# Coherent Differentiation

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# Intro: differentiation and addition

We have learned at school

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

And later

$f : E \to F$ (where $E$ and $F$ are, say, Banach spaces)
is differentiable at $x \in E$ if

$$f(x + u) = f(x) + (l \cdot u) + o(\|u\|)$$

where $l : E \to F$ linear bounded.
And then $l \in \mathcal{L}(E, F)$ is uniquely defined: $l = f'(x)$ is the differential (Jacobian etc) of $f$ at $x$.

Because $l \cdot u \in o(\|u\|) \Rightarrow l = 0$ when $l \in \mathcal{L}(E, F)$.

# Leibniz rule

Take $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (sufficiently regular) and define

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x,x) \end{aligned}$$

Then

$$\frac{dg(x)}{dx} = \frac{\partial f(x_1, x)}{\partial x_1}(x) + \frac{\partial f(x, x_2)}{\partial x_2}(x)$$

This generalizes the usual Leibniz rule $(uv)' = u'v + uv'$, $(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$ etc.

Differentiation is inherently related to addition.

In the Differential $\lambda$-calculus we have a differential application

$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash \mathrm{D}M \cdot N : A \Rightarrow B}$$

and a differential substitution defined by induction on $M$, such that if $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$ then

$$\Gamma, x : A \vdash \frac{\partial M}{\partial x} \cdot N : B$$

Differential reduction:

$$\mathrm{D}(\lambda x^A\, M) \cdot N \to \lambda x^A\, (\frac{\partial M}{\partial x} \cdot N)$$

where $\frac{\partial M}{\partial x} \cdot N$ is defined by induction on $M$.

The most important case in the definition of $\frac{\partial M}{\partial x} \cdot N$ is when $M = (P)Q$:

$$\frac{\partial (P)Q}{\partial x} \cdot N = (\frac{\partial P}{\partial x} \cdot N)Q + (\mathrm{D}P \cdot (\frac{\partial Q}{\partial x} \cdot N))Q$$

which combines

- the Leibniz Rule because $x$ can occur in $P$ *and in* $Q$
- and the Chain Rule because of the application (imagine $x$ occurs only in $Q$).

Reduction rule:

$$\mathrm{D}(\lambda x^A\, M) \cdot N \to \lambda x^A\, (\frac{\partial M}{\partial x} \cdot N)$$

so to have subject reduction it seems that we need

$$\frac{\Gamma \vdash M_0 : A \qquad \Gamma \vdash M_1 : A}{\Gamma \vdash M_0 + M_1 : A}$$

allowing to add any two terms of the same type.

# Consequence: non-determinism

If we have for instance a type $o$ of booleans with

$$\frac{}{\Gamma \vdash \mathbf{t} : o} \qquad \frac{}{\Gamma \vdash \mathbf{f} : o}$$

then we *must* accept $\mathbf{t} + \mathbf{f}$ as a valid term, with

$$\Gamma \vdash \mathbf{t} + \mathbf{f} : o$$

meaning that the language is essentially non-deterministic.

# In the semantics

So far, the categorical models $\mathbf{C}$ of the differential $\lambda$-calculus were (left-)additive categories.

Given $f, g \in \mathbf{C}(A, B)$, there is a morphism $f + g \in \mathbf{C}(A, B)$.

$\rightsquigarrow$ $\mathbf{C}$ is enriched over commutative monoids.

# This is not a fatality!

**Fact**

*Of course addition is required, but there is a (categorical, and then syntactical) way of controlling it, without giving up determinism.*

The possibility of such a theory appears in ...

# ... probabilistic coherence spaces (PCS)

A PCS is a pair $X = (|X|, \mathrm{P}X)$ where $|X|$ is a set and $\mathrm{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ satisfying some closure properties.

- $\mathrm{P}X$ is convex,
- downwards closed,
- closed under lubs of monotonic $\omega$-chains
- $+$ a technical condition to avoid $\infty$ coeffs.

They are a model of (probabilistic) $\lambda$-calculi, LL etc, but not of their differential extensions by lack of additivity.

# Derivatives in PCSs

In the associated category $\mathbf{Pcoh}_!$, 1 is an object such that $|1| = \{*\}$, $\mathrm{P}1 = [0,1]$ and a morphism $f \in \mathbf{Pcoh}_!(1,1)$ is a power series defining a function $[0,1] \to [0,1]$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{with} \quad \forall n\, a_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{n=0}^{\infty} a_n \leq 1$$

so that $f'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ has no reason to be a function $[0,1] \to [0,1]$.

**Example**

$f(x) = 1 - \sqrt{1-x}$, then $f'(x) = 1/(2\sqrt{1-x})$ is unbounded on $[x, 1)$.

However

**Fact**

*If* $x, u \in [0,1]$ *and* $x + u \in [0,1]$ *then we have*

$$f(x) + f'(x)u \leq f(x+u) \in [0,1]$$

because this sum is the beginning of the Taylor expansion which holds in this model:

$$f(x+u) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) u^n$$

and all coefficients are $\geq 0$. For any $f \in \mathbf{Pcoh}_!(X, Y)$.

# In a PCS, some sums are allowed…

Convex combinations: if $x, y \in \mathsf{P}X$ then $\frac{1}{3}x + \frac{2}{3}y \in \mathsf{P}X$.

Some non convex sums are also possible, for instance in the cartesian product $X \mathbin{\&} Y$, we have

$$(x, 0) + (0, y) = (x, y) \in \mathsf{P}(X \mathbin{\&} Y) = \mathsf{P}X \times \mathsf{P}Y$$

if $x \in \mathsf{P}X$ and $y \in \mathsf{P}Y$.

Other non convex allowed sums come from differentiation:

$$f(x) + f'(x) \cdot u \in \mathsf{P}Y$$

if $x, u \in \mathsf{P}X$ are such that $x + y \in \mathsf{P}X$ and $f : \mathsf{P}X \to \mathsf{P}Y$ is an "analytic function", that is a morphism $X \to Y$ in the Kleisli category $\mathbf{Pcoh}_!$.

## ... and some sums are forbidden!

For instance

$$\mathsf{P}(1 \oplus 1) = \{(x_0, x_1) \in \mathbb{R}_{\geq 0} \mid x_0 + x_1 \leq 1\}$$

in this object of booleans,

$$\mathbf{t} = (1,0), \mathbf{f} = (0,1) \in \mathsf{P}(1 \oplus 1) \quad \text{and} \quad \mathbf{t} + \mathbf{f} \notin \mathsf{P}(1 \oplus 1)\,.$$

or simply $1 \in [0,1]$ and $1 + 1 \notin [0,1]$.

# Fundamental observation

There is a functor $\mathbf{S} : \mathbf{Pcoh} \to \mathbf{Pcoh}$ which maps an object $X$ to an object $\mathbf{S}X$ such that

$$\mathsf{P}(\mathbf{S}X) = \{(x, u) \in \mathsf{P}X^2 \mid x + u \in \mathsf{P}X\}\,.$$

For instance $\mathsf{P}(\mathbf{S}1) = \{(x, u) \in [0, 1] \mid x + u \leq 1\}$.

We base our axiomatization on the existence of such a functor.

# Summable categories

## Definition (pre-summable category)

A *pre-summable* category is a tuple

$$(\mathcal{L}, \mathbf{S}, \pi_0, \pi_1, \sigma)$$

Where

- $\mathcal{L}$ is a category enriched over pointed sets (and the distinguished morphism is always denoted 0);
- $\mathbf{S} : \mathcal{L} \to \mathcal{L}$ is a functor which preserves the enrichment ($\mathbf{S}0 = 0$);
- and $\pi_0, \pi_1, \sigma : \mathbf{S}X \to X$ are natural transformations such that $\pi_0$ and $\pi_1$ are jointly monic.

If $f_0, f_1 \in \mathcal{L}(X, \mathbf{S}Y)$ satisfy $\pi_j\ f_0 = \pi_j\ f_1$ for $j = 0, 1$ then $f_0 = f_1$.

# Intuition

- $\mathbf{S}X$ is the objects of pairs $(x_0, x_1) \in X \times X$ such that $x_0 + x_1$ is well defined and $\in X$;
- $\pi_j : \mathbf{S}X \to X$ are the projections, $\pi_j(x_0, x_1) = x_j$;
- and $\sigma : \mathbf{S}X \to X$ maps $(x_0, x_1)$ to $x_0 + x_1$.

## Some terminology

We assume to have such a structure $(\mathcal{L}, \mathbf{S}, \pi_0, \pi_1, \sigma)$

### Definition (summability, witness, sum)

We say that $f_0, f_1 \in \mathcal{L}(X, Y)$ are summable if there is $h \in \mathcal{L}(X, \mathbf{S}Y)$ such that $\pi_j\ h = f_j$ for $j = 0, 1$.

Fact: when such an $h$ exists it is unique ($\pi_0, \pi_1$ are jointly monic), we set $\langle f_0, f_1 \rangle_{\mathbf{S}} = h$, it is the witness of summability of $f_0$ and $f_1$.

And then we set $f_0 + f_1 = \sigma \langle f_0, f_1 \rangle_{\mathbf{S}}$, the sum of $f_0$ and $f_1$.

## Some simple observations

- $\pi_0, \pi_1$ are summable with $\langle \pi_0, \pi_1 \rangle_{\mathbf{S}} = \mathsf{Id}_{\mathbf{S}X}$ and $\pi_0 + \pi_1 = \sigma$.
- If $f_0, f_1 \in \mathcal{L}(X, Y)$ are summable and $l \in \mathcal{L}(U, X)$ and $r \in \mathcal{L}(Y, V)$ then $r\, f_0\, l, r\, f_1\, l$ are summable with

$$\langle r\, f_0\, l, r\, f_1\, l \rangle_{\mathbf{S}} = \mathbf{S} r\, \langle f_0, f_1 \rangle_{\mathbf{S}}\, l$$
$$r\, f_0\, l + r\, f_1\, l = r\, (f_0 + f_1)\, l$$

by naturality of $\pi_0$, $\pi_1$ and $\sigma$.

