A relative PCF-definability result for strongly stable functions and some corollaries*

Thomas Ehrhard
Institut de Mathématiques de Luminy
C.N.R.S. U.P.R. 9016

Abstract

We prove that, in the hierarchy of simple types based on the type of natural numbers, any finite strongly stable function is equal to the application of the semantics of a PCF-definable functional to some strongly stable (generally not PCF-definable) functionals of type two. Applying a logical relation technique, we derive from this result that the strongly stable model of PCF is the extensional collapse of its sequential algorithms model.

Introduction

The strongly stable model of PCF presented in [BE94, Ehr93] is not fully abstract. In other words, although all the functions of type one of this model are sequential and thus PCF-definable, there exist types whose semantics contain finite elements which are not the semantics of any PCF-term (by a theorem of Milner [Mil77]). Typical examples of such non definable functions are the three counter-examples presented at the end of [Cur93a]. All these counter-examples are of type two (their types are of the shape $(\iota^k \to \iota) \to \iota$, where ι is the basic type of natural numbers). Of course, by applying PCF-terms to such non PCF-definable functionals of type two, one gets generally non PCF-definable functionals at any type. A very natural question is then: are all non PCF-definable functionals of this last shape? The object of this paper is to give a positive answer to this question.

We shall use this result for relating closely the strongly stable model of PCF to its sequential algorithms model. For this purpose, we shall define a logical relation between the two models with the intended meaning that a strongly stable function is related to a sequential algorithm if they compute the same thing. Using the relative definability result, we shall prove that, at any type, any (finite) strongly stable functional is related to at least one sequential algorithm[1]. We interpret this result as meaning that any strongly stable functional of the strongly stable model of PCF is sequentially computable (although not necessarily PCF-computable). By some rather standard argument, we shall derive from this result that the equational theory induced on PCF by its strongly stable model contains the theory induced by its sequential algorithms model.

Then we prove a stronger result by investigating further properties of this logical relation. It induces, at any type, a partial equivalence relation on the sequential algorithms of that type: two

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[1]The converse is not true: there are sequential algorithms which do not compute any function, by lack of extensionality. A classical example is the functional which maps the *strict* left "and" algorithm to "true" and the *strict* right "and" algorithm to "false". These two "and" algorithms are different as sequential algorithms but equal as strongly stable function, so they cannot be mapped to different values by a functional.

sequential algorithms are equivalent if they both compute a strongly stable function, and if these strongly stable functions are equal. We show that this partial equivalence relation is identical to the standard "collapsing" logical partial equivalence relation. As a consequence, the strongly stable model of finitary PCF (that is, PCF with the type of booleans as ground type) is isomorphic to the extensional collapse of its sequential algorithms model. Last, we generalize this result to the strongly stable and sequential algorithms models of (non finitary) PCF.

In order to make the presentation reasonably short, we only recall the material about strong stability and sequential algorithms. So the reader is assumed to be familiar with the usual notions of denotational semantics and domain theory: PCF and its models (see [Plo77]), stable semantics, dI-domains, traces, sequentiality (see for instance the preliminary sections of [Ehr96]).

First of all, we shall give some motivation for considering "derived" notions such as strongly stable functions or sequential algorithms for the purpose of modeling sequential behaviors of functions and functionals. Actually, there is a nice notion of *sequential functions*, which can be defined in various equivalent ways (see for instance [Vui74, Mil77, KP78]). So why do not we build a cartesian closed category of reasonably behaved domains (for instance, dI-domains, or better, qualitative domains) and sequential *functions*? The following semi-formal discussion aims at convincing the reader that such an attempt must fail, at least if we are seeking for a category satisfying requirements that we consider as reasonable.

More precisely, we prove that there is no CCC $\mathcal{C}$ satisfying the following conditions:

i) Any object of $\mathcal{C}$ is a pair $A = (A_*, A^*)$ where A_* is a partially ordered set having a least element denoted by $\perp$, and A^* is a set of *cells* on A_*: A is endowed with a binary relation ϵ on $A_* \times A^*$ such that $\perp \epsilon \alpha$ never holds, and if $x \epsilon \alpha$ and $x' \geq x$, then $x' \epsilon \alpha$. Intuitively, $x \epsilon \alpha$ means that x fills the cell α.

ii) Any morphism from A to B is a monotone *function* $A_* \to B_*$ satisfying the following *sequentiality* condition: for any $x \in A_*$ and any $\beta \in B^*$ such that $f(x) \not\epsilon \beta$, either for any $x' \geq x$ one has $f(x') \not\epsilon \beta$, or there exists $\alpha \in A^*$ such that $x \not\epsilon \alpha$, and for any $x' \geq x$, if $f(x') \epsilon \beta$, then $x' \epsilon \alpha$. This is a rephrasement of the standard notion of sequentiality we mentioned above.

iii) Let T denote the terminal object of $\mathcal{C}$. We assume that the function

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(\mathrm{T},\mathrm{A}) & \to & A_* \\ f & \mapsto & f(\perp) \end{array}$$

is *onto*. As a consequence, $\mathcal{C}$ has enough points, and all the categorical operations (projections, pairing, evaluation and exponential transpose) are set-theoretic (that is, are defined like in the category of sets and functions). Observe also that $\mathrm{T}_* = \{\perp\}$ and that, for any objects A and B of $\mathcal{C}$, $(A \times B)_*$ is in bijective correspondence with $A_* \times B_*$ and $(A \Rightarrow B)_*$ is in one to one correspondence with $\mathrm{Hom}_{\mathcal{C}}(\mathrm{A},\mathrm{B})$. We assume furthermore that the bijective correspondence between $(A \times B)_*$ and $A_* \times B_*$ is an order isomorphism, considering $A_* \times B_*$ as endowed with the product order.

iv) If A and B are objects of $\mathcal{C}$, we assume that for any element γ of $(A \times B)^*$ there exists $\alpha \in A^*$ such that

$$\forall (x,y) \in A_* \times B_* \quad (x,y) \epsilon \gamma \quad \text{iff} \quad x \epsilon \alpha$$

and then we write $\gamma = 1\alpha$, or there exists $\beta \in B^*$ such that

$$\forall (x,y) \in A_* \times B_* \quad (x,y) \epsilon \gamma \quad \text{iff} \quad y \epsilon \beta$$

and then we write $\gamma = 2\beta$. This assumption is the most important one. It expresses that a cell of $A \times B$ is a cell of A or a cell of B.

v) Our last assumption is rather innocent. $\mathcal{C}$ must have an object of booleans Bool such that Bool_* is the standard flat domain of booleans $\{\bot, \text{true}, \text{false}\}$ and Bool^* has only one element $*$, which is filled by an element a of Bool_* iff a is different from $\bot$. Furthermore, for any $n \in \mathbf{N}$, any sequential function $(\text{Bool}_*)^n \to \text{Bool}_*$ is a morphism of $\mathcal{C}$.

Assume that such a category $\mathcal{C}$ exists. Let $A = \text{Bool} \Rightarrow \text{Bool}$, the object of morphisms from Bool to Bool in $\mathcal{C}$. We know by iii) that A_* is $\text{Hom}_{\mathcal{C}}(\text{Bool}, \text{Bool})$, but we don't know anything about its order relation. We claim however that $\lambda u.\bot$ is its least element. Actually, let $f \in A_*$. Let $g : (\text{Bool}_*)^2 \to \text{Bool}_*$ be defined by

$$g(v, u) = \begin{cases} \bot & \text{if } v = \bot \\ f(u) & \text{otherwise} \end{cases} .$$

Then g is obviously sequential, and hence is a morphism in $\mathcal{C}$ by v). By iii), its exponential transpose $g' : \text{Bool} \to A$ in $\mathcal{C}$ maps $\bot$ to $\lambda u.\bot$ and true to f, and hence $\lambda u.\bot \leq f$ in A_* because g' is monotone. So the least element $\bot$ of A_* is $\lambda u.\bot$.

Now consider the evaluation function

$$\begin{array}{rcl} \text{Ev} : A_* \times \text{Bool}_* & \to & \text{Bool}_* \\ (f, x) & \mapsto & f(x) \end{array}$$

which is a morphism in $\mathcal{C}$ by iii) and hence satisfies the sequentiality condition by ii). We have $\text{Ev}(\bot, \bot) = \bot$, but there are clearly elements (f, x) of $A_* \times \text{Bool}_*$ such that $f(x) \neq \bot$. Hence there exists $\gamma \in (A \times \text{Bool})^*$ such that, for any $(f, x) \in A_* \times \text{Bool}_*$, if $f(x) \neq \bot$, then $(f, x) \,\epsilon\, \gamma$. By our assumption iv) about $\mathcal{C}$, there are two possibilities for γ: the case $\gamma = 2*$ where $*$ is the single element of Bool^* is impossible, since $\text{Ev}(\lambda u.\text{true}, \bot) \,\epsilon\, *$ but $(\lambda u.\text{true}, \bot) \not\epsilon\, 2*$ (observe that $\lambda u.\text{true} \in \text{A}_*$). Hence we must have $\gamma = 1\alpha$ for some $\alpha \in A^*$ that we choose now once and for all. Take any $f \in A_*$ such that $f \neq \bot$. There exists $x \in \text{Bool}_*$ such that $f(x) \neq \bot$, that is $\text{Ev}(f, x) \,\epsilon\, *$, and hence $(f, x) \,\epsilon\, \gamma$, that is $f \,\epsilon\, \alpha$. To summarize:

$$\forall f \in A_* \quad f \neq \bot \Rightarrow f \,\epsilon\, \alpha \tag{1}$$

and from this fact we shall derive a contradiction.

Let $h : (\text{Bool}_*)^3 \to \text{Bool}_*$ be the conditional function defined by

$$h(x, y, z) = \begin{cases} \bot & \text{if } z = \bot \\ x & \text{if } z = \text{true} \\ y & \text{if } z = \text{false} \end{cases}$$

which, by v), is a morphism $\text{Bool}^3 \to \text{Bool}$ in $\mathcal{C}$ since it is a sequential function. Let $h' : \text{Bool}^2 \to A$ be the morphism of $\mathcal{C}$ obtained by curryfying h with respect to its last argument. We have $h'(\bot, \bot) = \lambda z.\bot = \bot \not\epsilon\, \alpha$, and $h'(\text{true}, \bot) \neq \bot$, hence $h'(\text{true}, \bot) \,\epsilon\, \alpha$ by (1) and similarly $h'(\bot, \text{true}) \,\epsilon\, \alpha$. And this is impossible, since g' is a morphism of $\mathcal{C}$ and hence has to satisfy the sequentiality condition.

The reader acquainted with sequential algorithms can observe that most of our argument could be adapted to sequential algorithms as well, and especially, there also exists in that case a cell

α satisfying (1) ("initial cell" in the terminology of concrete data structures). However, in the framework of sequential algorithms, *it is not true* that $h'(\perp,\perp) = \perp$: the sequential algorithm $h'(\perp,\perp) : \mathrm{Bool} \to \mathrm{Bool}$ asks for the value of its argument, and, whatever be the value of this argument, does not yield any result, whereas the sequential algorithm $\perp$ does not even ask for the value of its argument. So in the case of sequential algorithms, we have $h'(\perp,\perp) \in \alpha$ and there is no contradiction anymore.

1 Types and terms

We consider the hierarchy of finite types based on the type ι of natural numbers. Types are defined as follows:

$$\sigma = \iota \mid \sigma \to \sigma \quad .$$

As usual, if $\sigma_1, \ldots, \sigma_n$ and σ are types, the type $\sigma_1 \to (\sigma_2 \to \ldots(\sigma_n \to \sigma)\ldots)$ is denoted by $\sigma_1, \ldots, \sigma_n \to \sigma$. This expression denotes σ if $n = 0$. If σ and τ are types and if $n \geq 0$ is a natural number, then $\sigma^n \to \tau$ is a shorthand for $\sigma, \ldots, \sigma \to \tau$ (n occurrences of σ).

We define the *degree* of a type σ as a natural number $\|\sigma\|$ by induction on types as usual:

$$\|\iota\| = 0 \quad \text{and} \quad \|\sigma \to \tau\| = \mathrm{Max}(\|\sigma\| + 1, \|\tau\|)$$

so that one has

$$\|\sigma_1 \to \ldots \to \sigma_k \to \iota\| = \mathop{\mathrm{Max}}_{1 \leq j \leq k} (\|\sigma_j\| + 1) \ .$$

We call *simple second order type* any type of the hierarchy of finite types of the shape $(\iota^n \to \iota) \to \iota$ $(n \geq 1)$.

The programming language we consider is a version of PCF (see [Plo77]) which has the type of natural numbers as unique ground type (just for shortening the base cases of inductions). We decide to represent boolean values by natural numbers as follows: 0 corresponds to "true" and "false" is represented by any non-zero natural number.

