12- Conjunctive algebras

In the previous chapter, we studied disjunctive algebras, which we introduced as a result of the decomposition of the implication with a disjunction and a negation. In particular, we saw that this decomposition canonically corresponds to the $\mathrm{L}^{\parr}$ calculus, into which the λ-calculus can be embedded. Notably, the so-defined λ-calculus is equipped with a call-by-name evaluation strategy, as in the Krivine abstract machine for the λ_c-calculus. We showed that this correspondence has a direct algebraic counterpart, since disjunctive algebras are in fact particular cases of implicative algebras.

We shall now study the dual case of structures resulting of the decomposition of the arrow into primitive negations and conjunctions. We mentioned in particular that Girard's decomposition of the arrow in linear logic can be expressed in terms of the multiplicative law of conjunction, written $\otimes$, by:

$$A \to B \triangleq \neg(A \otimes \neg B)$$

The connective $\otimes$ is indeed related to the disjunction $\parr$ by duality through the laws $\neg(A \parr B) = \neg A \otimes \neg B$ and $\neg(A \otimes B) = \neg A \parr \neg B$. The typing rules for this connective in linear logic are given by:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma \mid A \otimes B \vdash \Delta} \qquad\qquad \frac{\Gamma \vdash A \mid \Delta \quad \Gamma \vdash B \mid \Delta}{\Gamma \vdash A \otimes B \mid \Delta}$$

which are again dual to the rules for the disjunction.

We shall now follow the same process as in the previous chapter, but with the conjunction $\otimes$ as a primitive connective. First, we will present $\mathrm{L}^{\otimes}$, the fragment of Munch-Maccagnoni's L calculus [126] which corresponds to the connectives $\neg$ and $\otimes$. We will observe that this fragment allows for the encoding of a call-by-value λ-calculus. Next, we will give the realizability interpretation *à la* Krivine for this calculus. Then, based on the structure of this realizability model, we will introduce the notion of conjunctive structure. We will show that these structures are dual to the disjunctive structures we formerly introduced. Again, we will show how to embed terms and contexts of $\mathrm{L}^{\otimes}$ into conjunctive structures. Finally, we will define the notion of a separator for conjunctive structures, leading to the definition of conjunctive algebras. We shall prove that any disjunctive algebra induces a conjunctive algebra by duality.

Unfortunately, we did not achieve to prove the converse, namely that disjunctive algebras could be obtained by duality from conjunctive algebras. In fact, beyond that, we are lacking some basic results to be able to manipulate elements of conjunctive structures in the same computational fashion as in implicative or disjunctive algebras. As a consequence, we do not prove that disjunctive algebras can be recovered from conjunctive algebras by duality. As such, our study of conjunctive algebras thus remains incomplete. We shall come back to this aspect in the conclusion of this chapter.

12.1 A call-by-value decomposition of the arrow

We begin with the presentation of the fragment of L induced by the positive connectives $\otimes, \neg^{+}$ and $, \exists$. Next we shall see the realizability interpretation it induces, with the purpose of justifying afterwards the definition of conjunctive structures. Again, since this calculus has a lot of similarities with the call-by-value $\lambda\mu\tilde{\mu}$-calculus (see Section 4.5) in addition to being dual to $L^{\parr}$, we shall try to be concise in this section.

12.1.1 The $L^{\otimes}$ calculus

The $L^{\otimes}$ calculus is thus a subsystem of L. It corresponds exactly to the restriction of L to its positive fragment induced by the connectives $\otimes, \neg$ and $, \exists$. The syntax of terms, contexts and commands is given by:

Contexts	e^-	$::=$	$\alpha \mid \mu(x,y).c \mid \mu[\alpha].c \mid \mu x.c$
Terms	t^+	$::=$	$x \mid (t,t) \mid [e] \mid \mu\alpha.c$
Commands	c	$::=$	$\langle t^+ \| e^- \rangle$

We write $\mathcal{T}_0$, $\mathcal{E}_0$, C_0 for the sets of closed terms, contexts and commands. In this framework, values are defined by:

Values $\quad V ::= x \mid (V,V) \mid [e^-]$

Observe in particular that any (negative) context is a value. We denote by $\mathcal{V}_0$ the set of closed values. The syntax is really close to the one of $L^{\parr}$ (it has the same constructors, but terms are now positive while contests are negative), we recall the meanings of the different constructions:

- (t^+, t^+) are pairs of positive terms;
- $\mu(x_1, x_2).c$, which binds the variables x_1, x_2, is the dual destructor;
- $[e^-]$ is a constructor for the negation, which allows us to embed a negative context intro a positive term;
- $\mu[x].c$, which binds the variable x, is the dual destructor;
- $\mu\alpha.c$ and $\mu x.c$ correspond respectively to $\mu\alpha$ and $\tilde{\mu}x$ in the $\lambda\mu\tilde{\mu}$-calculus.

Remark 12.1 (Notations)**.** As we explained in the previous chapter, in L [126] is considered a syntax where a notation $\bar{x}$ is used to distinguish between the positive variable x (that can appear in the left-member $\langle x|$ of a command) and the co-variable $\bar{x}$ (resp. in the right member $|x\rangle$ of a command). The positive variable that we write x is also written x in [126], while the negative co-variable α is denoted by $\bar{\alpha}$. ⌟

The reduction rules correspond to the intuition one could have from the syntax of the calculus: all destructors and binders reduce in front of the corresponding values, while pairs of terms are expanded if needed. The rules are given by:

$$\begin{array}{rcl} \langle \mu\alpha.c \| e \rangle & \to_\beta & c[e/\alpha] \\ \langle [e] \| \mu[\alpha].c \rangle & \to_\beta & c[e/\alpha] \\ \langle V \| \mu x.c \rangle & \to_\beta & c[V/x] \\ \langle (V,V') \| \mu(x,x').c \rangle & \to_\beta & c[V/x, V'/x'] \\ \langle (t,u) \| e \rangle & \to_\beta & \langle t \| \mu x.\langle u \| \mu y.\langle (x,y) \| e \rangle\rangle\rangle \end{array} \qquad \begin{array}{rcl} c & \to_\eta & \langle \mu\alpha.c \| \alpha \rangle \\ c & \to_\eta & \langle [\alpha] \| \mu[\alpha].c \rangle \\ c & \to_\eta & \langle x \| \mu x.c \rangle \\ c & \to_\eta & \langle (x_1,x_2) \| \mu(x_1,x_2).c \rangle \end{array}$$

where $(t,u) \notin V$ in the last β-reduction rule.

Lastly, we shall present the type system of $L^{\otimes}$. Second-order formulas are defined from the positive connectives by:

Formulas $\quad A, B := X \mid A \otimes B \mid \neg A \mid \exists X.A$

$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : A \vdash \Delta}{\langle t \| e \rangle : \Gamma \vdash \Delta}\ (\textsc{Cut})$$

$$\frac{(\alpha : A) \in \Delta}{\Gamma \mid \alpha : A \vdash \Delta}\ (ax\vdash) \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A \mid \Delta}\ (\vdash ax)$$

$$\frac{c : \Gamma \vdash \Delta, x : A}{\Gamma \mid \mu x.c : A \vdash \Delta}\ (\mu\vdash) \qquad \frac{c : \Gamma, \alpha : A \vdash \Delta}{\Gamma \vdash \mu\alpha.c : A \mid \Delta}\ (\vdash\mu)$$

$$\frac{c : (\Gamma, x : A, x' : B \vdash \Delta)}{\Gamma \mid \mu(x,x').c : A \otimes B \vdash \Delta}\ (\otimes\vdash) \qquad \frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \vdash u : B \mid \Delta}{\Gamma \vdash (t,u) : A \otimes B \mid \Delta}\ (\vdash\otimes)$$

$$\frac{c : \Gamma, x : A \vdash \Delta}{\Gamma \mid \mu[\alpha].c : \neg A}\ (\neg\vdash) \qquad \frac{\Gamma \mid e : A \vdash \Delta}{\Gamma \vdash [e] : \neg A \vdash \Delta}\ (\vdash\neg)$$

$$\frac{\Gamma \vdash e : A \mid \Delta \quad X \notin FV(\Gamma,\Delta)}{\Gamma \mid e : \exists X.A \vdash \Delta}\ (\exists_l) \qquad \frac{\Gamma \vdash V : A[B/X] \mid \Delta}{\Gamma \vdash V : \exists X.A}\ (\exists_r)$$

Figure 12.1: Typing rules for the $L^{\otimes}$-calculus

We still work with two-sided sequents, where typing contexts are defined as finite lists of bindings between variable and formulas:

$$\Gamma ::= \varepsilon \mid \Gamma, x : A \qquad\qquad \Delta ::= \varepsilon \mid \Delta, \alpha : A$$

Sequents are again of three kinds, as in the $\lambda\mu\tilde{\mu}$-calculus and $L^{⅋}$:

- $\Gamma \vdash t : A \mid \Delta$ for typing terms,
- $\Gamma \mid e : A \vdash \Delta$ for typing contexts,
- $c : \Gamma \vdash \Delta$ for typing commands.

The type system is given in Figure 12.1, where each connective corresponds to a left and a right rule.

Remark 12.2 (Existential quantifier)**.** As in the type system of $L^{⅋}$, we do not associate the existential quantifier to a constructor. Indeed, since our primary motivation is the definition of conjunctive structures, in which this quantifier will simply be expressed by arbitrary joins, it would be irrelevant to add a constructor now. In turn, observe that we restrict the introduction of the existential quantifier to values. ⌟

12.1.2 Embedding of the λ-calculus

Guided by the expected definition of the arrow:

$$A \to B \;\triangleq\; \neg(A \otimes \neg B)$$

we can follow Munch-Maccagnoni's paper [126, Appendix E], to embed the λ-calculus into $L^{\otimes}$.