**Remark (main tool)**

Use the fact that $\pi_0, \pi_1$ are jointly monic.

We introduce a few axioms to make this "partial addition" behave as expected.

# Commutativity

**Axiom (Commutativity)**

$\pi_1, \pi_0$ are summable and $\pi_1 + \pi_0 = \sigma$.

**Fact (consequences)**

$\langle \pi_1, \pi_0 \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}X, \mathbf{S}X)$ *is an involution.*

*If* $f_0, f_1 \in \mathcal{L}(X, Y)$ *are summable then* $f_1, f_0$ *are summable with* $f_1 + f_0 = f_0 + f_1$*.*

Intuitively: $\langle \pi_1, \pi_0 \rangle_{\mathbf{S}}(x_0, x_1) = (x_1, x_0)$.

# Neutrality

**Axiom (Neutrality)**

For any $f \in \mathcal{L}(X, Y)$, $f$ and $0$ are summable and $f + 0 = f$.

In particular we have two injections

$$\iota_0 = \langle \mathsf{Id}_X, 0 \rangle_{\mathbf{S}}, \iota_1 = \langle 0, \mathsf{Id}_X \rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}X)$$

Intuitively $\iota_0(x) = (x, 0)$ and $\iota_1(x) = (0, x)$.

Associativity is more tricky. We split the condition in two pieces.

**Axiom (Witness)**

Let $f_{ij} \in \mathcal{L}(X, Y)$ for $i, j \in \{0, 1\}$ be 4 morphisms such that

- $f_{j0}, f_{j1}$ are summable for $j = 0, 1$
- and $f_{00} + f_{01}, f_{10} + f_{11}$ are summable

then $\langle f_{00}, f_{01}\rangle_{\mathbf{S}}, \langle f_{10}, f_{11}\rangle_{\mathbf{S}}$ are summable.

So there is a witness for this summability:

$$\langle\langle f_{00}, f_{01}\rangle_{\mathbf{S}}, \langle f_{10}, f_{11}\rangle_{\mathbf{S}}\rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}^2 Y) \, .$$

# The canonical flip

**Fact**

*There is exactly one morphism* $\mathsf{c} \in \mathcal{L}(\mathbf{S}^2X, \mathbf{S}^2X)$ *such that*

$$\forall i, j \in \{0,1\} \quad \pi_i \ \pi_j \ \mathsf{c} = \pi_j \ \pi_i \ .$$

$$\mathsf{c} = \langle\langle \pi_0 \ \pi_0, \pi_0 \ \pi_1 \rangle_{\mathbf{S}}, \langle \pi_1 \ \pi_0, \pi_0 \ \pi_0 \rangle_{\mathbf{S}} \rangle_{\mathbf{S}}$$

exists by the previous axioms.

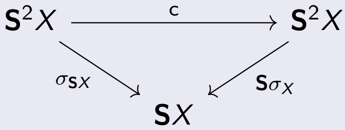
**Fact**

$c^2 = \mathsf{Id}_{\mathbf{S}^2X}.$

Intuitively $\mathsf{c}((x_{00}, x_{01}), (x_{10}, x_{11})) = ((x_{00}, x_{10}), (x_{01}, x_{11}))$.

## Axiom (Associativity)

The following diagram commutes




## Remark (Intuition)

The sum of witnesses is performed componentwise:

$$\langle x_{00}, x_{01} \rangle_{\mathbf{S}} + \langle x_{10}, x_{11} \rangle_{\mathbf{S}} = \langle x_{00} + x_{10}, x_{01} + x_{11} \rangle_{\mathbf{S}}$$

## Fact (consequence)

*If* $f_{ij} \in \mathcal{L}(X, Y)$ *for* $i, j \in \{0, 1\}$ *are such that*

- $f_{j0}, f_{j1}$ *are summable for* $j = 0, 1$
- *and* $f_{00} + f_{01}, f_{10} + f_{11}$ *are summable*

*then*

- $f_{0j}, f_{1j}$ *are summable for* $j = 0, 1$
- *and* $f_{00} + f_{10}, f_{01} + f_{11}$ *are summable*

*and* $(f_{00} + f_{01}) + (f_{10} + f_{11}) = (f_{00} + f_{10}) + (f_{01} + f_{11})$*.*

Associativity follows taking $f_{10} = 0$.

# Partially additive category

The category $\mathcal{L}$ becomes a partially additive category in the sense of partial monoids.

**Remark**

Partially additive categories do not suffice for our goal: the functor $\mathbf{S}$ will be crucial for differentiation!

## $\mathbf{S}$ is a monad

We have already $\zeta = \iota_0 = \langle \mathrm{Id}_X, 0\rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}X)$.

Using the axioms we also have
$\theta = \langle \pi_0\ \pi_0, \pi_1\ \pi_0 + \pi_0\ \pi_1\rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}^2X, \mathbf{S}X)$.

**Fact**

$(\mathbf{S}, \zeta, \theta)$ *is a monad on* $\mathcal{L}$*.*

Intuitively

$$\theta_X : \mathbf{S}^2X \to \mathbf{S}X$$
$$((x_{00}, x_{01}), (x_{10}, x_{11})) \mapsto (x_{00}, x_{10} + x_{01})$$

Notice that we forget $x_{11}$.

# When $\mathcal{L}$ is an SMC

## Distributivity

In all the situations we have in mind, $\mathcal{L}$ is a symmetric monoidal category with tensor product $\otimes$ and tensor unit 1.

In that case one expects $\otimes$ to distribute over $+$, when defined. This requires an additional

**Axiom (Distributivity)**

$0 \otimes g = 0$
and
if $f_0, f_1$ are summable then

- $f_0 \otimes g, f_1 \otimes g$ are summable
- and $f_0 \otimes g + f_1 \otimes g = (f_0 + f_1) \otimes g$.

# Strength

In particular $\pi_0 \otimes \mathsf{Id}_Y, \pi_1 \otimes \mathsf{Id}_Y \in \mathcal{L}(\mathbf{S}X \otimes Y, X \otimes Y)$ are summable so we have strengths

$$\varphi^0_{X,Y} = \langle \pi_0 \otimes \mathsf{Id}_Y, \pi_1 \otimes \mathsf{Id}_Y \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}X \otimes Y, \mathbf{S}(X \otimes Y))$$
$$\varphi^1_{X,Y} = \langle \mathsf{Id}_X \otimes \pi_0, \mathsf{Id}_X \otimes \pi_1 \rangle_{\mathbf{S}} \in \mathcal{L}(X \otimes \mathbf{S}Y, \mathbf{S}(X \otimes Y))$$

turn $\mathbf{S}$ into a commutative monad.

Intuitively

$$\varphi^0_{X,Y}((x_0, x_1) \otimes y) = (x_0 \otimes y, x_1 \otimes y)$$
$$\varphi^1_{X,Y}(x \otimes (y_0, y_1)) = (x \otimes y_0, x \otimes y_1)$$

# Commutativity of the monad

We have actually something stronger:

$$
\begin{array}{ccccc}
\mathbf{S}X \otimes \mathbf{S}Y & \xrightarrow{\varphi^0_{X,\mathbf{S}Y}} & \mathbf{S}(X \otimes \mathbf{S}Y) & \xrightarrow{\mathbf{S}\,\varphi^1_{X,Y}} & \mathbf{S}^2(X \otimes Y) \\
\downarrow{\scriptstyle \varphi^1_{\mathbf{S}X,Y}} & & & & \downarrow{\scriptstyle c_{X\otimes Y}} \\
\mathbf{S}(\mathbf{S}X \otimes Y) & & \xrightarrow{\qquad\mathbf{S}\,\varphi^0_{X,Y}\qquad} & & \mathbf{S}^2(X \otimes Y)
\end{array}
$$

Intuitively

$$(x_0, x_1) \otimes (y_0, y_1) \mapsto ((x_0 \otimes y_0, x_0 \otimes y_1), (x_1 \otimes y_0, x_1 \otimes y_1))$$
$$(x_0, x_1) \otimes (y_0, y_1) \mapsto ((x_0 \otimes y_0, x_1 \otimes y_0), (x_0 \otimes y_1, x_1 \otimes y_1))$$

# Induced symmetric monoidal structure

We then have

$$
\begin{array}{ccccc}
\mathbf{S}X \otimes \mathbf{S}Y & \xrightarrow{\varphi^0_{X,\mathbf{S}Y}} & \mathbf{S}(X \otimes \mathbf{S}Y) & \xrightarrow{\mathbf{S}\,\varphi^1_{X,Y}} & \mathbf{S}^2(X \otimes Y) \\
\downarrow{\scriptstyle \varphi^1_{\mathbf{S}X,Y}} & & & & \downarrow{\scriptstyle \mathsf{c}_{X\otimes Y}} \\
\mathbf{S}(\mathbf{S}X \otimes Y) & & \xrightarrow{\qquad\mathbf{S}\,\varphi^0_{X,Y}\qquad} & & \mathbf{S}^2(X \otimes Y) \\
 & & & & \downarrow{\scriptstyle \theta} \\
 & & & & \mathbf{S}(X \otimes Y)
\end{array}
$$

$$\mathsf{L}_{X,Y} : \mathbf{S}X \otimes \mathbf{S}Y \to \mathbf{S}(X \otimes Y)$$

Intuitively

$$\mathsf{L}_{X,Y} : ((x_0, x_1) \otimes (y_0, y_1)) \mapsto (x_0 \otimes y_0, x_0 \otimes y_1 + x_1 \otimes y_0)$$

# Differential structure

As in differential LL, we consider differentiation as a structure of the exponential.