A typical syntax for this language is the following (types are written as superscript when needed):

$$\begin{array}{rcll}
M & = & u^\sigma & \text{variables} \\
& \mid & n^\iota & \text{numerical constants, } n = 0, 1, 2 \ldots \\
& \mid & \mathrm{p}^{\iota\to\iota} \mid \mathrm{s}^{\iota\to\iota} & \text{predecessor and successor operators} \\
& \mid & \mathrm{If}^{\iota\to\iota\to\iota\to\iota} & \text{conditional operator} \\
& \mid & (M^{\sigma\to\tau} N^\sigma)^\tau & \text{application} \\
& \mid & (\lambda x^\sigma . M^\tau)^{\sigma\to\tau} & \text{abstraction} \\
& \mid & \mathrm{Y}^{(\sigma\to\sigma)\to\sigma} & \text{fixpoint functional.}
\end{array}$$

We do not give the operational semantics of PCF, which may be found in [Plo77].

A model $\mathcal{M}$ of PCF consists of a cartesian closed category, still denoted by $\mathcal{M}$, together with an object $\iota_\mathcal{M}$ of $\mathcal{M}$ for interpreting the type of natural numbers. From these, we can associate to any type σ an object $[\sigma]^\mathcal{M}$ of $\mathcal{M}$ as follows: $[\iota]^\mathcal{M} = \iota_\mathcal{M}$ and $[\sigma \to \tau]^\mathcal{M} = [\sigma]^\mathcal{M} \Rightarrow [\tau]^\mathcal{M}$ (objects of morphisms in $\mathcal{M}$). The model $\mathcal{M}$ has also to provide an interpretation for the basic operators of PCF (numerical constants, p, s, If and Y). They are points in the interpretation of the corresponding types. For instance, the interpretation of p is a point in $[\iota \to \iota]^\mathcal{M}$. Remember that a point of an object A of $\mathcal{M}$ is a morphism from the terminal object of $\mathcal{M}$ to A, and that the points of $A \Rightarrow B$ are in bijective correspondence with the morphisms from A to B in the category $\mathcal{M}$. Using these data, we can associate to any term M of PCF a morphism in $\mathcal{M}$. More precisely, if M is a PCF term of type τ with free variables among the list $l = \langle x_1^{\sigma_1}, \ldots, x_k^{\sigma_k} \rangle$, using a rather standard categorical machinery (see for instance [LS86, Cur93a]), we can associate to M a morphism $[M]_l^{\mathcal{M}}$ from $\prod_{i=1}^{k} [\sigma_i]^{\mathcal{M}}$ to $[\tau]^{\mathcal{M}}$. All these data must be such that, for any terms M and N of the same type, whose variables are among a common list of variables l, if $M = N$ (in the equational theory of PCF, including η-conversion), then $[M]_l^{\mathcal{M}} = [N]_l^{\mathcal{M}}$. By the way, β and η-conversion are automatically satisfied because $\mathcal{M}$ is cartesian closed.

2 Strong stability

Let us first introduce some notations.

Let E be a set. We denote by $\mathcal{P}^*_{\mathrm{fin}}(E)$ the set of all finite and non-empty subsets of E. We write $x \subseteq^*_{\mathrm{fin}} E$ when x is a finite and non-empty subset of E.

We denote by $\#E$ the cardinality of E.

Let E and F be two sets. If $C \subseteq E \times F$, we denote by C_1 or C_E the first projection of C and C_2 or C_F its second projection. We say that C is a *pairing* of E and F if $C_1 = E$ and $C_2 = F$.

The disjoint union of E and F will be denoted by $E + F$ and represented by $G = (\{1\} \times E) \cup (\{2\} \times F)$. If $C \subseteq G$, we denote by $C_1 = \{a \in E \mid (1, a) \in C\}$ its first component and by $C_2 = \{b \in F \mid (2, b) \in C\}$ its second component.

We say that E is a *multisection* or simply a *section* of F and we write $E \lhd F$ if

$$\forall a \in E \exists b \in F \ a \in b \quad \text{and} \quad \forall b \in F \exists a \in E \ a \in b \ .$$

This means that $E \subseteq \bigcup F$ and that $E \cap b$ is non empty for all $b \in F$.

Similarly, if both E and F are subsets of a partially ordered set $(V, \leq)$, we say that E is *Egli-Milner below* F and write $E \sqsubseteq F$ if

$$\forall a \in E \exists b \in F \ a \leq b \quad \text{and} \quad \forall b \in F \exists a \in E \ a \leq b \ .$$

2.1 dI-domains with coherence

We describe briefly the cartesian closed category of *dI-domains with coherence* that we introduced in [BE94]. We refer to this article for proofs of the results stated here. We shall not need this general framework until Section 4. Actually, we shall consider first a very special (but very well behaved) class of dI-domains with coherence: the *hypercoherences*.

Definition 1 *A* dI-domain with coherence *is a couple $(D, \mathcal{C}(D))$ (often abusively simply written D) where D is a dI-domain, and $\mathcal{C}(D)$ is a set of finite and non empty subsets of D satisfying:*

- *For any $x \in D$, $\{x\} \in \mathcal{C}(D)$.*
- *For any $A \in \mathcal{C}(D)$, for any finite and non-empty subset B of D, if $B \sqsubseteq A$, then $B \in \mathcal{C}(D)$.*
- *For any family $A_1, \ldots, A_n$ $(n \geq 1)$ of directed subsets of D, if for any $x_1 \in A_1, \ldots, x_n \in A_n$ one has $\{x_1, \ldots, x_n\} \in \mathcal{C}(D)$, then $\{\bigvee A_1, \ldots, \bigvee A_n\} \in \mathcal{C}(D)$.*

We define now the morphisms of this category.

Definition 2 *Let D and D' be dI-domains with coherence. A* strongly stable function *from D to D' is a continuous function $f : D \to D'$ such that, for any $A \in \mathcal{C}(D)$, $f(A) \in \mathcal{C}(D')$ and $f(\bigwedge A) = \bigwedge f(A)$.*

This definition is motivated by Proposition 4 which relates strong stability to the standard Vuillemin-Milner-Kahn-Plotkin notion of sequentiality in a special case which, roughly speaking, corresponds to the interpretations of "data types" in the strongly stable model. Again, we refer to [BE94] for more details.

The category of dI-domains with coherence and strongly stable functions will be denoted by **DIC**.

Proposition 1 *The category* **DIC** *is cartesian closed. Let D and D' be dI-domains with coherence.*

1. *The cartesian product of D and D' is $(E, \mathcal{C}(E))$, where E is the cartesian product of the dI-domains D and D', endowed with the product order, and a subset of E is in $\mathcal{C}(E)$ iff its first projection is in $\mathcal{C}(D)$, and its second projection is in $\mathcal{C}(D')$. This cartesian product will be denoted by $D \times D'$ (or $(D, \mathcal{C}(D)) \times (D', \mathcal{C}(D'))$).*

2. *The object of morphisms from D to D' is the set E of all strongly stable functions from D to D', stably ordered, which turns out to be a dI-domain. Let $f_1, \dots, f_n$ be strongly stable functions from D to D' (with $n \geq 1$). Then $\{f_1, \dots, f_n\} \in \mathcal{C}(E)$ iff for any family $x_1, \dots, x_m \in D$ such that $\{x_1, \dots, x_m\} \in \mathcal{C}(D)$ and any pairing K of $\{1, \dots, n\}$ and $\{1, \dots, m\}$, one has $\{f_i(x_j) \mid (i,j) \in K\} \in \mathcal{C}(D')$ and $(\bigwedge_{i=1}^n f_i)(\bigwedge_{j=1}^n x_j) = \bigwedge_{(i,j)\in K} f_i(x_j)$. This object of morphisms will be denoted by $D \Rightarrow D'$ or $(D, \mathcal{C}(D)) \Rightarrow (D', \mathcal{C}(D'))$.*

The operations on morphisms associated to these constructions: projections, pairing, evaluation, transposition (curryfication) are defined in the usual set-theoretical way (just like in the category of sets and functions).

By continuity of the f_i's and by the last requirement of Definition 1, the characterization of $\mathcal{C}(E)$ given in 2 above still holds if we require the x_j's to be compact (isolated).

We shall use the following technical lemma.

Lemma 1 *Let D and D' be two dI-domains with coherence. Let $f : D \to D'$ be a strongly stable function. Let $g : D \to D'$ be continuous and such that $g \leq f$ in the* stable *order. Then g is strongly stable.*

The proof is easy. See for instance [BE94].

2.2 Hypercoherences

In [Ehr93], we introduced the model of *hypercoherences* as a simplified framework where strong stability makes sense. We recall here the basic definitions and the properties of this model that we use in the sequel. For proofs and details, we refer to the previously mentioned article.

Definition 3 *A* hypercoherence *X is a pair $(|X|, \Gamma(X))$ where $|X|$ is a denumerable set (the* web*) and $\Gamma(X)$ is a subset of $\mathcal{P}^*_{\mathrm{fin}}(X)$ (the* atomic coherence*) such that, for any $a \in |X|$, one has $\{a\} \in \Gamma(X)$.*

If X is a hypercoherence, we denote by $\Gamma^*(X)$ and call *strict atomic coherence of* X the set of all elements of $\Gamma(X)$ which are not singletons (observe that X can be described by $\Gamma^*(X)$ as well as by $\Gamma(X)$).

A *qualitative domain* (see [Gir86]) is a dI-domain where any prime element is atomic. It can be seen as a set (of sets) closed under subsets and directed unions, and containing the empty set (the order relation being of course the inclusion). The *web* $|Q|$ of the qualitative domain Q is the set of all elements a such that $\{a\} \in Q$, that is, the set of all prime elements of Q.

To any hypercoherence, we associate as follows a dI-domain (more precisely, a qualitative domain) with coherence. Observe that not all qualitative domains with coherence can be obtained in this way.

Definition 4 *Let X be a hypercoherence. We define* $\mathrm{qD}(X)$ *and* $\mathcal{C}(X)$ *as follows :*

$$\mathrm{qD}(X) = \{x \subseteq |X| \mid \forall u \subseteq^*_{\mathrm{fin}} |X| \;\; u \subseteq x \Rightarrow u \in \Gamma(X)\}$$

and

$$\mathcal{C}(X) = \{A \subseteq^*_{\mathrm{fin}} \mathrm{qD}(X) \mid \forall u \subseteq^*_{\mathrm{fin}} |X| \;\; u \lhd A \Rightarrow u \in \Gamma(X)\} \;.$$

$\mathrm{qD}(X)$ *will be called the* qualitative domain generated by X *and its elements are called the* states *of* $\mathrm{qD}(X)$*, and* $\mathcal{C}(X)$ *will be called the* state coherence generated by X*. The set of finite states of* $\mathrm{qD}(X)$ *will be denoted by* $\mathrm{qD}_{\mathrm{fin}}(X)$*.*

It is clear that $\mathrm{qD}(X)$ is always a qualitative domain, and its web is $|X|$ by our only requirement about hypercoherences (remember that qualitative domains are dI-domains). Moreover, $(\mathrm{qD}(X), \mathcal{C}(X))$ is a dI-domain with coherence.

Definition 5 *Let X and Y be hypercoherences. A* strongly stable function *from X to Y is a strongly stable function from* $(\mathrm{qD}(X), \mathcal{C}(X))$ *to* $(\mathrm{qD}(Y), \mathcal{C}(Y))$ *(in* **DIC***).*

Observe that, if X is a hypercoherence, any bounded, non-empty and finite subset of $\mathrm{qD}(X)$ is in $\mathcal{C}(X)$ (this holds more generally for dI-domains with coherence). For this reason, any strongly stable function $X \to Y$ is stable from $\mathrm{qD}(X)$ to $\mathrm{qD}(Y)$, and thus we can use *traces* to represent strongly stable functions faithfully.

We denote by **HC** the category of hypercoherences and strongly stable functions.

Let X and Y be hypercoherences. Let $X \times Y$ be the hypercoherence defined by $|X \times Y| = |X| + |Y|$ and $w \in \Gamma(X \times Y)$ if $w \subseteq^*_{\mathrm{fin}} |X \times Y|$ and

$$(w_2 = \emptyset \Rightarrow w_1 \in \Gamma(X)) \text{ and } (w_1 = \emptyset \Rightarrow w_2 \in \Gamma(Y)) \;.$$

Let $X \Rightarrow Y$ be the hypercoherence Z whose web is the set of all (x, b) where $x \in \mathrm{qD}(X)$ is finite and $b \in |Y|$, and whose atomic coherence is given by: $w \in \Gamma(Z)$ if $w \in \mathcal{P}^*_{\mathrm{fin}}(|Z|)$ and

$$w_1 \in \mathcal{C}(X) \Rightarrow (w_2 \in \Gamma(Y) \;\text{ and }\; (\#w_2 = 1 \Rightarrow \#w_1 = 1)) \;.$$

Proposition 2 *The category* **HC** *is cartesian closed. If X and Y are two hypercoherences, their cartesian product is $X \times Y$ and the object of morphisms from X to Y is $X \Rightarrow Y$.*

Moreover, **HC** *is a sub-cartesian-closed category of* **DIC***.*

The latter statement means that

$$(\mathrm{qD}(X \times Y), \mathcal{C}(X \times Y)) \cong (\mathrm{qD}(X), \mathcal{C}(X)) \times (\mathrm{qD}(Y), \mathcal{C}(Y))$$

naturally and that similarly

$$(\mathrm{qD}(X \Rightarrow Y), \mathcal{C}(X \Rightarrow Y)) \cong (\mathrm{qD}(X), \mathcal{C}(X)) \Rightarrow (\mathrm{qD}(Y), \mathcal{C}(Y))$$

naturally. The inverse of this latter isomorphism maps a strongly stable function to its trace. For this reason, we shall often confuse strongly stable functions between hypercoherences with their traces.

