathrm{Cub}}}$. Consider the commutative diagrams:

$$\begin{tikzcd}
\underline{\mathrm{Cub}}(X) \arrow[r, "\underline{\mathrm{Cub}}(f)"] \arrow[d] & \underline{\mathrm{Cub}}(Y) \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

We have just proved that the vertical maps belong to $\mathcal{W}(\mathcal{S})$. Since $\underline{\mathrm{Cub}}(f)$ is a weak equivalence of $\mathcal{RTS}$, i.e. it belongs to the smallest Grothendieck localizer $\mathcal{W}(\varnothing) \subset \mathcal{W}(\mathcal{S})$, one deduces by the two-out-of-three property that the bottom map f belongs to $\mathcal{W}(\mathcal{S})$ as well. Hence we obtain the inclusion $\mathcal{W}_{\underline{\mathrm{Cub}}} \subset \mathcal{W}(\mathcal{S})$. Since $\underline{\mathrm{Cub}}(p_x)$ is an automorphism of $C_1[x] \cup C_1[x]$, one has $\mathcal{S} \subset \mathcal{W}_{\underline{\mathrm{Cub}}}$, and therefore $\mathcal{W}(\mathcal{S}) \subset \mathcal{W}_{\underline{\mathrm{Cub}}}$. □

Corollary 6.6. *(Compare with [8, Corollary 8.7]) The Bousfield localization of the left determined model structure of $\mathcal{RTS}$ with respect to the functor $\underline{\mathrm{Cub}}$ exists.*

Proof. The combinatorial model category $\mathcal{RTS}$ is left proper since all objects are cofibrant. We want to Bousfield localize with respect to a set of maps $\mathcal{S}$. Hence the proof is complete. □

Notation 6.7. Let us write $\mathbf{L}_{\underline{\mathrm{Cub}}}\mathcal{CTS}$ ($\mathbf{L}_{\underline{\mathrm{Cub}}}\mathcal{RTS}$ resp.) for the Bousfield localization of $\mathcal{CTS}$ ($\mathcal{RTS}$ resp.) by the functor $\underline{\mathrm{Cub}}$.

Proposition 6.8. *A regular transition system is fibrant in $\mathbf{L}_{\underline{\mathrm{Cub}}}\mathcal{RTS}$ if and only if it is fibrant in $\mathbf{L}_{\underline{\mathrm{Cub}}}\mathcal{CTS}$.*

Proof. The proof is similar to the proof of Theorem 5.5. □

Proposition 6.9. *(Compare with [8, Theorem 8.11 (1)(2)(3)]) The category*

$$\mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$$

of $\mathcal{S}$-injective regular transition systems is a small-orthogonality class and a full reflective subcategory of $\mathcal{RTS}$. Write $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}} : \mathcal{RTS} \to \mathcal{RTS}$ for the reflection. The unit map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ belongs to $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ for any regular transition system X.

Proof. By Proposition 6.2, being $\mathcal{S}$-injective is equivalent to being $\mathcal{S}$-orthogonal. By [2, Theorem 1.39], the subcategory $\mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$ is then a reflective subcategory of $\mathcal{RTS}$. For any regular transition system X, the map $X \to \mathbf{1}$ factors as a composite $X \to F(X) \to \mathbf{1}$ where the left-hand map belongs to $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ and the right-hand map belongs to $\mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$ by using the small object argument in the locally presentable category $\mathcal{RTS}$. Then $F(X)$ is $\mathcal{S}$-orthogonal by Proposition 6.2. We deduce that the map $X \to F(X)$ factors uniquely as a composite $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X) \to F(X)$ by the property of the adjunction. But the map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ factors uniquely as a composite $X \to F(X) \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ since the map $X \to F(X)$ belongs to $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ and since $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ is $\mathcal{S}$-orthogonal. Hence the functor F and $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}$ are isomorphic. □

The next proposition compares the functor $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}} : \mathcal{RTS} \to \mathcal{RTS}$ with the functor $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}} : \mathcal{CTS} \to \mathcal{CTS}$ defined in an analogous way in [8]:

Proposition 6.10. *Let X be a regular transition system. Then one has the natural isomorphism*

$$\mathrm{CSA}_2(\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)) \cong \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X).$$

Proof. The map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X) \to \mathrm{CSA}_2(\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X))$ is bijective on actions by Proposition 4.2. Hence the labelling map of $\mathrm{CSA}_2(\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X))$ is one-to-one since the labelling map of $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)$ is one-to-one by Proposition 6.2. Since the map $X \to \mathbf{1}$ factors as a composite

$$X \longrightarrow \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X) \longrightarrow \mathrm{CSA}_2(\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)) \longrightarrow \mathbf{1},$$

and since $\mathrm{CSA}_2(\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X))$ is $\mathcal{S}$-injective and regular, the latter satisfies the same universal property as $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$. Hence the proof is complete. □

Theorem 6.11. *(Compare with [8, Theorem 8.10]) A map of regular transition systems $f : X \to Y$ is a weak equivalence of the Bousfield localization $\mathbf{L}_{\mathrm{Cub}}\mathcal{RTS}$ of $\mathcal{RTS}$ by the set of maps $\mathcal{S}$ if and only if the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f) : \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X) \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ is an isomorphism.*

Proof. We already saw in the proof of Theorem 6.5 that every map of

$$\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$$

is a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}$. This implies that for all morphisms of regular transition systems $f : X \to Y$, if $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f)$ is an isomorphism, then f belongs to $\mathcal{W}(\mathcal{S})$. Conversely, let us suppose that $f : X \to Y$ is a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}$. Then $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f) : \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X) \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ is a map of regular transition systems between two $\mathcal{S}$-injective regular transition systems. By Proposition 6.2, both $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ and $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ satisfy CSA1 and are $\mathcal{S}^{cof}$-orthogonal. By [8, Proposition 7.7], both $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ and $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ are fibrant in $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS}$, and therefore fibrant in $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}$ by Proposition 6.8. In other terms, $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f) : \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X) \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ is a weak equivalence between two cofibrant-fibrant objects of the Bousfield localization. Hence, $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f)$ is a weak equivalence of the left determined model structure $\mathcal{RTS}$. By Proposition 6.2, both $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ and $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ satisfy CSA1. By Proposition 5.8, one deduces that $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f)$ is an isomorphism. □

Proposition 6.12 and Theorem 6.13 help to understand the difference between the weak equivalences of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS}$ and of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}$.

Proposition 6.12. *For all cubical transition systems X, the map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)$ is bijective on states and onto on actions. There exists a cubical transition system X_0 such that the map $X_0 \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X_0)$ is not onto on transitions. For all regular transition systems Y, the map $Y \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ is onto on states, on actions and on transitions. There exists a regular transition system Y_0 such that the map $Y_0 \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y_0)$ is not bijective on states.*

Proof. This is a corollary of Proposition A.2 and Proposition A.6 of Appendix A. □

Theorem 6.13. *There exists a strict inclusion of sets*

$$\Big\{\textit{weak equivalences of } \underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS} \textit{ between regular t.s.}\Big\} \subset \Big\{\textit{weak equivalences of } \underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}\Big\}.$$

In other terms, if $f : X \to Y$ is a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS}$ between two regular transition systems, then f is a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}$. There exists a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{RTS}$ which is not a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS}$.

Proof. Let $f : X \to Y$ be a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\mathcal{CTS}$ between two regular transition systems. Then by [8, Theorem 8.10], the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(f)$ is an isomorphism. The map $\mathrm{CSA}_2(\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(f))$ is therefore an isomorphism. So, by Proposition 6.10, $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(f)$ is an isomorphism. Hence by Theorem 6.11, f is a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\mathcal{RTS}$.