With this definition, a stack $u \cdot e$ in $A \to B$ (that is with u a term of type A and e a context of type B) is naturally embedded as a term $(u,[e])$, which is turn into the context $\mu[\alpha].\langle (u,[e]) \| \alpha \rangle$ which indeed inhabits the "arrow" type $\neg(A \otimes \neg B)$. Starting from this, the rest of the definitions are direct:

$$\begin{array}{rcl} \mu(x,[\alpha]).c & \triangleq & \mu(x,x').\langle x' \| \mu[\alpha].c \rangle \\ \lambda x.t & \triangleq & [\mu(x,[\alpha]).\langle t \| \alpha \rangle] \\ t \cdot e & \triangleq & \mu[\alpha].\langle (t,[e]) \| \alpha \rangle \\ t\, u & \triangleq & \mu\alpha.\langle t \| u \cdot \alpha \rangle \end{array}$$

These shorthands allow for the expected typing rules:

Proposition 12.3. *The following typing rules are admissible:*

$$\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \dfrac{\Gamma \vdash u : A \mid \Delta \quad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid u \cdot e : A \to B \vdash \Delta} \qquad \dfrac{\Gamma \vdash t : A \to B \mid \Delta \quad \Gamma \vdash u : A \mid \Delta}{\Gamma \vdash t\, u : B \mid \Delta}$$

Proof. Each case is directly derivable from $L^{\otimes}$ type system. We abuse the notations to denote by (*def*) a rule which simply consists in unfolding the shorthands defining the λ-terms.

- **Case** $\mu(x,[\alpha]).c$:

$$\dfrac{\dfrac{\dfrac{\dfrac{c : (\Gamma, x : A \vdash \Delta, \alpha : B)}{\Gamma \vdash \mu[x].c : \neg A \mid \Delta, \beta : B}\,(\mu\vdash) \quad \dfrac{}{\Gamma, x : A, x' : \neg B \vdash x' : \neg B \mid \Delta}\,(\vdash ax)}{\langle x' \| \mu[\alpha].c\rangle : (\Gamma, x : A, x' : \neg B \vdash \Delta)}\,(\textsc{Cut})}{\Gamma \mid \mu(x,x').\langle x' \| \mu[\alpha].c\rangle : A \otimes \neg B \vdash \Delta}\,(\otimes\vdash)}{\Gamma \mid \mu(x,[\alpha]).c : A \otimes \neg B \vdash \Delta}\,(def)$$

- **Case** $\lambda x.t$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, x : A \vdash t : B \mid \Delta \quad \dfrac{}{\Gamma \mid \beta : B \vdash \Delta, \beta : B}\,(ax\vdash)}{\langle t \| \beta\rangle : (\Gamma, x : A \vdash \beta : B, \Delta)}\,(\textsc{Cut})}{\Gamma \mid \mu(x,[\beta]).\langle t \| \beta\rangle : A \otimes \neg B \vdash \Delta}}{\Gamma \vdash [\mu(x,[\beta]).\langle t \| \beta\rangle] : \neg(A \otimes \neg B) \mid \Delta}\,(\vdash\neg)}{\Gamma \vdash \lambda x.t : A \to B \mid \Delta}\,(def)$$

- **Case** $u \cdot e$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash u : A \vdash \Delta \quad \dfrac{\Gamma \mid e : B \vdash \Delta}{\Gamma \vdash [e] : \neg B \mid \Delta}\,(\vdash\neg)}{\Gamma \vdash (u,[e]) : A \otimes \neg B \mid \Delta}\,(\vdash\otimes) \quad \dfrac{}{\Gamma \mid \alpha : (A \otimes \neg B) \vdash \Delta, \alpha : (A \otimes \neg B)}\,(ax\vdash)}{\langle (u,[e]) \| \alpha\rangle : (\Gamma \vdash \Delta, \alpha : A \otimes \neg B)}\,(\textsc{Cut})}{\Gamma \mid \mu[\alpha].\langle (u,[e]) \| \alpha\rangle : \neg(A \otimes \neg B) \vdash \Delta}\,(\neg\vdash)}{\Gamma \mid u \cdot e : A \to B \vdash \Delta}\,(def)$$

- **Case** $t\,u$:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash t : A \to B \mid \Delta \quad \dfrac{\Gamma \vdash u : A \mid \Delta \quad \dfrac{}{\Gamma \mid \alpha : B \vdash \Delta, \alpha : B}}{\Gamma \mid u \cdot \alpha : A \to B \vdash \Delta, \alpha : B}}{\langle t \| u \cdot \alpha\rangle : (\Gamma \vdash \Delta, \alpha : B)}\,(\textsc{Cut})}{\Gamma \vdash \mu\alpha.\langle t \| u \cdot \alpha\rangle : B \mid \Delta}\,(\vdash\mu)}{\Gamma \vdash t\,u : B \mid \Delta}\,(def)$$

□

Besides, the usual rules of β-reduction for the call-by-value evaluation strategy are simulated through the reduction of $L^{\otimes}$:

Proposition 12.4 (β-reduction)**.** *We have the following reduction rules:*

$$\begin{array}{rcl} \langle t\,u \| e\rangle & \to_\beta & \langle t \| u \cdot e\rangle \\ \langle \lambda x.t \| u \cdot e\rangle & \to_\beta & \langle u \| \mu x.\langle t \| e\rangle\rangle \\ \langle V \| \mu x.c\rangle & \to_\beta & c[V/x] \end{array}$$

Proof. The third rule is included in $L^{\otimes}$ reduction system, the first follows from:

$$\langle tu \| e \rangle = \langle \mu\alpha.\langle t \| u \cdot \alpha \rangle \| e \rangle \to_\beta \langle t \| u \cdot e \rangle$$

For the second rule, we first check that we have:

$$\langle (V,[e]) \| \mu(x,[\alpha]).c \rangle = \langle (V,[e]) \| \mu(x,x').\langle x' \| \mu[\alpha].c \rangle \rangle \to_\beta \langle [e] \| \mu[\alpha].c[V/X] \rangle \to_\beta c[V/x][e/\alpha]$$

from which we deduce:

$$\begin{aligned}
\langle \lambda x.t \| u \cdot e \rangle &= \langle [\mu(x,[\alpha]).\langle t \| \alpha \rangle] \| \mu[\alpha].\langle (u,[e]) \| \alpha \rangle \rangle \\
&\to_\beta \langle (u,[e]) \| \mu(x,[\alpha]).\langle t \| \alpha \rangle \rangle \\
&\to_\beta \langle u \| \mu y.\langle (y,[e]) \| \mu(x,[\alpha]).\langle t \| \alpha \rangle \rangle \rangle \\
&\to_\beta \langle u \| \mu x.\langle t \| e \rangle \rangle
\end{aligned}$$

□

Therefore, $L^{\otimes}$ allows us to recover the full computation strength of the call-by-value $\lambda\mu\tilde{\mu}$-calculus. We shall now see that it is suitable for a realizability interpretation which is very similar to the corresponding interpretation for the call-by-value $\lambda\mu\tilde{\mu}$-calculus (see Section 4.5.4).

12.1.3 A realizability model based on the $L^{\otimes}$-calculus

We briefly recall the definitions necessary to the realizability interpretation *à la* Krivine of $L^{\otimes}$. Most of the properties being the same as for $L^{⅋}$ or any of the several interpretations we gave in the previous chapters, we spare the reader from a useless copy-paste and go straight to the point.

A *pole* is defined as usual as any subset of C_0 closed by anti-reduction. We write $⫫$ for the pole, and $t ⫫ e$ for the orthogonality relation it induces. As it is common in call-by-value realizability model (see Section 4.5.4), formulas as interpreted as *truth values of values*, which we call *primitive truth values. Falsity values* are then defined by orthogonality to the corresponding primitive truth values, and *truth values* are defines by orthogonality to falsity values. Therefore, an existential formula $\exists X.A$ is interpreted by the union over all the possible instantiations for the primitive truth value of the variable X by a set $S \in \mathcal{P}(\mathcal{V}_0)$. As it is usual in Krivine realizability, in order to ease the definition we assume that for each subset S of $\mathcal{P}(\mathcal{V}_0)$, there is a constant symbol $\dot{S}$ in the syntax. The interpretation is given by:

$$\begin{array}{rcl}
|\dot{S}|_V & \triangleq & S \\
|A \otimes B|_V & \triangleq & \{(t,u) : t \in |A|_V \wedge u \in |B|_V\} \\
|\neg A|_V & \triangleq & \{[e] : e \in \|A\|\} \\
|\exists X.A|_V & \triangleq & \bigcup_{S \in \mathcal{P}(\mathcal{V}_0)} |A\{X := \dot{S}\}|_V \\
\|A\| & \triangleq & \{e : \forall V \in |A|_V, V ⫫ e\} \\
|A| & \triangleq & \{t : \forall e \in \|A\|, t ⫫ e\}
\end{array}$$

We define again *valuations*, which we write ρ, as functions mapping each second-order variable to a primitive falsity value $\rho(X) \in \mathcal{P}(\mathcal{V}_0)$. In this framework, we say that a *substitution*, which we denote by σ, is a function mapping each variable x to a closed value $V \in \mathcal{V}_0$ and each variable α to a closed context $e \in \mathcal{E}_0$:

$$\sigma ::= \varepsilon \mid \sigma, x \mapsto V \mid \sigma, \alpha \mapsto e$$

We write $\sigma \Vdash \Gamma$ and we say that a substitution σ realizes a context Γ, when for each binding $(x : A) \in \Gamma$, we have $\sigma(x) \in |A|_V$. Similarly, we say that σ realizes a context Δ if for each binding $(\alpha : A) \in \Delta$, we have $\sigma(\alpha) \in \|A\|$.