In the sequel, we shall consider a hypercoherence which will play a central role : the hypercoherence N of flat natural numbers. Its web is $\mathbf{N}$, the set of natural numbers, and $\Gamma(\mathrm{N})$ is the set of all singletons of $|\mathrm{N}| = \mathbf{N}$. One easily checks that, up to an order isomorphism, $\mathrm{qD}(\mathrm{N}) = \mathbf{N}_\perp$, the flat domain of natural numbers, and so, more generally, for any $n \in \mathbf{N}$, $\mathrm{qD}(\mathrm{N}^n) = \mathbf{N}_\perp^n$.

Observe that a finite and non-empty subset w of $|X \Rightarrow \mathrm{N}|$ belongs to $\Gamma^*(X \Rightarrow \mathrm{N})$ iff $w_1 \notin \mathcal{C}(X)$.

The following lemma plays a central role in the theory of strong stability (see [CE94, Ehr96] for other applications). It cannot be generalized to arbitrary dI-domains with coherence (or even to arbitrary qualitative domains with coherence).

Lemma 2 *Let X be a hypercoherence. Let $A \in \mathcal{C}(X)$ and let $n = \#A$. There exists a set $G \in \mathcal{C}(\mathrm{N}^n)$ with $\#G = n$ and a strongly stable function $\varphi : \mathrm{N}^n \to X$ such that $\varphi(G) = A$. Moreover, if all the elements of A are finite, there exists a finite such function φ.*

Such a set G will be called a *Berry set.*

Proof: For any natural number $n \geq 1$ we define a family $\{\gamma_j^n\}_{j=1,\ldots,n}$ of elements of $\mathrm{qD}(\mathrm{N}^n)$ as follows :

$$\gamma_j^n = \{(1, n-j+1), \ldots, (j-1, n-1)\} \cup \{(j+1, 1), \ldots, (n, n-j)\} .$$

It is easily checked that the set $\{\gamma_j^n\}_{j=1,\ldots,n}$ is in $\mathcal{C}(\mathrm{N}^n)$, but that no proper subset of this set of cardinality strictly greater than 1 is in $\mathcal{C}(\mathrm{N}^n)$.

Let $A = \{x_1, \ldots, x_n\}$ be any element of $\mathcal{C}(X)$ with $n = \#A$. Let $x_0 = \bigcap_{i=1}^n x_i$. Let us define the following set t :

$$t = \{(\emptyset, a) \mid a \in x_0\} \cup \{(\gamma_i^n, a) \mid i = 1, \ldots, n \text{ and } a \in x_i \setminus x_0\} .$$

Then we claim that $t \in \mathrm{qD}(\mathrm{N}^n \Rightarrow X)$. Actually, let u be any finite and non-empty subset of t, and assume that $u_1 \in \mathcal{C}(\mathrm{N}^n)$. Observe first that, by construction of t, one has : $\forall a \in u_2\ \exists x \in A \quad a \in x$. We consider three cases :

- $\emptyset \in u_1$. Let $a \in u_2$ be such that $(\emptyset, a) \in u$. Then we have $a \in x_0$ and hence $\forall x \in A \quad a \in x$, hence $u_2 \lhd A$, hence $u_2 \in \Gamma(X)$. If furthermore u_2 is the singleton $\{a\}$, then u_1 must obviously be the singleton $\{\emptyset\}$.
- $\emptyset \notin u_1$ and u_1 is a singleton $\{\gamma_i^k\}$ (for some $i \in \{1, \ldots, k\}$). Then $u_2 \subseteq x_i$ and hence $u_2 \in \Gamma(X)$.
- $\emptyset \notin u_1$ and $\#u_1 \geq 2$. Then we know that $u_1 = \{\gamma_i^n\}_{i=1,\ldots,n}$ since $u_1 \in \mathcal{C}(\mathrm{N}^n)$. Let $i \in \{1, \ldots, n\}$. Let $a \in u_2$ be such that $(\gamma_i^n, a) \in u$. Then we have $a \in x_i$ by construction of t and hence $u_2 \lhd A$, thus $u_2 \in \Gamma(X)$. Furthermore, if u_2 is a singleton $\{a\}$, then we must have, for any $i \in \{1, \ldots, n\}$, $a \in x_i \setminus x_0$ which is clearly impossible.

Let φ be the strongly stable function whose trace is t. We have, for any $i \in \{1, \ldots, n\}$,

$$\varphi(\gamma_i^n) = (x_i \setminus x_0) \cup x_0 = x_i$$

and we are done. ∎

From this lemma, we derive another characterization of coherent families of strongly stable functions.

Lemma 3 *Let X and Y be two hypercoherences, and let $\mathcal{F}$ be a finite and non empty set of strongly stable functions $X \to Y$. Assume that for any natural number k and any finite strongly stable function $\varphi : \mathrm{N}^k \to X$ the set $\mathcal{F} \circ \varphi = \{f \circ \varphi \mid f \in \mathcal{F}\}$ belongs to $\mathcal{C}\left(\mathrm{N}^k \Rightarrow Y\right)$. Then $\mathcal{F} \in \mathcal{C}(X \Rightarrow Y)$.*

Proof:

Let U be a finite and non-empty section of $\mathcal{F}$ and assume that $U_1 \in \mathcal{C}(X)$. Let $n = \#U_1$, by Lemma 2, there exists a Berry set G of N^n and a finite strongly stable function $\varphi : \mathrm{N}^n \to X$ such that $\varphi(G) = U_1$. Let

$$W = \{(f \circ \varphi, y) \mid f \in \mathcal{F},\ y \in G \text{ and } \exists b\, (\varphi(y), b) \in f \cap U\} \quad .$$

We check first that W is a pairing of $\mathcal{F} \circ \varphi$ and G. Let $g \in \mathcal{F} \circ \varphi$. Let $f \in \mathcal{F}$ be such that $g = f \circ \varphi$. Let $(x, b) \in f \cap U$ (remember that U is a section of $\mathcal{F}$). We have $x \in U_1 = \varphi(G)$, so let $y \in G$ be such that $x = \varphi(y)$. We have $(g, y) \in W$. Conversely, let $y \in G$. We have $\varphi(y) \in U_1$, so let b be such that $(\varphi(y), b) \in U$. Let $f \in \mathcal{F}$ be such that also $(\varphi(y), b) \in f$. We have $(f \circ \varphi, y) \in W$. Setting $C = \mathrm{Ev}(W) = \{g(y) \mid (g, y) \in W\}$, we have

$$C \in \mathcal{C}(Y) \quad \text{and} \quad \bigcap C = \left(\bigcap(\mathcal{F} \circ \varphi)\right)\left(\bigcap G\right) \tag{2}$$

since, by hypothesis (remember that φ is finite), $\mathcal{F} \circ \varphi \in \mathcal{C}(\mathrm{N}^n \Rightarrow Y)$.

A similarly straightforward verification shows that U_2 is a section of C, so $U_2 \in \Gamma(Y)$. Assume furthermore that U_2 is a singleton $\{b\}$. Let $(g, y) \in W$. Let $f \in \mathcal{F}$ be such that $g = f \circ \varphi$ and $c \in |Y|$ be such that $(\varphi(y), c) \in f \cap U$. As $U_2 = \{b\}$ we have $c = b$, hence $b \in g(y)$. Hence $b \in \bigcap C$. So

$$\forall f \in \mathcal{F} \quad b \in (f \circ \varphi)\left(\bigcap G\right) = f\left(\bigcap U_1\right)$$

by (2), since $\bigcap(\mathcal{F} \circ \varphi) \subseteq f \circ \varphi$ for all $f \in \mathcal{F}$. Let $x \in U_1$. Let $f \in \mathcal{F}$ be such that $(x, b) \in f$. Since x is minimal such that $b \in f(x)$ and since $b \in f(\bigcap U_1)$ and $\bigcap U_1 \subseteq x$, we have $x = \bigcap U_1$ so that U_1 is a singleton. So $U \in \Gamma(X \Rightarrow Y)$ and hence $\mathcal{F} \in \mathcal{C}(X \Rightarrow Y)$. ∎

In the proof of Theorem 1, we shall use the following immediate consequence of the previous lemma.

Proposition 3 *Let $\mathcal{F}$ be a finite and non empty set of strongly stable functions from X to Y. If $\mathcal{F} \notin \mathcal{C}(X \Rightarrow Y)$, there exists a natural number n such that, for any $k \geq n$, there exists a finite strongly stable function $\varphi : \mathrm{N}^k \to X$ such that $\mathcal{F} \circ \varphi \notin \mathcal{C}(Y)$.*

3 Relative definability

Definition 6 *Let q be a natural number. Let τ be any type in the hierarchy of finite types based on the type of natural numbers. Let t be an element of* $\mathrm{qD}\left([\tau]^{\mathbf{HC}}\right)$. *One says that t is q-*PCF-definable *if there exists a term M of type τ with all free variables among a list $l = \langle u_1^{\sigma_1}, \ldots, u_p^{\sigma_p}\rangle$ of* *variables such that for all j, $\|\sigma_j\| \leq q$, and there exist some elements $s_1 \in \mathrm{qD}\left([\sigma_1]^{\mathbf{HC}}\right), \ldots, s_p \in \mathrm{qD}\left([\sigma_p]^{\mathbf{HC}}\right)$ such that*

$$t = [M]_l^{\mathbf{HC}}(s_1, \ldots, s_p) \quad .$$

Of course, this definition makes sense for any model of PCF.

For instance, in [Plo77], Plotkin proved that, in the standard Scott model of PCF, any compact element of the interpretation of any type is 1-PCF-definable. We prove here a similar result for the strongly stable model of PCF.

Observe that if $x_1 \in [\sigma_1]^{\mathbf{HC}}, \ldots, x_n \in [\sigma_n]^{\mathbf{HC}}$ are q-PCF-definable, and if $M : \sigma$ is a PCF term whose all free variables are among the list $l = \langle u_1^{\sigma_1}, \ldots, u_n^{\sigma_n} \rangle$, then $[M]_l^{\mathbf{HC}}(x_1, \ldots, x_n)$ is q-PCF-definable. We shall use tacitly this remark in the following proof.

Theorem 1 *Let σ be any type in the hierarchy of finite types based on the type of natural numbers. Any finite element of $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ is 2-PCF-definable.*

Proof: We proceed by induction on the degree of types. For types of degree 0 or 1, the result is obvious (remember that any sequential function is PCF-definable).

For the inductive step, consider a type σ such that $\|\sigma\| \geq 2$. This type can be written as

$$\sigma = (\sigma_1^1, \ldots, \sigma_{l_1}^1 \to \iota), \ldots, (\sigma_1^k, \ldots, \sigma_{l_k}^k \to \iota) \to \iota$$

with $k \geq 1$ and possibly, for some j's, $l_j = 0$. But at least one of the l_j's is different from 0. Let t be a finite state of $[\sigma]^{\mathbf{HC}}$.

For $j = 1, \ldots, k$, let us denote by X_j the hypercoherence $\prod_{q=1}^{l_j} [\sigma_q^j]^{\mathbf{HC}}$, so that t may be considered as a state of the following hypercoherence:

$$\left(\prod_{j=1}^{k} (X_j \Rightarrow \mathrm{N}) \right) \Rightarrow \mathrm{N} \quad .$$

The set t (which is finite) can be written

$$t = \{(\vec{t^1}, a_1), \ldots, (\vec{t^n}, a_n)\}$$

where $a_1, \ldots, a_n$ are natural numbers, and $\vec{t^1} = (t_1^1, \ldots, t_k^1), \ldots, \vec{t^n} = (t_1^n, \ldots, t_k^n)$ where, for each $i = 1, \ldots, n$ and each $j = 1, \ldots, k$, t_j^i is a finite state in the hypercoherence $X_j \Rightarrow \mathrm{N}$.