Now we want to find a weak equivalence g of $\underline{\mathbf{L}}_{\mathrm{Cub}}\mathcal{RTS}$ which is not a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\mathcal{CTS}$. One has

$$\omega(C_2[x,x]) = (\{0,1\}^2, \{(x,1),(x,2)\})$$

by Proposition 2.6 with $x \in \Sigma$. Consider the set $\{0,1\}^2 \times \{-,+\}$ and let us make the identifications $(0,0,-) = (0,0,+) = I$ and $(1,1,-) = (1,1,+) = F$. Write S for the quotient. Let $W = (S, \{u, v^-, v^+\})$. For $\alpha \in \{-,+\}$, consider the map $\phi^\alpha : \omega(C_2[x,x]) \to W$ of $\mathbf{Set}^{\{s\}\cup\Sigma}$ induced by the mappings $(\epsilon_1,\epsilon_2) \mapsto (\epsilon_1,\epsilon_2,\alpha)$ for $(\epsilon_1,\epsilon_2) \in \{0,1\}^2$, $(x,1) \mapsto u$ and $(x,2) \mapsto v^\alpha$. Consider the ω-final lift $\overline{W}$ of the cone of maps $\phi^-, \phi^+ : \omega(C_2[x,x]) \rightrightarrows W$. By Theorem 3.3, the weak transition system $\overline{W}$ is cubical. The only higher dimensional transitions of $\overline{W}$ are the four transitions $(I,u,v^\pm,F)$ and $(I,v^\pm,u,F)$. Hence the unique state ν such that the tuples (I,u,ν) and $(\nu,v^\pm,F)$ are transitions of $\overline{W}$ is $\nu = (1,0,\pm)$. It turns out that the unique state ν' such that the tuples $(I,v^\pm,\nu')$ and (ν',u,F) are transitions of $\overline{W}$ is $\nu' = (0,1,\pm)$. One deduces that $\overline{W}$ is regular. There exists a map of cubical transition systems $g : \overline{W} \to C_2[x,x]$ defined as follows: it takes the state $(\epsilon_1,\epsilon_2,\pm)$ to (ϵ_1,ϵ_2) for $(\epsilon_1,\epsilon_2) \in, \{0,1\}^2$, the action u to $(x,1)$ and the actions v^- and v^+ to $(x,2)$. It is easy to see that one has the isomorphisms

$$\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(\overline{W}) \cong C_2[x,x] \cong \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(C_2[x,x]),$$

hence g is a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\mathcal{RTS}$ by Theorem 6.11. Since g is not bijective on states, the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(f)$ is not bijective on states by Proposition 6.12. Therefore the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(f)$ is not an isomorphism. Hence g is not a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\mathcal{CTS}$ by [8, Theorem 8.10]. $\square$

We can now completely elucidate this model structure thanks to the following result:

Theorem 6.14. *(Compare with [8, Theorem 8.11 (4)(5)]) The left adjoint* $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}} : \underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{RTS} \to \mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$ *induces a left Quillen equivalence between* $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{RTS}$ *and* $\mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$ *equipped with the discrete model structure (all maps are cofibrations and fibrations and the weak equivalences are the isomorphisms).*

Proof. For any fibrant object X of $\mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$, the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X) \to X$ is an isomorphism and X is cofibrant in $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{RTS}$. For any cofibrant object Y of $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{RTS}$, Y is fibrant in $\mathbf{inj}_{\mathcal{RTS}}(\mathcal{S})$ and the map $Y \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(Y)$ is a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{RTS}$ by Proposition 6.9 and by Lemma 6.4. This is the definition of a Quillen equivalence. □

Theorem 6.11 does not mean that two regular transition systems are weakly equivalent if and only if they are isomorphic. Indeed, for any regular transition system X, the unit map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$, by identifying the actions of X with their labelling, modifies the geometric structure of X by forcing identifications of states (see Proposition 6.12). Roughly speaking, this map removes all non-discernable transitions. This behaviour is slightly different from the one of the unit map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)$. Once again by Proposition 6.12, the unit map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)$ also identifies the actions of a cubical transition system X by their labelling, but the latter map is constant on states, and not necessarily onto on transitions. It may create new transitions which are actually not observable and which are killed by applying the functor $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$.

7. Fibrant regular and cubical transition systems

The purpose of this last section is to describe completely the fibrant regular and cubical transition systems. We already know by Proposition 6.8 that the fibrant regular transition systems are exactly the fibrant cubical ones which are regular. Thus, we just have to give a combinatorial characterization of the fibrant objects of $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{CTS}$. Corollary 7.16 encompasses the results of [8] and [9].

Definition 7.1. *A cubical transition system X is* combinatorially fibrant *if for any $n \geq 1$, any state α and β and any actions $u_1, v_1, \dots, u_n, v_n$ such that*

$\mu(u_i) = \mu(v_i)$ for $1 \leq i \leq n$, if the tuple $(\alpha, u_1, \dots, u_n, \beta)$ is a transition of X, then the tuple $(\alpha, v_1, \dots, v_n, \beta)$ is a transition of X as well.

Proposition 7.2. *Let $X = (S, \mu : L \to \Sigma, T)$ be a combinatorially fibrant cubical transition system. Write $\mathrm{Path} : \mathcal{CTS} \to \mathcal{CTS}$ for the right adjoint of the cartesian cylinder $\mathrm{Cyl} : \mathcal{CTS} \to \mathcal{CTS}$. Then the cubical transition system $\mathrm{Path}(X)$ has S as its set of states and $L \times_\Sigma L$ as its set of actions, the labelling map is the composite map $\mu : L \times_\Sigma L \to L \to \Sigma$ and a tuple $(\alpha, (u_1^0, u_1^1), \dots, (u_n^0, u_n^1), \beta)$ of $S \times (L \times_\Sigma L)^n \times S$ is a transition of $\mathrm{Path}(X)$ if and only if there exist $\epsilon_1, \dots, \epsilon_n \in \{0, 1\}$ such that the tuple $(\alpha, u_1^{\epsilon_1}, \dots, u_n^{\epsilon_n}, \beta)$ is a transition of X.*

Proof. Let us recall that the cartesian cylinder $\mathrm{Cyl} : \mathcal{CTS} \to \mathcal{CTS}$ is the restriction of an endofunctor of $\mathcal{WTS}$ defined in the same way. The functor $\mathrm{Cyl} : \mathcal{WTS} \to \mathcal{WTS}$ has a right adjoint $\mathrm{Path}^{\mathcal{WTS}} : \mathcal{WTS} \to \mathcal{WTS}$ defined on objects as follows [8, Proposition 5.8]: for a weak transition system $X = (S, \mu : L \to \Sigma, T)$, the weak transition system $\mathrm{Path}^{\mathcal{WTS}}(X)$ has the same set of states S, the set of actions is $L \times_\Sigma L$ and a tuple

$$(\alpha, (u_1^-, u_1^+), \dots, (u_n^-, u_n^+), \beta)$$

with $n \geq 1$ is a transition of $\mathrm{Path}^{\mathcal{WTS}}(X)$ if and only if the 2^n tuples $(\alpha, u_1^\pm, \dots, u_n^\pm, \beta)$ are transitions of X. The right adjoint of the functor $\mathrm{Cyl} : \mathcal{CTS} \to \mathcal{CTS}$ is equal to the composite functor

$$\mathrm{Path} : \mathcal{CTS} \subset \mathcal{WTS} \xrightarrow{\mathrm{Path}^{\mathcal{WTS}}} \mathcal{WTS} \longrightarrow \mathcal{CTS},$$

where the right-hand functor from $\mathcal{WTS}$ to $\mathcal{CTS}$ is the coreflection.