So we assume moreover that

- $\mathcal{L}$ is cartesian ($\top$: terminal object, $X_0 \mathbin{\&} X_1$: product, $\mathrm{pr}_i \in \mathcal{L}(X_0 \mathbin{\&} X_1, X_i)$, $\langle f_0, f_1 \rangle \in \mathcal{L}(Y, X_0 \mathbin{\&} X_1)$ if $f_i \in \mathcal{L}(Y, X_0)$).
- $\mathcal{L}$ is equipped with a resource modality $(!\_, \mathrm{der}, \mathrm{dig}, \mathrm{m}^0, \mathrm{m}^2)$

$$\mathrm{der}_X \in \mathcal{L}(!X, X) \quad \mathrm{dig}_X \in \mathcal{L}(!X, !!X) \quad \text{comonad structure}$$

$$\mathrm{m}^0 \in \mathcal{L}(1, !\top) \quad \mathrm{m}^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \mathbin{\&} Y))$$

Seely isos, strong sym. monoidality

# Preservation of products

We need a further property about **S**.

## Axiom (Product)

The functor **S** preserves cartesian products, more precisely:

$$\langle \mathbf{S}\mathsf{pr}_0, \mathbf{S}\mathsf{pr}_1 \rangle \in \mathcal{L}(\mathbf{S}\,(X_0 \mathbin{\&} X_1)\,, \mathbf{S}X_0 \mathbin{\&} \mathbf{S}X_1)$$

is an iso.

This holds in all the LL-based examples we have in mind, because in these examples **S** is a right adjoint.

# The differentiation operator

In this setting (resource category with a summability structure), a *differential structure* is a natural transformation

$$\partial_X \in \mathcal{L}(!\mathbf{S}X, \mathbf{S}!X)$$

satisfying some properties.

**Remark (main idea)**

Given $f \in \mathcal{L}_!(X, Y)$, this will allow to define

$$\mathbf{D}f = (\mathbf{S}f)\ \partial_X \in \mathcal{L}_!(\mathbf{S}X, \mathbf{S}Y)$$

which will (intuitively) be the map $(x, u) \mapsto (f(x), f'(x) \cdot u)$.

We list the conditions to be satisfied by $\partial_X$

# Second derivative: intuition

Let $f \in \mathcal{L}_!(X, Y)$, we have $\mathbf{D}f \in \mathcal{L}_!(\mathbf{D}X, \mathbf{D}Y)$

$$\mathbf{D}f(x, u) = (f(x), \frac{f(x)}{dx} \cdot u)$$

We can apply $\mathbf{D}$ to $\mathbf{D}f$, we get

$$\mathbf{D}^2 f((x, u), (y, v)) = (\mathbf{D}f(x, u), \frac{d\mathbf{D}f(x, u)}{d(x, u)} \cdot (y, v))$$

Remember $\mathbf{D}f(x,u) = (f(x), f'(x)\cdot u)$.

By standard rules of calculus:

$$\frac{d\mathbf{D}f(x,u)}{d(x,u)}\cdot(y,v) = \frac{\partial\mathbf{D}f(x,u)}{\partial x}\cdot y + \frac{\partial\mathbf{D}f(x,u)}{\partial u}\cdot v$$

$$\begin{aligned}\frac{\partial\mathbf{D}f(x,u)}{\partial x}\cdot y &= \frac{\partial}{\partial x}(f(x), f'(x)\cdot u)\cdot y \\ &= (f'(x)\cdot y, f''(x)\cdot(u,y)) \\ \frac{\partial\mathbf{D}f(x,u)}{\partial u}\cdot v &= \frac{\partial}{\partial u}(f(x), f'(x)\cdot u)\cdot v \\ &= (0, f'(x)\cdot v)\end{aligned}$$

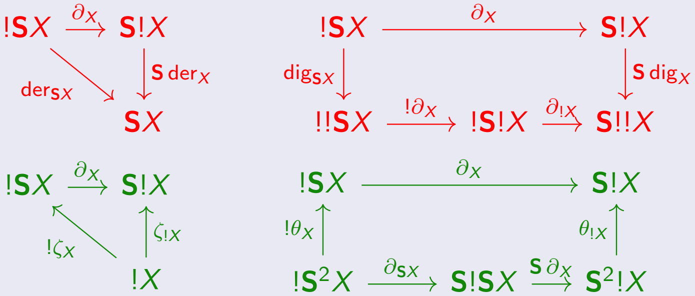
Finally we have, intuitively

$$\mathbf{D}^2 f((x,u),(y,v)) = ((f(x), f'(x)\cdot u),\\ (f'(x)\cdot y, f''(x)\cdot(u,y) + f'(x)\cdot v)$$

Notice that in the first 3 components, we have only 1st order derivatives.

## Axiom (Chain Rule + Linearity)

$\partial$ is a distributive law between the monad $\mathbf{S}$ and the comonad $!\_$ in the following sense.




See John Power and Hiroshi Watanabe, *Combining a monad and a comonad*, TCS 2002 for this kind of dist. law.

# Intuition for the dist. law

The first two diagrams allow to define a functor

$$\begin{aligned}\mathbf{D} : \mathcal{L}_! &\to \mathcal{L}_! \\ X &\mapsto \mathbf{S}X \\ (f : !X \to Y) &\mapsto ((\mathbf{S}f)\ \partial_X : !\mathbf{S}X \to \mathbf{S}Y)\end{aligned}$$

Intuitively, and in probabilistic coherence spaces for instance:

- $f \in \mathcal{L}_!(X, Y)$ means that $f$ is an analytic function $X \to Y$
- $\mathbf{D}f \in \mathbf{Pcoh}_!(X, Y)$ is the $(x, u) \mapsto (f(x), f'(x) \cdot u)$

so this functoriality means that the chain rule holds.

And that the differential of a linear morphism is the morphism itself: $\mathbf{D}(f\ \mathrm{der}_X) = (\mathbf{S}f)\ \mathrm{der}_{\mathbf{S}X}$ for $f \in \mathcal{L}(X, Y)$.

The two next diagrams allow to lift the monad $(\mathbf{S}, \zeta, \theta)$ to $\mathcal{L}_!$.

For $\theta_X \in \mathcal{L}_!(\mathbf{S}^2 X, \mathbf{S} Y) = \mathcal{L}(!\mathbf{S}^2 X, \mathbf{S} Y)$: we take $\theta_X$ $\mathsf{der}_X$.

These diagrams allow to prove that $\theta$ is a natural transformation on $\mathcal{L}_!$. If $f \in \mathcal{L}_!(X, Y)$:

$$\begin{array}{ccc} \mathbf{D}^2 X & \xrightarrow{\theta_X} & \mathbf{D} X \\ {\scriptstyle \mathbf{D}^2 f}\downarrow & & \downarrow{\scriptstyle \mathbf{D} f} \\ \mathbf{D}^2 Y & \xrightarrow{\theta_Y} & \mathbf{D} Y \end{array}$$

And similarly $\zeta$ is natural in $\mathcal{L}_!$.

# Intuition: linearity of the differential

Remember:

$$\theta_X((x_0,u_0),(x_1,u_1)) = (x_0, u_0 + x_1)$$

$$\mathbf{D}^2 f((x_0,u_0),(x_1,u_1)) = ((f(x_0), f'(x_0)\cdot u_0), \\ (f'(x_0)\cdot x_1, f''(x_0)\cdot(u_0,x_1) + f'(x_0)\cdot u_1)$$

The commutation means:

$$\mathbf{D} f(x_0, u_0 + x_1) = (f(x_0), f'(x_0)\cdot u_0 + f'(x_0)\cdot x_1)$$

that is $f'(x_0)\cdot(u_0 + x_1) = f'(x_0)\cdot u_0 + f'(x_0)\cdot x_1$.

Naturality of $\zeta$ in $\mathcal{L}_!$: $f'(x)\cdot 0 = 0$.

# Locality

To represent one of the differential situation we are interested in, this distributive law has to satisfy additional axioms: *Locality*, *Leibniz* and *Schwarz*.

## Axiom (Locality)

$$
\begin{array}{ccc}
!\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \\
& {\scriptstyle !\pi_0} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\
& !X &
\end{array}
$$

Only for $\pi_0$, not for $\pi_1$!

# Intuition

Again we use $\pi_0$ for $\pi_0\,\mathrm{der}_X \in \mathcal{L}_!(\mathbf{D}X, X)$.

The diagram means that $\pi_0$ is natural in $\mathcal{L}_!$. If $f \in \mathcal{L}_!(X, Y)$:

$$\begin{array}{ccc} \mathbf{D}X & \xrightarrow{\pi_0} & X \\ {\scriptstyle \mathbf{D}f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbf{D}Y & \xrightarrow{\pi_0} & Y \end{array}$$

This corresponds to the intuition that

$$\mathbf{D}f(x, u) = (f(x), f'(x) \cdot u)$$

## Remark

$\pi_1 \in \mathcal{L}_!(\mathbf{D}X, X)$ also exists but is fundamentally not natural in $\mathcal{L}_!$ (of course $\pi_1$ is natural in $\mathcal{L}$).

# Leibniz

Is expressed as a "monoidality" condition (to simplify we assume $\mathbf{S}(X \,\&\, Y) = \mathbf{S}X \,\&\, \mathbf{S}Y$)

## Axiom (Leibniz)

$$
\begin{array}{ccccc}
!\mathbf{S}X \otimes !\mathbf{S}Y & \xrightarrow{\partial_X \otimes \partial_Y} & \mathbf{S}!X \otimes \mathbf{S}!Y & \xrightarrow{\mathsf{L}_{!X,!Y}} & \mathbf{S}(!X \otimes !Y) \\
{\scriptstyle \mathsf{m}^2_{\mathbf{S}X,\mathbf{S}Y}}\downarrow & & & & \downarrow{\scriptstyle \mathbf{S}\mathsf{m}^2_{X,Y}} \\
!\mathbf{S}\,(X \,\&\, Y) & & \xrightarrow{\qquad\quad \partial_{X\&Y} \qquad\quad} & & \mathbf{S}!(X \,\&\, Y)
\end{array}
$$

$+$ a "0-ary version".

# Intuition

Given $f \in \mathcal{L}_!(X \mathbin{\&} Y, Z)$, this commutation gives us an expression for $\mathbf{D}f$ in terms of the two differentials $\partial_X$ and $\partial_Y$.