For a given vector of arguments $\vec{f} \in \mathrm{qD}\left(\prod_{j=1}^{k} (X_j \Rightarrow \mathrm{N})\right)$, we know that there is at most one $i \in \{1, \ldots, n\}$ such that $\vec{t^i} \leq \vec{f}$. The problem of computing $t(\vec{f})$ is thus twofold:

1) first, we must restrict our attention to an unique $i_0 \in \{1, \ldots, n\}$ (which of course depends on $\vec{f}$), being sure that if $\vec{t^i} \leq \vec{f}$ for some (again, necessarily unique) i, then this i is equal to i_0,

2) and then we must test whether $\vec{t^{i_0}} \leq \vec{f}$.

As we shall see at the end of the proof, the second step is not very difficult, using second order strongly stable functionals. More precisely, we shall exhibit for all $i \in \{1, \ldots, n\}$ a 2-PCF-definable functional of codomain N mapping $\vec{f}$ to $\{0\}$ if $\vec{t^i} \leq \vec{f}$ and to $\emptyset$ otherwise. Since these functionals are "semi-decision" procedures, they cannot be used naively for solving 1), testing successively for

$i = 1, 2, \ldots, n$ whether $\vec{t^i} \leq \vec{f}$. We must be more intelligent and "guess" the value of the relevant i_0. This can be done using second order strongly stable functionals as we shall see in the first part of the proof.

Let $I \subseteq \{1, \ldots, n\}$ be of cardinality strictly greater than 1. We know that the set $\{(\vec{t^i}, a_i) \mid i \in I\}$ is in $\Gamma^* \left([\sigma]^{\mathbf{HC}}\right)$. So

$$\{\vec{t^i} \mid i \in I\} \notin \mathcal{C}\left(\prod_{j=1}^{k} (X_j \Rightarrow \mathrm{N})\right)$$

and thus there exists $j \in \{1, \ldots, k\}$ such that $\{t^i_j \mid i \in I\} \notin \mathcal{C}(X_j \Rightarrow \mathrm{N})$. Let us choose such a j, and let us denote it by j^I.

Applying Proposition 3, let us choose $n^I \in \mathbf{N}$ such that for any $k \geq n^I$ there exists $\varphi : \mathrm{N}^k \to X_{j^I}$ such that $\{t^i_{j^I} \circ \varphi \mid i \in I\} \notin \mathcal{C}\left(\mathrm{N}^k \Rightarrow \mathrm{N}\right)$.

To each subset I of $\{1, \ldots, n\}$, let us associate *injectively* a natural number $\hat{I}$. Let $\nu = \mathrm{Max}\{n^I \mid I \subseteq \{1, \ldots, n\}$ and $\#I \geq 2\}$ (it is here that we use the hypothesis that t is finite). For $I \subseteq \{1, \ldots, n\}$ of cardinality ≥ 2, let $\varphi^I : \mathrm{N}^\nu \to X_{j^I}$ be a finite strongly stable function such that $\{t^i_{j^I} \circ \varphi^I \mid i \in I\} \notin \mathcal{C}(\mathrm{N}^\nu \Rightarrow \mathrm{N})$.

For each $j \in \{1, \ldots, k\}$, let $\varphi_j : \mathrm{qD}(\mathrm{N} \times \mathrm{N}^\nu) \to \mathrm{qD}(X_j)$ be the following function, defined by gluing together the functions φ^I for all sets I such that $j^I = j$ (we identify $\mathrm{qD}(\mathrm{N} \times \mathrm{N}^\nu)$ with $\mathrm{qD}(\mathrm{N}) \times \mathrm{qD}(\mathrm{N}^\nu)$):

$$\varphi_j(z, y) = \begin{cases} \varphi^I(y) & \text{if } z = \{\hat{I}\} \text{ and } j^I = j \\ \emptyset & \text{otherwise} \end{cases}$$

or equivalently by the following trace:

$$\{((\{\hat{I}\}, y), c) \mid I \subseteq \{1, \ldots, n\},\ \#I \geq 2,\ j^I = j \text{ and } (y, c) \in \varphi^I\}\ .$$

It is easily checked that, for any $j \in \{1, \ldots, k\}$, the function φ_j is strongly stable $\mathrm{N}^{\nu+1} \to X_j$, because the φ^I's are strongly stable. Observe also that the φ_j's are finite, since the φ^I's are, and since there are only a finite number of possible sets I.

For $j \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, n\}$, let $s^i_j = t^i_j \circ \varphi_j$, it is a finite strongly stable function $\mathrm{N}^{\nu+1} \to \mathrm{N}$. We denote by $\vec{s^i}$ the vector of functions $(s^i_1, \ldots, s^i_k)$.

Let $s = \{(\vec{s^1}, 1), \ldots, (\vec{s^n}, n)\}$. Let I be any subset of $\{1, \ldots, n\}$ of cardinality strictly greater than 1. We have $\{t^i_{j^I} \circ \varphi^I \mid i \in I\} \notin \mathcal{C}(\mathrm{N}^\nu \Rightarrow \mathrm{N})$, hence $\{s^i_{j^I} \mid i \in I\} \notin \mathcal{C}(\mathrm{N}^{\nu+1} \Rightarrow \mathrm{N})$. Actually, let $U = \{(y_q, b_q) \mid q = 1, \ldots, p\}$ be a section of $\{t^i_{j^I} \circ \varphi^I \mid i \in I\}$ such that $U \notin \Gamma(\mathrm{N}^\nu \Rightarrow \mathrm{N})$, which means that $U_1 \in \mathcal{C}(\mathrm{N}^\nu)$ and that $U_2 \notin \Gamma^*(\mathrm{N})$, and that at least one of these two sets is not a singleton. For $i \in I$, let $Q_i = \{q \mid (y_q, b_q) \in t^i_{j^I} \circ \varphi^I\}$. For each $i \in I$, Q_i is non empty, and the union of the Q_i's is $\{1, \ldots, p\}$. Let $i \in I$ and $q \in Q_i$. We have $b_q \in (t^i_{j^I} \circ \varphi^I)(y_q) = (t^i_{j^I} \circ \varphi_{j^I})(\{\hat{I}\}, y_q)$. Let $(z^i_q, y^i_q) \leq (\{\hat{I}\}, y_q)$ be minimal such that $b_q \in (t^i_{j^I} \circ \varphi_{j^I})(z^i_q, y^i_q)$. Let $U' = \{((z^i_q, y^i_q), b_q) \mid i \in I$ and $q \in Q_i\}$. This set is a section of $\{t^i_{j^I} \circ \varphi_{j^I} \mid i \in I\}$. Moreover, we have $U'_2 = U_2$ and $U'_1 \sqsubseteq \{\{\hat{I}\}\} \times U_1 \in \mathcal{C}(\mathrm{N}^{\nu+1})$, so that $U'_1 \in \mathcal{C}(\mathrm{N}^{\nu+1})$. Consequently, if $U' \in \Gamma(\mathrm{N}^{\nu+1} \Rightarrow \mathrm{N})$, U' must be a singleton $\{((z, y), b)\}$. Let us assume that this is the case, so that also $U_2 = \{b\}$. Since $U \notin \Gamma(\mathrm{N}^\nu \Rightarrow \mathrm{N})$, U_1 is not a singleton. Assume first that $z = \{\hat{I}\}$. Let $q \in \{1, \ldots, p\}$ and let $i \in I$ be such that $q \in Q_i$. We have $b \in (t^i_{j^I} \circ \varphi_{j^I})(z, y) = (t^i_{j^I} \circ \varphi^I)(y)$ and hence, since $y \subseteq y_q$ and since y_q is *minimal* such that $b \in (t^i_{j^I} \circ \varphi^I)(y_q)$, we have $y_q = y$, hence U_1 is a singleton, contradiction. Assume now that $z = \emptyset$. Let $q \in \{1, \ldots, p\}$ and let $i \in I$ be such that $q \in Q_i$. We have $b \in (t^i_{jI} \circ \varphi_{jI})(\emptyset, y) = t^i_{jI}(\emptyset)$, and hence, since $b \in (t^i_{jI} \circ \varphi^I)(y_q)$ with y_q *minimal*, we must have $y_q = \emptyset$, contradiction again. So U' is a section of $\{s^i_{jI} \mid i \in I\}$ such that $U' \notin \Gamma\,(\mathrm{N}^\nu \Rightarrow \mathrm{N})$ and so $\{s^i_{jI} \mid i \in I\} \notin \mathcal{C}\,(\mathrm{N}^{\nu+1} \Rightarrow \mathrm{N})$ as announced.

Thus $\{(\vec{s^i}, i) \mid i \in I\} \in \Gamma^*\left((\mathrm{N}^{\nu+1} \Rightarrow \mathrm{N})^k \Rightarrow \mathrm{N}\right)$ and this holds for any $I \subseteq \{1, \ldots, n\}$ with $\#I \geq 2$, so $s \in \mathrm{qD}\left((\mathrm{N}^{\nu+1} \Rightarrow \mathrm{N})^k \Rightarrow \mathrm{N}\right)$.

Let Φ be the strongly stable function whose trace is s (it is a function from $\mathrm{qD}\left((\mathrm{N}^{\nu+1} \Rightarrow \mathrm{N})^k\right)$ to $\mathrm{qD}\,(\mathrm{N})$). Let us define a function $F : \prod_{j=1}^k (\mathrm{qD}\,(X_j \Rightarrow \mathrm{N})) \to \mathrm{qD}\,(\mathrm{N})$ by

$$F(f_1, \ldots, f_k) = \begin{cases} \{a_i\} & \text{if } \Phi(f_1 \circ \varphi_1, \ldots, f_k \circ \varphi_k) = \{i\} \text{ and } f_1 \geq t^i_1, \ldots, f_k \geq t^i_k, \\ \emptyset & \text{otherwise.} \end{cases}$$

One has $F = t$. Actually, let $\vec{f} \in \mathrm{qD}\left(\prod_{j=1}^k (X_j \Rightarrow \mathrm{N})\right)$. There are two cases:

- Either there exists a (necessarily unique) $i \in \{1, \ldots, n\}$ such that $\vec{f} \geq \vec{t^i}$. Then for $j = 1, \ldots, k$, we have $f_j \circ \varphi_j \geq t^i_j \circ \varphi_j = s^i_j$ and hence $\Phi(f_1 \circ \varphi_1, \ldots, f_k \circ \varphi_k) = \{i\}$, so $F(\vec{f}) = \{a_i\} = t(\vec{f})$.
- Or there is no such i. Then we clearly have $F(\vec{f}) = \emptyset = t(\vec{f})$.

It remains to check that F is 2-PCF-definable.

Let $j \in \{1, \ldots, k\}$. Let us set $l = l_j$, $\sigma_q = \sigma^j_q$, $\varphi = \varphi_j$ and $X = X_j$ (we use these notations until the end of the proof). We observe first that the function $\lambda f \,.\, f \circ \varphi$ is 2-PCF-definable. This is due to the fact that $\varphi = \langle \psi_1, \ldots, \psi_l \rangle$ with, for $q \in \{1, \ldots, l\}$, $\psi_q \in \mathrm{qD}_{\mathrm{fin}}\left([\iota^{\nu+1} \to \sigma_q]^{\mathbf{HC}}\right)$, and ψ_q is 2-PCF-definable by inductive hypothesis since $\|\iota^{\nu+1} \to \sigma_q\| = \mathrm{Max}(1, \|\sigma_q\|) \leq \mathrm{Max}(1, \|\sigma\| - 2) < \|\sigma\|$ (remember that $\|\sigma\| \geq 2$). So the function

$$\lambda(f_1, \ldots, f_k).\Phi(f_1 \circ \varphi_1, \ldots, f_k \circ \varphi_k)$$

is 2-PCF-definable, as Φ is a strongly stable functional of type 2.

