1- Logic

1.1 Theory

A famous character of a well-known book[1] once said to a young student of his:

> *The truth. [...] It is a beautiful and terrible thing, and should therefore be treated with great caution.*

While inattentive readers of this best-seller might have missed the significance of this declaration, it makes no doubt that this wise character intended to point out the fact that *truth* is a concept that is not as well-defined as one believes. This thesis being somewhat centered on the notions of *truth* and *proofs*, our starting point will be the definition of these key notions. In spite of a long faith in a total and absolute truth that mathematics ought to contain, belief of which Leibniz's quest for a *calculus ratiocinator* and Hilbert's second problem[2] only were the top of the iceberg, one of the major lesson from the 20$^{\text{th}}$ century in logic is that the notion of mathematical truth is deeply relative to its context and not uniquely defined.

In the next sections, we shall present two very different notions of truth. Considering again the example of geometry[2], two concepts are to be opposed. On the one hand, the *theory* of Euclidean geometry is an axiomatization intended to give a faithful representation of the world, expressed by means of Euclide's postulates. On the other hand, a *model* of this theory is a particular structure in which all the axioms of the theory hold. As explained in the introduction, a given axiomatization might be satisfied by several models. From these concepts are derived two different notions of truth:

- *provability*, a syntactic notion, expresses the existence of a proof in a theory,
- *validity*, a semantic notion, expresses the validation of a formula by a particular model of the theory.

Let us contemplate the case of Euclid's parallel postulate to illustrate the distinction between these notions. The parallel postulate is independent from other Euclid's postulates, that is to say that in the theory where only the first two postulates (cf. introduction) are assumed, the parallel postulate is neither provable nor disprovable. Notwithstanding, there exists at the same time a model in which it is *valid* (euclidian geometry) and different models in which it is not (non-euclidian geometries).

We shall start this section by introducing different concepts that are necessary to the definition of the concept of theory in Section 1.1.1, and pursue with the definition of a model in Section 1.2.

1.1.1 Language

Roughly, we can say that a theory is given by a *language*, which defines formulas and thus the expressiveness of the theory; and by the set of *theorems*, the formulas that are considered as true. Presented

[1]We deliberately choose to leave the precise reference apart from our bibliography, such an item would indubitably put the scientific rigor of this manuscript in question.

[2]See the introduction.

this way, truth corresponds to true formulas, which seems—and is—terribly tautological. The interesting point resides in defining which are the true formulas, and especially in how we define them. But before refining our notion of theory, let us first examine some examples of languages.

Example 1.1 (Propositional logic)**.** The language of propositional logic consists in propositions that are formed themselves by other propositions and the use of logical connectives. Specifically, we assume given a denumerable set $\mathcal{A}$ of *atomic formulas* and we define the *propositions* (or formulas), that are denoted by capital letters A, B, by:

$$A, B ::= X \mid \neg A \mid A \Rightarrow B \mid A \wedge B \mid A \vee B \qquad (X \in \mathcal{A})$$

where $\neg A$ reads "*not A*", $A \Rightarrow B$ reads "*A implies B*", $A \wedge B$ reads "*A and B*", and $A \vee B$ reads "*A or B*". We often consider that we have two particular atomic formulas in $\mathcal{A}$: *true*, that we write $\top$, and *false* that we write $\bot$, and if so, $\neg A$ is defined as $A \Rightarrow \bot$. It may be observed that our choice of connectives is arbitrary in the sense that we could have defined formulas from less or more connectives, or more generally from a signature of logical connectives. ⌟

While propositional logic can tracked to the 3rd century B.C.[3], the development of predicate logic, that can be considered as the next major advancement in logic, is much more recent and due to Frege in the 1870s. Intuitively, propositional logic only allows for declarative sentences such as *"I am a cat"* or *"Plato is a cat"* (or logical composition of declarative sentences, as in *"I am a cat" implies "I like fish"*), but it does not allow to identify the common structure "*be a cat*". Neither does it relate the *"I"* which is a cat and the *"I"* which likes fish. Less does it permit to express something like *"If x is a cat then x likes fish"*. The statement *"x is a cat"* or "$\mathrm{Cat}(x)$" is what is called a predicate, depending on a variable x, and more generally denoted by $P(x)$. The main achievement of Frege was to introduce this notion, together with the concept of *quantification*, allowing to specify the quantity of individuals for which a statement holds. The *universal quantification*, written $\forall$, denotes the fact that a statement holds *for all* individuals: $\forall x. \mathrm{Cat}(x)$ is *"for all x, x is a cat"*. The *existential quantification*, written $\exists$, denotes the *existence of (at least) one* individual for which the statement holds: $\exists x. \mathrm{Cat}(x)$ is *"there exists x such that x is a cat"*. The resulting language is called the language of predicate logic or language of first-order logic.

Example 1.2 (First-order logic)**.** The language of *first-order logic* is defined from two different syntactic categories:

- *terms* or *first-order expressions*, that are built from a fixed set $\mathcal{V}$ of variables and a fixed signature Σ_1 of functions symbols with their arities[4]:

$$e_1, e_2 \ ::= \ x \mid f(e_1, \ldots, e_k) \qquad (x \in \mathcal{V}, f \in \Sigma_1)$$

- *formulas*, that are defined from a fixed signature Σ_2 of predicate symbols with their arities:

$$A, B \ ::= \ P(e_1, \ldots, e_k) \mid \forall x.A \mid \exists x.A \mid A \Rightarrow B \mid A \wedge B \mid A \vee B \qquad (P \in \Sigma_2)$$

⌟

It is worth noting that this language strictly subsumes the language of propositional logic, where atomic formulas are nothing more than predicates of arity 0.

[3]More precisely, to the stoic Chrysippus, according to the Stanford Encyclopedia of Philosophy: `https://plato.stanford.edu/archives/spr2016/entries/logic-ancient/`.

[4]Such a signature can formally be defined as a pair $\Sigma_1 = (\mathcal{F}, \mathrm{ar})$ where $\mathcal{F}$ is a denumerable set of functions symbols and ar is a function $\mathcal{F} \to \mathbb{N}$ which assigns to each function its arity, *i.e.* the number of arguments it takes.

Example 1.3 (First-order arithmetic)**.** The language of first-order arithmetic is a special case of a first-order language, where the signature for first-order expressions contains a constant 0 (function of arity 0), a symbol S (of arity 1) to denote the successor, as well as two function symbols + and × denoting respectively the addition and the multiplication of natural numbers. As for the formulas, they are defined with the two quantifiers of first-order logic and one unique predicate symbol = to denote the equality of terms. The resulting syntax, where $\mathcal{V}$ is the set of variables, is given by:

$$\begin{array}{llll} \textbf{Terms} & e_1, e_2 & ::= & x \mid 0 \mid s(e) \mid e_1 + e_2 \mid e_1 \times e_2 \qquad (x \in \mathcal{V}) \\ \textbf{Formulas} & A, B & ::= & e_1 = e_2 \mid \top \mid \bot \mid \forall x.A \mid \exists x.A \mid A \Rightarrow B \mid A \wedge B \mid A \vee B \end{array}$$

⌟

These languages are called *first-order* because quantification is only authorized over first-order terms (natural numbers in the case of arithmetic). As we shall use further in this manuscript second-order or higher-order logic, let us give some more insight on this point.