Let $(u, v) \in L \times_\Sigma L$. Since u is used in X, there exists a transition (α, u, β) of X. Since $\mu(u) = \mu(v)$ and since X is combinatorially fibrant, the triple (α, v, β) is a transition of X. This means that the couple $(u, v) \in L \times_\Sigma L$ is used by the transition $(\alpha, (u, v), \beta)$ of $\mathrm{Path}^{\mathcal{WTS}}(X)$. We deduce that all actions of $\mathrm{Path}^{\mathcal{WTS}}(X)$ are used. Consider a transition

$$(\alpha, (u_1^-, u_1^+), \dots, (u_n^-, u_n^+), \beta)$$

of $\mathrm{Path}^{\mathcal{WTS}}(X)$ with $n \geq 2$. Let $1 \leq p \leq n-1$. Since X is cubical, there exists a state γ such that the tuples $(\alpha, u_1^-, \dots, u_p^-, \gamma)$ and $(\gamma, u_{p+1}^-, \dots, u_n^-, \beta)$

are two transitions of X. But X is combinatorially fibrant. This implies that all tuples $(\alpha, u_1^\pm, \dots, u_p^\pm, \gamma)$ and $(\gamma, u_{p+1}^\pm, \dots, u_n^\pm, \beta)$ are transitions of X. Therefore the two tuples

$$(\alpha, (u_1^-, u_1^+), \dots, (u_p^-, u_p^+), \gamma), (\gamma, (u_{p+1}^-, u_{p+1}^+), \dots, (u_n^-, u_n^+), \beta)$$

are transitions of $\mathrm{Path}^{\mathcal{WTS}}(X)$. This means that the weak transition system $\mathrm{Path}^{\mathcal{WTS}}(X)$ satisfies the Intermediate state axiom. We have just proved that if X is combinatorially fibrant, then the weak transition system $\mathrm{Path}^{\mathcal{WTS}}(X)$ is cubical: in other terms, one has $\mathrm{Path}(X) = \mathrm{Path}^{\mathcal{WTS}}(X)$ in this case. Finally and because X is combinatorially fibrant, all tuples $(\alpha, u_1^\pm, \dots, u_n^\pm, \beta)$ are transitions of X if and only if there exist $\epsilon_1, \dots, \epsilon_n \in \{0, 1\}$ such that the tuple $(\alpha, u_1^{\epsilon_1}, \dots, u_n^{\epsilon_n}, \beta)$ is a transition of X. This completes the proof. □

Proposition 7.3. *If the cubical transition system X is combinatorially fibrant, then so is the cubical transition system* $\mathrm{Path}(X)$.

Proof. Let $X = (S, \mu : L \to \Sigma, T)$ be a combinatorially fibrant cubical transition system. Let

$$(\alpha, (u_1^-, u_1^+), \dots, (u_n^-, u_n^+), \beta), (\alpha, (v_1^-, v_1^+), \dots, (v_n^-, v_n^+), \beta)$$

be two tuples of $S \times (L \times_\Sigma L)^n \times S$ with $n \geq 1$ and $\mu(u_i^-, u_i^+) = \mu(v_i^-, v_i^+)$ for $1 \leq i \leq n$. Let us suppose that $(\alpha, (u_1^-, u_1^+), \dots, (u_n^-, u_n^+), \beta)$ is a transition of $\mathrm{Path}(X)$. Then the tuple $(\alpha, u_1^-, \dots, u_n^-, \beta)$ is a transition of X. But for all $1 \leq i \leq n$, one has

$$\mu(u_i^-) = \mu(u_i^+) = \mu(u_i^-, u_i^+) = \mu(v_i^-, v_i^+) = \mu(v_i^-) = \mu(v_i^+).$$

So, all tuples $(\alpha, v_1^\pm, \dots, v_n^\pm, \beta)$ are transitions of X because X is combinatorially fibrant. This implies that the tuple $(\alpha, (v_1^-, v_1^+), \dots, (v_n^-, v_n^+), \beta)$ is a transition of $\mathrm{Path}(X)$. This is the definition of combinatorial fibrancy applied to $\mathrm{Path}(X)$. □

Proposition 7.4. *Let X be a cubical transition system. If X is combinatorially fibrant, then it is injective with respect to any map of the form $f \star \gamma^\epsilon$ for $\epsilon = 0, 1$ for any cofibration of cubical transition systems f.*

Proof. Let $f : A \to B$ be a map of cubical transition systems. Let L be the set of actions of X. By adjunction, the cubical transition system X is injective with respect to $f \star \gamma^\epsilon$ if and only if the map $\pi^\epsilon : \mathrm{Path}(X) \to X$ satisfies the RLP with respect to f. Let us recall that the map $\pi^\epsilon : \mathrm{Path}(X) \to X$ is the identity on states and the projection on the $(\epsilon + 1)$-th component $L \times_\Sigma L \to L$ on actions by Proposition 7.2. Consider a diagram of solid arrows of cubical transition systems:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & \mathrm{Path}(X) \arrow[d, "\pi^\epsilon"] \\ B \arrow[r, "\psi"] \arrow[ur, dashed, "\ell"] & X. \end{tikzcd}$$

Since the right vertical map is onto on actions and the left vertical map is one-to-one on actions, there exists a set map $\widetilde{\ell} : L_B \to L \times_\Sigma L$ from the set of actions of B to the set of actions of $\mathrm{Path}(X)$ such that the following diagram of sets is commutative, L_A being the set of actions of A (note that $\widetilde{\pi^\epsilon}$ is the projection on the $(\epsilon + 1)$-th component):

$$\begin{tikzcd} L_A \arrow[r, "\phi"] \arrow[d, "f"'] & L \times_\Sigma L \arrow[d, "\pi^\epsilon"] \\ L_B \arrow[r, "\psi"] \arrow[ur, dashed, "\widetilde{\ell}"] & L. \end{tikzcd}$$

Let $\ell : B \to \mathrm{Path}(X)$ defined on states by $\ell(\alpha) = \psi(\alpha)$ and on actions by $\ell(u) = \widetilde{\ell}(u)$. The diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & \mathrm{Path}(X) \arrow[d, "\pi^\epsilon"] \\ B \arrow[r, "\psi"] \arrow[ur, "\ell"] & X \end{tikzcd}$$

is commutative since its right vertical map is the identity on states. It just remains to prove that $\ell : B \to \mathrm{Path}(X)$ is a well-defined map of cubical transition systems. Let $(\alpha, u_1, \dots, u_n, \beta)$ be a transition of B. It suffices to prove that the tuple

$$(\alpha, \widetilde{\ell}(u_1), \dots, \widetilde{\ell}(u_n), \beta)$$

is a transition of $\mathrm{Path}(X)$ to complete the proof. Without lack of generality, we can suppose that $\epsilon = 0$, which means that $\widetilde{\ell}(u) = (\psi(u), \chi(u))$. One obtains

$$(\alpha, \widetilde{\ell}(u_1), \dots, \widetilde{\ell}(u_n), \beta) = (\alpha, (\psi(u_1), \chi(u_1)), \dots, (\psi(u_n), \chi(u_n)), \beta).$$

Since ψ maps the transitions of B to transitions of X, the tuple

$$(\alpha, \psi(u_1), \dots, \psi(u_n), \beta)$$

is a transition of X. Since $\mu(\psi(u)) = \mu(u) = \mu(\chi(u))$ for all actions u of B, and since X is combinatorially fibrant, the tuple

$$(\alpha, (\psi(u_1), \chi(u_1)), \dots, (\psi(u_n), \chi(u_n)), \beta)$$

is then a transition of $\mathrm{Path}(X)$ by Proposition 7.2. □

Proposition 7.5. *Let X be a cubical transition system. If X is combinatorially fibrant, then it is injective with respect to the maps of $\mathcal{S}^{cof}$.*

Proof. Let $x \in \Sigma$. Consider a diagram of solid arrows of cubical transition systems

$$\begin{tikzcd} C_1[x] \sqcup C_1[x] \arrow[r, "\phi"] \arrow[d, "p_x^{cof}"'] & X \\ Z_x^{x_1,x_2}, \arrow[ur, dashed, "\ell"'] & \end{tikzcd}$$

where x_1 and x_2 are the two actions of $C_1[x] \sqcup C_1[x]$ and where $Z_x^{x_1,x_2}$ is the cubical transition system depicted in Figure 2. Define ℓ on states by $\ell(\alpha) = \phi(\alpha)$, and on actions by $\ell(x_i) = \phi(x_i)$ for $i = 1, 2$ and $\ell(x) = \phi(x_1)$. Let $(\alpha_i, \phi(x_i), \beta_i)$ for $i = 1, 2$ be the images by ϕ of the two transitions of $C_1[x] \sqcup C_1[x]$. Since X is combinatorially fibrant, the two triples $(\alpha_i, \phi(x_{3-i}), \beta_i)$ for $i = 1, 2$ are two transitions of X. The map ℓ is therefore a well-defined map of cubical transition systems. □