We conclude the proof by showing that, for any $i \in \{1, \ldots, n\}$, the function

$$\begin{aligned} G : \mathrm{qD}\,(X \Rightarrow \mathrm{N}) &\to \mathrm{qD}\,(\mathrm{N}) \\ f &\mapsto \begin{cases} \{0\} & \text{if } f \geq t \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

(where $t = t^i_j$) is 2-PCF-definable. Let us write t as $\{(y^1, c^1), \ldots, (y^m, c^m)\}$ with $c^p \in \mathbf{N}$ and $y^p \in \mathrm{qD}\,(X)$ for each $p \in \{1, \ldots, m\}$. For $p \in \{1, \ldots, m\}$ and $b \in y^p$, let us denote by $S_{p,b}$ the obviously strongly stable finite function $\mathrm{qD}\,(\mathrm{N}) \to \mathrm{qD}\,(X)$ whose trace is

$$S_{p,b} = \{(\emptyset, c) \mid c \in y^p \setminus \{b\}\} \cup \{(\{0\}, b)\}\ ,$$

or equivalently

$$S_{p,b}(x) = \begin{cases} y^p & \text{if } x = \{0\} \\ y^p \setminus \{b\} & \text{otherwise} \end{cases}\ .$$

Then $S_{b,p} = \langle S^1_{b,p}, \ldots, S^l_{b,p} \rangle$ with $S^q_{b,p} \in \mathrm{qD}_{\mathrm{fin}}\left([\iota \to \sigma_q]^{\mathbf{HC}}\right)$ for each $q \in \{1, \ldots, l\}$. For any such q, one has $\|\iota \to \sigma_q\| = \mathrm{Max}(1, \|\sigma_q\|) < \|\sigma\|$ and thus, by inductive hypothesis, each of these finite functions $S^q_{b,p}$ is 2-PCF-definable. Similarly, the y^p's are 2-PCF definable. Let $\Sigma : \mathrm{qD}\,(\mathrm{N} \Rightarrow \mathrm{N}) \to \mathrm{qD}\,(\mathrm{N})$ be the function defined by

$$\Sigma(h) = \begin{cases} \{0\} & \text{if } h(\emptyset) = \emptyset \text{ and } h(\{0\}) = \{0\} \\ \emptyset & \text{otherwise} \end{cases}$$

in other words, the trace of Σ is $\{(\{(\{0\}, 0)\}, 0)\}$. It is easily seen that Σ is a strongly stable functional. (But Σ is not PCF-definable, because it is not monotone wrt. the extensional ordering of functions.) For $f \in \mathrm{qD}\,(X \Rightarrow \mathrm{N})$ one has (using the functional Σ for checking minimality of the y^p's)

$$G(f) = \begin{cases} \{0\} & \text{if, for any } p \in \{1, \ldots, m\} \text{ one has: } f(y^p) = \{c^p\} \\ & \text{and, for any } b \in y^p,\ \Sigma(Z \circ f \circ S_{b,p}) = \{0\} \\ \emptyset & \text{otherwise} \end{cases}$$

where $Z : \mathrm{qD}\,(\mathrm{N}) \to \mathrm{qD}\,(\mathrm{N})$ is the PCF-definable *strict* constant function taking 0 as unique value, whose trace is $\{(\{m\}, 0) \mid m \in \mathbf{N}\}$. So G is 2-PCF-definable as the y_p's and the $S^q_{b,p}$'s are, and this concludes the proof of the theorem. ■

Let $\sigma = (\iota^{k_1} \to \iota), \ldots, (\iota^{k_n} \to \iota) \to \iota$ be an arbitrary second order type, and let $\tau = (\iota^{k+1} \to \iota) \to \iota$ where $k = \mathrm{Max}(k_1, \ldots, k_n)$. Then there exist two PCF closed terms $A : \sigma \to \tau$ and $B : \tau \to \sigma$ such that, (possibly up to η-conversion), $\lambda F.B(A(F)) = \lambda F.F$. From this, it results that any element of $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ is of the shape $[B]^{\mathbf{HC}}(y)$ for some $y \in [\tau]^{\mathbf{HC}}$. As a consequence, when using Theorem 1, we can always assume that the free variables of the PCF term whose existence is asserted by the theorem are of *simple* second order type.

The remainder of the paper is devoted to deriving some consequences of Theorem 1, concerning the connections between the strongly stable model and the sequential algorithm model of PCF.

4 Sequential algorithms

Let us describe briefly the model of sequential algorithms. We just give here some definitions and state, without proofs, some results. We mainly refer to [Ehr96] for details. Our sequential algorithms are not different from the sequential algorithms presented in a game-theoretic framework by Curien in [Cur93b]. The connection between the "abstract" setting that we present here and the "concrete" settings (CDS's and games) is presented in details by Bucciarelli in [Buc93]. The abstract presentation is very convenient for our purpose.

Definition 7 *A* sequential structure *is a tuple $E = (E_*, E^*, \epsilon_E, \vdash_E)$, where E_* is a dI-domain, E^* is a set of* cells *containing a distinguished element $\perp$, ϵ_E is a binary relation (called* filling relation*) on $E_* \times E^*$, linear in its first component (that is, for any $\alpha \in E^*$ and any bounded subset A of E_*, if $\bigvee A \;\epsilon_E\; \alpha$ then there exists $x \in A$ such that $x \;\epsilon_E\; \alpha$ and, when A is furthermore finite and non-empty, if $x \;\epsilon_E\; \alpha$ for all x in A, then $\bigwedge A \;\epsilon_E\; \alpha$) and such that $x \;\epsilon_E\; \perp$ never holds, and $\vdash_E$ is a binary relation (called* enabling relation*) on $E_* \times E^*$ satisfying*

- *For any $x, x' \in E_*$, if x and x' are upper bounded, and if, for any $\alpha \in E^*$, $x \;\epsilon_E\; \alpha$ iff $x' \;\epsilon_E\; \alpha$, then $x = x'$.*
- *If $x \vdash_E \alpha$, then $x \not\epsilon_E\; \alpha$.*
- *$x \vdash_E \perp$ always holds.*

- *If $x \epsilon_E \alpha$, then there exists $x' \leq x$ such that $x' \vdash_E \alpha$.*
- *If $x \vdash_E \alpha$ and $x' \geq x$ satisfies $x' \not\epsilon_E \alpha$, then $x' \vdash_E \alpha$.*
- *If $A \subseteq E_*$ is directed and $\bigvee A \vdash_E \alpha$, then there exists $x \in A$ such that $x \vdash_E \alpha$.*
- *If neither $x \epsilon_E \alpha$ nor $x \vdash_E \alpha$, then there exists $\beta \in E^*$ such that $x \vdash_E \beta$ and, for any $x' \geq x$, if $x' \vdash_E \alpha$, then $x' \epsilon_E \beta$.*

We shall consider two notions of morphisms between sequential structures: *sequential functions* and *sequential algorithms.*

Definition 8 *Given two sequential structures E and F, a* sequential function *from E to F is a continuous function $f : E_* \to F_*$ such that, for any $x \in E_*$ and any $\beta \in F^*$ such that $f(x) \vdash_F \beta$, there exists $\alpha \in E^*$ such that $x \vdash_E \alpha$ and for any $x' \geq x$, if $f(x') \in \beta$, then $x' \in \alpha$.*

If $x \in E_*$, let us denote by E_x the set of all $\alpha \in E^*$ such that $x \vdash_E \alpha$.

Let us remind also that a subset A of E_* is said to be *coherent* if it is finite and non-empty and, for any $\alpha \in E^*$, if $x \epsilon_E \alpha$ holds for any $x \in A$, then $\bigwedge A \epsilon_E \alpha$. We denote by $\mathcal{C}^{\mathrm{L}}(E)$ the set of all coherent subsets of E_*. (Remember that in a dI-domain, all non-empty sets have a glb.) Then, $(E_*, \mathcal{C}^{\mathrm{L}}(E))$ is a dI-domain with coherence.

The following proposition is at the origin of the idea of strong stability.

Proposition 4 *A function $f : E_* \to F_*$ is sequential iff it is continuous and satisfies, for any $A \in \mathcal{C}^{\mathrm{L}}(E)$, $f(A) \in \mathcal{C}^{\mathrm{L}}(F)$ and $f(\bigwedge A) = \bigwedge f(A)$, that is, iff f is a strongly stable function from $(E_*, \mathcal{C}^{\mathrm{L}}(E))$ to $(F_*, \mathcal{C}^{\mathrm{L}}(F))$.*

Sequential algorithms are sequential functions together with a kind of "Skolem function" for the $\forall\exists$ condition in the definition of sequentiality. Here is a precise definition.

Definition 9 *A* sequential algorithm *from E to F is a pair (f, φ) where f is a continuous function from E_* to F_* and $(\varphi_x)_{x \in E_*}$ is a family of functions $\varphi_x : F_{f(x)} \to E_x$ such that*

- *$\varphi_x(\perp) = \perp$*
- *For any $x \in E_*$ and any $\beta \in F_{f(x)}$, and for any $x' \geq x$, if $f(x') \epsilon_F \beta$, then $x' \epsilon_E \varphi_x(\beta)$.*
- *For any $x \in E_*$ and any $\beta \in F_{f(x)}$, and for any $x' \geq x$, if $x' \not\epsilon_E \varphi_x(\beta)$, then $\varphi_{x'}(\beta) = \varphi_x(\beta)$.*
- *If $A \subseteq E_*$ is directed, if $\beta \in F_{f(\bigvee A)}$ with $\varphi_{\bigvee A}(\beta) \neq \perp$, then there exists $x \in A$ such that $\beta \in F_{f(x)}$ and $\varphi_x(\beta) = \varphi_{\bigvee A}(\beta)$.*

Sometimes, the function f will be called the *extensional component* of the algorithm (f, φ). In concrete settings (see [Cur93b]), sequential algorithms are strategies in games. They are naturally ordered under inclusion. The corresponding order relation in the abstract setting is described by the following definition.

Definition 10 *Let (f, φ) and (g, ψ) be two sequential algorithms from E to F. One says that (f, φ) is* stably less *than (g, ψ) if f is less than g in the pointwise order, and for any $x \in E_*$ and any $\beta \in F_{f(x)}$, if $\varphi_x(\beta) \neq \perp$, then $g(x) \not\epsilon_F \beta$ and $\psi_x(\beta) = \varphi_x(\beta)$.*

One checks easily that, if $(f,\varphi) \leq (g,\psi)$, then $f \leq g$ in the stable order of functions.

The following proposition relates sequential functions to sequential algorithms.

Proposition 5 *If (f,φ) is a sequential algorithm from E to F, then f is a sequential function. Conversely, if f is a sequential function from E to F, there exists φ such that (f,φ) is a sequential algorithm from E to F.*

The category of sequential structures and sequential algorithms is cartesian closed. The product of E and F is the sequential structure G where $G_* = E_* \times F_*$ (with the product order), and G^* is the disjoint union of E^* and F^*, with the $\perp$'s collapsed. The filling relation is given by $(x,y) \in_G (1,\alpha)$ iff $x \in_E \alpha$ (similarly for the second component of the product), and similarly for the enabling relation. We denote by $E \times F$ this cartesian product. One has the following isomorphism in the category **DIC**:

$$((E \times F)_*, \mathcal{C}^{\mathrm{L}}(E \times F)) \simeq (E_*, \mathcal{C}^{\mathrm{L}}(E)) \times (F_*, \mathcal{C}^{\mathrm{L}}(F))$$

The function space of E and F is a sequential structure G that we do not want to describe here in details. We just need to know that G_* is the set of all sequential algorithms from E to F, endowed with the stable order of sequential algorithms, which actually turns out to be a dI-domain. We denote by $E \Rightarrow F$ this function space.

In this category, we define a model of PCF by choosing an object of natural numbers ω as follows: ω_* is the standard flat domain of natural numbers $\{\perp, 0, 1, \ldots\}$ and ω^* is $\{\perp, *\}$, the filling relation being defined by $x \in_\omega \alpha$ iff $x \neq \perp$ and $\alpha = *$, and the enabling relation by $x \vdash_\omega \alpha$ iff $\alpha = \perp$ or ($\alpha = *$ and $x = \perp$). One checks easily that $(\omega_*, \mathcal{C}^{\mathrm{L}}(\omega)) \cong \mathrm{N}$ in **DIC** and hence more generally $({\omega^n}_*, \mathcal{C}^{\mathrm{L}}(\omega^n)) \cong \mathrm{N}^n$. This isomorphism will often be considered as an equality in the sequel. PCF primitives are interpreted in a natural way. For instance, the If primitive is interpreted by the sequential algorithm $(f,\varphi) : \omega^3 \to \omega$ given by

$$f(a,b,c) = \begin{cases} \perp & \text{if } a = \perp \\ b & \text{if } a = 0 \\ c & \text{otherwise} \end{cases}$$

and

$$\varphi_{(a,b,c)}(*) = \begin{cases} (1,*) & \text{if } a = \perp \\ (2,*) & \text{if } a = 0 \text{ and } b = \perp \\ (3,*) & \text{if } a \notin \{\perp, 0\} \text{ and } c = \perp \end{cases}$$

Concerning fixpoint operators, let us just say that they can be defined in a fairly standard way (see the end of [BE93] for details). More precisely, for any sequential structure E, we have a fixpoint sequential algorithm $(F,\Phi) : (E \Rightarrow E) \to E$ satisfying

$$F(f,\varphi) = \bigvee_{n=0}^{\infty} f^n(\perp) \quad .$$

Let us denote by **SEQ** this model of PCF.

5 The heterogeneous logical relation

We relate **SEQ** to **HC**. For this purpose, we define a binary logical relation between the two models. More precisely, we define $\mathsf{S} \subseteq {[\sigma]^{\mathbf{SEQ}}}_* \times \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ by induction on σ as follows:

- If $\sigma = \iota$, $y \mathsf{S} x$ iff $x = y$.
- If $\sigma = \tau \to \tau'$, $(g, \psi) \mathsf{S} f$ iff for any $x \in \mathrm{qD}\left([\tau]^{\mathbf{HC}}\right)$ and any $y \in [\tau]^{\mathbf{SEQ}}{}_*$, if $y \mathsf{S} x$, then $g(y) \mathsf{S} f(x)$.