Remark 1.4 (Order of a language)**.** Let us informally define Prop as the "set" of propositions. Intuitively, we could think of Prop as being the set that only contains *true* and *false*: Prop = $\{\top, \bot\}$. In the case of arithmetic, first-order individuals corresponds to natural numbers in $\mathbb{N}$. A predicate $P(x_1, \ldots, x_k)$ is thus a function from $\mathbb{N}^k$ to Prop. Alternatively, one can think of a predicate $P(x)$ as a set P of naturals number, with $P(x) \equiv x \in P$. This way, second-order individuals are sets in $\mathcal{P}(\mathbb{N})$, third-order individuals are sets of sets in $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, fourth-order sets of sets of sets, etc... : n^{th}-order individuals are elements of $\underbrace{\mathcal{P}(\cdots\mathcal{P}(}_{n-1}\mathbb{N})\cdots)$. With this intuition in mind, we say that a n^{th}-order language is a language that allows for quantifications ranging over n^{th}-order individuals. For instance:

- zero-order logic is just propositional logic, since it does not allow any quantification,
- first-order logic is indeed predicate logic, which allows for quantifications over terms and expresses properties about natural numbers,
- second-order logic corresponds to a language with quantifications ranging over predicates and expresses properties about sets of natural numbers,
- etc... ⌟

Up to now, in each example we only defined a language, whose symbols were not given any particular logical signification. Specifically, we said for instance that "=" denoted the equality, that "+" denoted the addition or that $s(0)$ was the successor of 0, so that any reader should be inclined to think of $s(0)$ as 1 and to 1 + 1 as 2. But there is no formal reason to do so!

In other words, we do not have any relation yet between $s(0) + s(0)$ and $s(s(0))$. We can write $s(0) + s(0) = s(s(0))$ just like we can write $s(0) = 0$ or $\top \Rightarrow \bot$, because in both cases the language is expressive enough. But we still need to give some kind of meaning to these symbols, and a least to define what we consider as true statements. To put it differently, we need to define what is the logical content of a theory.

We can now refine our notion of theory. A theory consists in three elements, namely:

- a *language*, which delimits the expressiveness of the theory;
- *axioms*, a minimal set[5] of closed formulas taken as true;
- a *deductive system*, which allows to deduce theorems from the axioms.

By minimal, we mean that none of the axioms should be proved from the other one using the deductive system, which we shall now define. By closed, we mean that a formula can only contain variables that are bound by some quantifier. For instance, $\forall x.\exists y.y = x + x$ is a closed formula but $\exists y.y = x + x$ is not since x is *free*. Formally, we define by induction the set of free variables $FV(A)$ of a formula A and say that a formula A is *closed* if $FV(A) = \emptyset$.

[5]These sets will mostly be finite in this manuscript.

Definition 1.5 (Free variables)**.** The sets of free variables of first-order terms and formulas are inductively defined by :

$$\begin{array}{rclcrcl} FV(x) & \triangleq & \{x\} & & FV(f(e_1,...,e_k)) & \triangleq & FV(e_1) \cup \ldots \cup FV(e_k) \\ FV(A \Rightarrow B) & \triangleq & FV(A) \cup FV(B) & & FV(P(e_1,\ldots,e_k)) & \triangleq & FV(e_1) \cup \ldots \cup FV(e_k) \\ FV(A \wedge B) & \triangleq & FV(A) \cup FV(B) & & FV(\forall x.A) & \triangleq & FV(A)\backslash\{x\} \\ FV(A \vee B) & \triangleq & FV(A) \cup FV(B) & & FV(\exists x.A) & \triangleq & FV(A)\backslash\{x\} \end{array}$$

⌟

Similarly, we define $A[e/x]$, which reads *"the formula A in which x is substituted by e"*, that we will use in the next section.

Definition 1.6 (Substitution)**.** The substitution of a variable x by an expression e is defined by induction over terms:

$$\begin{array}{rcll} y[e/x] & \triangleq & e & (\text{if } x = y) \\ y[e/x] & \triangleq & y & (\text{if } x \neq y) \\ (f(e_1,...,e_k))[e/x] & \triangleq & f(e_1[e/x],\ldots,e_k[e/x]) & \end{array}$$

and formulas:

$$\begin{array}{rcll} (P(e_1,\ldots,e_k))[e/x] & \triangleq & P(e_1[e/x],\ldots,e_k[e/x]) & \\ (A \Rightarrow B)[e/x] & \triangleq & A[e/x] \Rightarrow B[e/x] & \\ (A \wedge B)[e/x] & \triangleq & A[e/x] \wedge B[e/x] & \\ (A \vee B)[e/x] & \triangleq & A[e/x] \vee B[e/x] & \\ (\forall y.A)[e/x] & \triangleq & \forall y.(A[e/x]) & (\text{if } x \neq y, y \notin FV(e)) \\ (\forall y.A)[e/x] & \triangleq & \forall y.A & (\text{otherwise}) \\ (\exists y.A)[e/x] & \triangleq & \exists y.(A[e/x]) & (\text{if } x \neq y, y \notin FV(e)) \\ (\exists y.A)[e/x] & \triangleq & \exists y.A & (\text{otherwise}) \end{array}$$

Observe that in the case where the variable x corresponds to the variable bound by a quantifier (*e.g.* $\forall x.A$), the substitution is erased. ⌟

1.1.2 Deductive system

The aim of a deductive system is to capture the notion of logical consequence in a theory. There exist numerous deductive systems doing so, of which the most known are Hilbert's deduction system, natural deduction and Gentzen's sequent calculus. We will implicitly present Hilbert's system in Chapter 10, and we will introduce sequent calculus in Chapter 4. Let us focus now on the system of natural deduction, that we present with explicit contexts. Assume that we have a fixed language, for instance the language of first-order logic. We call *context* any list (possibly empty) of formulas written $\Gamma \equiv A_1,\ldots,A_n$. Formally, this corresponds to the simple following grammar:

$$\Gamma ::= \varepsilon \mid \Gamma, A$$

and we define $FV(\Gamma)$ as the union of free variables in each formula:

$$FV(\varepsilon) \triangleq \varepsilon \qquad\qquad FV(\Gamma,A) \triangleq FV(\Gamma) \cup FV(A)$$

A *judgment* is a pair (Γ,A) written $\Gamma \vdash A$, where Γ is a context and A is a formula. Intuitively, the sequent $\Gamma \vdash A$ expresses that the formula A is a logical consequence of the hypotheses Γ. Sequents are deduced from each other by means of a *deductive system.* A deductive system is given by a set of *inference rules* , which are of the form:

$$\frac{J_1 \quad \ldots \quad J_n}{J}\ (bli)$$

Propositional logic

(Introduction rules)

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ (\mathrm{Ax}) \qquad \frac{}{\Gamma \vdash \top}\ (\top)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\Rightarrow_I)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge_I)$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee_I^1) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee_I^2)$$

(Elimination rules)

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot)$$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\Rightarrow_E)$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge_E^1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge_E^2)$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee_E)$$

First-order logic

$$\frac{\Gamma \vdash A \quad x \notin FV(\Gamma)}{\Gamma \vdash \forall x.A}\ (\forall_I) \qquad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[t/x]}\ (\forall_E)$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x.A}\ (\exists_I) \qquad \frac{\Gamma \vdash \exists x.A \quad \Gamma, A \vdash B \quad x \notin FV(\Gamma, B)}{\Gamma \vdash B}\ (\exists_E)$$

Figure 1.1: Natural deduction

where *bli* is the name of the rule, where the judgment J is the conclusion of the rule and where $J_1, \dots, J_n$ are its premises. The rules of natural deduction, given in Figure 1.1, are divided in two sorts of rules:

- *introduction rules*, that give the necessary premises to introduce a connective,
- *elimination rules*, that give a conclusion that is derivable from a connective.