Lemma 12.5 (Adequacy)**.** *Let Γ, Δ be typing contexts, ρ be a valuation and σ be a substitution which verifies that $\sigma \Vdash \Gamma[\rho]$ and $\sigma \Vdash \Delta[\rho]$. We have:*

1. *If V^+ is a value such that $\Gamma \vdash V^+ : A \mid \Delta$, then $V^+[\sigma] \in |A[\rho]|_V$.*
2. *If e is a context such that $\Gamma \mid e : A \vdash \Delta$, then $e[\sigma] \in \|A[\rho]\|$.*
3. *If t is a term such that $\Gamma \vdash t : A \mid \Delta$, then $t[\sigma] \in |A[\rho]|$.*
4. *If c is a command such that $c : (\Gamma \vdash \Delta)$, then $c[\sigma] \in \perp\!\!\!\perp$.*

Proof. The proof is again an induction over typing derivations. The proof being very similar to the one for $L^{⅋}$ (Proposition 11.10), the call-by-value $\lambda\mu\tilde{\mu}$-calculus (Proposition 4.23) or L [126], we leave it to the reader. □

12.2 Conjunctive structures

We shall now introduce the notion of *conjunctive structure*. Following the methodology from the previous chapter, we begin by observing the existing commutations in the realizability models induced by $L^{\otimes}$. Since we are in a structure centered on positive connectives, we should pay attention to the commutations with joins:

Proposition 12.6 (Commutations)**.** *In any $L^{\otimes}$ realizability model (that is to say for any pole $\perp\!\!\!\perp$), the following equalities hold:*

1. *If $X \notin FV(B)$, then $|\exists X.(A \otimes B)|_V = |(\exists X.A) \otimes B|_V$.*
2. *If $X \notin FV(A)$, then $|\exists X.(A \otimes B)|_V = |A \otimes (\exists X.B)|_V$.*
3. $|\neg(\exists X.A)|_V = \bigcap_{S \in \mathcal{P}(\mathcal{V}_0)} |\neg A\{X := \dot{S}\}|_V$

Proof.

1. Assume the $X \notin FV(B)$, then we have:

$$\begin{aligned}
|\exists X.(A \otimes B)|_V &= \bigcup_{S \in \mathcal{P}(\mathcal{V}_0)} |A\{X := \dot{S}\} \otimes B|_V \\
&= \bigcup_{S \in \mathcal{P}(\mathcal{V}_0)} \{(V_1, V_2) : V_1 \in |A\{X := \dot{S}\}|_V \wedge V_2 \in |B|_V\} \\
&= \{(e_1, e_2) : e_1 \in \bigcup_{S \in \mathcal{P}(\mathcal{V}_0)} |A\{X := \dot{S}\}|_V \wedge e_2 \in |B|_V\} \\
&= \{(e_1, e_2) : e_1 \in |\exists X.A|_V \wedge e_2 \in \|B\|\} \quad = \quad |(\exists X.A) \otimes B|_V
\end{aligned}$$

2. Identical.
3. The proof is again a simple unfolding of the definitions:

$$\begin{aligned}
|\neg(\exists X.A)|_V = \{[t] : t \in |\exists X.A|\} \quad &= \quad \{[t] : t \in \bigcap_{S \in \mathcal{P}(\mathcal{V}_0)} |A\{X := \dot{S}\}|\} \\
= \bigcap_{S \in \mathcal{P}(\mathcal{V}_0)} \{[t] : t \in |A\{X := \dot{S}\}]|\} &= \bigcap_{S \in \mathcal{P}(\mathcal{V}_0)} |\neg A\{X := \dot{S}\}|_V
\end{aligned}$$

□

Since we are interested in primitive truth values, which are logically ordered by inclusion (in particular, the existential quantifier is interpreted by unions, thus joins), in terms of algebraic structures, the previous proposition advocates for the equalities:

1. $\displaystyle\bigvee_{b \in B} (a \otimes b) = a \otimes (\bigvee_{b \in B} b)$ 2. $\displaystyle\bigvee_{b \in B} (b \otimes a) = (\bigvee_{b \in B} b) \otimes a$ 3. $\displaystyle\neg \bigvee_{a \in A} a = \bigwedge_{a \in A} \neg a$

Definition• 12.7 (Conjunctive structure)**.** A *conjunctive structure* is a complete join-semilattice $(\mathcal{A}, \preccurlyeq)$ equipped with a binary operation $(a, b) \mapsto a \otimes b$, called the *conjunction of* $\mathcal{A}$, and a unary operation $a \mapsto \neg a$ called the *negation of* $\mathcal{A}$, that fulfill the following axioms:

1. Negation is anti-monotonic in the sense that for all $a, a' \in \mathcal{A}$:

   (Variance) $$\text{if } a \preccurlyeq a' \quad \text{then} \quad \neg a' \preccurlyeq \neg a$$

2. Conjunction is monotonic in the sense that for all $a, a', b, b' \in \mathcal{A}$:

   (Variance) $$\text{if } a \preccurlyeq a' \;\text{ and }\; b \preccurlyeq b' \quad \text{then} \quad a \otimes b \preccurlyeq a' \otimes b'$$

3. Arbitrary meets distributes over both operands of conjunction, in the sense that for all $a \in \mathcal{A}$ and for all subsets $B \subseteq \mathcal{A}$:

   (Distributivity) $$\bigcurlyvee_{b \in B} (a \otimes b) = a \otimes (\bigcurlyvee_{b \in B} b) \qquad \bigcurlyvee_{b \in B} (b \otimes a) = (\bigcurlyvee_{b \in B} b) \otimes a$$

4. Negation of an arbitrary join is equal to the meet of the set of negated elements, in the sense that for all subsets $A \subseteq \mathcal{A}$:

   (Commutation) $$\neg \bigcurlyvee_{a \in A} a = \bigcurlywedge_{a \in A} \neg a$$

┘

Remark 12.8. Recall that a complete join-semilattice is a complete lattice (Theorem 9.3). Therefore, conjunctive structures also have arbitrary meets. The novelty, in comparison with implicative and disjunctive structures, is that the definition of conjunctive separators will make use of arbitrary meets (while the properties of distributivity and commutation are given for arbitrary joins). This mismatch is at the origin of most of the difficulties that we will meet in the sequel. ┘

As in the cases of implicative and disjunctive structures, the commutations imply that:

Proposition 12.9. *If $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ is a conjunctive structure, then the following hold for all $a \in \mathcal{A}$:*

1.• $\bot \otimes a = \bot$

2.• $a \otimes \bot = \bot$

3.• $\neg \bot = \top$

Proof. Using proposition 9.4 and the axioms of conjunctive structures, one can prove:

1. $\bot \otimes a = (\bigcurlyvee \emptyset) \otimes a = \bigcurlyvee_{x, a \in \mathcal{A}} \{x \otimes a : x \in \emptyset\} = \bigcurlyvee \emptyset = \bot$
2. Identical.
3. $\neg \bot = \neg(\bigcurlyvee \emptyset) = \bigcurlywedge_{x \in \mathcal{A}} \{\neg x : x \in \emptyset\} = \bigcurlywedge \emptyset = \top$

□

12.2.1 Examples of conjunctive structures

12.2.1.1 Dummy structure

Following the constraints given by the lemma above, we have at least one way to define a dummy structure:

Example• 12.10 (Dummy conjunctive structure)**.** Given a complete lattice L, the following definitions give rise to a dummy structure that fulfills the axioms of Definition 11.13:

$$a \otimes b \triangleq \bot \qquad \neg a \triangleq \top \qquad (\forall a, b \in \mathcal{A})$$

The verification of the different axioms is straightforward. ┘

12.2.1.2 Complete Boolean algebras

Example• 12.11 (Complete Boolean algebras)**.** Let $\mathcal{B}$ be a complete Boolean algebra. It embodies a conjunctive structure, that is defined by:

- $\mathcal{A} \triangleq \mathcal{B}$
- $a \preccurlyeq b \triangleq a \preccurlyeq b$
- $a \otimes b \triangleq a \wedge b$ $\quad(\forall a, b \in \mathcal{A})$
- $\neg a \triangleq \neg a$

The different axioms are direct consequence of proposition 9.7. ⌟

12.2.2 Conjunctive structure of classical realizability

As for the disjunctive case, we can abstract the structure of the realizability interpretation of $L^{\otimes}$ into a structure of the form $(\mathcal{T}_0, \mathcal{E}_0, \mathcal{V}_0, (\cdot,\cdot), [\cdot], \perp\!\!\!\perp)$, where $\mathcal{V}_0 \subseteq \mathcal{T}_0$ is the distinguished subset of values, $(\cdot,\cdot)$ is a map from ${\mathcal{T}_0}^2$ to $\mathcal{T}_0$ (whose restriction to $\mathcal{V}_0$ has values in $\mathcal{V}_0$), $[\cdot]$ is an operation from $\mathcal{E}_0$ to $\mathcal{V}_0$, and $\perp\!\!\!\perp \subseteq \mathcal{T}_0 \times \mathcal{E}_0$ is a relation. From this sextuple we can define:

- $\mathcal{A} \triangleq \mathcal{P}(\mathcal{V}_0)$
- $a \preccurlyeq b \triangleq a \subseteq b$
- $a \otimes b \triangleq (a,b) = \{(V_1,V_2) : V_1 \in a \wedge V_2 \in b\}$
- $\neg a \triangleq [a^{\perp\!\!\!\perp}] = \{[e] : e \in a^{\perp\!\!\!\perp}\}$

$(\forall a, b \in \mathcal{A})$

Proposition 12.12. *The quadruple* $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ *is a conjunctive structure.*

Proof. We show that the axioms of Definition 12.7 are satisfied.