Given $((x, y), (u, v)) \in \mathbf{S}(X \mathbin{\&} Y)$,
that is, $(x, u) \in \mathbf{S}X$ and $(y, v) \in \mathbf{S}Y$,

$$\frac{df(x,y)}{d(x,y)} \cdot (u, v) = \frac{\partial f(x,y)}{\partial x} \cdot u + \frac{\partial f(x,y)}{\partial y} \cdot v$$

In the diagram, $+$ is implemented by $\mathsf{L}_{!X,!Y}$.

# Schwarz

**Axiom (Schwarz)**

$$\begin{array}{ccccc}
!\mathbf{S}^2 X & \xrightarrow{\partial_{\mathbf{S}X}} & \mathbf{S}!\mathbf{S}X & \xrightarrow{\mathbf{S}\partial_X} & \mathbf{S}^2 !X \\
{\scriptstyle !\mathsf{c}_X}\downarrow & & & & \downarrow{\scriptstyle \mathsf{c}_{!X}} \\
!\mathbf{S}^2 X & \xrightarrow{\partial_{\mathbf{S}X}} & \mathbf{S}!\mathbf{S}X & \xrightarrow{\mathbf{S}\partial_X} & \mathbf{S}^2 !X
\end{array}$$

If $f \in \mathcal{L}_!(X, Y)$ then

$$\mathbf{D}^2 f = (\mathbf{S}^2 f)\,(\mathbf{S}\,\partial_X)\ \partial_{\mathbf{S}X}$$

so this diagram means that c is natural in $\mathcal{L}_!$:

$$\begin{array}{ccc} \mathbf{D}^2 X & \xrightarrow{\mathbf{D}^2 f} & \mathbf{D}^2 Y \\ {\scriptstyle \mathsf{c}_X}\downarrow & & \downarrow{\scriptstyle \mathsf{c}_Y} \\ \mathbf{D}^2 X & \xrightarrow{\mathbf{D}^2 f} & \mathbf{D}^2 Y \end{array}$$

where we use also $\mathsf{c}_X$ for $\mathsf{c}_X\ \mathsf{der}_{\mathbf{S}^2 X}$.

# Intuition

Remember that

$$\begin{aligned} c_X((x_0, u_0), (x_1, u_1)) &= ((x_0, x_1), (u_0, u_1)) \\ \mathbf{D}^2 f((x_0, u_0), (x_1, u_1)) &= ((f(x_0), f'(x_0) \cdot u_0), \\ &\qquad (f'(x_0) \cdot x_1, f''(x_0) \cdot (u_0, x_1) + f'(x_0) \cdot u_1) \end{aligned}$$

so this naturality means that

$$\begin{aligned} \mathbf{D}^2 f((x_0, x_1), (u_0, u_1)) &= ((f(x_0), f'(x_0) \cdot x_1), \\ &\qquad (f'(x_0) \cdot u_0, f''(x_0) \cdot (u_0, x_1) + f'(x_0) \cdot u_1) \end{aligned}$$

$$\begin{aligned} &((f(x_0), f'(x_0) \cdot x_1), (f'(x_0) \cdot u_0, f''(x_0) \cdot (x_1, u_0) + f'(x_0) \cdot u_1) \\ &\quad = ((f(x_0), f'(x_0) \cdot x_1), (f'(x_0) \cdot u_0, f''(x_0) \cdot (u_0, x_1) + f'(x_0) \cdot u_1) \end{aligned}$$

So taking $u_1 = 0$ we get

$$f''(x_0) \cdot (x_1, u_0) = f''(x_0) \cdot (u_0, x_1)$$

which is the crucial property that the second derivative is a symmetric bilinear function, often called *Schwarz lemma*.

# Coherent Differentiation (II)

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# Short recap

## Summability structure

- $\mathcal{L}$ is a category with 0-morphisms
- $\mathbf{S} : \mathcal{L} \to \mathcal{L}$ is a 0-preserving functor
- $\pi_0, \pi_1, \sigma : \mathbf{S}X \to X$ are natural transformations
- $\pi_0, \pi_1$ are jointly monic
- $f_0, f_1 \in \mathcal{L}(X, Y)$ are summable if there is $\langle f_0, f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}Y)$ with $\pi_i \langle f_0, f_1 \rangle_{\mathbf{S}} = f_i$, and then $f_0 + f_1 = \sigma \langle f_0, f_1 \rangle_{\mathbf{S}}$.

$+$ axioms to turn $\mathcal{L}(X, Y)$ into a partial commutative monoid.

In particular $\mathsf{c} \in \mathcal{L}(\mathbf{S}^2 X, \mathbf{S}^2 X)$ the standard flip with $\pi_i \; \pi_j \; \mathsf{c} = \pi_j \; \pi_i$.

$\mathbf{S}$ inherits a monad structure $(\mathbf{S}, \zeta, \theta)$.

# Differentiation

$\mathcal{L}$ is assumed to be a resource category (cartesian SMC with a resource comonad aka. exponential, with Seely strong monoidality).

The differential structure is a natural transformation $\partial_X \in \mathcal{L}(!\mathbf{S}X, \mathbf{S}!X)$ which satisfies some futher commutations:

- it is a distributive law between the monad $\mathbf{S}$ and the comonad $!\_$: Chain Rule and Linearity (of the derivative)
- Locality
- Leibniz
- Schwarz.

Then one defines the Differentiation Functor $\mathbf{D} : \mathcal{L}_! \to \mathcal{L}_!$ by $\mathbf{D}X = \mathbf{S}X$ and if $f \in \mathcal{L}_!(X, Y) = \mathcal{L}_!(!X, Y)$ then $\mathbf{D}f = (\mathbf{S}f)\ \partial_X \in \mathcal{L}_!(\mathbf{D}X, \mathbf{D}Y)$.

# Canonical structure

# A special, very common, case

We assume that $\mathcal{L}$ is monoidal closed (convenient though not strictly necessary) so that the functor

$$\_ \otimes \mathbf{I}$$

where $\mathbf{I} = 1 \,\&\, 1$, has a right adjoint $\mathbf{S}_{\mathbf{I}}$.

$$\mathbf{S}_{\mathbf{I}} X = (\mathbf{I} \multimap X)$$
$$\mathbf{S}_{\mathbf{I}} f = (\mathbf{I} \multimap f) \in \mathcal{L}(\mathbf{I} \multimap X, \mathbf{I} \multimap Y)$$

for $f \in \mathcal{L}(X, Y)$.

**Remark**

We still assume that $\mathcal{L}$ has zero-morphisms.

# Two natural questions

**Remark**

In (probabilistic) coherence spaces, **S** is defined exactly in that way.

- When does this definition give rise to a summability structure?
- What does the differential structure boil down to in this setting?

We have three morphisms

$$\overline{\pi}_0 = \langle \mathsf{Id}, 0 \rangle \in \mathcal{L}(1, \mathsf{I})$$
$$\overline{\pi}_1 = \langle 0, \mathsf{Id} \rangle \in \mathcal{L}(1, \mathsf{I})$$
$$\Delta = \langle \mathsf{Id}, \mathsf{Id} \rangle \in \mathcal{L}(\mathsf{I}, \mathsf{I})$$

which induce natural transformations $\pi_0, \pi_1, \sigma \in \mathcal{L}(\mathbf{S}_{\mathsf{I}} X, X)$ by "precomposition".

For instance $\pi_0$ is

$$(\mathsf{I} \multimap X) \xrightarrow{\sim} (\mathsf{I} \multimap X) \otimes 1 \xrightarrow{\mathsf{Id} \otimes \overline{\pi}_0} (\mathsf{I} \multimap X) \otimes \mathsf{I} \xrightarrow{\mathsf{ev}} X$$

# Summability as a property

## Definition

$\mathcal{L}$ is canonically summable if $(\mathbf{S}_{\mathsf{l}}, \pi_0, \pi_1, \sigma)$ defined in that way are a summability structure.

## Remark (a property of $\mathcal{L}$, not a structure)

This is a property of $\mathcal{L}$, not an additional structure on $\mathcal{L}$.

In particular we need $\overline{\pi}_0, \overline{\pi}_1$ to be jointly epic.

# What do summability and sums become?

Remember that $\mathcal{L}(X, Y) \simeq \mathcal{L}(1, X \multimap Y)$.

**Fact**

*$x_0, x_1 \in \mathcal{L}(1, X)$ are summable if there is $h \in \mathcal{L}(\mathsf{I}, X)$ such that*

$$x_i = h\,\overline{\pi}_i$$

*and then $x_0 + x_1 = h\,\Delta \in \mathcal{L}(1, X)$.*

## Canonical Witness Axiom

If $f_0, f_1 \in \mathcal{L}(\mathsf{I}, X)$ are such that $f_0\,\Delta, f_1\,\Delta \in \mathcal{L}(\mathsf{I}, X)$ are summable, then so are $f_0, f_1$. That is, up to $\mathcal{L}(\mathsf{I}, X) \simeq \mathcal{L}(1, \mathsf{I} \multimap X)$:

if $f_0, f_1, f \in \mathcal{L}(\mathsf{I}, X)$ are such that

$$f_i\,\Delta = f\,\overline{\pi}_i \text{ for } i = 0, 1$$

then there is $h \in \mathcal{L}(\mathsf{I} \otimes \mathsf{I}, X)$ such that

$$f_i\,\lambda = h\,(\overline{\pi}_i \otimes \mathsf{I}) \in \mathcal{L}(1 \otimes \mathsf{I}, X)$$

where $\lambda$ is the can. isom. $1 \otimes \mathsf{I} \to \mathsf{I}$.

**Remark**

Then $f\,\rho = h\,(\mathsf{I} \otimes \Delta)$. Because $\overline{\pi}_0, \overline{\pi}_1$ are jointly epic.