For instance, the elimination for the connective $\Rightarrow$ is none other than the Aristotelian principle of *modus ponens*:

$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ (\Rightarrow_E)$$

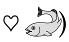
expressing that knowing $A \Rightarrow B$ and A, one can deduce B. Some rules (the axiom rule, the introduction of $\forall$ and the elimination of $\exists$) also have a side-condition to restrict their scope. For example, the rule (Ax) only applies if the formula A appears in the list Γ of hypotheses, while the introduction rule for $\forall$ applies only if the variable x does not occur freely in Γ (intuitively, x refers to any arbitrary term).

Succession of inferences are then arranged in the form of a *derivation tree*, whose root is traditionally located at the bottom. A sequent $\Gamma \vdash A$ is said to be *derivable* if there exists a derivation tree whose root is this sequent. This derivation tree is also called *proof tree* or simply *proof*.

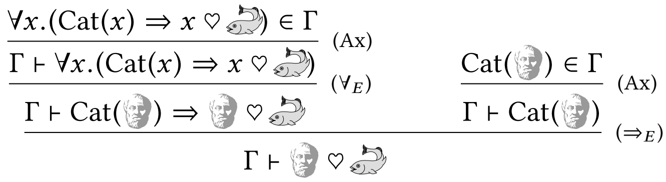
Example 1.7 (Plato likes fish)**.** Let us illustrate how natural deduction works by constructing the derivation tree corresponding to the syllogism: "*Plato is a cat, all cats likes fish thus Plato likes fish*". We define two predicates $\mathrm{Cat}(x)$ and $x \heartsuit y$ by :

$$\mathrm{Cat}(x) \triangleq \text{“ } x \text{ is a cat”} \qquad\qquad x \heartsuit y \triangleq \text{“ } x \text{ likes } y\text{”}$$

and denote Plato by  and "fish" by . Our hypothesis, which will constitute the context Γ, are then defined by:

$\Gamma = \mathrm{Cat}($$), \forall x.(\mathrm{Cat}(x) \Rightarrow x \heartsuit$ $)$

All this being set, we are now ready to give the expected derivation:



This proof tree reflects the structure of the expected proof. From bottom to top (and right to left), this proof can be read[6]:

- Plato likes fish by application of the modus ponens ($\Rightarrow_E$), since "if Plato is a cat then Plato likes fish" and "Plato is a cat",
- the latter holds because it is an hypothesis (Ax),
- the former holds because it is in fact true for any individual ($\forall_E$): "for all x, if x is a cat then x likes fish",
- this last statement is an hypothesis (Ax).

We enjoin the reader desirous of getting more familiar with the manipulation of proof trees to do the following exercises:

1. Introduce a predicate $\mathrm{Fish}(x) \triangleq$ *"x is a fish"*. Then generalize the hypothesis as *"any cat like any fish"* and consider some fish to prove that Plato likes it.
2. Give a different derivation of the same judgment.
3. Change the hypothesis *"Plato is a cat"* by *"Plato does not like fish"* and prove that *"Plato is not a cat"*. ⌟

1.1.2.1 Intuitionistic and classical logic

Alternatively, one can think of an inference rule as a logical axiom. Indeed, the choice of inference rules is not inconsequential and all deductive systems are not equivalent. Natural deduction, as we presented it, is said to be intuitionistic or constructive, because it only entails constructive principle. For instance, to construct a proof of a disjunction $A \vee B$, we need to actually choose between its left-hand side A or its right-hand side B. As a consequence, the De Morgan law:

$$\neg(A \wedge B) \Rightarrow (\neg A) \vee (\neg B)$$

is not provable[7] in natural deduction with an empty context. Intuitively, this is due to the fact that the knowledge of $\neg(A \wedge B)$ only provides us with the information that *"A and B"* is not true, it does not tell us whether A or B (or both) is false. Hence we have no way to prove $(\neg A) \vee (\neg B)$, which requires to give either a proof of $\neg A$ or a proof of $\neg B$. Similarly, the principle of *excluded-middle*:

$$A \vee (\neg A)$$

[6] This corresponds to the way the proof tree is build. The natural way of constructing a "hand-written" proof would be just the opposite, from top to bottom: *We know that for any individual x, if x is a cat, then x likes fish. In particular, if Plato is a cat, then he likes fish. But we also know that Plato is a cat, hence he likes fish.*

[7] The De Morgan law is not "false" in the sense that its negation is provable (which is not), but it is indeed not provable (we will prove this in Section 1.2). Such an affirmation might seem puzzling at first sight (how can we prove the unprovability of a formula?), but it is one of the biggest motivation to the introduction of a semantical truth through models.

is not provable[8] for all formulas, since it requires to effectively know whether A is true or not. If we can prove one of A or $\neg A$, we can obviously prove $A \vee (\neg A)$, if not we are stuck.

On the opposite, *classical logic* allows for instance to deduce a proof of $A \vee B$ from a *reductio ad absurdum*: supposing that neither A nor B, one might obtain a proof of false ($\bot$) which is absurd, and conclude that the hypothesis was false, hence A or B is true. This formally corresponds to the addition of an extra logical axiom, which is usually chosen amongst these three principles:

$A \vee (\neg A)$	$(\neg\neg A) \Rightarrow A$	$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
(Excluded-middle)	*(Double-negation elimination)*	*(Peirce's law)*

None of these axioms is provable in intuitionistic natural deduction, and they are logically equivalent in the sense that any one of them is deducible from any other one[9]. It is worth saying that in spite of our presentation—which is mostly intuitionistic in this chapter—, classical logic is the logic the wo.man in the street is accustomed to. In particular, most of mathematicians consider the double-negation elimination or the excluded-middle as valid principles for reasoning and proving theorems.