Proposition 7.6. *Let $X = (S, \mu : L \to \Sigma, T)$ and $X' = (S', \mu' : L' \to \Sigma, T')$ be two cubical transition systems. The binary product $X \times X'$ has $S \times S'$ as its set of states, $L \times_\Sigma L' = \{(x, x') \in L \times L', \mu(x) = \mu'(x')\}$ as its set of actions and the labelling map $\mu \times_\Sigma \mu' : L \times_\Sigma L' \to \Sigma$. A tuple $((\alpha, \alpha'), (u_1, u'_1), \ldots, (u_n, u'_n), (\beta, \beta'))$ is a transition of $X \times X'$ if and only if $\mu(u_i) = \mu'(u'_i)$ for $1 \leq i \leq n$ with $n \geq 1$, the tuple $(\alpha, u_1, \ldots, u_n, \beta)$ is a transition of X and $(\alpha', u'_1, \ldots, u'_n, \beta')$ a transition of X'.*

Proof. The binary product is the same in $\mathcal{CTS}$ and in $\mathcal{WTS}$ because $\mathcal{CTS}$ is a small-injectivity class of $\mathcal{WTS}$. The theorem is then a consequence of [8, Proposition 5.5]. $\square$

Proposition 7.7. *Let X be a cubical transition system. If X is combinatorially fibrant, then it is injective with respect to any map of the form $f \star \gamma$ for any map of cubical transition systems f which is onto on states.*

Proof. Let $f : A \to B$ be a map of cubical transition systems. By adjunction, the cubical transition system X is injective with respect to $f \star \gamma$ if and only if the map $\pi : \mathrm{Path}(X) \to X \times X$ satisfies the RLP with respect to f. Consider a diagram of solid arrows of cubical transition systems:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & \mathrm{Path}(X) \arrow[d, "\pi"] \\ B \arrow[r, "\psi=(\psi_0,\psi_1)"'] \arrow[ur, dashed, "\ell"] & X \times X. \end{tikzcd}$$

Since the set map $f : A^0 \to B^0$ is onto by hypothesis, for any state α of B, there exists $s(\alpha) \in A^0$ such that $f(s(\alpha)) = \alpha$. Let $\ell : B \to \mathrm{Path}(X)$ defined on states by $\ell(\alpha) = \phi(s(\alpha))$ and on actions by $\ell(u) = \psi(u)$ (since X is combinatorially fibrant, the map $\pi : \mathrm{Path}(X) \to X \times X$ is the identity on actions by Proposition 7.2). We are going to prove that ℓ is a well-defined map of cubical transition systems and that it is a lift of the diagram above.

<u>ℓ is a lift for the sets of actions.</u> One has the following diagram of solid

arrows between the sets of actions:

$$\begin{tikzcd}
L_A \arrow[r, "\phi"] \arrow[d, "f"'] & L_X \times_\Sigma L_X \arrow[d, equal] \\
L_B \arrow[r, "\psi"] \arrow[ur, dashed, "\psi"] & L_X \times_\Sigma L_X.
\end{tikzcd}$$

It is evident that the two triangles commute since the square of solid arrows commutes.

<u>ℓ is a lift for the sets of states.</u> One has the diagram of solid arrows between the sets of states:

$$\begin{tikzcd}
A^0 \arrow[r, "\phi"] \arrow[d, "f"] & X^0 \arrow[d, "\Delta"] \\
B^0 \arrow[r, "\psi=(\psi_0,\psi_1)"] \arrow[u, dashed, bend left, "s"] \arrow[ur, dashed, "\phi\circ s"] & X^0 \times X^0,
\end{tikzcd}$$

where $\Delta : s \mapsto (s, s)$ is the codiagonal map. For any state β of B^0, one has

$$\begin{aligned}
&\psi(\beta) \\
&= \psi(f(s(\beta))) && \text{since } s \text{ is a section of } f \\
&= \pi(\phi(s(\beta))) && \text{since } \psi \circ f = \pi \circ \phi \\
&= (\phi(s(\beta)), \phi(s(\beta))) && \text{by Proposition 7.6.}
\end{aligned}$$

Hence we obtain $\psi_0 = \psi_1 = \phi \circ s$ on states, and therefore $\Delta \circ \phi \circ s = \psi$ on states. We deduce that the bottom triangle commutes on states. For any state α of A^0, one has

$$\begin{aligned}
&\Delta(\phi(s(f(\alpha)))) \\
&= \psi(f(s(f(\alpha)))) && \text{since } \Delta \circ \phi = \psi \circ f \\
&= \psi(f(\alpha)) && \text{since } s \text{ is a section of } f \\
&= \Delta(\phi(\alpha)) && \text{because the square above is commutative.}
\end{aligned}$$

Hence we obtain $\phi \circ s \circ f = \phi$ on states. We obtain that the top triangle commutes.

ℓ maps a transition of B to a transition of $\mathrm{Path}(X)$. Let

$$(\alpha, u_1, \dots, u_n, \beta)$$

be a transition of B. Then one has

$$\begin{aligned}(\ell(\alpha), \ell(u_1), \dots, \ell(u_n), \ell(\beta)) &= (\phi(s(\alpha)), \psi(u_1), \dots, \psi(u_n), \phi(s(\beta))) \\ &= (\psi_0(\alpha), \psi(u_1), \dots, \psi(u_n), \psi_0(\beta)).\end{aligned}$$

The tuple $(\psi_0(\alpha), \psi_0(u_1), \dots, \psi_0(u_n), \psi_0(\beta))$ is a transition of X since it is the image by the composite map of cubical transition systems $\psi_0 : B \to X \times X \to X$ of the transition $(\alpha, u_1, \dots, u_n, \beta)$ of B. Therefore by Proposition 7.2 applied with $\epsilon_1 = \dots = \epsilon_n = 0$, the tuple

$$(\ell(\alpha), \ell(u_1), \dots, \ell(u_n), \ell(\beta))$$

is a transition of $\mathrm{Path}(X)$ since X is combinatorially fibrant. This means that

$$\ell : B \longrightarrow \mathrm{Path}(X)$$

is a well-defined map of cubical transition systems. □

Proposition 7.8. *Let X be a cubical transition system. If X is combinatorially fibrant, then it is injective with respect to any map of the form $(f \star \gamma) \star \gamma$ for any map of cubical transition systems f.*

Proof. Let $f : A \to B$ be a map of cubical transition systems. The map $f \star \gamma$ goes from $(B \sqcup B) \sqcup_{A \sqcup A} \mathrm{Cyl}(A)$ to $\mathrm{Cyl}(B)$. Since the forgetful functor from $\mathcal{CTS}$ to $\mathbf{Set}$ taking a cubical transition system to its underlying set of states is colimit-preserving, the set of states of the source of $f \star \gamma$ is $B^0 \sqcup_{A^0} B^0$. Hence the map $f \star \gamma$ is onto on states. Then by Proposition 7.7, X is injective with respect to $(f \star \gamma) \star \gamma$. □

Notation 7.9. Let I and S be two sets of maps of a locally presentable category $\mathcal{K}$. Let $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ be a cylinder. Denote by $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I)$ the set of maps defined as follows:

- $\Lambda^0_{\mathcal{K}}(\mathrm{Cyl}, S, I) = S \cup (I \star \gamma^0) \cup (I \star \gamma^1)$

- $\Lambda^{n+1}_{\mathcal{K}}(\mathrm{Cyl}, S, I) = \Lambda^n_{\mathcal{K}}(\mathrm{Cyl}, S, I) \star \gamma$
- $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I) = \bigcup_{n\geq 0} \Lambda^n_{\mathcal{K}}(\mathrm{Cyl}, S, I)$.