## Theorem

*If $\overline{\pi}_0, \overline{\pi}_1$ are jointly epic, then $(\mathbf{S}_{\mathsf{I}}, \pi_0, \pi_1, \sigma)$ (as defined above) is a summability structure on $\mathcal{L}$ iff the Canonical Witness Axiom holds.*

# $\mathsf{I}$ is a commutative comonoid

Thanks to the axioms we can define

$$\widetilde{\mathsf{L}} \in \mathcal{L}(\mathsf{I}, \mathsf{I} \otimes \mathsf{I})$$

uniquely characterized by

$$\widetilde{\mathsf{L}}\,\overline{\pi}_0 = \overline{\pi}_0 \otimes \overline{\pi}_0 \text{ and } \widetilde{\mathsf{L}}\,\overline{\pi}_1 = \overline{\pi}_0 \otimes \overline{\pi}_1 + \overline{\pi}_1 \otimes \overline{\pi}_0$$

**Fact**

$(\mathsf{I}, \mathsf{pr}_0 \in \mathcal{L}(\mathsf{I}, 1), \widetilde{\mathsf{L}})$ *is a commutative comonoid in* $\mathcal{L}$*.*

$\mathsf{pr}_0 \in \mathcal{L}(\mathsf{I} = (1 \,\&\, 1)\,, 1)$ is the first projection.

# The commutative monad structure of $\mathbf{S}_{\mathsf{I}}$

We have seen that $\mathbf{S}_{\mathsf{I}}$ has a structure of commutative monad.

**Fact**

*The monad* $(\mathbf{S}_{\mathsf{I}}, \zeta, \theta)$ *is induced by the commutative comonoid structure* $(\mathsf{pr}_0, \widetilde{\mathsf{L}})$ *of* $\mathsf{I}$.

For instance $\theta = \mathsf{cur}\, f : (\mathsf{I} \multimap (\mathsf{I} \multimap X)) \to (\mathsf{I} \multimap X)$ where $f$ is

$$
\begin{array}{ccc}
(\mathsf{I} \multimap (\mathsf{I} \multimap X)) \otimes \mathsf{I} & \xrightarrow{\mathsf{Id} \otimes \widetilde{\mathsf{L}}} & (\mathsf{I} \multimap (\mathsf{I} \multimap X)) \otimes \mathsf{I} \otimes \mathsf{I} \\
 & & \downarrow \scriptstyle{\mathsf{ev} \otimes \mathsf{Id}} \\
X & \xleftarrow{\quad\quad\mathsf{ev}\quad\quad} & (\mathsf{I} \multimap X) \otimes \mathsf{I}
\end{array}
$$

# Differentiation as a !-coalgebra (canonical case)

# $!\_$ and its coalgebras

We assume that $\mathcal{L}$ is a cartesian resource category (cartesian product &, exponential comonad $!\_$, Seely isos etc).

A !-coalgebra structure on $X \in \mathcal{L}$ is a $d \in \mathcal{L}(X, !X)$ such that

$$\begin{array}{ccc} X & \xrightarrow{d} & !X \\ & \searrow^{\mathsf{Id}} & \downarrow \mathsf{der}_X \\ & & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{d} & !X \\ d \downarrow & & \downarrow \mathsf{dig}_X \\ !X & \xrightarrow{!d} & !!X \end{array}$$

These colgebras form the Eilenberg-Moore category $\mathcal{L}^!$ where

$f \in \mathcal{L}^!((X, d), (Y, e))$ if $\begin{array}{ccc} X & \xrightarrow{d} & !X \\ f \downarrow & & \downarrow !f \\ Y & \xrightarrow{e} & !Y \end{array}$ in $\mathcal{L}$.

## $\mathcal{L}^!$ is cartesian

Due to the fact that $\mathcal{L}$ is a resource category ($\otimes$, &, Seely isos):

**Fact**

*$\mathcal{L}^!$ is cartesian, with terminal object*

$$(1, \mu^0 : 1 \to !1)$$

*and the cartesian product of $(P_0, d_0), (P_1, d_1) \in \mathcal{L}^!$ is $(P_0 \otimes P_1, \mu^2 (d_0 \otimes d_1))$*

$$P_0 \otimes P_1 \xrightarrow{d_0 \otimes d_1} !P_0 \otimes !P_1 \xrightarrow{\mu^2} !(P_0 \otimes P_1)$$

*Projection $\mathrm{pr}_0^{\otimes}$ (and similarly for $\mathrm{pr}_1^{\otimes}$):*

$$P_0 \otimes P_1 \xrightarrow{d_0 \otimes d_1} !P_0 \otimes !P_1 \xrightarrow{\mathrm{der}_{P_0} \otimes !0} P_0 \otimes !\top \simeq P_0$$

Uses the lax symmetric monoidality structure $(\mu^0, \mu^2)$ of $!\_$.

# Chain Rule and coalgebra

## Fact

*There is a bijective correspondence between*

- *the !-coalgebra structures on* $\mathsf{I}$
- *and the distributive laws between* $\mathbf{S}_{\mathsf{I}}$ *and* $!\_$ *in the sense of the Chain Rule:*

$$
\begin{array}{ccc}
!\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \\
& \searrow^{\mathrm{der}_{\mathbf{S}X}} & \downarrow{\scriptstyle \mathbf{S}\,\mathrm{der}_X} \\
& & \mathbf{S}X
\end{array}
\qquad
\begin{array}{ccccc}
!\mathbf{S}X & & \xrightarrow{\qquad\partial_X\qquad} & & \mathbf{S}!X \\
\downarrow{\scriptstyle \mathrm{dig}_{\mathbf{S}X}} & & & & \downarrow{\scriptstyle \mathbf{S}\,\mathrm{dig}_X} \\
!!\mathbf{S}X & \xrightarrow{!\partial_X} & !\mathbf{S}!X & \xrightarrow{\partial_{!X}} & \mathbf{S}!!X
\end{array}
$$

# Coalgebra $\rightsquigarrow$ Chain Rule

Suppose we are given $\delta \in \mathcal{L}(\mathsf{I}, !\mathsf{I})$, then for any object $X$ we can define $\partial_X = \mathrm{cur}\, f \in \mathcal{L}(!\mathbf{S}_\mathsf{I} X = !(\mathsf{I} \multimap X), \mathbf{S}_\mathsf{I} !X = (\mathsf{I} \multimap !X))$ where $f$ is

$$!(\mathsf{I} \multimap X) \otimes \mathsf{I} \xrightarrow{\mathsf{Id} \otimes \delta} !(\mathsf{I} \multimap X) \otimes !\mathsf{I} \xrightarrow{\mu^2} !((\mathsf{I} \multimap X) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !X$$

$\mu^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \otimes Y))$ is the lax-monoidality structure of $!\_$ wrt. $\otimes$.

# Chain Rule $\rightsquigarrow$ Coalgebra

Conversely assume we are given $\partial_X \in \mathcal{L}(!\mathbf{S}_{\mathsf{I}} X, \mathbf{S}_{\mathsf{I}} !X)$ for each $X \in \mathcal{L}$, we have in particular, taking $X = \mathsf{I}$:

$$\mathsf{I} \xrightarrow{\lambda_{\mathsf{I}}{}^{-1}} 1 \otimes \mathsf{I} \xrightarrow{\mu^0 \otimes \mathsf{Id}} !1 \otimes \mathsf{I} \xrightarrow{!(\mathsf{cur}\,\lambda_{\mathsf{I}}) \otimes \mathsf{Id}} !(\mathsf{I} \multimap \mathsf{I}) \otimes \mathsf{I}$$

$$!(\mathsf{I} \multimap \mathsf{I}) \otimes \mathsf{I} \xrightarrow{\partial_{\mathsf{I}} \otimes \mathsf{Id}} (\mathsf{I} \multimap !\mathsf{I}) \otimes \mathsf{I} \xrightarrow{\mathsf{ev}} !\mathsf{I}$$

where $\mu^0 \in \mathcal{L}(1, !1)$ is the “unit” of the lax-monoidality and $\lambda_{\mathsf{I}} \in \mathcal{L}(1 \otimes \mathsf{I}, \mathsf{I})$ (the canonical iso).

# A natural question

So assume we are given a coalgebra structure $\delta \in \mathcal{L}(\mathsf{I}, !\mathsf{I})$.

What conditions must satisfy $\delta$ for ensuring that the corresponding distributive law $(\partial_X)_{X \in \mathcal{L}}$ satisfies the additional conditions

- Linearity (second part of the dist. law)
- Local
- Leibniz
- Schwarz?

The answer is surprisingly simple.

# Linearity and Leibniz

Linearity and Leibniz boil down to

$$
\begin{array}{ccc}
\mathsf{I} & \xrightarrow{\delta} & !\mathsf{I} \\
{\scriptstyle \mathsf{pr}_0}\downarrow & & \downarrow{\scriptstyle !\mathsf{pr}_0} \\
1 & \xrightarrow{\mu^0} & !1
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{I} & & \xrightarrow{\qquad\qquad\delta\qquad\qquad} & & !\mathsf{I} \\
{\scriptstyle \widetilde{\mathsf{L}}}\downarrow & & & & \downarrow{\scriptstyle !\widetilde{\mathsf{L}}} \\
\mathsf{I} \otimes \mathsf{I} & \xrightarrow{\delta\otimes\delta} & !\mathsf{I} \otimes !\mathsf{I} & \xrightarrow{\mu^2} & !(\mathsf{I} \otimes \mathsf{I})
\end{array}
$$

that is

$$
\mathsf{pr}_0 \in \mathcal{L}^!((\mathsf{I}, \delta), \overbrace{(1, \mu^0)}^{\text{term. obj.}})
$$

$$
\widetilde{\mathsf{L}} \in \mathcal{L}^!((\mathsf{I}, \delta), \underbrace{(\mathsf{I}, \delta) \otimes (\mathsf{I}, \delta)}_{\text{cart. prod.}})
$$

## comonoid from the coalgebra

This means that we have

$$
\begin{array}{ccc}
\mathsf{I} & \xrightarrow{\delta} & !\mathsf{I} \\
\downarrow{\scriptstyle \mathsf{pr}_0} & & \downarrow{\scriptstyle !0} \\
1 & \xleftarrow{(\mathsf{m}^0)^{-1}} & !\top
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{I} & & \xrightarrow{\delta} & & !\mathsf{I} \\
\downarrow{\scriptstyle \widetilde{\mathsf{L}}} & & & & \downarrow{\scriptstyle \Delta} \\
\mathsf{I} \otimes \mathsf{I} & \xleftarrow{\mathsf{der}_\mathsf{I} \otimes \mathsf{der}_\mathsf{I}} & !\mathsf{I} \otimes !\mathsf{I} & \xleftarrow{(\mathsf{m}^2)^{-1}} & !(\mathsf{I} \,\&\, \mathsf{I})
\end{array}
$$

because $\mathcal{L}^!$ is cartesian.