Remark 1.8. The Curry-Howard correspondence, that will be presented in Section 2.3, makes this idea of *constructivism* even stronger: it associates to each proof a program whose computation corresponds to the proof. Originally formulated in an intuitionistic setting, it was then extended to a classical framework thanks to a clever interpretation of Peirce's law. All this manuscript is dedicated to the study of classical proofs through this interpretation. ⌟

1.1.3 Theory

Given by a language together with a deductive system and a set of axioms, a theory $\mathscr{T}$ allows to deduce theorems by means of logical consequences. Formally, a *demonstration* or *proof* of a formula A in the theory $\mathscr{T}$ is a derivation whose conclusion is of the form $\Gamma \vdash A$, where Γ is a (finite) set of axioms of $\mathscr{T}$. When such a demonstration exists, A is called a *theorem* of $\mathscr{T}$. The theory $\mathscr{T}$ is said to be incoherent or inconsistent whenever the formula $\bot$ is a theorem of $\mathscr{T}$ (or, equivalently, when any formula is a theorem of $\mathscr{T}$). Otherwise, the theory is said to be *coherent* or *consistent*. Furthermore, a theory $\mathscr{T}$ is said to be *complete* if for each formula A, either A is a theorem of $\mathscr{T}$ either its negation $\neg A$ is.

Example 1.9 (Intuitionistic logic)**.** The theory of intuitionistic propositional logic NJ is the theory obtain from the propositional rules of natural deduction (see Figure 1.1) with no further axioms. ⌟

Example 1.10 (Relations)**.** A *relation* corresponds to a predicate $\mathcal{R}(x,y)$ of arity 2, that we rather write $x \mathcal{R} y$. Numerous generic properties about relations can be defined in first-order logic, amongst which:

(R1)	*Reflexivity* :	$\forall x. x \mathcal{R} x$
(R2)	*Transitivity* :	$\forall x.\forall y.\forall z. x \mathcal{R} y \Rightarrow y \mathcal{R} z \Rightarrow x \mathcal{R} z$
(R3)	*Anti-symmetry* :	$\forall x.\forall y. x \mathcal{R} y \Rightarrow y \mathcal{R} x \Rightarrow x = y$
(R4)	*Symmetry* :	$\forall x.\forall y. x \mathcal{R} y \Rightarrow y \mathcal{R} x$
(R5)	*Totality* :	$\forall x.\forall y. x \mathcal{R} y \vee y \mathcal{R} x$

A relation is called a *pre-order*, and often written $\leq$, if it is reflexive and transitive *i.e.* if (R1),(R2) are theorems of the ambient theory. If (R3) is also a theorem (the pre-order is anti-symmetric), it is called an *order*. An order is total if it satisfies the condition (R5). An *equivalence* is a relation for which (R1),(R2) and (R4) holds. ⌟

[8] We will give a formal argument of this statement in Section 1.2.2. In fact, we will even prove that the excluded-middle is independent from intuitionistic logic, that is to say that neither the excluded-middle nor its negation are provable.

[9] Proving the equivalence is a nice and classical exercise.

Example 1.11 (Theory of equality)**.** The theory of equality, in the language of first-order arithmetic, corresponds to the following axioms:

(E1)	$\forall x.(x = x)$
(E2)	$\forall x.\forall y.\forall z.(x = y \wedge x = z \Rightarrow y = z)$
(E3)	$\forall x.\forall y.(x = y \Rightarrow s(x) = s(y))$
(E4)	$\forall x.\forall y.\forall z.(x = y \Rightarrow x + z = y + z)$
(E5)	$\forall x.\forall y.\forall z.(x = y \Rightarrow z + x = z + y)$
(E6)	$\forall x.\forall y.\forall z.(x = y \Rightarrow x \times z = y \times z)$
(E7)	$\forall x.\forall y.\forall z.(x = y \Rightarrow z \times x = z \times y)$

Observe that the first two axioms (E1) and (E2) imply that the relation of equality is reflexive, transitive, symmetric and anti-symmetric. ⌟

If equalities as $1 = 1$ or $1 + 2 = 1 + 2$ are simple consequences of the axioms (E1-E7), the equality $1 + 1 = 2$ (*i.e.* $s(0) + s(0) = s(s(0))$) is still not provable. Indeed, such an equality relies on properties of the addition and not of the equality. Similarly, $1 \times 1 = 1$ relies on properties of the multiplication. These properties are expressed by Peano axioms, which define the theory of first-order arithmetic.

Example 1.12 (Peano arithmetic)**.** The theory of *Peano arithmetic*, that we write (PA), is obtained by adding to the theory of equality the six axioms below:

(PA1)	$\forall x.(0 + x = x)$
(PA2)	$\forall x.\forall y.(s(x) + y = s(x + y))$
(PA3)	$\forall x.(0 \times x = 0)$
(PA4)	$\forall x.\forall y.(s(x) \times y = (x \times y) + y)$
(PA5)	$\forall x.\forall y.(s(x) = s(y) \Rightarrow x = y)$
(PA6)	$\forall x.(s(x) \neq 0)$

as well as the *axioms of induction*:

(PA7) $\qquad \forall z_1 \ldots z_n (A[x/0] \wedge \forall x.(A \Rightarrow A[s(x)/x]) \Rightarrow \forall x.A)$

for each formula A whose free variables are $x, z_1, \ldots, z_n$. ⌟

Finally, we have now at our disposal a theory in which we can indeed assert that $1 + 1 = 2$

Theorem 1.13 (1+1=2)**.** $\mathrm{PA} \vdash s(0) + s(0) = s(s(0))$

Proof. We only sketch the proof in english, and let any circumspect reader derive the formal proof tree. The axiom PA2 implies that $s(0) + s(0) = s(0 + s(0))$ and PA1 implies that $0 + s(0) = s(0)$. Using the axiom (E3) of equality, we deduced that $s(0 + s(0)) = s(s(0))$, and we conclude by transitivity of the equality (E2). □

It is easy to check that expected properties of arithmetic are provable with these axioms, for instance that the successor corresponds indeed to the addition of 1 (*i.e.* $s(0)$):

$$\mathrm{PA} \vdash \forall x.x + s(0) = s(x)$$

or that the principle of strong induction holds:

$$\mathrm{PA} \vdash \forall x.(\forall y.(y < x \Rightarrow A(y)) \Rightarrow A(x)) \Rightarrow \forall x A(x)$$

1.1.3.1 Gödel's incompleteness

Unfortunately for Leibniz's and Hilbert's dream of an absolute truth, the notion of provability does not meet this expectancy. Indeed, this syntactic concept of truth does not allow to decide of the truth of all statements: some statements are neither provable nor provable. More precisely, as soon as a theory $\mathscr{T}$ is expressive enough, either there is a closed formula G such that $\mathscr{T} \nvdash G$ and $\mathscr{T} \nvdash \neg G$ or the theory is incoherent. This is known as Gödel first incompleteness theorem [61], who managed to adapt the old liar's paradox:

"I am a liar"

to the theory of arithmetic. Roughly, Gödel defined an encoding $\ulcorner \cdot \urcorner$ of the formulas and demonstrations of first-order arithmetic to natural numbers[10] This encodings allows to convert the statement *"A is a theorem of $\mathscr{T}$"* into the statement *"x is the code of a theorem of $\mathscr{T}$"*, which can be expressed as an arithmetic formula. This permits the definition of the following formula G:

$$G \triangleq \neg Th(\ulcorner G \urcorner) \qquad (\text{“}\ulcorner G \urcorner \textit{ is not the code of a theorem of } \mathscr{T}\text{”}).$$

If $\mathscr{T}$ is coherent, $\mathscr{T}$ can not prove G, otherwise G would be a theorem and $\mathscr{T}$ would prove $\ulcorner G \urcorner$ is not the code of a theorem of $\mathscr{T}$. Neither can $\mathscr{T}$ prove $\neg G$, *i.e.* $\ulcorner G \urcorner$ is the code of a theorem, since G would not be a theorem and $\mathscr{T}$ would be inconsistent.