Theorem 7.10. *Let X be a cubical transition system. If X is combinatorially fibrant, then it is fibrant.*

Proof. Let X be a combinatorially fibrant cubical transition system. By Proposition 7.4 and Proposition 7.5, it is $\Lambda^0(\mathrm{Cyl}, \mathcal{S}^{cof}, \mathcal{I})$-injective. Let $f : A \to B$ be a map of cubical transition systems. Let $\epsilon \in \{0, 1\}$. The map $f \star \gamma^\epsilon$ goes from $B \sqcup_A \mathrm{Cyl}(A)$ to $\mathrm{Cyl}(B)$. Since the forgetful functor from $\mathcal{CTS}$ to $\mathbf{Set}$ taking a cubical transition system to its underlying set of states is colimit-preserving, the set of states of the source of $f \star \gamma^\epsilon$ is B^0. Hence $f \star \gamma^\epsilon$ is bijective on states. Therefore all maps of $\Lambda^0(\mathrm{Cyl}, \mathcal{S}^{cof}, \mathcal{I})$ are bijective on states. Then, by Proposition 7.7, X is $\Lambda^1(\mathrm{Cyl}, \mathcal{S}^{cof}, \mathcal{I})$-injective. The cubical transition system X is $\Lambda^n(\mathrm{Cyl}, \mathcal{S}^{cof}, \mathcal{I})$-injective for all $n \geq 2$ by Proposition 7.8. Hence X is fibrant in the Bousfield localization of $\mathcal{CTS}$ by the cofibrations of $\mathcal{S}^{cof}$ by [8, Corollary 6.8] and [10, Theorem 4.6]. But Bousfield localizing by $\mathcal{S}^{cof}$ is the same as Bousfield localizing by $\mathcal{S}$, which is the same as Bousfield localizing by the cubification functor. Hence the proof is complete. $\square$

Notation 7.11. Let $x \in \Sigma$. The two maps from $C_1[x]$ to $\uparrow x \uparrow$ are denoted by c_x^ϵ for $\epsilon = 0, 1$. One has $p_x = c_x^0 \sqcup c_x^1$ for all $x \in \Sigma$.

Proposition 7.12. *Let $x \in \Sigma$. Consider the pushout diagram of $\mathcal{CTS}$*

$$\begin{tikzcd}
C_1[x] \arrow[r, "c_x^0"] \arrow[d, "\gamma^0_{C_1[x]}"'] & \uparrow x \uparrow \arrow[d] \\
\mathrm{Cyl}(C_1[x]) \arrow[r] & \mathrm{Cyl}(C_1[x]) \sqcup_{0,0} \uparrow x \uparrow \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The composite

$$\theta_x : C_1[x] \sqcup C_1[x] \xrightarrow{\gamma^1_{C_1[x]} \sqcup c_x^1} \mathrm{Cyl}(C_1[x]) \sqcup \uparrow x \uparrow \longrightarrow \mathrm{Cyl}(C_1[x]) \sqcup_{0,0} \uparrow x \uparrow$$

is a trivial cofibration of $\underline{\mathbf{L}_{\mathrm{Cub}}}\, \mathcal{CTS}$.

$$\left\{\begin{array}{c} C_1[x] \sqcup C_1[x] \\ \alpha \xrightarrow{x_1} \beta \\ \\ \gamma \xrightarrow{x_2} \delta \end{array}\right. \xrightarrow{\theta_x} \left\{\begin{array}{c} \mathrm{Cyl}(C_1[x]) \sqcup_{0,0} \uparrow x \uparrow \\ \begin{tikzcd} \alpha \arrow[r, bend left, "x_1"] \arrow[r, bend right, "x_2"'] & \beta \end{tikzcd} \\ \\ \gamma \xrightarrow{x_2} \delta \end{array}\right.$$

Figure 3: Cofibration θ_x with $\mu(x_1) = \mu(x_2) = x$

Proof. The map θ_x is depicted in Figure 3. It is bijective on actions, therefore it is a cofibration. One has $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(C_1[x] \sqcup C_1[x]) \cong \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\mathrm{Cyl}(C_1[x]) \sqcup_{0,0} \uparrow x \uparrow) \cong \uparrow x \uparrow$. Hence it is a weak equivalence of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS}$ by [8, Theorem 8.10]. $\square$

Proposition 7.13. *In the following, the notation $\sqcup_{\substack{0_n = 0_n \\ 1_n = 1_n}}$ means the identification of the initial states (the final states resp.) of the two summands. Let $n \geq 2$ and $x_1, \dots, x_n \in \Sigma$. Then the map*

$$\eta_{x_1,\dots,x_n} : \partial C_n[x_1, \dots, x_n] \sqcup_{\substack{0_n = 0_n \\ 1_n = 1_n}} C_n[x_1, \dots, x_n] \longrightarrow C_n[x_1, \dots, x_n] \sqcup_{\substack{0_n = 0_n \\ 1_n = 1_n}} C_n[x_1, \dots, x_n]$$

induced by the inclusion $\partial C_n[x_1, \dots, x_n] \subset C_n[x_1, \dots, x_n]$ is a trivial cofibration of $\underline{\mathbf{L}_{\mathrm{Cub}}}\,\mathcal{CTS}$.

Proof. The map $\eta_{x_1,\dots,x_n}$ is bijective on actions: the set of actions is

$$\{(x_1, 1), \dots, (x_n, n)\} \times \{0, 1\},$$

with for example 0 for the left-hand term and 1 for the right-hand term. Hence it is a cofibration. The map $\eta_{x_1,\dots,x_n}$ is also bijective on states: the set of states is a set denoted by $\{0,1\}^n \sqcup_{\substack{0_n = 0_n \\ 1_n = 1_n}} \{0,1\}^n$, which means the quotient of the coproduct $\{0,1\}^n \sqcup \{0,1\}^n$ by the identifications of 0_n (1_n resp.) of the left-hand term with 0_n (1_n resp.) of the right-hand term. Since the map $X \to \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)$ is bijective on states for all cubical transition systems

X, the map of cubical transition systems

$$\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\eta_{x_1,\dots,x_n}) : \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}\left(\partial C_n[x_1,\dots,x_n] \sqcup_{\substack{0_n = 0_n \\ 1_n = 1_n}} C_n[x_1,\dots,x_n]\right) \longrightarrow \underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}\left(C_n[x_1,\dots,x_n] \sqcup_{\substack{0_n = 0_n \\ 1_n = 1_n}} C_n[x_1,\dots,x_n]\right)$$

is bijective on states as well. The set of actions of the source and target of $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\eta_{x_1,\dots,x_n})$ is $\{x_1,\dots,x_n\}$. Since $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\eta_{x_1,\dots,x_n})$ is one-to-one on action by [8, Remark 8.8], it is bijective on actions. By [10, Proposition 4.4], the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\eta_{x_1,\dots,x_n})$ is one-to-one on transitions by. To see that the map $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\eta_{x_1,\dots,x_n})$ is also onto on transitions, it suffice to see that the $n!$ n-transitions of the left-hand n-cube of the target are the $n!$ tuples $(0_n, x_{\sigma(1)},\dots,x_{\sigma(n)}, 1_n)$ which are actually transitions of the source because of the identifications of the two initial states and the two final states. So $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(\eta_{x_1,\dots,x_n})$ is an isomorphism. Therefore by [8, Theorem 8.10], the map $\eta_{x_1,\dots,x_n}$ is a weak equivalence of $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{CTS}$. □

Proposition 7.14. *A cubical transition system is combinatorially fibrant if and only if it is injective with respect to θ_x and $\eta_{x_1,\dots,x_n}$ for all $x, x_1,\dots,x_n \in \Sigma$.*