**Remark**

As a consequence, a canonically summable resource category where $!\_$ is the free exponential (roughly speaking, a Lafont category which is canonically summable) has exactly one differential structure (in our sense).

Related to a result of Blute, Cockett, Lemay and Seely (in additive resource categories).

Locality corresponds to

$$\begin{array}{ccc} \mathsf{I} & \xrightarrow{\delta} & !\mathsf{I} \\ {\scriptstyle \iota_0}\uparrow & & \uparrow{\scriptstyle !\iota_0} \\ 1 & \xrightarrow{\mu^0} & !1 \end{array}$$

that is $\iota_0 \in \mathcal{L}^!((1, \mu^0), (\mathsf{I}, \delta))$.

And Schwarz straightforwardly holds.

## Remark: the Kleisli category of $\mathbf{S}_\mathsf{I}$

It turns out to be exactly the same thing as the category $\mathcal{L}[(\mathsf{I},\delta)]$ of free comodules of the coalgebra $(\mathsf{I},\delta)$.

**Theorem (Girard)**

*If $\mathcal{L}$ is a model of LL then $\mathcal{L}[(\mathsf{I},\delta)]$ is a model of LL. Very likely conjecture: it is also a summable differential model of LL.*

The objects of $\mathcal{L}[(\mathsf{I},\delta)]$ are those of $\mathcal{L}$.

$f \in \mathcal{L}[(\mathsf{I},\delta)](X,Y)$ if $f = (f_0, f_1) \in \mathcal{L}(X,Y)$ is a summable pair of morphisms. Composition:

$$(g_0, g_1)\,(f_0, f_1) = (g_0\, f_0, g_0\, f_1 + g_1\, f_0)\,.$$

Intuition: "$g_1\, f_1 = 0$", $\mathcal{L}[(\mathsf{I},\delta)]$ is a kind of infinitesimal extension of $\mathcal{L}$.

# To summarize

In the canonical case, for a closed resource category $\mathcal{L}$:

1. summability boils down to the Canonical Witness Axiom about $\mathsf{I} = 1 \,\&\, 1$ (+ the fact that $\overline{\pi}_0, \overline{\pi}_1$ are jointly epic);
2. and the differential structure boils down to a coalgebra structure on $\mathsf{I}$

such that the morphisms $\mathsf{pr}_0 \in \mathcal{L}(\mathsf{I}, 1)$, $\iota_0 \in \mathcal{L}(1, \mathsf{I})$ and $\widetilde{\mathsf{L}} \in \mathcal{L}(\mathsf{I}, \mathsf{I} \otimes \mathsf{I})$ are coalgebra morphisms.

Remember that these 3 morphisms arise from the summability assumptions.

# Concrete instance I: Coherence Spaces

A coherence space is

$$E = (|E|, \frown_E)$$

where $|E|$ is a set and $\frown_E$ is a binary symmetric and reflexive relation on $|E|$.

The domain of cliques:

$$\mathrm{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x \ a \frown_E a'\}$$

ordered by $\subseteq$.

# Morphisms

- $|E \multimap F| = |E| \times |F|$
- $(a, b) \frown_{E \multimap F} (a', b')$ if

$$a \frown_E a' \Rightarrow (b \frown_F b' \text{ and } b = b' \Rightarrow a = a')$$

And then

$$\mathbf{Coh}(E, F) = \mathrm{Cl}(E \multimap F)$$

# Some notations for **Coh**

- Identity: $\mathsf{Id}_E = \{(a,a) \mid a \in |E|\}$
- Composition: if $s \in \mathbf{Coh}(E,F)$ and $t \in \mathbf{Coh}(F,G)$ then

$$t\,s = \{(a,c) \in |E| \times |G| \mid \exists b \in |F|\ (a,b) \in s \text{ and } (b,c) \in t\} \in \mathbf{Coh}(E,G)$$

- Application to a clique: if $s \in \mathbf{Coh}(E,F)$ and $x \in \mathsf{Cl}(E)$ then $s \cdot x = \{b \in |F| \mid a \in x \text{ and } (a,b) \in s\} \in \mathsf{Cl}(F)$.

## **Coh** is cartesian

- Terminal object $\top = (\emptyset, \emptyset)$.
- Cartesian product $|E_0 \mathbin{\&} E_1| = \{0\} \times |E_0| \cup \{1\} \times |E_1|$
  $(i, a) \frown_{E_0 \& E_1} (j, b)$ if $i = j \Rightarrow a \frown_{E_i} b$.
- The projections are

  $$\mathrm{pr}_i = \{((i, a), a) \mid i \in \{0, 1\} \text{ and } a \in |E_i|\} \in \mathbf{Coh}(E_0 \mathbin{\&} E_1, E_i)\,.$$

If $t_i \in \mathbf{Coh}(F, E_i)$ then

$$\begin{aligned}\langle t_0, t_1 \rangle &= \{(b, (i, a)) \mid i \in \{0, 1\} \text{ and } (b, a) \in t_i\} \\ &\in \mathbf{Coh}(F, E_0 \mathbin{\&} E_1)\,.\end{aligned}$$

### Remark

$\mathsf{Cl}(\top) = \{\emptyset\}$ and $\mathsf{Cl}(E_0) \times \mathsf{Cl}(E_1) \simeq \mathsf{Cl}(E_0 \mathbin{\&} E_1)$ by

$$(x_0, x_1) \mapsto \{0\} \times x_0 \cup \{1\} \times x_1\,.$$

## **Coh** is monoidal closed

- Unit $1 = (\{*\}, =)$.
- Tensor product $|E_0 \otimes E_1| = |E_0| \times |E_1|$ and $(a_0, a_1) \frown_{E_0 \otimes E_1} (a'_0, a'_1)$ if $a_i \frown_{E_i} a'_i$ for $i = 0, 1$.
- If $t_i \in \mathbf{Coh}(E_i, F_i)$ for $i = 0, 1$ then

$$t_0 \otimes t_1 = \{((a_0, a_1), (b_0, b_1)) \mid (a_i, b_i) \in t_i \text{ for } i = 0, 1\} \\ \in \mathbf{Coh}(E_0 \otimes E_1, F_0 \otimes F_1)\,.$$

Monoidal closedness:

$$\mathbf{Coh}(G \otimes E, F) \simeq \mathbf{Coh}(G, E \multimap F)\,.$$

## **Coh** as a resource category

- $|!E|$ = the set of all finite multisets $m = [a_1, \ldots, a_n]$ with $a_i \in |E|$ and $\forall i, j\ a_i \frown_E a_j$. It is a uniform exponential.
- $m \frown_{!E} m'$ if $\forall a \in m, a' \in m'\ m \frown_E m'$.
- And it $t \in \mathbf{Coh}(E, F)$ then

$$
\begin{aligned}
!t = \{([a_1, \ldots, a_n], [b_1, \ldots, b_n]) \mid \\
& n \in \mathbb{N},\ [a_1, \ldots, a_n] \in |!E| \\
& \text{and } (a_i, b_i) \in t \text{ for } i = 1, \ldots, n\} \\
\in \mathbf{Coh}(!E, !F)\,.
\end{aligned}
$$

**Remark**

This is the free exponential. There is another one where $|!E|$ is made of sets instead of multisets; it is not compatible with the differential structure.

## **Coh** is canonically summable

- **Coh** has 0-morphisms: $0 = \emptyset \in \mathbf{Coh}(E, F)$.
- $\mathsf{I} = 1 \mathbin{\&} 1$ so that $|\mathsf{I}| = \{0, 1\}$ and $0 \frown_{\mathsf{I}} 1$.
- The injections $\overline{\pi}_i = \{(*, i)\} \in \mathbf{Coh}(1, \mathsf{I})$ are jointly epic.

**Remark**

$s \in \mathbf{Coh}(\mathsf{I}, E)$ is fully determined by the pair

$$s \cdot \{0\}, s \cdot \{1\} \in \mathsf{Cl}(E)$$

such that

$$s \cdot \{0\} \cap s \cdot \{1\} = \emptyset\,.$$

# The Can. Witness Axiom holds in **Coh**

Let $t_0, t_1, t \in \mathbf{Coh}(\mathsf{I}, E)$ such that

$$t_i \, \Delta = t \, \overline{\pi}_i \text{ for } i = 0, 1 \,.$$

This means $t_i \cdot \{0, 1\} = t \cdot \{i\}$ for $i = 0, 1$. That is:

$$t_0 \cdot \{0, 1\} \cup t_1 \cdot \{0, 1\} \in \mathsf{Cl}(E) \text{ and } t_0 \cdot \{0, 1\} \cap t_1 \cdot \{0, 1\} = \emptyset \,.$$

Then let $s = \{((i, j), a) \mid (i, a) \in t_j\} \subseteq |\mathsf{I} \otimes \mathsf{I} \multimap E|$, we have

$$s \in \mathbf{Coh}(\mathsf{I} \otimes \mathsf{I}, E) \,.$$

The functor $\mathbf{S}_\mathsf{I} : \mathbf{Coh} \to \mathbf{Coh}$ is given by

$$\mathbf{S}_\mathsf{I} E = (\mathsf{I} \multimap E)$$

so that $|\mathbf{S}_\mathsf{I} E| = \{0, 1\} \times |E|$ with

$$(i, a) \frown_{\mathbf{S}_\mathsf{I} E} (i', a') \text{ if } a \frown_E a' \text{ and } i \neq i' \Rightarrow a \neq a'.$$

Hence

$$\mathrm{Cl}(\mathbf{S}_\mathsf{I} E) \simeq \{(x_0, x_1) \in \mathrm{Cl}(E)^2 \mid x_0 \cup x_1 \in \mathrm{Cl}(E) \text{ and } x_0 \cap x_1 = \emptyset\}.$$

**Remark**

Two cliques $x_0, x_1$ of $E$ are summable if $x_0 \cup x_1 \in \mathrm{Cl}(E)$ and $x_0 \cap x_1 = \emptyset$. In that case we use $x_0 + x_1$ for $x_0 \cup x_1$.