To Hilbert's claim *"For us mathematicians there is no 'Ignorabimus'[...] we shall know!"*, Gödel's theorem somehow answers: *"No, my dear, we won't !"*.

Theorem 1.14 (First incompleteness theorem). *If $\mathscr{T}$ is coherent and contains PA, then $\mathscr{T}$ is incomplete.*

1.2 Models

We shall now contemplate a semantic notion of truth, namely the satisfiability by a *model.* As explained in the introduction, while a theory specifies the axioms and rules that are to be satisfied, giving an axiomatic representation of the world, a model $\mathcal{M}$ of a theory $\mathscr{T}$ is the given of one possible world in which all the theorems of $\mathscr{T}$ are satisfied. If the distinction between the syntax and the semantics of a sentence can be traced back to older works[1], model theory as the study of the interpretation of a language by means of set-theoretic structures is mostly based on Alfred Tarski's truth definition [153].

Given a theory $\mathscr{T}$, that is to say a language $\mathscr{L}$ together with a set of axioms and deduction rules, a model is the given of a universe in which the language $\mathscr{L}$ is interpreted and of a relation of satisfiability such that the interpretation of each theorem of $\mathscr{T}$ is satisfied. Let us examine a simple example before giving a formal definition.

Example 1.15. Consider the language of first-order arithmetic (Example 1.3), in a theory without axioms (*i.e.* theorems are logical tautologies), and consider the statement:

$$\forall x.(0 + x = x)$$

which is the first axiom (PA1) of Peano arithmetic. In this context, it is not an theorem, hence it can be either true or false in a model. The first natural interpretation we might come with is to choose as universe the set $\mathbb{N}$ of natural numbers, to interpret '0' by the natural 0, '+' by the addition of natural

[10]You can think of this as an enumeration of every possible formulas and demonstrations. It corresponds to something like 0 is the code for $\top$,1 is the code for $\bot$,..., 42 is the code for the proof of the conjunction of formulas of code 5 and 7, etc... and $\ulcorner \forall x.\forall y.x + y = 27 \urcorner = 137668$. The key point is that every formula and demonstration have a code.

[11]Besides the aforementioned works on non-Euclidean geometries, Frege's works can be pointed out: he formally introduced the distinction between the character x and the quoted 'x' to distinguish between the signified and the signifier.

numbers and ‘ $='$ by the equality on natural numbers. We write $\mathbb{N} \vDash A$ to denote that $\mathbb{N}$ satisfies the formula A, and we define the satisfiability of the universal quantifier by:

$$\mathbb{N} \vDash \forall x.A(x) \qquad \text{if and only if} \qquad \text{for all } n \in \mathbb{N}, \mathbb{N} \vDash A(n)$$

Then (PA1) is true with respect to this interpretation, since for any natural number n, $\mathbb{N} \vDash 0 + n = n$.

Now, we could also give a different interpretation. Consider the set $\mathcal{W}$ of (finite) words defined on the usual alphanumeric alphabet ‘$0-9, a-z$’. We interpret 0 by the character 0, + by the concatenation of words and = by the equality. We define the satisfiability of the universal quantifier in a similar way:

$$\mathcal{W} \vDash \forall x.A(x) \qquad \text{if and only if} \qquad \text{for all } w \in \mathcal{W}, \mathcal{W} \vDash A(w)$$

Then (PA1) is false with respect to this interpretation: indeed, if we consider for instance the word ‘*abc*’, we have $0 + abc = 0abc \neq abc$, *i.e.* $\mathcal{W} \nvDash 0 + abc = abc$. Thus $\mathcal{W}$ does not satisfies (PA1): $\mathcal{W} \nvDash \forall x.(0 + x = x)$. ⌟

Formally, given a language $\mathscr{L}$, a pair $(\mathcal{M},\mathcal{I})$ is said to be an $\mathscr{L}$-structure if $\mathcal{I}$ maps the symbols of $\mathscr{L}$ to appropriate elements of $\mathcal{M}$: function symbols are mapped to functions (of the corresponding arity) and predicates are mapped to functional relations. $\mathcal{M}$ is called the universe of the structure, and $\mathcal{I}$ its interpretation function.

Definition 1.16 (Model)**.** Given a $\mathscr{L}$-structure, a *formula* $A(m_1,\ldots,m_n)$ *with parameters in* $\mathcal{M}$ is defined as a formula $A(x_1,\ldots,x_n)$ whose free variables $x_1,\ldots,x_n$ have been substituted by elements $m_1,\ldots,m_n$ of $\mathcal{M}$. Finally, a $\mathscr{L}$-structure $(\mathcal{M},\mathcal{I})$ is said to be a *model* of a theory $\mathscr{T}$ if there is a relation of *satisfiability* over formulas with parameters in $\mathcal{M}$, such that every theorem of $\mathscr{T}$ are satisfied by $\mathcal{M}$. This relation is often denoted by $\mathcal{M} \vDash A$ and reads A is *valid* (or true) in $\mathcal{M}$ or $\mathcal{M}$ satisfies A. ⌟

In practice, the relation of satisfiability is defined primitively on atomic formulas and then by induction on the structure of a formula. If the definition is adequate with the deductive system, then the resulting relation defines indeed a model.

Definition 1.17 (Adequacy)**.** Let $\mathscr{L}$ be a language, $\mathscr{T}$ be a theory based on this language and $\mathcal{M}$ be an $\mathscr{L}$-structure.

- A judgment $\Gamma \vdash A$ in $\mathscr{T}$ is *adequate* (w.r.t. to the model $\mathcal{M}$) if the validity of the premises ($\mathcal{M} \vDash \Gamma$) entails the validity of the conclusion ($\mathcal{M} \vDash A$).
- More generally, we say that an inference rule

$$\frac{J_1 \quad \cdots \quad J_n}{J_0}$$

is adequate (w.r.t. to the model $\mathcal{M}$) if the adequacy of all judgments $J_1,\ldots,J_n$ implies the adequacy of the typing judgment J_0. ⌟

Proposition 1.18. *If all the axioms of a theory $\mathscr{T}$ are valid in a structure $\mathcal{M}$, and if all its rules of inference are adequate, then $\mathcal{M}$ is a model of $\mathscr{T}$.*

Proof. Indeed, if there is a proof of a formula A in $\mathscr{T}$, this proof is build out of axioms and inferences rules. Since axioms are valid in $\mathcal{M}$ and inference rules are adequate w.r.t. $\mathcal{M}$, by induction we get that adequate judgments at every floors of the tree. In particular, the root of the proof tree ($\mathscr{T} \vdash A$) is adequate, that is to say that $\mathcal{M} \vdash A$ is valid. This is true for every theorem of $\mathscr{T}$, hence $\mathcal{M}$ is a model of $\mathscr{T}$. □

In particular, if $\mathscr{T}$ is not coherent (*i.e.* $\mathscr{T} \vdash \bot$), then $\bot$ is valid in any model $\mathcal{M}$. By contraposition, this gives us a semantic criterion of coherency.