Proof. Let X a combinatorially fibrant cubical transition system. Then X is fibrant by Theorem 7.10. Since the maps θ_x and $\eta_{x_1,\dots,x_n}$ for all

$$x, x_1,\dots,x_n \in \Sigma$$

are trivial cofibrations by Proposition 7.12 and Proposition 7.13, X is injective with respect to these maps. Conversely, let X be a cubical transition system which is injective with respect to θ_x and $\eta_{x_1,\dots,x_n}$ for all $x, x_1,\dots,x_n \in \Sigma$. Let (α, x_1, β) be a transition of X and let x_2 an action of X such that $\mu(x_1) = \mu(x_2)$. The injectivity of X with respect to $\theta_{\mu(x_1)}$ proves that the triple (α, x_2, β) is a transition of X. Let $(\alpha, x_1,\dots,x_n,\beta)$ be a transition of X with $n \geq 2$. Let $y_1,\dots,y_n$ be n actions of X with $\mu(x_i) = \mu(y_i)$ for $1 \leq i \leq n$. The injectivity of X with respect to $\eta_{\mu(x_1),\dots,\mu(x_n)}$ proves that the triple $(\alpha, y_1,\dots,y_n,\beta)$ is a transition of X. So, X is combinatorially fibrant. □

Corollary 7.15. *Let X be a cubical transition system. If X is fibrant, then it is combinatorially fibrant.*

Proof. Let X be a fibrant cubical transition system. Then it is injective with respect to any trivial cofibration of $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{CTS}$. By Proposition 7.12, Proposition 7.13 and Proposition 7.14, it is then combinatorially fibrant. $\square$

Corollary 7.16. *A cubical transition system X is fibrant in $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{CTS}$ if and only it is combinatorially fibrant.*

Corollary 7.17. *Every $\mathcal{S}$-injective cubical transition system is fibrant in $\underline{\mathbf{L}}_{\mathrm{Cub}}\,\mathcal{CTS}$.*

Proof. Let $(\alpha, u_1, \dots, u_n, \beta)$ and $(\alpha, v_1, \dots, v_n, \beta)$ as in the statement of Theorem 7.16. Since X is $\mathcal{S}$-injective, the labelling map μ is one-to-one by Proposition 6.2. Therefore $u_i = v_i$ for $1 \leq i \leq n$. $\square$

In particular, all cubical transition systems of the form $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{CTS}}(X)$ and all regular transition systems of the form $\underline{\mathbf{L}}_{\mathcal{S}}^{\mathcal{RTS}}(X)$ are fibrant because they are $\mathcal{S}$-injective.

A. Proof of Proposition 6.12

Proposition A.1. *Let $x \in \Sigma$. Every pushout of $p_x : C_1[x] \sqcup C_1[x] \to \uparrow x \uparrow$ in $\mathcal{CTS}$ is bijective on states, and onto on actions. There exists a pushout of p_x which is not onto on transitions.*

Proof. The category $\mathcal{CTS}$ is a full coreflective category of $\mathcal{WTS}$, which means that the colimits in $\mathcal{CTS}$ are calculated in $\mathcal{WTS}$. Therefore the forgetful functors taking a cubical transition system to their sets of states and actions are colimit-preserving. Since p_x is bijective on states (onto on actions resp.), any pushout of p_x in $\mathcal{CTS}$ is therefore bijective on states (onto on actions resp.).

Let $x \in \Sigma$. One has $\omega(C_3[x,x,x]) = (\{0,1\}^3, \{(x,1),(x,2),(x,3)\})$ by Proposition 2.6. Consider the quotient set

$$S = \{0,1\}^3 \times \{-,+\} / \left((0,0,0,-) = (0,0,0,+) = I \right. \\ \left. \text{and } (1,1,1,-) = (1,1,1,+) = F\right).$$

Let

$$W = (S, \{u_1, u^0, u^1, u_3\}) \in \mathbf{Set}^{\{s\}\cup\Sigma}$$

with $\mu(u_1) = \mu(u^0) = \mu(u^1) = \mu(u_3) = x$. For $\alpha \in \{-,+\}$, consider the map

$$\phi^\alpha : \omega(C_3[x,x,x]) \to W$$

of $\mathbf{Set}^{\{s\}\cup\Sigma}$ induced by the mappings $(\epsilon_1, \epsilon_2, \epsilon_3) \mapsto (\epsilon_1, \epsilon_2, \epsilon_3, \alpha)$ for

$$(\epsilon_1, \epsilon_2, \epsilon_3) \in \{0,1\}^3, (x,1) \mapsto u_1, (x,2) \mapsto u^\alpha, (x,3) \mapsto u_3.$$

Consider the ω-final lift $\overline{W}$ of the cone of maps

$$\phi^-, \phi^+ : \omega(C_3[x,x,x]) \rightrightarrows W.$$

By Theorem 3.3, the weak transition system $\overline{W}$ is cubical. Finally, consider the pushout diagram of cubical transition systems:

$$\begin{tikzcd}
C_1[x] \sqcup C_1[x] \arrow[r] \arrow[d, "p_x"'] & \overline{W} \arrow[d] \\
\uparrow x \uparrow \arrow[r] & \overline{\overline{W}} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the top horizontal arrow sends the 1-transition $(0,(x,1),1)$ of the left-hand copy of $C_1[x]$ to $((1,0,0,-), u^-, (1,1,0,-))$ and the 1-transition $(0,(x,1),1)$ of the right-hand copy of $C_1[x]$ to $((1,0,0,+), u^+, (1,1,0,+))$. We claim that the map of cubical transition systems

$$\overline{W} \longrightarrow \overline{\overline{W}}$$

is not surjective on transitions. Indeed $\overline{W}$ contains the transitions

$$(I, u_1, u^\alpha, u_3, F)$$

for $\alpha \in \{-,+\}$, and the four transitions

$$(I, u_1, (1,0,0,-)), ((1,0,0,-), u^-, u_3, F),$$
$$(I, u_1, u^+, (1,1,0,+)), ((1,1,0,+), u_3, F).$$

The cubical transition system $\overline{W}$ does not contain any transition from

$$(1,0,0,-)$$

to

$$(1,1,0,+).$$

In the pushout $\overline{\overline{\overline{W}}}$, the identification $u^- = u^+$ is made. Therefore from the five preceding transitions, one obtains by using the composition axiom a transition $((1,0,0,-),u^-,(1,1,0,+))$. □

Proposition A.2. *Every map of* $\mathbf{cell}_{\mathcal{CTS}}(\mathcal{S})$ *is bijective on states and onto on actions. There exists a map of* $\mathbf{cell}_{\mathcal{CTS}}(\mathcal{S})$ *which is not onto on transitions.*

Proof. A map of cubical transition systems is onto on actions if and only if it satisfies the RLP with respect to the maps $\varnothing \to \underline{x}$ for any $x \in \Sigma$. As a consequence, the class of maps of cubical transition systems which are onto on actions is accessible and accessibly embedded in the category of maps of cubical transition systems by [19, Proposition 3.3]. Hence any map of $\mathbf{cell}_{\mathcal{CTS}}(\mathcal{S})$ is onto on actions. All maps of $\mathcal{S}$ are bijective on states. Since the state set functor from $\mathcal{CTS}$ to $\mathbf{Set}$ is colimit-preserving, all maps of $\mathbf{cell}_{\mathcal{CTS}}(\mathcal{S})$ are bijective on states. The last assertion is a corollary of Proposition A.1. □

Proposition A.3. *Let* $x \in \Sigma$. *Every pushout of* $p_x : C_1[x] \sqcup C_1[x] \to \uparrow\! x\! \uparrow$ *in* $\mathcal{RTS}$ *is onto on states, on actions and on transitions.*