1. Anti-monotonicity. Let $a, a' \in \mathcal{A}$, such that $a \preccurlyeq a'$ ie $a \subseteq a'$. Then $a'^{\perp\!\!\!\perp} \subseteq a^{\perp\!\!\!\perp}$ and thus
$$\neg a' = \{[t] : t \in a'^{\perp\!\!\!\perp}\} \subseteq \{[t] : t \in a^{\perp\!\!\!\perp}\} = \neg a$$
i.e. $\neg a' \preccurlyeq \neg a$.
2. Covariance of the conjunction. Let $a, a', b, b' \in \mathcal{A}$ such that $a' \subseteq a$ and $b' \subseteq b$. Then we have
$$a \otimes b = \{(t,u) : t \in a \wedge u \in b\} \subseteq \{(t,u) : t \in a' \wedge u \in b'\} = a' \otimes b'$$
i.e. $a \otimes b \preccurlyeq a' \otimes b'$
3. Distributivity. Let $a \in \mathcal{A}$ and $B \subseteq \mathcal{A}$, we have:
$$\bigvee_{b \in B} (a \otimes b) = \bigvee_{b \in B} \{(v,u) : t \in a \wedge u \in b\} = \{(t,u) : t \in a \wedge u \in \bigvee_{b \in B} b\} = a \otimes (\bigvee_{b \in B} b)$$
4. Commutation. Let $B \subseteq \mathcal{A}$, we have (recall that $\bigwedge_{b \in B} b = \bigcap_{b \in B} b$):
$$\bigwedge_{b \in B} \{\neg b\} = \bigwedge_{b \in B} \{[t] : t \in b^{\perp\!\!\!\perp}\} = \{[t] : t \in \bigwedge_{b \in B} b^{\perp\!\!\!\perp}\} = \{[t] : t \in (\bigvee_{b \in B} b)^{\perp\!\!\!\perp}\} = \neg(\bigvee_{b \in B} b)$$

□

12.2.3 Interpreting $L^{\otimes}$ terms

We shall now see how to embed $L^{\otimes}$ commands, contexts and terms into any conjunctive structure. For the rest of the section, we assume given a conjunctive structure $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$.

12.2.3.1 Commands

Following the same intuition as for the embedding of $L^{\parr}$ into disjunctive structures, we define the *commands* $\langle a \| b \rangle$ of the conjunctive structure $\mathcal{A}$ as the pairs (a, b), and we define the pole $\perp\!\!\!\perp$ as the ordering relation $\preccurlyeq$. We write $C_{\mathcal{A}} = \mathcal{A} \times \mathcal{A}$ for the set of commands in $\mathcal{A}$ and $(a, b) \in \perp\!\!\!\perp$ for $a \preccurlyeq b$.

We consider the same relation $\trianglelefteq$ over $C_{\mathcal{A}}$, which was defined by:

$$c \trianglelefteq c' \quad \triangleq \quad \text{if } c \in \perp\!\!\!\perp \text{ then } c' \in \perp\!\!\!\perp \qquad (\forall c, c' \in C_{\mathcal{A}})$$

Since the definition of commands only relies on the underlying lattice of $\mathcal{A}$, the relation $\trianglelefteq$ has the same properties as in disjunctive structures and in particular it defines a preorder (see Section 11.2.4.1).

12.2.3.2 Terms

The definitions of terms are very similar to the corresponding definitions for the dual contexts in disjunctive structures.

Definition• 12.13 (Pairing). For all $a, b \in \mathcal{A}$, we let $(a, b) \triangleq a \otimes b$. ⌟

Definition• 12.14 (Boxing). For all $a \in \mathcal{A}$, we let $[a] \triangleq \neg a$. ⌟

Definition• 12.15 (μ^+).

$$\mu^+.c \triangleq \bigcurlywedge_{a \in \mathcal{A}} \{a : c(a) \in \perp\!\!\!\perp\}$$

⌟

We have the following properties for μ^+:, whose proofs are trivial:

Proposition 12.16 (Properties of μ^+). *For any functions $c, c' : \mathcal{A} \to C_{\mathcal{A}}$, the following hold:*

1.• *If for all $a \in \mathcal{A}$, $c(a) \trianglelefteq c'(a)$, then $\mu^+.c' \preccurlyeq \mu^+.c$* *(Variance)*
2.• *For all $t \in \mathcal{A}$, then $t = \mu^+.(a \mapsto \langle t \| a \rangle)$* *($\eta$-expansion)*
3.• *For all $e \in \mathcal{A}$, then $\langle \mu^+.c \| e \rangle \trianglelefteq c(e)$* *($\beta$-reduction)*

Proof. 1. Direct consequence of Proposition 11.21.

2,3. Trivial by definition of μ^+.

□

12.2.3.3 Contexts

Dually to the definitions of the (positive) contexts μ^+ as a meet, we define the embedding of (negative) terms, which are all binders, by arbitrary joins:

Definition• 12.17 (μ^-). For all $c : \mathcal{A} \to C_{\mathcal{A}}$, we define:

$$\mu^-.c \triangleq \bigcurlyvee_{a \in \mathcal{A}} \{a : c(a) \in \perp\!\!\!\perp\}$$

⌟

Definition• 12.18 ($\mu^{()}$). For all $c : \mathcal{A}^2 \to C_{\mathcal{A}}$, we define:

$$\mu^{()}.c \triangleq \bigcurlyvee_{a, b \in \mathcal{A}} \{a \otimes b : c(a, b) \in \perp\!\!\!\perp\}$$

⌟

Definition• 12.19 ($\mu^{[]}$)**.** For all $c : \mathcal{A} \to C_{\mathcal{A}}$, we define:

$$\mu^{[]}.c \triangleq \bigcurlyvee_{a \in \mathcal{A}} \{\neg a : c(a) \in \perp\!\!\!\perp\}$$

⌟

Again, these definitions satisfy variance properties with respect to the preorder $\trianglelefteq$ and the order relation $\preccurlyeq$. Observe that the $\mu^{()}$ and μ^{-} binders, which are negative binders catching positive terms, are contravariant with respect to these relations while the $\mu^{[]}$ binder, which catches a negative context, is covariant.

Proposition 12.20 (Variance)**.** *For any functions c, c' with the corresponding arities, the following hold:*

1.• *If $c(a) \trianglelefteq c'(a)$ for all $a \in \mathcal{A}$, then $\mu^{-}.c' \preccurlyeq \mu^{-}.c$*
2.• *If $c(a,b) \trianglelefteq c'(a,b)$ for all $a, b \in \mathcal{A}$, then $\mu^{()}.c' \preccurlyeq \mu^{()}.c$*
3.• *If $c(a) \trianglelefteq c'(a)$ for all $a \in \mathcal{A}$, then $\mu^{[]}.c \preccurlyeq \mu^{[]}.c'$*

Proof. Direct consequences of Proposition 11.21. □

The η-expansion is also reflected by the ordering relation $\preccurlyeq$:

Proposition 12.21 (η-expansion)**.** *For all $t \in \mathcal{A}$, the following holds:*

1.• $\mu^{-}.(a \mapsto \langle t \| a \rangle) = t$
2.• $\mu^{()}.(a, b \mapsto \langle t \| (a, b) \rangle) \preccurlyeq t$
3.• $\mu^{[]}.(a \mapsto \langle t \| [a] \rangle) \preccurlyeq t$

Proof. Trivial from the definitions. □

The β-reduction is again reflected by the preorder $\trianglelefteq$ as the property of subject reduction:

Proposition 12.22 (β-reduction)**.** *For all $e, e_1, e_2, t \in \mathcal{A}$, the following holds:*

1.• $\langle \mu^{-}.c \| e \rangle \trianglelefteq c(e)$
2.• $\langle \mu^{()}.c \| (e_1, e_2) \rangle \trianglelefteq c(e_1, e_2)$
3.• $\langle \mu^{[]}.c \| [t] \rangle \trianglelefteq c(t)$

Proof. Trivial from the definitions. □

12.2.4 Adequacy

We shall now prove that the interpretation of $\mathrm{L}^{\otimes}$ is adequate with respect to its type system. Again, we extend the syntax of formulas to define second-order formulas with parameters by:

$$A, B ::= a \mid X \mid \neg A \mid A \otimes B \mid \exists X.A \qquad (a \in \mathcal{A})$$

This allows us to define an embedding of closed formulas with parameters into the conjunctive structure $\mathcal{A}$;

$$\begin{array}{rcll} a^{\mathcal{A}} & \triangleq & a & \text{(if } a \in \mathcal{A}\text{)} \\ (\neg A)^{\mathcal{A}} & \triangleq & \neg A^{\mathcal{A}} & \\ (A \otimes B)^{\mathcal{A}} & \triangleq & A^{\mathcal{A}} \otimes B^{\mathcal{A}} & \\ (\exists X.A)^{\mathcal{A}} & \triangleq & \curlyvee_{a \in \mathcal{A}} (A\{X := a\})^{\mathcal{A}} & \end{array}$$

As in the previous chapter, we define substitutions, which we write σ, as functions mapping variables (of terms, contexts and types) to element of $\mathcal{A}$:

$$\sigma ::= \varepsilon \mid \sigma[x \mapsto a] \mid \sigma[\alpha \mapsto a] \mid \sigma[X \mapsto a] \qquad (a \in \mathcal{A},\ x,X \text{ variables})$$

We say that a substitution σ realizes a typing context Γ, which write $\sigma \Vdash \Gamma$, if for all bindings $(x : A) \in \Gamma$ we have $\sigma(x) \preccurlyeq (A[\sigma])^{\mathcal{A}}$. Dually, we say that σ realizes Δ if for all bindings $(\alpha : A) \in \Delta$, we have $\sigma(\alpha) \succcurlyeq (A[\sigma])^{\mathcal{A}}$.