The commutative comonoid structure of $\mathsf{I}$ is given by

$$
\begin{aligned}
\mathsf{pr}_0 &= \{(0,*)\} \in \mathbf{Coh}(\mathsf{I},1) \\
\widetilde{\mathsf{L}} &= \{(0,(0,0)),(1,(1,0)),(1,(0,1))\} \in \mathbf{Coh}(\mathsf{I},\mathsf{I}\otimes\mathsf{I})\,.
\end{aligned}
$$

Remember it induces the monad structure $\zeta_E \in \mathbf{Coh}(E, \mathbf{S}_\mathsf{I} E)$ and $\theta_E \in \mathbf{Coh}(\mathbf{S}_\mathsf{I}^2 E, \mathbf{S}_\mathsf{I} E)$.

As expected

$$
\begin{aligned}
\theta_E : \mathbf{S}_\mathsf{I}^2 E &\to \mathbf{S}_\mathsf{I} E \\
((x_{00}, x_{01}), (x_{10}, x_{11})) &\mapsto (x_{00}, x_{10} + x_{01})
\end{aligned}
$$

up to

$$
\mathsf{Cl}(\mathbf{S}_\mathsf{I}^2 E) \simeq \{((x_{00}, x_{01}), (x_{10}, x_{11})) \mid x_{00} + x_{01} + x_{10} + x_{11} \in \mathsf{Cl}(E)\}\,.
$$

## The differential structure of **Coh**

We define $\delta \subseteq |\mathsf{I} \multimap !\mathsf{I}|$:

$$\delta = \{(0, n[0]) \mid n \in \mathbb{N}\} \cup \{(1, n[0] + [1]) \mid n \in \mathbb{N}\}$$

where $n[a] = [\overbrace{a, \ldots, a}^{n\times}]$. It is easy to check that $\delta \in \mathbf{Coh}(\mathsf{I}, !\mathsf{I})$.

## $\delta$ is a coalgebra

The main thing to check is

$$\begin{array}{ccc} \mathsf{I} & \xrightarrow{\delta} & !\mathsf{I} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle !\delta} \\ !\mathsf{I} & \xrightarrow{\mathsf{dig}_E} & !!\mathsf{I} \end{array}$$

that is, given $i \in \{0,1\}$ and $M \in \mathcal{M}_{\text{fin}}(\mathcal{M}_{\text{fin}}(\{0,1\}))$,

$$(i, M) \in !\delta\,\delta \Leftrightarrow (i, M) \in \mathsf{dig}_{\mathsf{I}}\ \delta$$

where
$\mathsf{dig}_E = \{(m, [m_1, \ldots, m_k]) \in |!E| \times |!!E| \mid m = m_1 + \cdots + m_k\}$.

# main case

The main case is when $i = 1$.

$(1, M) \in !\delta\,\delta$ means $\exists k \in \mathbb{N}$ such that

$$(k[0] + [1], M) \in !\delta$$

that is:

$$M = [m_1, \ldots, m_{k+1}] \text{ with } (0, m_i) \in \delta \text{ for } i = 1, \ldots k$$
$$\text{and } (1, m_{k+1}) \in \delta$$

that is: $\exists k \in \mathbb{N}\, \exists n_1, \ldots, n_{k+1} \in \mathbb{N}$

$$M = [n_1[0], \ldots, n_k[0], n_{k+1}[0] + [1]]$$

And $(1, M) \in \mathrm{dig_l}\ \delta$ means $\exists k \in \mathbb{N}$ such that

$$(k[0] + [1], M) \in \mathrm{dig_l}$$

that is:

$$M = [m_1, \ldots, m_l] \text{ with } m_1 + \cdots + m_l = k[0] + [1]$$

that is: $\exists l \in \mathbb{N}^+ \, \exists n_1, \ldots, n_l \in \mathbb{N}$

$$M = [n_1[0], \ldots, n_{l-1}[0], n_l[0] + [1]]$$

The diagram commutes!

# The differential distributive law

Remember that $\delta$ induces a distributive law $\partial_E = \mathrm{cur}\, u \in \mathbf{Coh}(!\mathbf{S}_{\mathsf{I}}E, \mathbf{S}_{\mathsf{I}}!E)$ where

$$u : !(\mathsf{I} \multimap E) \otimes \mathsf{I} \to !E$$

is

$$!(\mathsf{I} \multimap E) \otimes \mathsf{I} \xrightarrow{\mathsf{Id} \otimes \delta} !(\mathsf{I} \multimap E) \otimes !\mathsf{I} \xrightarrow{\mu^2} !((\mathsf{I} \multimap E) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !E$$

Notice that $\mu^2_{E,F} \in \mathbf{Coh}(!E \otimes !F, !(E \otimes F))$ is

$$\mu^2_{E,F} = \{(([a_1, \ldots, a_n], [b_1, \ldots, b_n]), [(a_1, b_1), \ldots, (a_n, b_n)]) \mid n \in \mathbb{N},\ [a_1, \ldots, a_n] \in |!E| \text{ and } [b_1, \ldots, b_n] \in |!F|\}$$

$$!(\mathsf{I} \multimap E) \otimes !\mathsf{I} \xrightarrow{\mu^2} !((\mathsf{I} \multimap E) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !E$$

is

$$\{((([(i_1, a_1), \ldots, (i_k, a_k)], [i_1, \ldots, i_k]), [a_1, \ldots, a_k]) \mid \\ k \in \mathbb{N}, i_1, \ldots, i_k \in \{0, 1\} \text{ and } [(i_1, a_1), \ldots, (i_k, a_k)] \in |!(\mathsf{I} \multimap E)|\}$$

and $[(i_1, a_1), \ldots, (i_k, a_k)] \in |!(\mathsf{I} \multimap E)|$ means that

$$\forall j, j' \ a_j \frown_E a_{j'} \text{ and } j \neq j' \Rightarrow a_j \neq a_{j'}.$$

$$!(\mathsf{I} \multimap E) \otimes \mathsf{I} \xrightarrow{\mathsf{Id} \otimes \delta} !(\mathsf{I} \multimap E) \otimes !\mathsf{I}$$

is

$$\{((p, 0), (p, n[0])) \mid n \in \mathbb{N},\ \ p \in \mathsf{Cl}(\mathsf{I} \multimap E)\}$$
$$\cup \{((p, 1), (p, n[0] + [1])) \mid n \in \mathbb{N},\ \ p \in \mathsf{Cl}(\mathsf{I} \multimap E)\}$$

so $u =$

$$!(\mathsf{I} \multimap E) \otimes \mathsf{I} \xrightarrow{\mathsf{Id} \otimes \delta} !(\mathsf{I} \multimap E) \otimes !\mathsf{I} \xrightarrow{\mu^2} !((\mathsf{I} \multimap E) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !E$$

is

$$\begin{aligned} u = \{ & ((([(0, a_1), \ldots, (0, a_k)], 0), [a_1, \ldots, a_k]) \mid \\ & \quad k \in \mathbb{N} \text{ and } [a_1, \ldots, a_k] \in |!E|\} \\ \cup \{ & ((([(0, a_1), \ldots, (0, a_k), (1, a_{k+1})], 1), [a_1, \ldots, a_{k+1}]) \mid \\ & \quad k \in \mathbb{N} \text{ and } [a_1, \ldots, a_{k+1}] \in |!E| \text{ and } a_{k+1} \notin \{a_1, \ldots, a_k\}\} \end{aligned}$$

## Expression of $\partial_E$

$$
\begin{aligned}
\partial_E = \{ & (([(0,a_1),\dots,(0,a_k)],(0,[a_1,\dots,a_k]) \mid \\
& \quad k \in \mathbb{N} \text{ and } [a_1,\dots,a_k] \in |!E|\} \\
\cup \{ & (([(0,a_1),\dots,(0,a_k),(1,a_{k+1})],(1,[a_1,\dots,a_{k+1}]) \mid \\
& \quad k \in \mathbb{N} \text{ and } [a_1,\dots,a_{k+1}] \in |!E| \text{ and } a_{k+1} \notin \{a_1,\dots,a_k\}\} \\
\in \ & \mathbf{Coh}(!(\mathsf{I} \multimap E), \mathsf{I} \multimap !E)\,.
\end{aligned}
$$

# The Kleisli category $\mathbf{Coh}_!$

Object: those of $\mathbf{Coh}$ and $\mathbf{Coh}_!(E, F) = \mathbf{Coh}(!E, F)$.
$s \in \mathbf{Coh}_!(E, F)$ induces a stable function

$$\begin{aligned} \widehat{s} : \mathsf{Cl}(E) &\to \mathsf{Cl}(F) \\ x &\mapsto \{b \in |F| \mid \exists m \in \mathcal{M}_{\mathrm{fin}}(x)\ (m, b) \in s\} \end{aligned}$$

**Remark**

Different $s$'s can induce the same stable function: $\widehat{s}$ forgets about the multiplicities in multisets.

If $s_1 = \{([a], b)\}$ and $s_2 = \{([a, a], b)\}$ then $\widehat{s_1} = \widehat{s_2}$.