Corollary 1.19 (Coherence)**.** *If a theory $\mathscr{T}$ has a model $\mathcal{M}$ such that $\bot$ is not valid in $\mathcal{M}$, then $\mathscr{T}$ is coherent.*

Unlike for provability, in a model any statement is necessarily[2] either satisfied or not. Nevertheless, the same theory can admit very different models, and a statement can be true in some of them, false in others. This justifies the introduction of the notion of completeness, which corresponds to the implication dual to soundness (which is the very definition of a model):

$$\begin{array}{ll} \text{(Soundness)} & \mathscr{T} \vdash A \;\Rightarrow\; \mathcal{M} \vDash A \\ \text{(Completeness)} & \mathcal{M} \vDash A \;\Rightarrow\; \mathscr{T} \vdash A \end{array}$$

Definition 1.20 (Completeness)**.** A theory $\mathscr{T}$ is said to be *complete* with respect to a class of models $\mathcal{M}$ if for all formula A, the satisfiability of A in $\mathcal{M}$ ($\mathcal{M} \vDash A$) for any such model $\mathcal{M}$ implies the provability of A in $\mathscr{T}$ ($\mathscr{T} \vdash A$). ⌟

We shall examine now some examples of models.

1.2.1 Truth tables

The easiest model of all for propositional logic is known since the antiquity, and consists in a truth table with only two elements[3] $\top$ and $\bot$. The interpretation of the different connectives is defined as internal laws, whose values are given by the following truth tables:

$p \wedge q$		
p \ q	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

$p \vee q$		
p \ q	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$p \Rightarrow q$		
p \ q	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

Formally, given a propositional theory $\mathscr{T}$ this corresponds to a model $\mathcal{M} = \{\top, \bot, \}$ such that the interpretation function maps every axioms (atomic propositions) to $\top$ and to the following definition of the satisfiability relation :

$$\begin{array}{lll} \mathcal{M} \vDash \top & & \\ \mathcal{M} \vDash A \wedge B & \text{if and only if} & \mathcal{M} \vDash A \text{ and } \mathcal{M} \vDash B \\ \mathcal{M} \vDash A \vee B & \text{if and only if} & \mathcal{M} \vDash A \text{ or } \mathcal{M} \vDash B \\ \mathcal{M} \vDash A \Rightarrow B & \text{if and only if} & \mathcal{M} \vDash A \text{ implies } \mathcal{M} \vDash B \\ \mathcal{M} \vDash \neg A & \text{if and only if} & \mathcal{M} \nvDash A \end{array}$$

This definition can be extended to judgments by defining:

$$\begin{array}{lll} \mathcal{M} \vDash A_1, \dots, A_n & \text{if and only if} & \mathcal{M} \vDash A_1 \wedge \cdots \wedge A_n \\ \mathcal{M} \vDash \Gamma \vdash A & \text{if and only if} & \mathcal{M} \vDash \Gamma \text{ implies } \mathcal{M} \vDash A \end{array}$$

and it is easy to check that all the inference rules for propositional logic in Figure 1.1 are adequate. Besides, it is worth noting that such a model always validates the excluded middle since:

$$\mathcal{M} \vDash A \vee (\neg A) \;\Leftrightarrow\; \mathcal{M} \vDash A \text{ or } \mathcal{M} \vDash (\neg A) \;\Leftrightarrow\; \mathcal{M} \vDash A \text{ or } \mathcal{M} \nvDash A$$

[12] This actually means that we consider our meta-theory to be classical, but for the sake of simplicity, we do not want to dwell on considerations about meta-theory here.

[13] Formally, we should call them *True* and *False* (or with any other names), which are elements of the model, so as to distinguish them from $\top$ and $\bot$, which are elements of the syntax and of whom they are the interpretations. We abuse the notations in the same way for the logical connectives.

1.2.2 Heyting algebra

Heyting algebras, named after the mathematician Arend Heyting, are a generalization of truth tables for intuitionistic logic. They allow to interpret propositions in a partially ordered set that has more than just two points, where the structure of ordering reflects the logical behavior of connectives. The main intuition can be resumed by the motto:

"*the higher an element is, the truer it is*"

In particular, if $x \leq y$ and x is "true", then so is "y". Reading this order the other way around, $x \leq y$ means than x is more precise (or contains more information, is more constrained) than y. Implicative algebras, that we will present in Chapter 10, are a generalization of Heyting algebras (and of this intuition).

Definition 1.21 (Lattice)**.** A *lattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that every pair of elements $(a, b) \in \mathcal{L}^2$ has a lower bound $a \wedge b$ and an upper bound $a \vee b$. ⌟

This defines two internal laws $\wedge, \vee : \mathcal{L}^2 \to \mathcal{L}$, of which we can show[4] that they fulfill the following properties:

- for all $a, b \in \mathcal{L}$, $a \wedge b = b \wedge a \quad \text{and} \quad a \vee b = b \vee a$ (*Commutativity*)
- for all $a, b, c \in \mathcal{L}$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad \text{and} \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (*Associativity*)
- for all $a, b \in \mathcal{L}$, $\forall a, b, a \wedge (a \vee b) = a = a \vee (a \wedge b)$ (*Absorption*)
- for all $a, b \in \mathcal{L}$, $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$ (*Consistency (w.r.t. $\leq$)*)

Definition 1.22 (Heyting algebra)**.** A *Heyting algebra* $\mathcal{H}$ is defined as a bounded lattice $(\mathcal{H}, \leq)$ such that for all a and b in $\mathcal{H}$ there is a greatest element x of $\mathcal{H}$ such that

$$a \wedge x \leq b$$

This element is denoted by $a \to b$, while the upper and lower bound of $\mathcal{H}$ are respectively written $\top$ and $\bot$. ⌟

It is worth noting that by definition we have:

$$a \wedge (a \to b) \leq b$$

that is, following our intuition, that b is "truer" than $a \wedge (a \to b)$. Indeed, if a and $a \to b$ are true, so should be b according to the rule of *modus ponens*. Besides, $a \wedge (a \to b)$ is indeed more precise than just b, in that it contains information that b has not.