Theorem 12.23 (Adequacy)**.** *The typing rules of $L^{\otimes}$ (Figure 12.1) are adequate with respect to the interpretation of terms (contexts,commands) and formulas: for all contexts Γ,Δ, for all formulas with parameters A and for all substitutions σ such that $\sigma \Vdash \Gamma$ and $\sigma \Vdash \Delta$, we have:*

1. *For any term t, if $\Gamma \vdash t : A \mid \Delta$, then $(t[\sigma])^{\mathcal{A}} \preccurlyeq A[\sigma]^{\mathcal{A}}$;*
2. *For any context e, if $\Gamma \mid e : A \vdash \Delta$, then $(e[\sigma])^{\mathcal{A}} \succcurlyeq A[\sigma]^{\mathcal{A}}$;*
3. *For any command c, if $c : (\Gamma \vdash \Delta)$, then $(c[\sigma])^{\mathcal{A}} \in \perp\!\!\!\perp$.*

Proof. By induction on the typing derivations. Since most of the cases are similar to the corresponding cases for the adequacy of the embedding of $L^{⅋}$ into disjunctive structures, we only give some key cases.

- **Case** $(\vdash \otimes)$. Assume that we have:

$$\frac{\Gamma \vdash t_1 : A_1 \mid \Delta \quad \Gamma \vdash t_2 : A_2 \mid \Delta}{\Gamma \vdash (t_1,t_2) : A_1 \otimes A_2 \mid \Delta}\ (\vdash \otimes)$$

By induction hypotheses, we have that $(t_1[\sigma])^{\mathcal{A}} \preccurlyeq (A_1[\sigma])^{\mathcal{A}}$ and $(t_2[\sigma])^{\mathcal{A}} \preccurlyeq (A_2[\sigma])^{\mathcal{A}}$. Therefore, by monotonicity of the $\otimes$ operator, we have:

$$((t_1,t_2)[\sigma])^{\mathcal{A}} = (t_1[\sigma],t_2[\sigma])^{\mathcal{A}} = (t_1[\sigma])^{\mathcal{A}} \otimes (t_2[\sigma])^{\mathcal{A}} \preccurlyeq (A_1[\sigma])^{\mathcal{A}} ⅋ (A_2[\sigma])^{\mathcal{A}} .$$

- **Case** $(\otimes \vdash)$. Assume that we have:

$$\frac{c : \Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta}{\Gamma \mid \mu(x_1,x_2).c : A_1 \otimes A_2 \vdash \Delta}\ (\otimes \vdash)$$

By induction hypothesis, we get that $(c[\sigma, x_1 \mapsto (A_1[\sigma])^{\mathcal{A}}, x_2 \mapsto (A_2[\sigma])^{\mathcal{A}}])^{\mathcal{A}} \in \perp\!\!\!\perp$. Then by definition we have

$$((\mu(x_1,x_2).c)[\sigma])^{\mathcal{A}} = \bigcurlyvee_{a,b \in \mathcal{A}} \{a ⅋ b : (c[\sigma, x_1 \mapsto a, x_2 \mapsto b])^{\mathcal{A}} \in \perp\!\!\!\perp\} \succcurlyeq (A_1[\sigma])^{\mathcal{A}} \otimes (A_2[\sigma])^{\mathcal{A}}.$$

- **Case** $(\exists \vdash)$. Assume that we have:

$$\frac{\Gamma \mid e : A \vdash \Delta \quad X \notin FV(\Gamma,\Delta)}{\Gamma \mid e : \exists X.A \vdash \Delta}\ (\exists \vdash)$$

By induction hypothesis, we have that for all $a \in \mathcal{A}$, $(e[\sigma])^{\mathcal{A}} \succcurlyeq ((A)[\sigma, x \mapsto a])^{\mathcal{A}}$. Therefore, we have that $(e[\sigma])^{\mathcal{A}} \succcurlyeq \curlyvee_{a \in \mathcal{A}} (A\{X := a\}[\sigma])^{\mathcal{A}}$.

- **Case** $(\vdash \exists)$. Similarly, assume that we have:

$$\frac{\Gamma \vdash t : A\{X := B\} \mid \Delta}{\Gamma \vdash t : \exists X.A \mid \Delta}\ (\vdash \exists)$$

By induction hypothesis, we have that $(t[\sigma])^{\mathcal{A}} \preccurlyeq (A[\sigma, X \mapsto (B[\sigma])^{\mathcal{A}}])^{\mathcal{A}}$. Therefore, we have that $(t[\sigma])^{\mathcal{A}} \preccurlyeq \curlyvee_{b \in \mathcal{A}} \{A\{X := b\}[\sigma]^{\mathcal{A}}\}$. ◻

12.2.5 Duality between conjunctive and disjunctive structures

We now show how disjunctive structures and conjunctive structures are connected by a form of duality. Per se, this connection only reflects the well-known duality between call-by-value and call-by-name [32]. In fact, the passage from one structure to the other exactly reflects the dual translation from the $\lambda\mu\tilde{\mu}$-calculus to itself [32, Section 7] which sends terms to contexts and vice-versa. This duality is also reflected in L [126] already in its syntax, in which the same constructors are used both for terms and contexts. Here, since the term t and the context e of a well-formed command are connected by $t^{\mathcal{A}} \preccurlyeq e^{\mathcal{A}}$, we materialize the duality by reversing the order relation. We know that reversing the order in a complete lattice yields a complete lattice in which meets and joins are exchanged (Proposition 9.5). Therefore, it only remains to prove that the axioms of disjunctive and conjunctive structures can be deduced through this duality one from each other.

12.2.5.1 From disjunctive to conjunctive structures

Let $(\mathcal{A}, \preccurlyeq, ⅋, \neg)$ be a disjunctive structure. We define:

- $\mathcal{A}^{\otimes} \triangleq \mathcal{A}^{⅋}$
- $a \triangleleft b \triangleq b \preccurlyeq a$
- $\curlywedge^{\otimes} \triangleq \curlyvee^{⅋}$
- $\curlyvee^{\otimes} \triangleq \curlywedge^{⅋}$
- $a \otimes b \triangleq a ⅋ b$
- $\neg a \triangleq \neg a$

$(\forall a, b \in \mathcal{A})$

As expected, we have that:

Theorem• 12.24. *The structure $(\mathcal{A}^{\otimes}, \triangleleft, \otimes, \neg)$ defined above is a conjunctive structure.*

Proof. We check that for all $a, a', b, b' \in \mathcal{A}$ and for all subsets $A \subseteq \mathcal{A}$, we have:

1.• If $a \triangleleft a'$ then $\neg a' \triangleleft \neg a$ (Variance)
2.• If $a \triangleleft a'$ and $b \triangleleft b'$ then $a \otimes b \triangleleft a' \otimes b'$. (Variance)
3.• $(\curlywedge^{\otimes}_{a\in A} a) \otimes b = \curlywedge^{\otimes}_{a\in A}(a \otimes b)$ and $b \otimes (\curlywedge^{\otimes}_{a\in A} a) = \curlywedge^{\otimes}_{a\in A}(b \otimes a)$ (Distributivity)
4.• $\neg(\curlyvee^{\otimes}_{a\in A} a) = \curlywedge^{\otimes}_{a\in A}(\neg a)$ (Commutation)

All the proof are trivial from the corresponding properties of disjunctive structures. □

12.2.5.2 From conjunctive to disjunctive structures

Let $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ be a conjunctive structure. We define:

- $\mathcal{A}^{⅋} \triangleq \mathcal{A}^{\otimes}$
- $a \triangleleft b \triangleq b \preccurlyeq a$
- $\curlywedge^{⅋} \triangleq \curlyvee^{\otimes}$
- $\curlyvee^{⅋} \triangleq \curlywedge^{\otimes}$
- $a ⅋ b \triangleq a \otimes b$
- $\neg a \triangleq \neg a$

$(\forall a, b \in \mathcal{A})$

Again, we have that:

Theorem• 12.25. *The structure $(\mathcal{A}^{\otimes}, \triangleleft, \otimes, \neg)$ defined above is a conjunctive structure.*

Proof. We check that for all $a, a', b, b' \in \mathcal{A}$ and for all subsets $A \subseteq \mathcal{A}$, we have:

1.• If $a \triangleleft a'$ then $\neg a' \triangleleft \neg a$. (Variance)
2.• If $a \triangleleft a'$ and $b \triangleleft b'$ then $a ⅋ b \triangleleft a' ⅋ b'$. (Variance)
3.• $(\curlywedge^{⅋}_{a\in A} a) ⅋ b = \curlywedge^{⅋}_{a\in A}(a ⅋ b)$ and $a ⅋ (\curlywedge^{⅋}_{b\in B} b) = \curlywedge^{⅋}_{b\in B}(a ⅋ b)$ (Distributivity)
4.• $\neg(\curlywedge^{⅋} A) = \curlyvee^{⅋}_{a\in A}(\neg a)$ (Commutation)