The goal of this section is to prove that this relation is *onto* in the sense that for any type τ and any $x \in \mathrm{qD}\left([\tau]^{\mathbf{HC}}\right)$, there exists $y \in [\tau]^{\mathbf{SEQ}}{}_*$ such that $y \mathsf{S} x$. Actually, we prove this result under the further assumption that x is finite.

Lemma 4 *For any type σ and any monotone families $(x_n)_{n\in\mathbf{N}}$ in $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ and $(y_n)_{n\in\mathbf{N}}$ in $[\sigma]^{\mathbf{SEQ}}{}_*$ such that $y_n \mathsf{S} x_n$ for all n, one has $\bigvee_{n\in\mathbf{N}} y_n \mathsf{S} \bigcup_{n\in\mathbf{N}} x_n$ (the relation S is closed).*

The proof is a straightforward induction on types.

Let us state the fundamental lemma of logical relations in that particular case.

Lemma 5 *For any closed term M of PCF, $[M]^{\mathbf{SEQ}} \mathsf{S} [M]^{\mathbf{HC}}$.*

Proof: Observe first that the primitives of the language are related by S. Concerning fixpoints, just apply Lemma 4. For application and abstraction, we use the fact that, in the model **SEQ**, the interpretation of application and abstraction are standard, as far as the *extensional* part of morphisms is concerned. For instance, the evaluation algorithm is an algorithm $(E \Rightarrow F) \times E \to F$ whose extensional part maps the couple $((f, \varphi), x)$ to $f(x)$. ∎

We prove now that the relation S is onto at type 2.

Proposition 6 *Let k be a natural number, and let $\sigma = (\iota^k \to \iota) \to \iota$. For any $f \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ there exists $(g, \psi) \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $(g, \psi) \mathsf{S} f$.*

Proof: Let $E = [\iota^k \to \iota]^{\mathbf{SEQ}}$ and $X = [\iota^k \to \iota]^{\mathbf{HC}}$, and let $F = [\iota^k]^{\mathbf{SEQ}}$ and $Y = [\iota^k]^{\mathbf{HC}}$. Let $\pi : E_* \to \mathrm{qD}(X)$ be defined by $\pi(f, \varphi) = f$. This definition makes sense since f is sequential and hence strongly stable. The function π is strongly stable from $(E_*, \mathcal{C}^{\mathrm{L}}(E))$ to $(\mathrm{qD}(X), \mathcal{C}(X))$, which are both objects of the category **DIC** of dI-domains with coherence. Actually, in the cartesian closed category **SEQ**, we have an evaluation sequential algorithm $(\mathrm{Ev}, \varepsilon) : E \times F \to \omega$. The extensional component of this algorithm is the evaluation function

$$\begin{aligned} \mathrm{Ev} : E_* \times F_* &\to \omega_* \\ ((f, \varphi), x) &\mapsto f(x) \end{aligned}$$

which is thus a sequential function from $E \times F$ to ω, and hence is a strongly stable function from $(E_*, \mathcal{C}^{\mathrm{L}}(E)) \times (F_*, \mathcal{C}^{\mathrm{L}}(F))$ to $(\omega_*, \mathcal{C}^{\mathrm{L}}(\omega))$. Its transpose $\mathrm{Ev}' : (E_*, \mathcal{C}^{\mathrm{L}}(E)) \to ((F_*, \mathcal{C}^{\mathrm{L}}(F)) \Rightarrow (\omega_*, \mathcal{C}^{\mathrm{L}}(\omega)))$, which maps (f, φ) to f is a morphism of **DIC**, that is a strongly stable function, and we are done since $(F_*, \mathcal{C}^{\mathrm{L}}(F)) \Rightarrow (\omega_*, \mathcal{C}^{\mathrm{L}}(\omega))$ is isomorphic to $(\mathrm{qD}(X), \mathcal{C}(X))$.

Now let $H : X \to \mathrm{N}$ be a strongly stable function. Then $H \circ \pi$ is a strongly stable function $(E_*, \mathcal{C}^{\mathrm{L}}(E)) \to \mathrm{N}$, and hence by Proposition 4, it is a sequential function from E to ω. Thus by Proposition 5, there exists Φ such that $(H \circ \pi, \Phi)$ is a sequential algorithm from E to ω. It turns out that $(H \circ \pi, \Phi) \mathsf{S} H$ and this is due to the obvious fact that, for $f \in \mathrm{qD}(X)$ and $(g, \psi) \in E_*$, one has $(g, \psi) \mathsf{S} f$ iff $f = g$. ∎

We conclude using Theorem 1 for lifting the result of Proposition 6 to any type of the hierarchy of finite types.

Theorem 2 *Let σ be a PCF-type. For any finite $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ there exists $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \mathrel{\mathsf{S}} x$.*

Proof: By Theorem 1, there exist $\sigma_1,\ldots,\sigma_n$, simple second order PCF-types, some states $x_1 \in \mathrm{qD}\left([\sigma_1]^{\mathbf{HC}}\right),\ldots,x_n \in \mathrm{qD}\left([\sigma_n]^{\mathbf{HC}}\right)$ and a closed PCF term $M : \sigma_1,\ldots,\sigma_n \to \sigma$ such that $x = [M]^{\mathbf{HC}}(x_1,\ldots,x_n)$. By Proposition 6, there exist $y_1 \in [\sigma_1]^{\mathbf{SEQ}}{}_*,\ldots,y_n \in [\sigma_n]^{\mathbf{SEQ}}{}_*$ such that $y_i \mathrel{\mathsf{S}} x_i$ for $i = 1,\ldots,n$. By Lemma 5, we have $[M]^{\mathbf{SEQ}}(y_1,\ldots,y_n) \mathrel{\mathsf{S}} [M]^{\mathbf{HC}}(x_1,\ldots,x_n) = x$ and we conclude. ∎

As a consequence of this last theorem, the relation S is *functional* in the sense that if $y \mathrel{\mathsf{S}} x$ and $y \mathrel{\mathsf{S}} x'$, then $x = x'$ (the proof of this fact is a simple induction on types).

As a corollary, we get immediately

Theorem 3 *Let M and N be two closed PCF terms of the same type. If M and N have the same semantics in* **SEQ**, *they have the same semantics in* **HC**.

6 Extensional collapse

Since the relation S is functional, for any type σ we can define a partial equivalence relation[2] $\sim$ on $[\sigma]^{\mathbf{SEQ}}{}_*$ by $y \sim y'$ iff there exists $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ such that $y \mathrel{\mathsf{S}} x$ and $y' \mathrel{\mathsf{S}} x$.

On the other hand, one can define a partial equivalence relation $\approx$ on $[\sigma]^{\mathbf{SEQ}}{}_*$ as a logical relation, by induction on types, as follows:

- at type ι, $\approx$ is the equality
- at type $\sigma \to \tau$, $(f,\varphi) \approx (f',\varphi')$ iff for any $y, y' \in [\sigma]^{\mathbf{SEQ}}{}_*$, if $y \approx y'$, then $f(y) \approx f'(y')$.

The quotient of the model[3] **SEQ** by this relation is the so-called *extensional collapse* of **SEQ**.

We aim at proving that these two partial equivalence relations are identical at any type. For this purpose we need first to prove a few lemmas.

The first lemma states, for any type σ, that the order relation and the coherence relation of $[\sigma]^{\mathbf{HC}}$ can be lifted along S.

Lemma 6 *Let $x, x' \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be finite and such that $x \subseteq x'$. There exist $y, y' \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \mathrel{\mathsf{S}} x$, $y' \mathrel{\mathsf{S}} x'$ and $y \leq y'$.*

Let $x_1,\ldots,x_n \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be finite and such that $x_1,\ldots,x_n \in \mathcal{C}\left([\sigma]^{\mathbf{HC}}\right)$. There exist $y_1,\ldots,y_n \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y_1,\ldots,y_n \in \mathcal{C}^{\mathrm{L}}\left([\sigma]^{\mathbf{SEQ}}{}_*\right)$, $y_1 \mathrel{\mathsf{S}} x_1,\ldots,y_n \mathrel{\mathsf{S}} x_n$ and furthermore $\bigwedge_{i=1}^n y_i \mathrel{\mathsf{S}} \bigcap_{i=1}^n x_i$.

Proof: Let $d : \mathrm{N} \to \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be the strongly stable function defined by the following finite trace:

$$\mathrm{tr}\,(d) = \{(\emptyset, b) \mid b \in x\} \cup \{(\{0\}, b) \mid b \in x' \setminus x\}\ .$$

So d is a finite element of $\mathrm{qD}\left([\iota \to \sigma]^{\mathbf{HC}}\right)$ and hence, by Theorem 2, there exists a sequential algorithm $(e,\varepsilon) \in [\iota \to \sigma]^{\mathbf{SEQ}}{}_*$ such that $(e,\varepsilon) \mathrel{\mathsf{S}} d$. We set $y = e(\bot)$ and $y' = e(0)$, so that $y \leq y'$, $y \mathrel{\mathsf{S}} x$ and $y' \mathrel{\mathsf{S}} x'$ as required.

[2]That is, a symmetric and transitive binary relation.

[3]For this kind of constructions, the categorical notion of model presented in Section 1 is not suitable. More convenient is the restricted notion of typed applicative structure described in [Mey82].

For the second part of the lemma, let $z_1,\ldots,z_n \in \mathrm{qD}(\mathrm{N}^n)$ be such that $\{z_1,\ldots,z_n\} \in \mathcal{C}(\mathrm{N}^n)$, and let $d : \mathrm{N}^n \to [\sigma]^{\mathbf{HC}}$ be strongly stable and such that $d(z_1) = x_1,\ldots,d(z_n) = x_n$ (we apply Lemma 2). As the x_i's are finite, d can be assumed to be a finite element of $\mathrm{qD}\left([\iota^n \to \sigma]^{\mathbf{HC}}\right)$, and hence by Theorem 2, there exists $(e,\varepsilon) \in [\iota^n \to \sigma]^{\mathbf{SEQ}}{}_*$ such that $(e,\varepsilon)\ \mathsf{S}\ d$. We set $y_1 = e(z_1),\ldots,y_n = e(z_n)$. Since e is sequential, we have $\{y_1,\ldots,y_n\} \in \mathcal{C}^{\mathrm{L}}\left([\sigma]^{\mathbf{SEQ}}\right)$ and $e(\bigcap_{i=1}^n z_i) = \bigwedge_{i=1}^n y_i$. So we conclude. ∎

The second lemma essentially states that S is "strongly stable" at any type. We need Lemma 6 for proving this result.

Lemma 7 *i) Let $y,y' \in [\sigma]^{\mathbf{SEQ}}{}_*$ and $x,x' \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be such that $y\ \mathsf{S}\ x$, $y'\ \mathsf{S}\ x'$ and $y \le y'$. Then $x \subseteq x'$.*

ii) Let $y_1,\ldots,y_n \in [\sigma]^{\mathbf{SEQ}}{}_*$ be such that $\{y_1,\ldots,y_n\} \in \mathcal{C}^{\mathrm{L}}\left([\sigma]^{\mathbf{SEQ}}{}_*\right)$, and let $x_1,\ldots,x_n \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be such that $y_1\ \mathsf{S}\ x_1,\ldots,y_n\ \mathsf{S}\ x_n$. Then $\{x_1,\ldots,x_n\} \in \mathcal{C}\left([\sigma]^{\mathbf{HC}}\right)$ and $\bigwedge_{i=1}^n y_i\ \mathsf{S}\ \bigcap_{i=1}^n x_i$.

Proof: We prove simultaneously i) and ii) by induction on σ. At type ι, the result is obvious. We consider now the type $\sigma \to \tau$. Let $(f,\varphi),(f',\varphi') \in [\sigma\to\tau]^{\mathbf{SEQ}}{}_*$ be such that $(f,\varphi) \le (f',\varphi')$, and let $g,g' \in \mathrm{qD}\left([\sigma\to\tau]^{\mathbf{HC}}\right)$ be such that $(f,\varphi)\ \mathsf{S}\ g$ and $(f',\varphi')\ \mathsf{S}\ g'$. We have to prove that $g \le g'$ in the stable order. So let $x,x' \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be such that $x \subseteq x'$. We have to prove that $g(x) = g(x') \cap g'(x)$. For this we can assume that x and x' are finite, by continuity of g and g'. By Lemma 6 there exist $y,y' \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \le y'$, $y\ \mathsf{S}\ x$ and $y'\ \mathsf{S}\ x'$. Since $f \le f'$ in the stable ordering of functions, we have $f(y) = f'(y) \wedge f(y')$. But we have $f(y)\ \mathsf{S}\ g(x)$, $f'(y)\ \mathsf{S}\ g'(x)$ and $f(y')\ \mathsf{S}\ g(x')$, and hence by inductive hypothesis (part ii), for τ) $f'(y)\wedge f(y')\ \mathsf{S}\ g'(x)\cap g(x')$ because $\{f'(y),f(y')\} \in \mathcal{C}^{\mathrm{L}}\left([\tau]^{\mathbf{SEQ}}\right)$, these two points being upper bounded by $f'(y')$. We conclude by functionality of S.