Proof. Consider a pushout diagram in $\mathcal{RTS}$:

$$\begin{tikzcd}
C_1[x] \sqcup C_1[x] \arrow[r] \arrow[d, "p_x"'] & X \arrow[d, "f"] \\
\uparrow\! x\! \uparrow \arrow[r] & X'. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The category $\mathcal{RTS}$ is a full reflective subcategory of $\mathcal{CTS}$. Therefore a colimit in $\mathcal{RTS}$ is calculated by taking the image by the reflection $\mathrm{CSA}_2 : \mathcal{CTS} \to \mathcal{RTS}$ of the colimit in $\mathcal{CTS}$. The canonical map $Z \to \mathrm{CSA}_2(Z)$

is onto on states and bijective on actions for all cubical transition systems Z by Proposition 4.2. Therefore by Proposition A.1, the map f is onto both on states and on actions. Let $X = (S, \mu : L \to \Sigma, T)$ and $X' = (S', \mu : L' \to \Sigma, T')$. Write $f(T)$ for the set of transitions of X' of the form $(f(\alpha), f(u_1), \dots, f(u_n), f(\beta))$ such that the tuple $(\alpha, u_1, \dots, u_n, \beta)$ belongs to T. One has $f(T) \subset T'$. Let $f(u)$ be an action of X'. Then there exists a transition (α, u, β) of X since X is cubical. Therefore the tuple $(f(\alpha), f(u), f(\beta))$ belongs to $f(T)$. This means that all actions of X' are used by a transition of $f(T)$. Let $(f(\alpha), f(u_1), \dots, f(u_n), f(\beta))$ be a transition of $f(T)$. Then $(\alpha, u_{\sigma(1)}, \dots, u_{\sigma(n)}, \beta)$ is a transition of X for all permutations σ of $\{1, \dots, n\}$. So the tuple $(f(\alpha), f(u_{\sigma(1)}), \dots, f(u_{\sigma(n)}), f(\beta))$ is a transition of $f(T)$. Let $n \geq 3$ and $p, q \geq 1$ with $p + q < n$. Let

$$\begin{gathered}(\alpha, u_1, \dots, u_n, \beta), (\alpha, u_1, \dots, u_p, \mu), (\mu, u_{p+1}, \dots, u_n, \beta), \\ (\alpha, u_1, \dots, u_{p+q}, \nu), (\nu, u_{p+q+1}, \dots, u_n, \beta)\end{gathered}$$

be five transitions of $f(T)$. Let

$$(\alpha, u_1, \dots, u_n, \beta) = (f(\gamma), f(v_1), \dots, f(v_n), f(\delta)).$$

There exist two states ϵ and η of X such that the five tuples

$$\begin{gathered}(\gamma, v_1, \dots, v_p, \epsilon), (\gamma, v_1, \dots, v_{p+q}, \eta), (\epsilon, v_{p+1}, \dots, v_n, \delta), \\ (\eta, v_{p+q+1}, \dots, v_n, \delta), (\epsilon, v_{p+1}, \dots, v_{p+q}, \eta)\end{gathered}$$

are transitions of X since X is cubical and by using the composition axiom in X. Therefore, the five tuples

$$\begin{gathered}(f(\gamma), f(v_1), \dots, f(v_p), f(\epsilon)), (f(\gamma), f(v_1), \dots, f(v_{p+q}), f(\eta)), \\ (f(\epsilon), f(v_{p+1}), \dots, f(v_n), f(\delta)), (f(\eta), f(v_{p+q+1}), \dots, f(v_n), f(\delta)), \\ (f(\epsilon), f(v_{p+1}), \dots, f(v_{p+q}), f(\eta))\end{gathered}$$

are transitions of $f(T)$. So the five tuples

$$\begin{gathered}(\alpha, u_1, \dots, u_p, f(\epsilon)), (\alpha, u_1, \dots, u_{p+q}, f(\eta)), \\ (f(\epsilon), u_{p+1}, \dots, u_n, \beta), (f(\eta), u_{p+q+1}, \dots, u_n, \beta), \\ (f(\epsilon), u_{p+1}, \dots, u_{p+q}, f(\eta))\end{gathered}$$

are transitions of $f(T)$. The point is that X' is regular. One deduces $f(\epsilon) = \mu$ and $f(\eta) = \nu$. One obtains

$$(\mu, u_{p+1}, \dots, u_{p+q}, \nu) = (f(\epsilon), f(v_{p+1}), \dots, f(v_{p+q}), f(\eta)) \in f(T).$$

Let $n \geq 2$ and $1 \leq p < n$. Let $(f(\alpha), f(u_1), \dots, f(u_n), f(\beta))$ be a transition of $f(T)$. Since X is cubical, there exists a state μ such that $(\alpha, u_1, \dots, u_p, \mu)$ and $(\mu, u_{p+1}, \dots, u_n, \beta)$ are two transitions of X. Since X' is cubical, there exists a state ν of X' such that $(f(\alpha), f(u_1), \dots, f(u_p), \nu)$ and $(\nu, f(u_{p+1}), \dots, f(u_n), f(\beta))$ are transitions of X'. Since X' is regular, one has $f(\mu) = \nu$. Therefore

$$(f(\alpha), f(u_1), \dots, f(u_p), \nu)$$

and

$$(\nu, f(u_{p+1}), \dots, f(u_n), f(\beta))$$

belong to $f(T)$. We have proved that the tuple $Y = (S', L' \to \Sigma, f(T))$ is a regular transition system. The map $X \to X'$ factors uniquely as a composite $X \to Y \to X'$. The map $\uparrow x \uparrow \to X'$ factors uniquely as a composite $\uparrow x \uparrow \to Y \to X'$. By the universal property of the pushout, one obtains $X' = Y$ and $T' = f(T)$. □

Proposition A.4. *A map of regular transition systems is onto on states if and only if it satisfies the RLP with respect to the map $\varnothing \to \{0\}$. The class of maps of regular transition systems which are onto on states is accessible and accessibly embedded in the category of maps of regular transition systems.*

Proof. The first assertion is obvious. The second assertion is then a consequence of [19, Proposition 3.3]. □

Proposition A.5. *A map of regular transition systems is onto on transitions if and only if it satisfies the RLP with respect to the maps $\varnothing \to C_n[x_1, \dots, x_n]$ for $n \geq 1$ and $x_1, \dots, x_n \in \Sigma$. The class of maps of regular transition systems which are onto on transitions is accessible and accessibly embedded in the category of maps of regular transition systems.*

Proof. Let $f : X \to Y$ be a map of regular transition systems which is onto on transitions. Consider a commutative diagram of weak transition systems with X and Y regular:

$$\begin{tikzcd}
& C_n[x_1,\dots,x_n] \arrow[r, "k_2"] & X \arrow[d, "f"] \\
C_n[x_1,\dots,x_n]^{ext} \arrow[r, "\subset"] \arrow[ur, dashed, "k_1"] \arrow[urr, dashed, "\ell"] & C_n[x_1,\dots,x_n] \arrow[r, "\phi"] & Y.
\end{tikzcd}$$

The lift ℓ exists since the map $f : X \to Y$ is onto on transitions by hypothesis. Since X is cubical, the map $\ell : C_n[x_1,\dots,x_n]^{ext} \to X$ factors as a composite

$$\ell : C_n[x_1,\dots,x_n]^{ext} \xrightarrow{k_1} C_n[x_1,\dots,x_n] \xrightarrow{k_2} X.$$

The point is that Y is regular. Thus, Y is orthogonal to the inclusion

$$C_n[x_1,\dots,x_n]^{ext} \subset C_n[x_1,\dots,x_n]$$

by [7, Theorem 5.6]. Therefore k_1 is the inclusion $C_n[x_1,\dots,x_n]^{ext} \subset C_n[x_1,\dots,x_n]$ and $\phi = f \circ k_2$. We deduce that f satisfies the RLP with respect to the maps $\varnothing \to C_n[x_1,\dots,x_n]$ for $n \geq 1$ and $x_1,\dots,x_n \in \Sigma$.