All the proof are trivial from the corresponding properties of conjunctive structures. □

12.3 Conjunctive algebras

12.3.1 Separation in conjunctive structures

We shall now define the notion of separator for conjunctive structures. To this end, we consider axioms (*i.e.* combinators) which correspond to the dual properties axiomatizing the disjunction $⅋$ in disjunctive algebras. Remember that in a conjunctive structure, the arrow is defined:

$$a \overset{\otimes}{\to} b \triangleq \neg(a \otimes \neg b) \qquad (\forall a, b \in \mathcal{A})$$

We thus define the following combinators:

$$\begin{array}{lcl} s_1^{\otimes} & \triangleq & \curlywedge_{a\in\mathcal{A}} [a \overset{\otimes}{\to} (a \otimes a)] \\ s_2^{\otimes} & \triangleq & \curlywedge_{a,b\in\mathcal{A}} [(a \otimes b) \overset{\otimes}{\to} a] \\ s_3^{\otimes} & \triangleq & \curlywedge_{a,b\in\mathcal{A}} [(a \otimes b) \overset{\otimes}{\to} (b \otimes a)] \\ s_4^{\otimes} & \triangleq & \curlywedge_{a,b,c\in\mathcal{A}} [(a \overset{\otimes}{\to} b) \overset{\otimes}{\to} (c \otimes a) \overset{\otimes}{\to} (c \otimes b)] \\ s_5^{\otimes} & \triangleq & \curlywedge_{a,b,c\in\mathcal{A}} [(a \otimes (b \otimes c)) \overset{\otimes}{\to} ((a \otimes b) \otimes c))] \end{array}$$

which leads us to the expected definition of a separator:

Definition• 12.26 (Separator)**.** Given a conjunctive algebra $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$, we call separator for $\mathcal{A}$ any subset $\mathcal{S} \subseteq \mathcal{A}$ that fulfills the following conditions for all $a, b \in \mathcal{A}$:

(1) If $a \in \mathcal{S}$ and $a \preccurlyeq b$ then $b \in \mathcal{S}$ (upward closure)

(2) $s_1^{\otimes}, s_2^{\otimes}, s_3^{\otimes}, s_4^{\otimes}$ and $s_5^{\otimes}$ are in $\mathcal{S}$ (combinators)

(3) If $a \overset{\otimes}{\to} b \in \mathcal{S}$ and $a \in \mathcal{S}$ then $b \in \mathcal{S}$ (closure under modus ponens)

A separator $\mathcal{S}$ is said to be *consistent* if $\bot \notin \mathcal{S}$. ⌟

Example• 12.27 (Complete Boolean algebras)**.** Once again, if $\mathcal{B}$ is a complete Boolean algebra, $\mathcal{B}$ induces a disjunctive structure in which it is easy to verify that the combinators $s_1^{\otimes}, s_3^{\otimes}, s_3^{\otimes}, s_4^{\otimes}$ and $s_5^{\otimes}$ are equal to $\top$. Therefore, the singleton $\{\top\}$ or any filter for $\mathcal{B}$ are valid separators for the induced conjunctive structure. ⌟

12.3.2 Conjunctive algebra from classical realizability

Remember that any model of classical realizability based on $L^{\otimes}$ induces a conjunctive structure, where:

- $\mathcal{A} \triangleq \mathcal{P}(\mathcal{V}_0)$
- $a \preccurlyeq b \triangleq a \subseteq b$
- $a \otimes b \triangleq (a,b) = \{(V_1,V_2) : V_1 \in a \wedge V_2 \in b\}$
- $\neg a \triangleq [a^{⫫}] = \{[e] : e \in a^{⫫}\}$

$(\forall a, b \in \mathcal{A})$

As in the implicative and disjunctive cases, the set of formulas realized by a closed term[1], that is to say:

$$\mathcal{S}_{⫫} \triangleq \{a \in \mathcal{P}(V_0^+) : a^{⫫⫫} \cap \mathcal{T}_0 \neq \emptyset\}$$

defines a valid separator. The condition (1) and (3) are clearly verified (for the same reasons as in the disjunctive and implicative cases), but we should verify that the formulas corresponding to the combinators are indeed realized. Let us then consider the following closed terms:

$$\begin{array}{lcl} TS_1 & \triangleq & \lambda a.(a,a) \\ TS_2 & \triangleq & \lambda(a,b).a \\ TS_3 & \triangleq & \lambda(a,b).(b.a) \end{array} \qquad \begin{array}{lcl} TS_4 & \triangleq & \lambda f.(\lambda(c,a).(c,fa)) \\ TS_5 & \triangleq & \lambda(a,(b,c)).((a,b),c \end{array}$$

[1] As in the $\lambda\mu\tilde{\mu}$-calculus (see Section 4.4.5) and $L^{⅋}$, proof-like terms in $L^{\otimes}$ simply correspond to closed terms.

where we use the shorthands:

$$\begin{array}{rcl} \lambda x.t & \triangleq & [\mu(x,[\alpha]).\langle t\|\alpha\rangle] \\ \lambda(a,b).t & \triangleq & \lambda x.\mu\alpha.\langle x\|\mu(a,b).\langle t\|\alpha\rangle\rangle \\ \lambda(a,(b,c)).t & \triangleq & \lambda(a,x).\mu\alpha.\langle x\|\mu(b,c).\langle t\|\alpha\rangle\rangle \end{array}$$

To show that these terms indeed realize the expected formulas, we need to introduce the additional rule for the universal quantifier and to give its realizability interpretation:

$$\frac{\Gamma \vdash V : A \mid \Delta \quad X \notin FV(\Gamma,\Delta)}{\Gamma \vdash V : \forall X.A}\,(\vdash\forall) \qquad \frac{\Gamma \vdash e : A[B/X] \mid \Delta}{\Gamma \mid e : \forall X.A \vdash \Delta}\,(\forall\vdash) \qquad |\forall X.A|_V \triangleq \bigcap_{S\subseteq\mathcal{P}(\mathcal{V}_0)} |A\{X := \dot{S}\}|_V$$

Lemma 12.28. *The typing rules above are adequate with respect to the realizability interpretation of $L^{\otimes}$.*

Proof. The proof, which relies on the value restriction for the right rule, is the same as for L or $L^{⅋}$. □

Proposition 12.29. *The previous terms have the following types in $L^{⅋}$:*

1. $\vdash TS_1 : \forall A.A \to (A\otimes A) \mid$
2. $\vdash TS_2 : \forall AB.(A\otimes B) \to A \mid$
3. $\vdash TS_3 : \forall AB.A\otimes B \to B\otimes A \mid$
4. $\vdash TS_4 : \forall ABC.(A\to B) \to (C\otimes A \to C\otimes B) \mid$
5. $\vdash TS_5 : \forall ABC.(A\otimes(B\otimes C)) \to ((A\otimes B)\otimes C) \mid$

Proof. Straightforward typing derivations in $L^{\otimes}$. □

We deduce that $\mathcal{S}_{\perp\!\!\!\perp}$ is a valid separator for the conjunctive structure, and thus that any realizability model based on $L^{\otimes}$ induces a conjunctive algebra:

Proposition 12.30. *The quintuple $(\mathcal{P}(\mathcal{V}_0),\preccurlyeq,\otimes,\neg,\mathcal{S}_{\perp\!\!\!\perp})$ as defined above is a conjunctive algebra.*

Proof. Conditions (1) and (3) are trivial. Condition (2) follows from the previous propositions and the adequacy of the realizability interpretation of $L^{\otimes}$, observing that by definition of the conjonctive structure, we have $|\forall X.A|_V = \bigcurlywedge_{a\in\mathcal{A}} |A\{X := \dot{a}\}|_V$. □

12.3.3 From disjunctive to conjunctive algebras

We shall now prove that any disjunctive algebra induces by duality a conjunctive algebra, using the construction we presented before to obtain a conjunctive structure from the underlying disjunctive structures. The key of this construction was to consider the reversed lattice, inversing thus meets and joins:

- $\mathcal{A}^{\otimes} \triangleq \mathcal{A}^{⅋}$
- $a \triangleleft b \triangleq b \preccurlyeq a$
- $\bigcurlywedge^{\otimes} \triangleq \bigcurlyvee^{⅋}$
- $\bigcurlyvee^{\otimes} \triangleq \bigcurlywedge^{⅋}$
- $a\otimes b \triangleq a ⅋ b$
- $\neg a \triangleq \neg a$

$(\forall a,b\in\mathcal{A})$

Since both structures have the same career and disjunction, we will adopt the following notation to distinguish the conjunctive and disjunctive arrows:

$$a \overset{⅋}{\to} b \triangleq \neg a ⅋ b \qquad\qquad a \overset{\otimes}{\to} b \triangleq \neg(a\otimes\neg b) \qquad\qquad (\forall a,b\in\mathcal{A})$$

The question is now to determine, given a separator $\mathcal{S}^{⅋}$ for the disjunctive structure, how to define a separator $\mathcal{S}^{\otimes}$ for the conjunctive structure. Since separator are upwards closed and the lattice underlying the disjunctive structure is reversed in the conjunctive one, we should consider a set which is downward closed with respect to the order $\preccurlyeq$. To this purpose, we use the only contravariant operation we have at hands, and we define $\mathcal{S}^{\otimes}$ as the pre-image of $\mathcal{S}^{⅋}$ through the negation:

$$\mathcal{S}^{\otimes} \triangleq \neg^{-1}(\mathcal{S}^{⅋}) = \{a\in\mathcal{A} : \neg a\in\mathcal{S}^{⅋}\}$$

By definition, we thus have the following lemma:

Lemma• 12.31. *For all* $a \in \mathcal{A}$, $a \in \mathcal{S}^{\otimes}$ *if and only if* $\neg a \in \mathcal{S}^{\parr}$.