Given a Heyting algebra, it suffices to define the interpretation of atomic formulas to get a model of propositional intuitionistic logic. Assume that every atomic formula A is mapped to a *truth value* $|A|$ that is an element of $\mathcal{H}$, so that every axiom is mapped to $\top$. In the case of the theory NJ, this requirement simply corresponds to the equation $|\top| = \top$. Then we can naturally extend the definition of $|\cdot|$ to meet all the formulas:

$$\begin{array}{rcl@{\qquad\qquad}rcl} |A \wedge B| & \triangleq & |A| \wedge |B| & |A \Rightarrow B| & \triangleq & |A| \to |B| \\ |A \vee B| & \triangleq & |A| \vee |B| & |\neg A| & \triangleq & |A| \to \bot \end{array}$$

and extend once again the definition to judgments by:

$$|A_1, \ldots, A_n| \triangleq |A_1| \wedge \ldots \wedge |A_n| \qquad |\Gamma \vdash A| \triangleq |\Gamma| \to |A|$$

[14]The lower bound $a \wedge b$ (resp. upper bound $a \vee b$) is define as the biggest (resp. lowest) element being lower (resp. bigger) than a and b: $a \wedge b \triangleq \min\{c \in \mathscr{L} : c \leq a \wedge c \leq b\}$. From this definition, it is an easy exercise to prove the expected properties.

Finally, satisfied formulas are defined as formulas whose truth value is $\top$:

$$\mathcal{H} \vDash A \qquad \text{if and only if} \qquad |A| = \top$$

It is easy to check that the rules of propositional logic are all adequate with this interpretation and thus that this indeed defines a model.

Proposition 1.23 (Soundness)**.** *If $\mathcal{H}$ is a Heyting algebra and A a formula, then the provability of A implies its validity in $\mathcal{H}$:*

$$(\vdash A) \quad \Rightarrow \quad (\mathcal{H} \vDash A).$$

But more interestingly, intuitionistic logic has the property of being complete with respect to Heyting algebras. This means that a formula that is satisfied by any Heyting algebra is provable in natural deduction.

Proposition 1.24 (Completeness)**.** *Let A be a formula. If for any Heyting algebra $\mathcal{H}$, A is valid ($\mathcal{H} \vDash A$) then A is provable:*

$$(\forall \mathcal{H}.\mathcal{H} \vDash A) \quad \Rightarrow \quad (\vdash A).$$

As a consequence, to know that a formula A is not provable in intuitionistic logic, it is enough to find one Heyting algebra in which it is not valid. Besides, if there is also one model in which it is valid, then the formula is independent: neither A nor its negation $\neg A$ are provable, and both theories obtained by defining A or its negation are coherent, since they admit a model.

This is for instance the case of the excluded-middle. Indeed, a truth table is a particular case of Heyting algebra reduced to two values $\bot$ and $\top$, so that we already know a model in which $A \vee (\neg A)$ is valid. We can easily construct a Heyting algebra in which it is not valid. Consider the lattice $\{0, 1/2, 1\}$, by definition of $\wedge, \vee, \Rightarrow, \neg$, we get:

$p \wedge q$			
p \ q	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$p \vee q$			
p \ q	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$p \to q$			
p \ q	0	1/2	1
0	1	1	1
1/2	0	1	1
1	0	1/2	1

p	$\neg p$
0	1
1/2	0
1	0

This defines a Heyting algebra $\mathcal{H}_{1/2}$, where we can observe that $1/2 \vee (\neg 1/2) = 1/2 \vee (1/2 \to 0) = 1/2 \vee 0 = 1/2$, which invalidates the excluded-middle. So that for any formula A mapped to $1/2$, the excluded-middle is not satisfied:

$$\mathcal{H}_{1/2} \nvDash A \vee (\neg A).$$

This concludes the proof of the independence of the excluded-middle from intuitionistic logic.

Last but not least, Heyting algebras also provide a model for first-order (intuitionistic) logic, provided that they are complete as a lattice.

Definition 1.25 (Complete lattice)**.** A lattice $\mathcal{L}$ is said *complete* when every subset A of $\mathcal{L}$ admits a supremum, written $\bigwedge A$, and an infimum, written $\bigvee A$. A Heyting algebra $\mathcal{H}$ is complete if it is complete as a lattice. ⌟

Given a complete Heyting algebra $\mathcal{H}$, it is possible to construct a model for first-order logic. The interpretation of predicates and quantifiers is defined as follows:

- any k-ary predicate $P(x_1, \ldots, x_k)$ is interpreted as a k-ary function $\dot{P} : \mathcal{H}^k \to \mathcal{H}$, so that the formulas with parameters $P(m_1, \ldots, m_k)$ is interpreted by:

$$|P(m_1, \ldots, m_k)| = \dot{P}(m_1, \ldots, m_k)$$

- the universal quantifier $\forall$ is interpreted as the infimum over all possible instantiation of its variable by an element of $\mathcal{H}$:
$$|\forall x.A(x)| = \bigwedge_{m\in\mathcal{H}} |A(m)|$$

- the existential quantifier $\exists$ is interpreted as the supremum over all possible instantiation of its variable by an element of $\mathcal{H}$:
$$|\exists x.A(x)| = \bigvee_{m\in\mathcal{H}} |A(m)|$$

Observe that once again, this definition matches our intuition: $\forall x.A(x)$ is interpreted as an element that is lower (and contains indeed more information) than every possible $A(m)$; when $\exists x.A(x)$ is interpreted as an element higher (and contains indeed less information) than every possible $A(m)$.

1.2.3 Kripke forcing

Kripke models, introduced by Saul Kripke [89, 90], give another semantics for intuitionistic logic. They are quite different of Heyting algebras in that they are not based on a lattice and, most importantly, because the relation of satisfiability is defined in a very different way. Besides, we will use an intuition based on Kripke forcing in Chapter 6 (to define the environment-passing style translation of a classical call-by-need calculus), which also motivates their presentation in this section.

Intuitively, a Kripke model is a universe containing different worlds. Every world contains a specific information, and this information can only be refined in the future of this world. Each world is thus connected to the possible worlds accessible from it, which all contain at least the same information. We shall present another metaphor due to Van Dalen [158] after giving the formal definition of Kripke models.

Definition 1.26 (Kripke model)**.** A *Kripke model* is a quadruple $\mathcal{M} = (\mathcal{W}, \leq, D, V)$ where:

- $\mathcal{W}$ is a set of possible worlds,
- $\leq$ is a pre-order and denotes the relation of accessibility between worlds,
- D is a function that maps every world w to the set $D(w)$ of terms defined in it,
- V is a function that maps a k-ary predicate $P(x_1,\ldots,x_k)$ and a world w to the set of tuple $(t_1,\ldots,t_k) \in D(w)^k$ such that $P(t_1,\ldots,t_k)$ is true in w.