Conversely, let us suppose that $f : X \to Y$ is a map of regular transition systems which satisfies the RLP with respect to the maps

$$\varnothing \to C_n[x_1,\dots,x_n]$$

for $n \geq 1$ and $x_1,\dots,x_n \in \Sigma$. Let $(\alpha, u_1,\dots,u_n,\beta)$ be a transition of Y. It yields a map $C_n[\mu(u_1),\dots,\mu(u_n)]^{ext} \to Y$. Since Y is cubical, this map factors as a composite $C_n[\mu(u_1),\dots,\mu(u_n)]^{ext} \subset C_n[\mu(u_1),\dots,\mu(u_n)] \to Y$. By hypothesis, the right-hand map factors as a composite

$$C_n[\mu(u_1),\dots,\mu(u_n)] \to X \xrightarrow{f} Y.$$

Thus, the map $C_n[\mu(u_1),\dots,\mu(u_n)]^{ext} \to Y$ factors as a composite

$$C_n[\mu(u_1),\dots,\mu(u_n)]^{ext} \to X \to Y.$$

Hence f is onto on transitions.

The last assertion is then a consequence of [19, Proposition 3.3]. $\square$

Proposition A.6. *Every map of* $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ *is onto on states, on actions and on transitions.*

Proof. A map of $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ is a transfinite composition of maps which are onto on states and on transitions by Proposition A.3. By Proposition A.4 and Proposition A.5, every map of $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$ is then onto on states and on transitions. Let $f : X \to Y$ be a map of $\mathbf{cell}_{\mathcal{RTS}}(\mathcal{S})$. Let u be an action of Y. Then there exists a transition (α, u, β) of Y. Consider the map $C_1[\mu(u)] \to Y$ taking the 1-transition of $C_1[\mu(u)]$ to (α, u, β). Then it factors as a composite $C_1[\mu(u)] \to X \to Y$. The image of the 1-transition of $C_1[\mu(u)]$ by the left-hand map yields a 1-transition (γ, v, δ) of X such that $(f(\gamma), f(v), f(\delta)) = (\alpha, u, \beta)$. Therefore $f(v) = u$ and f is onto on actions. □

B. Restricting an adjunction to a full reflective subcategory

The following proposition provides a tool to easily restrict the cylinder and the path functors of cubical transition systems to the reflective subcategory of regular ones. It is stated in a more general setting than the one of locally presentable categories.

Proposition B.1. *Let* $\mathcal{A} \subset \mathcal{K}$ *be two categories with* $\mathcal{A}$ *full and reflective. Let* $R : \mathcal{K} \to \mathcal{A}$ *be the reflection. Consider an adjunction* $F \dashv G : \mathcal{K} \to \mathcal{K}$. *Then the following conditions are equivalent:*

(i) $F(\mathcal{A}) \subset \mathcal{A}$ *and* $G(\mathcal{A}) \subset \mathcal{A}$.

(ii) There is a natural isomorphism $R(F(X)) \cong F(R(X))$ *for every* $X \in \mathcal{K}$.

If one of the two preceding conditions is satisfied, the restriction of F *to* $\mathcal{A}$ *is left adjoint to the restriction of* G *to* $\mathcal{A}$.

Proof. The last assertion easily follows from the sequence of isomorphisms

$$\mathcal{A}(F(A), B) \cong \mathcal{K}(F(A), B) \cong \mathcal{K}(A, G(B)) \cong \mathcal{A}(A, G(B))$$

for any $A, B \in \mathcal{A}$ and from the fact that $\mathcal{A}$ is a full subcategory of $\mathcal{K}$.

Let us prove now the implication $(i) \Rightarrow (ii)$. For any object X of $\mathcal{K}$ and any object A of $\mathcal{A}$, one has:

$$\begin{aligned}
&\mathcal{A}(R(F(X)), A) \\
&\cong \mathcal{K}(F(X), A) && \text{because } R \text{ is the left adjoint of } \mathcal{A} \subset \mathcal{K} \\
&\cong \mathcal{K}(X, G(A)) && \text{because } G \text{ is the right adjoint of } F \\
&\cong \mathcal{A}(R(X), G(A)) && \text{by adjunction and since } G(A) \in \mathcal{A} \\
&\cong \mathcal{K}(R(X), G(A)) && \text{because } \mathcal{A} \text{ is a full subcategory of } \mathcal{K} \\
&\cong \mathcal{K}(F(R(X)), A) && \text{because } G \text{ is the right adjoint of } F \\
&\cong \mathcal{A}(F(R(X)), A) && \text{because } \mathcal{A} \text{ is full in } \mathcal{K} \text{ and } F(\mathcal{A}) \subset \mathcal{A}.
\end{aligned}$$

By Yoneda applied in $\mathcal{A}$, one obtains the natural isomorphism $R(F(X)) \cong F(R(X))$.

Let us prove now the implication $(ii) \Rightarrow (i)$. Let A be an object of $\mathcal{A}$. Then the unit map $\eta_A : A \to R(A)$, which is an isomorphism since $A \in \mathcal{A}$, gives rise to the isomorphism $F(A) \cong F(R(A))$. By (ii), one then obtains the isomorphism $F(A) \cong R(F(A))$. Hence $F(A) \in \mathcal{A}$. We want to prove now that $G(A) \in \mathcal{A}$. One has the sequence of bijections

$$\begin{aligned}
&\mathcal{K}(G(A), G(A)) \\
&\cong \mathcal{K}(F(G(A)), A) && \text{because } G \text{ is the right adjoint of } F \\
&\cong \mathcal{A}(R(F(G(A)), A) && \text{because } R \text{ is the left adjoint of } \mathcal{A} \subset \mathcal{K} \\
&\cong \mathcal{K}(R(F(G(A)), A) && \text{because } \mathcal{A} \text{ is a full subcategory of } \mathcal{K} \\
&\cong \mathcal{K}(F(R(G(A)), A) && \text{because of } (ii) \\
&\cong \mathcal{K}(R(G(A)), G(A)) && \text{because } G \text{ is the right adjoint of } F.
\end{aligned}$$

This means that the identity of $G(A)$ factors as a composite

$$G(A) \stackrel{\eta_{G(A)}}{\longrightarrow} R(G(A)) \stackrel{r}{\longrightarrow} G(A),$$

i.e $r \circ \eta_{G(A)} = \mathrm{Id}_{G(A)}$. Hence $\eta_{G(A)}$ has a left inverse. We follow now the argument of [14]. By using the naturality of the unit $\eta : \mathrm{Id} \to R$, one obtains the commutative diagram

$$\begin{tikzcd} R(G(A)) \arrow[r, "r"] \arrow[d, "\eta_{R(G(A))}"'] & G(A) \arrow[d, "\eta_{G(A)}"'] \\ R(R(G(A)) \arrow[r, "Rr"] & R(G(A)). \end{tikzcd}$$

Since $r \circ \eta_{G(A)} = \mathrm{Id}_{G(A)}$, one has

$$Rr \circ R(\eta_{G(A)}) = R(r \circ \eta_{G(A)}) = R(\mathrm{Id}_{G(A)}) = \mathrm{Id}_{R(G(A))} .$$

For all objects Z of $\mathcal{K}$, the map $R(\eta_Z) : R(Z) \to R(R(Z))$ is an isomorphism by the universal property of the reflection R. With $Z = G(A)$, one obtains that $R(\eta_{G(A)})$ is an isomorphism. Therefore $Rr = R(\eta_{G(A)})^{-1}$ is an isomorphism. The map $\eta_{R(G(A))}$ is an isomorphism as well since $\eta_{R(G(A))} = R(\eta_{G(A)})$. Therefore

$$\eta_{G(A)} \circ \big(r \circ (Rr \circ \eta_{R(G(A))})^{-1}\big) = \mathrm{Id}_{R(G(A))} .$$

Hence $\eta_{G(A)}$ has a right inverse. Thus, $\eta_{G(A)} : G(A) \to R(G(A))$ is an isomorphism. Hence $G(A) \in \mathcal{A}$. $\square$

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