Besides, it is easy to show that the so-defined $\mathcal{S}^{\otimes}$ is indeed upward closed with respect to the reversed order:

Lemma• 12.32. *For all* $a, b \in \mathcal{A}$, *if* $a \triangleleft b$ *and* $a \in \mathcal{S}^{\otimes}$ *then* $b \in \mathcal{S}^{\otimes}$.

Proof. Straightforward: if $a \triangleleft b$ and $a \in \mathcal{S}^{\otimes}$, then $\neg a \in \mathcal{S}^{\parr}$ and $\neg a \preccurlyeq \neg b$, thus $\neg b \in \mathcal{S}^{\parr}$ and $b \in \mathcal{S}^{\otimes}$. □

Therefore, it remains to prove that $\mathcal{S}^{\otimes}$ contains the expected combinators, and that it is closed under modus ponens. For both proofs, the following proposition is fundamental:

Proposition• 12.33 (Contraposition)**.** *For all* $a, b \in \mathcal{A}$, *we have:*

$$a \overset{\otimes}{\to} b \in \mathcal{S}^{\otimes} \quad \Leftrightarrow \quad \neg a \overset{\parr}{\to} \neg b \in \mathcal{S}^{\parr}$$

Proof. Let $a, b \in \mathcal{A}$ be fixed. We do the proof directly by equivalence, since all the required equivalences hold for disjunctive algebras:

$$\begin{array}{rcll} a \overset{\otimes}{\to} b \in \mathcal{S}^{\otimes} & \Leftrightarrow & \neg(a \otimes \neg b) \in \mathcal{S}^{\otimes} & \text{(by definition)} \\ & \Leftrightarrow & \neg\neg(a \parr \neg b) \in \mathcal{S}^{\parr} & \text{(by definition)} \\ & \Leftrightarrow & (a \parr \neg b) \in \mathcal{S}^{\parr} & \text{(by DNE + Modus ponens)} \\ & \Leftrightarrow & (\neg\neg a \parr \neg b) \in \mathcal{S}^{\parr} & \text{(by DNI + }\parr\text{-compatible)} \\ & \Leftrightarrow & \neg a \overset{\parr}{\to} \neg b \in \mathcal{S}^{\parr} & \text{(by definition)} \end{array}$$

where DNE and DNI refer to the elimination and introduction of double negation (Proposition 11.58). The $\parr$-compatibility refers to the possibility of applying arrows of the shape $(a \to b) \in \mathcal{S}^{\parr}$ to get $(b \parr c) \in \mathcal{S}^{\parr}$ from $(a \parr c) \in \mathcal{S}^{\parr}$ (by application of $s_4^{\parr}$). The detailed proof is given in the Coq development. □

In particular, we can now deduce that $\mathcal{S}^{\otimes}$ is closed under modus ponens. The proof is straightforward from the previous lemma and Lemma 12.31.

Corollary• 12.34 (Modus Ponens)**.** *For all* $a, b \in \mathcal{A}$, *if* $a \in \mathcal{S}^{\otimes}$ *and* $a \overset{\otimes}{\to} b \in \mathcal{S}^{\otimes}$, *then* $b \in \mathcal{S}^{\otimes}$.

We now prove that $s_1^{\otimes}$, $s_1^{\otimes}$, $s_1^{\otimes}$, $s_1^{\otimes}$ and $s_1^{\otimes}$ belong to $\mathcal{S}^{\otimes}$. In each case, the proof somewhat consists in using the previous lemmas to be able to make use of the fact the dual combinator which is in $\mathcal{S}^{\parr}$.

Proposition• 12.35 ($s_1^{\otimes}$)**.** $s_1^{\otimes} \in \mathcal{S}^{\otimes}$

Proof. We want to show that $s_1^{\otimes} = \curlywedge^{\otimes}_{a \in \mathcal{A}} a \overset{\otimes}{\to} a \otimes a$ is in $\mathcal{S}^{\otimes}$. By definition of $\overset{\otimes}{\to}$ and commutation of the negation, we have $s_1^{\otimes} = \curlywedge^{\otimes}_{a \in \mathcal{A}} \neg(a \otimes \neg(a \otimes a)) = \neg \curlyvee^{\otimes}_{a \in \mathcal{A}} (a \otimes \neg(a \otimes a))$. To prove that the former is in the store, it suffices to prove that:

$$\neg\neg \curlyvee^{\otimes}_{a \in \mathcal{A}} (a \parr \neg(a \otimes a)) \in \mathcal{S}^{\parr} \qquad \textit{i.e.} \qquad \neg\neg \curlywedge^{\parr}_{a \in \mathcal{A}} (a \parr \neg(a \parr a)) \in \mathcal{S}^{\parr}$$

We conclude by double negation introduction (Proposition 11.58) and generalized modus ponens (Lemma 11.54) with $s_3^{\parr}$ and $s_1^{\parr}$. □

Proposition• 12.36 ($s_2^{\otimes}$)**.** $s_2^{\otimes} \in \mathcal{S}^{\otimes}$

Proof. We want to show that $s_2^{\otimes} = \curlywedge_{a,b \in \mathcal{A}} (a \otimes b) \overset{\otimes}{\to} a$ is in $\mathcal{S}^{\otimes}$. By definition of $\overset{\otimes}{\to}$ and commutation of the negation, we have $s_2^{\otimes} = \curlywedge_{a,b \in \mathcal{A}} \neg((a \otimes b) \otimes \neg a) = \neg \curlyvee^{\otimes}_{a,b \in \mathcal{A}} ((a \otimes b) \otimes \neg a)$. To prove that the former is in the store, it suffices to prove that:

$$\neg\neg \curlyvee^{\otimes}_{a \in \mathcal{A}} ((a \otimes b) \otimes \neg a) \qquad \textit{i.e.} \qquad \neg\neg \curlywedge^{\parr}_{a \in \mathcal{A}} ((a \otimes b) \otimes \neg a)$$

We conclude by double negation introduction (Proposition 11.58) and generalized modus ponens (Lemma 11.54) with $s_3^{\parr}$ and $s_2^{\parr}$. □

The three other proofs for $s_3^{\otimes}$, $s_4^{\otimes}$ and $s_5^{\otimes}$ are identical and left to the reader.

Proposition• 12.37. $s_3^{\otimes} \in \mathcal{S}^{\otimes}$

Proposition• 12.38. $s_4^{\otimes} \in \mathcal{S}^{\otimes}$

Proposition• 12.39. $s_5^{\otimes} \in \mathcal{S}^{\otimes}$

We can thus conclude that $\mathcal{S}^{\otimes}$ is indeed a separator for the conjunctive structure, or, in other words:

Theorem• 12.40. *The quintuple* $(\mathcal{A}^{\otimes}, \triangleleft, \otimes, \neg, \mathcal{S}^{\otimes})$ *defines a conjunctive algebra.*

12.4 Conclusion

12.4.1 On conjunctive algebras

First, we should say is that we are still missing many things in the understanding of conjunctive algebras. In particular, as such we are not able to prove the converse direction, that is that disjunctive algebras can be obtained from conjunctive algebras by duality. Neither are we in the position of defining a conjunctive tripos to study its connection with the implicative and disjunctive cases. The main reason for this is that in conjunctive structures, the application induced by the λ-calculus does not satisfy[2] the usual adjunction:

$$a \preccurlyeq b \to c \quad \Leftrightarrow \quad ab \preccurlyeq c$$

This property being crucial in most of the proofs we presented for implicative and disjunctive algebras, we are not able to follow the same track. In particular, the adjunction is central in the definition of the induced Heyting algebra (thus of the induced tripos).

In fact, the absence of this property is in itself a reassuring fact. Indeed, one of the lesson we learned from the $\lambda\mu\tilde{\mu}$-calculus, is that through the duality of computation, on the side of terms, the call-by-name evaluation strategy computes as the call-by-value evaluation strategy does on the side of contexts, and vice-versa. Therefore, it is not that surprising that the application (on the side of terms) does not satisfy the same properties in disjunctive and conjunctive structures. Actually, we can say more, namely that in a structure with all commutations (of the connectives with meets and joins), the adjunction holds[3]. But again, such a structure can only induce triposes[4] which are necessarily isomorphic to forcing triposes. As such, it is thus a feature for conjunctive structures not to satisfy the (call-by-name) adjunction.

We did not have the time to explore this question much in depth, but at first sight, it reminds us of the situation in Streicher's AKSs or Ferrer *et al.* $^{\mathcal{K}}$OCAs, where an adjunctor is needed for the equivalence to holds. In these particular settings, the problem is due to the fact that (call-by-name) falsity values are restricted to those which are closed under bi-orthogonality. It is worth notice that one of the usual interest of considering this particular shape of falsity values is related to value restriction (see [126] for a discussion on the topic). While we saw how to circumvent this difficulty in implicative and disjunctive structures, it might be the case that it is unavoidable in a call-by-value fashion. Anyway, if the necessity of an adjunctor has the downside of complicating proofs, it does not prevent from inducing triposes. Therefore, this could be on solution to obtain a notion of conjunctive tripos. Another solution may consist in defining another application for which the adjunction holds. To this purpose, one track to follow could be to observe the behavior of the usual application (in disjunctive structure) on elements of the conjunctive through the embedding given in Section 12.2.5.2.