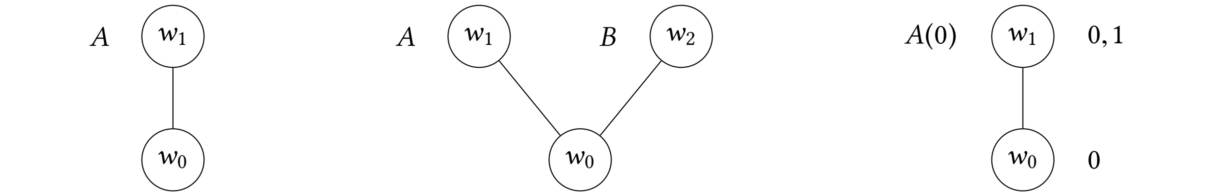
The set $\mathcal{W}$ is supposed to be given with a distinguished world $w_0 \in \mathcal{W}$ such that every other world are accessible from it:
$$\forall w' \in \mathcal{W}, w_0 \leq w'$$

Furthermore, D and V are required to be monotonic in the sense that if an element is defined (resp. an atomic formula holds) in a given world w, then it has to be defined in every world w' accessible from w. Formally, for all $w, w' \in W$ and any predicate P:

$$w \leq w' \Rightarrow D(w) \subseteq D(w') \qquad\qquad w \leq w' \Rightarrow V(P,w) \subseteq V(P,w')$$

┘

Given a Kripke model $\mathcal{M} = (\mathcal{W}, \leq, D, V)$, we define a relation $w \Vdash A$ that denotes the validity of the formula A in the world w. We say that the world w *forces* A and we call $\Vdash$ the *forcing relation.* This



$\mathcal{M} \nvDash A \vee \neg A$ $\qquad$ $\mathcal{M} \nvDash \neg(A \wedge B) \Rightarrow (\neg A \vee \neg B)$ $\qquad$ $\mathcal{M} \nvDash (\neg\forall x A(x)) \Rightarrow \exists x.\neg A(x)$

(a) Excluded-middle $\qquad$ (b) De Morgan's law $\qquad$ (c) $(\neg\forall x.A) \neq (\exists x.\neg A)$

Figure 1.2: Examples of Kripke counter-models

relation is defined by induction on the structure of formulas:

$$
\begin{array}{lcl}
w \Vdash P(t_1, \ldots t_k) & \triangleq & (t_1, \ldots, t_k) \in V(P, w) \\
w \Vdash A \wedge B & \triangleq & (w \Vdash A) \wedge (w \Vdash B) \\
w \Vdash A \vee B & \triangleq & (w \Vdash A) \vee (w \Vdash B) \\
w \Vdash A \Rightarrow B & \triangleq & \forall w' \geq w. w' \Vdash A \Rightarrow w' \Vdash B \\
w \Vdash \neg A & \triangleq & \forall w' \geq w. w' \nVdash A \\
w \Vdash \forall x.A(x) & \triangleq & \forall w' \geq w. \forall d \in D(w'), w' \Vdash A(d). \\
w \Vdash \exists x.A(x) & \triangleq & \exists d \in D(w), w \Vdash A(d) \\
w \Vdash \Gamma \vdash A & \triangleq & (\forall C \in \Gamma, w \Vdash C) \Rightarrow w \Vdash A
\end{array}
$$

Finally, we say that a model $\mathcal{M}$ satisfies a formula A (resp. a judgment $\Gamma \vdash A$) and write $\mathcal{M} \vDash A$ if and only if $w_0 \Vdash A$.

Remark 1.27. Van Dalen describes Kripke models using a different intuition. Rather than poorly reformulating his point of view, we quote his metaphor as such (see [158, pp.12-13]):

> *The basic idea is to mimic the mental activity of Brouwer's individual, who creates all of mathematics by himself. This idealized mathematician, also called creating subject by Brouwer, is involved in the construction of mathematical objects, and in the construction of proofs of statements. This process takes place in time. So at each moment he may create new elements, and at the same time he observes the basic facts that hold for his universe so far. In passing from one moment in time to the next, he is free how to continue his activity, so the picture of his possible activity looks like a partially ordered set (even like a tree). At each moment there is a number of possible next stages. These stages have become known as possible worlds. Observe that the 'truth' at a node w essentially depends on the future. This is an important feature in intuitionism (and in constructive mathematics, in general). The dynamic character of the universe demands that the future is taken into account. This is particularly clear for ∀. If we claim that "all dogs are friendly", then one unfriendly dog in the future may destroy the claim.*

⌟

This semantics is also sound and complete with respect to intuitionistic logic, and allows to define very simple models that do not satisfy classical principles. We give as examples in Figure 1.2 counter-models for the excluded-middle, the De Morgan's law and the equivalence between $\neg\forall x.A$ and $\exists x.\neg A$. Once again, thanks to the completeness of Kripke models, this is enough to prove that these principles (which all hold in the two-points Heyting algebra) are independent from intuitionistic logic.

1.2.4 The standard model of arithmetic

Lastly, we shall introduce briefly the standard model of arithmetic. This model is defined as the $\mathscr{L}$-structure (where $\mathscr{L}$ refers to the language of arithmetic) whose domain is the set $\mathbb{N}$ of natural numbers and in which each symbol of $\mathscr{L}$ is interpreted canonically (the symbol '0' is interpreted by 0, the symbol 's' by the function $n \mapsto n+1$, and so on). Abusing the notation, this $\mathscr{L}$-structure build on the set $\mathbb{N}$ is itself written $\mathbb{N}$. Formally, to each closed term t of the language $\mathscr{L}$ is associated a natural number $\mathrm{Val}(t)$, called the *value of* t. This value is defined inductively on the structure of t by:

$$
\begin{array}{rclcrcl}
\mathrm{Val}(0) & \triangleq & 0 & \quad & \mathrm{Val}(t+u) & \triangleq & \mathrm{Val}(t)+\mathrm{Val}(u) \\
\mathrm{Val}(s(t)) & \triangleq & \mathrm{Val}(t)+1 & & \mathrm{Val}(t\times u) & \triangleq & \mathrm{Val}(t)\,\mathrm{Val}(u)
\end{array}
$$

and satisfies that for all $n \in \mathbb{N}$, $\mathrm{Val}(\overline{n}) = n$, where $\overline{n} = s^n(0)$. The satisfiability relation $\mathbb{N} \vDash A$ is defined again by induction on the structure of A by:

$$
\begin{array}{lcl}
\mathbb{N} \vDash t = u & \triangleq & Val(t) = Val(u) \\
\mathbb{N} \nvDash \bot & & \\
\mathbb{N} \vDash A \Rightarrow B & \triangleq & \mathbb{N} \nvDash A \vee \mathbb{N} \vDash B \\
\mathbb{N} \vDash A \wedge B & \triangleq & \mathbb{N} \vDash A \wedge \mathbb{N} \vDash B \\
\mathbb{N} \vDash A \vee B & \triangleq & \mathbb{N} \vDash A \vee \mathbb{N} \vDash B \\
\mathbb{N} \vDash \forall x.A & \triangleq & \text{for all } n \in \mathbb{N},\ \mathbb{N} \vDash A[\overline{n}/x]
\end{array}
$$

It is easy to show that this indeed defines a model of Peano arithmetic, and in particular that it entails its consistency. Yet, it should be observed that this definition is infinitary, since the interpretation of $\forall x.A$ requires to know the interpretation of $A[n/x]$ for all $n \in \mathbb{N}$. This implies that the meta-theory in which we reason needs to account for mechanisms allowing to construct infinitary objects and to reason on them. For instance, this is not possible within Peano arithmetic, where all the objects are finite natural numbers. Hence Peano arithmetic *a priori* cannot prove its own consistency, at least by this way. Gödel actually closed the problem with his second incompleteness theorem, which states that a consistent theory $\mathscr{T}$ containing (PA) cannot prove its own consistency unless it is inconsistent.