Let $(f^1,\varphi^1),\ldots,(f^n,\varphi^n) \in [\sigma\to\tau]^{\mathbf{SEQ}}{}_*$ be such that the set $\{(f^1,\varphi^1),\ldots,(f^n,\varphi^n)\}$ belongs to $\mathcal{C}^{\mathrm{L}}\left([\sigma\to\tau]^{\mathbf{SEQ}}\right)$, and let $g^1,\ldots,g^n \in \mathrm{qD}\left([\sigma\to\tau]^{\mathbf{HC}}\right)$ be such that $(f^1,\varphi^1)\ \mathsf{S}\ g^1,\ldots,(f^n,\varphi^n)\ \mathsf{S}\ g^n$. We have to prove

(a) that $\bigwedge_{i=1}^n (f^i,\varphi^i)\ \mathsf{S}\ \bigcap_{i=1}^n g^i$

(b) and that $\{g^1,\ldots,g^n\} \in \mathcal{C}\left([\sigma\to\tau]^{\mathbf{HC}}\right)$.

Let $(f,\varphi) = \bigwedge_{i=1}^n (f^i,\varphi^i)$. Since evaluation is a sequential function $[\sigma\to\tau]^{\mathbf{SEQ}} \times [\sigma]^{\mathbf{SEQ}} \to [\tau]^{\mathbf{SEQ}}$, we have by Proposition 4

$$\forall y \in [\sigma]^{\mathbf{SEQ}}{}_* \quad \{f^i(y) \mid i = 1,\ldots,n\} \in \mathcal{C}^{\mathrm{L}}\left([\tau]^{\mathbf{SEQ}}\right) \quad \text{and} \quad f(y) = \bigwedge_{i=1}^n f^i(y) \tag{3}$$

For proving (a), we consider the function $g : \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right) \to \mathrm{qD}\left([\tau]^{\mathbf{HC}}\right)$ (*a priori* not necessarily strongly stable) defined by $g(x) = \bigcap_{i=1}^n g^i(x)$. It is clear that g is continuous. For proving that g is strongly stable, by Lemma 1, it suffices to show that $g \le g^i$ (i arbitrary) in the stable order. So let $x,x' \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be such that $x \subseteq x'$. We have to prove that $g(x) = g(x') \cap g^i(x)$. For this we can assume that x and x' are finite, by continuity of g and g^i. By Lemma 6, there exist $y,y' \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \le y'$, $y\ \mathsf{S}\ x$ and $y'\ \mathsf{S}\ x'$. We have $f^i(y)\ \mathsf{S}\ g^i(x)$, and so by inductive hypothesis at type τ and by (3), we have $f(y) \mathsf{S} g(x)$ and similarly $f(y') \mathsf{S} g(x')$. Since $f \leq f^i$ in the stable order, we have $f(y) = f(y') \wedge f^i(y)$ and hence by inductive hypothesis and functionality of S, we conclude that $g(x) = g(x') \cap g^i(x)$ and hence $g \leq g^i$ in the stable order. So g is strongly stable and furthermore $g \leq \bigcap_{i=1}^n g^i$, but since the stable order is included in the extensional order, we actually have $g = \bigcap_{i=1}^n g^i$. Observe that we have also just seen that $f \mathsf{S} g$, that is $\bigwedge_{i=1}^n (f^i, \varphi^i) \mathsf{S} \bigcap_{i=1}^n g^i$ so (a) is proved.

Now we prove (b). Let $x_1, \ldots, x_m \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be such that $\{x_1, \ldots, x_m\} \in \mathcal{C}\left([\sigma]^{\mathbf{HC}}\right)$, and let K be a pairing of $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$. We have to prove that $\{g^i(x_j) \mid (i,j) \in K\} \in \mathcal{C}\left([\tau]^{\mathbf{HC}}\right)$ and that $(\bigcap_{i=1}^n g^i)(\bigcap_{j=1}^m x_j) = \bigcap_{(i,j)\in K} g^i(x_j)$. For this we can assume all the x_j's to be finite, because the g^i's are continuous. By Lemma 6, let $y_1, \ldots, y_n \in [\sigma]^{\mathbf{SEQ}}{}_*$ be such that $y_1 \mathsf{S} x_1, \ldots, y_n \mathsf{S} x_n$, $\{y_1, \ldots, y_n\} \in \mathcal{C}^{\mathrm{L}}\left([\sigma]^{\mathbf{SEQ}}\right)$ and $\bigwedge_{i=1}^n y_i \mathsf{S} \bigcap_{i=1}^n x_i$. Since for $(i,j) \in K$ we have $f^i(y_j) \mathsf{S} g^i(x_j)$, we have by inductive hypothesis $\{g^i(x_j) \mid (i,j) \in K\} \in \mathcal{C}\left([\tau]^{\mathbf{HC}}\right)$, since $\{f^i(y_j) \mid (i,j) \in K\} \in \mathcal{C}^{\mathrm{L}}\left([\tau]^{\mathbf{SEQ}}\right)$ by sequentiality of the evaluation function $[\sigma \to \tau]^{\mathbf{SEQ}} \times [\sigma]^{\mathbf{SEQ}} \to [\tau]^{\mathbf{SEQ}}$. For the same reason, we have $\bigwedge_{(i,j)\in K} f^i(y_j) = f(\bigwedge_{j=1}^m y_j)$ where $(f, \varphi) = \bigwedge_{i=1}^n (f^i, \varphi^i)$. We have $\bigwedge_{(i,j)\in K} f^i(y_j) = f(\bigwedge_{j=1}^m y_j) \mathsf{S} g(\bigcap_{j=1}^m x_j)$ (since $(f, \varphi) \mathsf{S} g$) and $\bigwedge_{(i,j)\in K} f^i(y_j) \mathsf{S} \bigcap_{(i,j)\in K} g^i(x_j)$ by inductive hypothesis and we conclude by functionality of S that $\bigcap_{(i,j)\in K} g^i(x_j) = (\bigcap_{i=1}^n g^i)(\bigcap_{j=1}^m x_j)$ and (b) is proven. ∎

Given a model $\mathcal{M}$ of PCF and a morphism $r : [\iota]^{\mathcal{M}} \to [\iota]^{\mathcal{M}}$ in this model, we can define at any type σ a morphism $r^\sigma : [\sigma]^{\mathcal{M}} \to [\sigma]^{\mathcal{M}}$ (or equivalently a point of $[\sigma \to \sigma]^{\mathcal{M}}$) by induction on σ as follows:

- $r^\iota = r$
- $r^{(\sigma\to\tau)} = [\lambda u^{\sigma\to\tau} \lambda v^\sigma (w_1^{\tau\to\tau}(u(w_2^{\sigma\to\sigma} v)))]^{\mathcal{M}}_{\langle w_1, w_2\rangle}(r^\tau, r^\sigma)$

Lemma 8 *Let $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ and $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ be such that $y \mathsf{S} x$. There exists an increasing sequence $(x_i)_{i\in\mathbf{N}}$ of finite elements of $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ and an increasing sequence $(y_i)_{i\in\mathbf{N}}$ of elements of $[\sigma]^{\mathbf{HC}}{}_*$ such that $\bigcup_{i=1}^n x_i = x$, $\bigvee_{i=1}^n y_i = y$, and, for all $i \in \mathbf{N}$, $y_i \mathsf{S} x_i$.*

Proof: Let r_n be the strongly stable function $\mathrm{N} \to \mathrm{N}$ whose trace is $\{(\{i\}, i) \mid 0 \leq i < n\}$. The family $(r_n)_{n\in\mathbf{N}}$ is increasing and $\bigvee_{n\in\mathbf{N}} r_n = \mathrm{Id}$. One extends each r_n to any types σ as ${r_n}^\sigma$, and we get an increasing sequence $({r_n}^\sigma)_{n\in\mathbf{N}}$ of morphisms such that $\bigvee_{n\in\mathbf{N}} {r_n}^\sigma = \mathrm{Id}$ (by monotonicity and continuity of the interpretations of terms). Furthermore, the functions ${r_n}^\sigma$ have *finite* range and take only *finite* values, as easily checked by induction on types.

On the other hand, it is clear that $R_n = (r_n, \rho^n)$ is a sequential algorithm from ω to ω, if we set $\rho^n_\perp(*) = *$ and $\rho^n_y(*) = \perp$ if $y \neq \perp$. Moreover, the sequence $(R_n)_{n\in\mathbf{N}}$ is increasing and has the identity sequential algorithm as lub. Of course, $R_n \mathsf{S} r_n$. Again, we extend R_n to any type σ as ${R_n}^\sigma$, and the sequence $({R_n}^\sigma)_{n\in\mathbf{N}}$ is monotone and has the identity as lub. Furthermore, for any σ and any $n \in \mathbf{N}$, we have ${R_n}^\sigma \mathsf{S} {r_n}^\sigma$ by Lemma 5.

So by setting $x_n = {r_n}^\sigma(x)$ and $y_n = {R_n}^\sigma(y)$, we define two sequences satisfying the required conditions. ∎

Theorem 4 *Let $y, y' \in [\sigma]^{\mathbf{SEQ}}{}_*$. Then $y \sim y'$ iff $y \approx y'$.*

Proof: We prove the two directions in a single induction on σ. At type ι, the result is obvious. Let us consider the type $\sigma \to \tau$.

Let $(f, \varphi), (f', \varphi') \in [\sigma \to \tau]^{\mathbf{SEQ}}{}_*$ be such that $(f, \varphi) \sim (f', \varphi')$. Let $g \in \mathrm{qD}\left([\sigma \to \tau]^{\mathbf{HC}}\right)$ be such that $(f, \varphi) \mathrel{\mathsf{S}} g$ and $(f', \varphi') \mathrel{\mathsf{S}} g$. Let $y, y' \in [\sigma]^{\mathbf{SEQ}}{}_*$ be such that $y \approx y'$. By inductive hypothesis we have $y \sim y'$, so let $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be such that $y \mathrel{\mathsf{S}} x$ and $y' \mathrel{\mathsf{S}} x$. We have $f(y) \mathrel{\mathsf{S}} g(x)$ and $f'(y') \mathrel{\mathsf{S}} g(x)$ and hence $f(y) \sim f'(y')$ and hence, by inductive hypothesis, $f(y) \approx f'(y')$.

Next, let $(f, \varphi), (f', \varphi') \in [\sigma \to \tau]^{\mathbf{SEQ}}{}_*$ be such that $(f, \varphi) \approx (f', \varphi')$. We have to define a strongly stable function h such that $(f, \varphi) \mathrel{\mathsf{S}} h$ and $(f', \varphi') \mathrel{\mathsf{S}} h$. We define first the restriction g of this function to finite arguments. Let $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be finite. By Theorem 2, there exists $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \mathrel{\mathsf{S}} x$. We have $y \sim y$ and hence $y \approx y$, by inductive hypothesis. But by symmetry and transitivity of $\approx$, we have $(f, \varphi) \approx (f, \varphi)$, so $f(y) \approx f(y)$ and by inductive hypothesis, $f(y) \sim f(y)$ so there exists $z \in \mathrm{qD}\left([\tau]^{\mathbf{HC}}\right)$ such that $f(y) \mathrel{\mathsf{S}} z$ and this z is unique by functionality of S. Moreover, z does not depend on the choice of y: if y' satisfies also $y' \mathrel{\mathsf{S}} x$, then $y \sim y'$, hence by inductive hypothesis, $y \approx y'$, hence, as $(f, \varphi) \approx (f, \varphi)$, we have $f(y) \approx f(y')$ and then, by inductive hypothesis, $f(y) \sim f(y')$ and since $f(y) \mathrel{\mathsf{S}} z$ and S is functional, we have $f(y') \mathrel{\mathsf{S}} z$. So we can set $g(x) = z$. The function g is completely characterized by the following property: for any finite $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ and any $y \in [\sigma]^{\mathbf{SEQ}}{}_*$, if $y \mathrel{\mathsf{S}} x$, then $f(y) \mathrel{\mathsf{S}} g(x)$ (and then, also, $f'(y) \mathrel{\mathsf{S}} g(x)$, as $f(y) \approx f'(y)$ and thus $f(y) \sim f'(y)$ by inductive hypothesis).