[2]The left to right implication is trivially satisfied, the not satisfied implication it the right to left one.

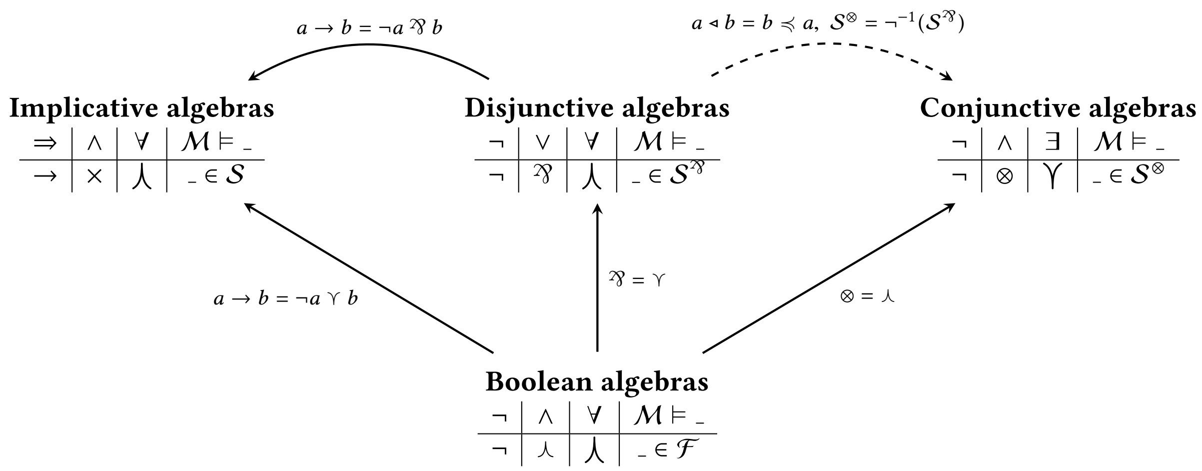
[3]This only a sufficient condition, but we conjecture having extra-commutations to obtain the adjunction is also necessary.

[4]To be precise, since we were not able to define conjunctive triposes, we should rather say that a conjunctive structure with all the commutations would probably induce disjunctive structures with the same commutations. These disjunctive structures would only induce triposes isomorphic to forcing triposes. Yet, we believe that in the case where a canonical notion conjunctive triposes could be defined, the very same would happen.

12.4.2 On the algebraization of Krivine classical realizability

In the last three chapters, we have shown that the underlying algebraic structures of classical realizability can be reified into algebras whose structures depend on the choice of logical connectives. Realizability models based on the λ_c-calculus, whose type system is defined with an arrow as logical connective, are particular instances of implicative algebras; models based on $L^{\parr}$, whose type system is defined with a disjunction and a negation as logical connectives, are particular cases of disjunctive algebras; models based on $L^{\otimes}$, whose connectives are a conjunction and a negation, are particular cases of conjunctive algebras. We highlighted the fact that the choice of connective (and therefore the corresponding algebraic structure) was related to the choice of a strategy of evaluation for the λ-calculus: call-by-name naturally corresponds to implicative and disjunctive algebras, while conjunctive algebras canonically embodies a call-by-value λ-calculus.

In the continuity of classical realizability, one of the main features of these algebraic structures is to give different semantics to the logical connectives $\wedge, \vee$ and to the quantifiers. For instance, the conjunction $a \wedge b$ is interpreted by the product type $a \times b$ in implicative algebras; whereas the universal quantification $\forall X.A(X)$ is interpreted by a meet $\bigcurlywedge_{b \in \mathcal{A}} A(b)$. This distinction between both interpretations leaves the door open to the definition of triposes that reflect Krivine realizability models [98, 99, 100, 101]. In particular, these models are more general than the models one can obtain by means of a forcing construction. It is worth noting that in the construction of realizability triposes from an implicative algebra $\mathcal{A}$, the structure of Heyting algebra which is obtained through the quotient $(\mathcal{A}^I/\mathcal{S}[I], \vdash_{\mathcal{S}[I]})$ (and therefore, the hyperdoctrine and the tripos) ignores the former order relation $\preccurlyeq$ and the former meets and joins $\curlywedge, \curlyvee$. More, whenever the underlying algebraic structure $\mathcal{A}$ has too many commutation properties, then the connective $\times$ (resp $+$) becomes equivalent to $\wedge$ (resp $\vee$). As a consequence, everything happens as if they were the same in $\mathcal{A}$, that is as if $\mathcal{A}$ were a Boolean algebra: the induced tripos is isomorphic to a forcing tripos. Schematically, the situation can be summed up by the following diagram[5]:



In this diagram, plain arrows $A \to B$ indicate that the structure A is a particular case of B, while the dashed one $A \dashrightarrow B$ means that B can be obtained from A through a construction. We annotate the arrow with the key definitions in the passage from one structure to another.

As we explained in Chapters 10 and 11, the left part of this diagram can be reflected at the level of the induced triposes. Indeed, if a structure A is particular case of a class of structures B (*i.e.* for an arrow $A \to B$ above), then the tripos $\mathcal{T}_A$ that A induces is also a particular case of tripos $\mathcal{T}_B$: formally, this is reflected by a surjective map $\mathcal{T}_B(I) \twoheadrightarrow \mathcal{T}_A(I)$ for all $I \in \mathbf{Set}^{op}$ (see the diagram in Section 10.4.4.1).

Up to now, the conclusion from the last chapters is that implicative algebras appear as the more

[5]Where we write $\mathcal{M} \vDash \_$ to represent the criterion of validity and where $\mathcal{F}$ denotes a filter of Boolean algebra.

general setting. Nonetheless, we did not achieve yet a complete study of conjunctive algebras. In particular, we are lacking the definition of an application (from the point of view of λ-calculus) satisfying the adjunction necessary to obtain a Heyting algebra (and thus a tripos). Besides, we are also missing an arrow in the previous diagram, from conjunctive to disjunctive algebra. We conjecture that there should be a way to prove that from any conjunctive algebra can be obtained a disjunctive algebra through the same duality, that is by reversing the order (see Section 12.2.5.2) and taking as (disjunctive) separator the preimage $\neg^{-1}(\mathcal{S}^{\otimes})$ of the (conjunctive) separator. In particular, we believe that the induced triposes should be proved to be isomorphic. In addition to giving a proof to support the claim that implicative algebras provide us with the more general framework, such a result would have a particular significance, showing that call-by-name and call-by-value calculi induce equivalent realizability models.

In a long-term perspective, several directions of investigation emerge. First, implicative algebras appear as a promising new tool from a model-theoretic point of view. They indeed provide us with a framework whose ground structure is as simple as Boolean algebras, while carrying all the computational power of the λ-calculus. In particular, they seem easier to manipulate than Krivine's realizability algebras while providing us with the same expressiveness. Since Krivine's realizability models seem to bring novel possibilities with respect to the traditional models of set theory, implicative algebras might be the more convenient structure to develop the model-theoretic analysis of classical realizability.

Second, we saw that implicative algebras identify types and programs, somewhat performing the last step of unification in the proofs-as-programs correspondence. As such, implicative algebras are tailored to the second-order λ_c-calculus, that is to say the second-order classical logic, but they clearly scale to high-order classical logic. On its computational facet, following the leitmotiv of the second part of this thesis, it raises the question of extending the calculus with side-effects. For instance, we wonder how our interpretations for the (call-by-need) $\overline{\lambda}_{[lv\tau\star]}$-calculus or—which is more ambitious—for dLPA$^{\omega}$ may be interpreted algebraically. In particular, an interpretation of dLPA$^{\omega}$ in terms of implicative algebras might help us to answer the questions we raised in Section 8.5 about the structure of the induced model. Especially, we could hope to take advantage of the criteria of collapsing so as to determine whether dLPA$^{\omega}$ allows for realizability models which are not equivalent to forcing constructions.

Furthermore, in the continuity of the study of disjunctive and conjunctive algebras, it would be interesting to determine how much of these structures can be combined without collapsing to a forcing situation. To put it differently, we saw that an implicative (resp. disjunctive) algebra in which arbitrary meets and joins distribute over all the connectives can only induce a tripos which is isomorphic to a forcing tripos. Yet, it is not clear whether it is possible to define an algebra which is both disjunctive and conjunctive without collapsing to a Boolean algebra. Such a structure would make sense to model the call-by-push-value paradigm [109], whose evaluation is directed by the polarity of terms (and thus requires a syntax with connectives of both polarities). Among other things, call-by-push value has shown to be a conducing setting for the study of side-effects in the realm of the proofs-as-programs correspondence.

Last, all along this manuscript we have been using several times Krivine realizability as a tool to prove properties for different calculi. Even if this perspective is at first sight fuzzier than the previous ones, it could be interesting to determine whether the reasoning process—*i.e.* defining a realizability interpretation and proving its adequacy in order to finally deduce theorems (mainly normalization and consistence properties)—can be transposed algebraically. In other words, we wonder whether, given a calculus, one could hope to define an embedding of this given calculus into an implicative algebra, next prove the adequacy of the embedding; then consider, for instance, the "separator" of normalizing terms to prove the normalization of the calculus. In itself, such an approach would probably be very closed from the usual one, but having a unifying framework might bring us some benefits.

For all these reasons, I am convinced that implicative algebras have a bright future ahead. We hope that this thesis would have done its bit towards a broader diffusion of their potentialities and features. *I have a dream that one day, we will all compute with formulas as if they were λ-terms...*