Let us prove that g is monotone. Let $x, x' \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be finite and such that $x \subseteq x'$. By Lemma 6, there exist $y, y' \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \mathrel{\mathsf{S}} x$, $y' \mathrel{\mathsf{S}} x'$ and $y \leq y'$. We have $f(y) \leq f(y')$, $f(y) \mathrel{\mathsf{S}} g(x)$ and $f(y') \mathrel{\mathsf{S}} g(x')$. Hence, by Lemma 7, we have $g(x) \subseteq g(x')$. Now let us check that g is strongly stable (for families of $\mathcal{C}\left([\sigma]^{\mathbf{HC}}\right)$ whose elements are finite). Let $x_1, \ldots, x_n \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ be finite and such that $\{x_1, \ldots, x_n\} \in \mathcal{C}\left([\sigma]^{\mathbf{HC}}\right)$. By Lemma 6, there exist $y_1, \ldots, y_n \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $\{y_1, \ldots, y_n\} \in \mathcal{C}^{\mathrm{L}}\left([\sigma]^{\mathbf{SEQ}}\right)$, $y_1 \mathrel{\mathsf{S}} x_1, \ldots, y_n \mathrel{\mathsf{S}} x_n$ and $\bigwedge_{i=1}^n y_i \mathrel{\mathsf{S}} \bigcap_{i=1}^n x_i$. By sequentiality of f we have $\{f(y_1), \ldots, f(y_n)\} \in \mathcal{C}^{\mathrm{L}}\left([\tau]^{\mathbf{SEQ}}\right)$ and $f(\bigwedge_{i=1}^n y_i) = \bigwedge_{i=1}^n f(y_i)$. Since $f(y_1) \mathrel{\mathsf{S}} g(x_1), \ldots, f(y_n) \mathrel{\mathsf{S}} g(x_n)$, we have by Lemma 7 that $\{g(x_1), \ldots, g(x_n)\} \in \mathcal{C}\left([\tau]^{\mathbf{HC}}\right)$ and $\bigwedge_{i=1}^n f(y_i) \mathrel{\mathsf{S}} \bigcap_{i=1}^n g(x_i)$. We also have that $\bigwedge_{i=1}^n f(y_i) = f(\bigwedge_{i=1}^n y_i) \mathrel{\mathsf{S}} g(\bigcap_{i=1}^n x_i)$ and hence $g(\bigcap_{i=1}^n x_i) = \bigcap_{i=1}^n g(x_i)$ by functionality of S.

To conclude, it remains to extend g to non finite elements. Let $h : \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right) \to \mathrm{qD}\left([\tau]^{\mathbf{HC}}\right)$ be defined by

$$h(x) = \bigcup \{g(x_0) \mid x_0 \subseteq x \text{ and } x_0 \text{ finite}\} \ .$$

By monotonicity of g, h is well defined and continuous. One checks easily that h is strongly stable. Let us prove that $(f, \varphi) \mathrel{\mathsf{S}} h$. Let $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ and $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ be such that $y \mathrel{\mathsf{S}} x$. By Lemma 8, we can find an increasing sequence $(x_i)_{i \in \mathbf{N}}$ of finite elements of $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ and an increasing sequence $(y_i)_{i \in \mathbf{N}}$ of elements of $[\sigma]^{\mathbf{SEQ}}{}_*$ such that $\bigcup_{i=1}^n x_i = x$, $\bigvee_{i=1}^n y_i = y$, and, for all $i \in \mathrm{N}$, $y_i \mathrel{\mathsf{S}} x_i$. We have $f(y) = \bigvee_{i=1}^\infty f(y_i)$ and $h(x) = \bigcup_{i=1}^\infty g(x_i)$ (because the x_i's are finite), and hence, by Lemma 4, $f(y) \mathrel{\mathsf{S}} h(x)$. Hence $(f, \varphi) \mathrel{\mathsf{S}} h$ and similarly $(f', \varphi') \mathrel{\mathsf{S}} h$, so $(f, \varphi) \sim (f', \varphi')$ and we conclude. ∎

If V is a set and R is a partial equivalence relation on V, the relation R is an equivalence relation on the set V' of elements of V which are related to themselves by R. We call *quotient* of V by the partial equivalence R, and denote by V/R, the quotient V'/R.

For any type σ, the relation S induces an injective mapping from $[\sigma]^{\mathbf{SEQ}}{}_*/\sim$ to $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$, which is surjective onto finite elements. So by the theorem above, this mapping can be considered as a function c_σ from $[\sigma]^{\mathbf{SEQ}}{}_*/\approx$ to $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$, and it can be checked that this family of functions (c_σ) is a model morphism (that is, commutes to the interpretation of terms) from the extensional collapse of the sequential algorithm model to the strongly stable model. Moreover, if $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ and $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ are such that $y \mathrel{\mathsf{S}} x$, and if y is compact, it is easily shown, using lemma 8 and the functionality of S, that x is finite. One can also prove that conversely, if $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ is finite, there exists $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ *compact*[4]such that $y \mathrel{\mathsf{S}} x$ (for this, one can use for instance the *finite retractions* introduced in definition 11 below). So c_σ induces a bijection between the equivalence classes of compact sequential algorithms of type σ and the finite strongly stable functions of the same type. Hence if, instead of considering the hierarchy of simple types based on the type of natural numbers, we consider a hierarchy based on some finite approximation of the type of natural numbers (for instance, the booleans), the strongly stable model is the extensional collapse of the sequential algorithms model as in that case, for any type σ, all the elements of $\mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ are finite and all the elements of $[\sigma]^{\mathbf{SEQ}}{}_*$ are compact. This result can be extended to the hierarchy based on natural numbers. This is done by Jaap van Oosten in [vO97] and by John Longley in [Lon98] in realizability settings.

We give now a simple proof of this general result in our sequential algorithms setting.

Clearly, it suffices to extend theorem 2, showing that for *any* $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$, there exists $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \mathrel{\mathsf{S}} x$. For this purpose, the main ingredients are theorem 2 and König's lemma.

For starting with, let us remind some quite standard technical material from [Ehr96] (with a non-standard terminology).

Definition 11 *Let E be a sequential structure. A* retraction *on E is a monotone map $p : E_* \to E_*$ which is stably less than the identity. A retraction p on E is* finite *if moreover:*

- *$p(E_*)$ is finite*
- *for any $y \in E_*$, there are only finitely many cells of E which are filled by $p(y)$ (and hence $p(y)$ is compact).*

A retraction p is continuous and satisfies $p \circ p = p$.

It is proven in [Ehr96] (lemma 12) that, if p is a retraction on E, then (p, π^p) is a sequential algorithm, where π^p is given by

$$\pi^p_y(\alpha) = \begin{cases} \alpha & \text{if } \alpha \text{ is not filled by } y\text{, but is filled by some element of } p(E_*) \\ \bot & \text{otherwise} \end{cases}$$

Then (p, π^p) is less than the identity sequential algorithm (lemma 12), and if p_1 and p_2 are finite retractions on E such that $p_1 \leq p_2$, then $(p_1, \pi^{p_1}) \leq (p_2, \pi^{p_2})$ (lemma 13 of [Ehr96]).

Let p be a retraction on E and q be a retraction on F. Then the map

$$\begin{array}{rcl} [p,q] : (E \Rightarrow F)_* & \to & (E \Rightarrow F)_* \\ (f,\varphi) & \mapsto & (q,\pi^q) \circ (f,\varphi) \circ (p,\pi^p) \end{array}$$

[4]Observe however that if y compact, one can perfectly have $y \sim y'$ for some non compact y'.

is a retraction on $E \Rightarrow F$, which is finite as soon as p and q are finite (see lemma 22 of [Ehr96]). Moreover the map $(p,q) \mapsto [p,q]$ is clearly monotone with respect to the pointwise order which, for retractions, is equivalent to the stable order.

Let σ be a PCF type and let p be a retraction on $[\sigma]^{\mathbf{SEQ}}$. Let r be a strongly stable map from $[\sigma]^{\mathbf{HC}}$ to itself. We write $p \mathsf{S}' r$ if $p(y) \mathsf{S} r(x)$ for all $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ and $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$ such that $y \mathsf{S} x$.

Let τ be another PCF type, q be a retraction on $[\tau]^{\mathbf{SEQ}}$ and s be a strongly stable map from $[\tau]^{\mathbf{HC}}$ to itself such that $q \mathsf{S}' s$. Let $[r,s]$ be the following strongly stable map:

$$\begin{array}{rcl} [r,s] : \mathrm{qD}\left([\sigma \to \tau]^{\mathbf{HC}}\right) & \to & \mathrm{qD}\left([\sigma \to \tau]^{\mathbf{HC}}\right) \\ g & \mapsto & s \circ g \circ r \end{array}$$

One checks easily that $[p,q] \mathsf{S}' [r,s]$.

In the proof of lemma 8, we have endowed each hypercoherence $[\sigma]^{\mathbf{HC}}$ with an increasing family $({r_n}^\sigma)_{n\in\mathbf{N}}$ of functions from $[\sigma]^{\mathbf{HC}}$ to itself, which has the identity as lub, and such that each ${r_n}^\sigma$ takes only finite values and has finite range. Moreover, remember that ${r_n}^{\sigma\to\tau} = [{r_n}^\sigma, {r_n}^\tau]$.

Hence, since each function r_n (see the proof of lemma 8) is a finite retraction on ω, the considerations above show that one can endow each sequential structure $[\sigma]^{\mathbf{SEQ}}$ with an increasing family $(p^{\sigma,n})_{n\in\mathbf{N}}$ of finite retractions such that $p^{\sigma,n} \mathsf{S}' {r_n}^\sigma$ for all $n \in \mathbf{N}$: take $p^{\iota,n} = r_n$ and $p^{\sigma\to\tau,n} = [p^{\sigma,n}, p^{\tau,n}]$, and remember that the function $(p,q) \mapsto [p,q]$ is monotone.

Now let σ be a PCF type and let $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$. For any natural number n, let $x_n = {r_n}^\sigma(x)$.

Let U_n be the set of all elements y of $p^{\sigma,n}([\sigma]^{\mathbf{SEQ}}{}_*)$ which satisfy $y \mathsf{S} x_n$. This set is finite, as $p^{\sigma,n}$ is a *finite* retraction. It is also non-empty: let $z \in [\sigma]^{\mathbf{SEQ}}{}_*$ be such that $z \mathsf{S} x_n$ (such a z exists by theorem 2 as x_n is finite). Then one has $p^{\sigma,n}(z) \mathsf{S} {r_n}^\sigma(x_n) = x_n$.

Consider now the set $\mathcal{U}$ of all the sequences $(y_1,\ldots,y_n)$ such that $y_i \in U_i$ for all $i \le n$, and $y_1 \le \ldots \le y_n$. Endowed with the prefix order, this set is a tree, which is finitely branching as the sets U_i are all finite.

Moreover, for any $n \in \mathbf{N}$, there exists a sequence $(y_1,\ldots,y_n) \in \mathcal{U}$ (take any $y_n \in U_n$ and set $y_i = p^{\sigma,i}(y_n)$ for $i \le n$) so $\mathcal{U}$ is infinite.

By König's lemma, $\mathcal{U}$ has an infinite branch. In other words there exists an infinite sequence $(y_1, y_2, \ldots)$ such that $(y_1, y_2, \ldots, y_n) \in \mathcal{U}$ for all $n \in \mathbf{N}$. For all $i \in \mathbf{N}$, we have $y_i \mathsf{S} x_i$ and $y_i \le y_{i+1}$. By lemma 4, $\bigvee_{i\in\mathbf{N}} y_i \mathsf{S} \bigvee_{i\in\mathbf{N}} x_i = x$ and we are done.

We can summarize as follows the results proven above.

Theorem 5 *For any type σ of PCF and any $x \in \mathrm{qD}\left([\sigma]^{\mathbf{HC}}\right)$, there exists $y \in [\sigma]^{\mathbf{SEQ}}{}_*$ such that $y \mathsf{S} x$.*

The family of morphisms (c_σ) is an isomorphism between the extensional collapse of the sequential algorithms model of PCF and its strongly stable model.

7 Concluding remarks and acknowledgments

We proved theorem 1 some years ago, but we observed only recently that it could be applied for comparing the equational theories of the models **SEQ** and **HC**. The idea of using logical relations for this purpose comes from discussions with A. Bucciarelli (the proofs follow the pattern presented in [Buc97]). The idea that the strongly stable model could be the extensional collapse of the sequential algorithms model was suggested by a remark of S. Abramsky, after his reading of [Ehr96].

It results actually from [Ehr96] that **HC** is the extensional collapse of a model of "extensional" sequential algorithms, which differs from the standard model of sequential algorithms we consider here.

I want to thank John Longley who pointed out to me that, in an earlier version of this paper, the final statement concerning the extensional collapse was not a direct consequence of the results previously proved, as the generalization of theorem 2 based on the use of König's lemma was absent from that version.

Theorem 5, combined with the fact that the model of sequential algorithms is fully abstract for the extension PCFC of PCF by a "catch and throw" operator (see [CCF94]), seems to indicate that there should exist a sub-language of PCFC admitting a fully abstract semantics in **HC**. Roughly speaking, it suffices likely to add to PCF all the closed terms M of PCFC of simple second order types which satisfy $[M]^{\mathbf{SEQ}} \approx [M]^{\mathbf{SEQ}}$. But this is not a very explicit definition, and it would be interesting to define such a sub-language in terms of a (preferably finite) set of second order natural intensional primitives.

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