# Differentiation on $\mathbf{Coh}_!$

Given $t \in \mathbf{Coh}_!(E, F) = \mathbf{Coh}(!E, F)$, remember that

$$\mathbf{D}t = (\mathbf{S}t)\ \partial_E \in \mathbf{Coh}(!\mathbf{S}_\mathsf{I} E = (\mathsf{I} \multimap E), \mathbf{S}_\mathsf{I} !E = (\mathsf{I} \multimap !F))\,.$$

Notice that

$$\mathbf{S}t = \{((i, m), (i, b)) \mid i \in \{0, 1\} \text{ and } (m, b) \in t\}\,.$$

So

$$\begin{aligned}\mathbf{D}t = \{([(0,a_1),\ldots,(0,a_k)],(0,b)) \mid ([a_1,\ldots,a_k],b)\in t)\} \\ \cup \{([(0,a_1),\ldots,(0,a_k),(1,a_{k+1})],(1,b)) \mid \\ ([a_1,\ldots,a_k,a_{k+1}],b)\in t) \text{ and } a_{k+1}\notin\{a_1,\ldots,a_k\}\}\end{aligned}$$

# The stable derivative

Remember that

$$\mathsf{Cl}(\mathbf{S}_{\mathsf{I}}E) \simeq \{(x,u) \mid x \cup u \in \mathsf{Cl}(E) \text{ and } x \cap u = \emptyset\}\,.$$

In that way we get the stable function

$$\begin{aligned}\widehat{\mathbf{D}t} : \mathsf{Cl}(\mathbf{S}_{\mathsf{I}}E) &\to \mathsf{Cl}(\mathbf{S}_{\mathsf{I}}F)\\ (x,u) &\mapsto (\widehat{t}(x), t'(x)\cdot u)\end{aligned}$$

where

$$t'(x)\cdot u = \{b \in |F| \mid \exists m \in \mathcal{M}_{\mathrm{fin}}(x),\ a \in u\ (m+[a], b) \in t\}$$

## Remark

If $t_i = \{(i[a], a)\}$ for $i = 1, 2$ we get

$$t_1'(\emptyset) \cdot \{a\} = \{a\}$$
$$t_2'(\emptyset) \cdot \{a\} = \emptyset$$

whereas $\widehat{t_1} = \widehat{t_2}$. The derivative is not associated with the stable function itself.

In some sense this derivative "does not see mutiplicities". This can be remedied using non-uniform coherence spaces.

# Coherent Differentiation (III)

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# The joint “epicness” axiom is necessary

Remember that we required $\mathcal{L}$ to satisfy the following.

The morphisms $\overline{\pi}_0, \overline{\pi}_1 : 1 \to \mathsf{I} = 1 \,\&\, 1$ are jointly epic, that is: if $f_0, f_1 : \mathsf{I} \to X$ satisfy

$$f_0\,\overline{\pi}_i = f_1\,\overline{\pi}_i \quad \text{for} \quad i = 0, 1$$

then $f_0 = f_1$.

This is not always true.

# The category of pointed sets

Pointed set: a set $X$ together with a distinguished $0_X \in X$.

Morphisms: functions $f : X \to Y$ s.t. $f(0_X) = 0_Y$.

This category is cartesian:
$X \mathbin{\&} Y = X \times Y$ with $0_{X \& Y} = (0_X, 0_Y)$.

And monoidal closed

$$X \otimes Y = \{(x, y) \in X \times Y \mid x = 0_X \Leftrightarrow y = 0_Y\}$$

and $0_{X \otimes Y} = (0_X, 0_Y)$ (smash product). The $\otimes$-unit is $1 = \{*, 0_1\}$.

**Remark**

It is even a resource category: set $!X = \{(0, 0)\} \cup \{1\} \times X$, and $0_{!X} = (0, 0)$.

Then the injections $\overline{\pi}_0, \overline{\pi}_1 : 1 \to \mathsf{I} = 1 \mathbin{\&} 1$ are given by

$$\overline{\pi}_0(0_1) = (0_1, 0_1) \quad \overline{\pi}_0(*) = (*, 0_1)$$
$$\overline{\pi}_1(0_1) = (0_1, 0_1) \quad \overline{\pi}_1(*) = (0_1, *)$$

If $f, g : 1 \mathbin{\&} 1 \to Y$ satisfy $f\,\overline{\pi}_i = g\,\overline{\pi}_i$ for $i = 0, 1$ we can still have $f(*, *) \neq g(*, *)$.

# The CWA is necessary

# Canonical Witness Axiom

**Reminder**

Joint epicness of $\overline{\pi}_0, \overline{\pi}_1$ and CWA are the only conditions we need in the canonical case to get a summability structure.

If $f_0, f_1 \in \mathcal{L}(\mathsf{I}, X)$ are such that $f_0\,\Delta, f_1\,\Delta \in \mathcal{L}(\mathsf{I}, X)$ are summable, then so are $f_0, f_1$. That is, up to $\mathcal{L}(\mathsf{I}, X) \simeq \mathcal{L}(1, \mathsf{I} \multimap X)$:

if $f_0, f_1, f \in \mathcal{L}(\mathsf{I}, X)$ are such that

$$f_i\,\Delta = f\,\overline{\pi}_i : 1 \to X \text{ for } i = 0, 1$$

then there is $h \in \mathcal{L}(\mathsf{I} \otimes \mathsf{I}, X)$ such that

$$f_i\,\lambda = h\,(\overline{\pi}_i \otimes \mathsf{I}) \in \mathcal{L}(1 \otimes \mathsf{I}, X)$$

where $\lambda$ is the can. isom. $1 \otimes \mathsf{I} \to \mathsf{I}$.

# Normed vector spaces

The CWA doe not always hold.

Let $\mathcal{N}$ be the category

- whose objects are the finite-dimensional $\mathbb{R}$-vector spaces $V$ equipped with a norm $\|\_\|_V$
- and a morphism $f : V \to W$ is a linear map such that $\forall v \in V \; \|f(v)\|_W \leq \|v\|_V$, that is $\|f\| \leq 1$,

where

$$\|f\| = \sup_{\|v\|_V \leq 1} \|f(v)\|_W \, .$$

$\mathcal{N}$ is cartesian with $\|(u, v)\|_{V \& W} = \max(\|u\|, \|v\|)$ for $(u, v) \in V \& W = V \times W$.

$\mathcal{N}$ is an SMCC with $\|v \otimes w\|_{V \otimes W} = \|v\|\|w\|$ for $v \in V$ and $w \in W$.

The unit of $\otimes$ is $1 = \mathbb{R}$ with $\|r\| = |r|$.

$V \multimap W$ is the space of all linear maps $V \to W$ with the norm $\|f\|_{V \multimap W} = \|f\|$ already defined.

Joint epicness axiom holds in $\mathcal{N}$.

Then the functor $\mathbf{S}_{\mathsf{I}} V$ (induced by I) is given by $\mathbf{S}_{\mathsf{I}} V = (\mathsf{I} \multimap V) = V \times V$ and

$$\|(u,v)\|_{\mathbf{S}_{\mathsf{I}} V} = \sup_{a,b \in [-1,1]} \|au + bv\|_V$$

So $u, v \in V$ are summable if $\forall a, b \in [-1,1] \quad \|au + bv\|_V \leq 1$.

In $\mathbb{R}$:

- $-1/2$ and $1/2$ are summable since $\|(-1/2, 1/2)\|_{\mathbf{S}_{\mathsf{I}} \mathbb{R}} = 1$
- $-1/2 + 1/2 = 0$ and $1$ are summable
- but $1/2$ and $1$ are not summable.

$\rightsquigarrow$ the CWA does not hold in $\mathcal{N}$.

**Remark**

CWA expresses not only associativity of (partial) $+$ but also some form of positivity of the elements of $\mathcal{L}(X, Y)$.

# Recap of the differential structure

Assume that $\mathcal{L}$ is a canonically summable resource category, that is:

- $\overline{\pi}_0, \overline{\pi}_1 \in \mathcal{L}(1, \mathsf{I} = 1 \mathbin{\&} 1)$ are jointly epic
- and the CWA holds.

Remember that $\mathsf{I}$ has a commutative comonoid structure given by

$$\mathsf{pr}_0 : \mathsf{I} \to 1 \qquad \widetilde{\mathsf{L}} : \mathsf{I} \to \mathsf{I} \otimes \mathsf{I}$$

with

$$\widetilde{\mathsf{L}}\,\overline{\pi}_0 = \overline{\pi}_0 \otimes \overline{\pi}_0 \qquad \widetilde{\mathsf{L}}\,\overline{\pi}_1 = \overline{\pi}_0 \otimes \overline{\pi}_1 + \overline{\pi}_1 \otimes \overline{\pi}_0$$

Remember that $+$ is just a notation for a composition with $\Delta = \langle \mathsf{Id}_1, \mathsf{Id}_1 \rangle : 1 \to \mathsf{I}$.

# Differential structure

In this setting, a differential structure is a !-coalgebra structure $\delta \in \mathcal{L}(\mathsf{I}, !\mathsf{I})$ such that

- $\mathsf{pr}_0 \in \mathcal{L}^!((\mathsf{I}, \delta), (1, \mu^0))$
- $\widetilde{\mathsf{L}} \in \mathcal{L}^!((\mathsf{I}, \delta), (\mathsf{I}, \delta) \otimes (\mathsf{I}, \delta) = (\mathsf{I} \otimes \mathsf{I}, \mu^2\,(\delta \otimes \delta)))$
- $\overline{\pi}_0 \in \mathcal{L}^!((1, \mu^0), (\mathsf{I}, \delta))$.

**Theorem**

*If $\mathcal{L}$ is a Lafont resource category which is canonically summable, then $\mathcal{L}$ has exactly one differential structure.*

*Idea: $\delta$ is uniquely determined by $(\mathsf{pr}_0, \widetilde{\mathsf{L}})$.*

Lafont resource category: for each $X \in \mathcal{L}$, $!X$ is the free commutative comonoid "cogenerated" by $X$.

# Coherence spaces

A coherence space is

$$E = (|E|, \frown_E)$$

where $|E|$ is a set and $\frown_E$ is a binary symmetric and reflexive relation on $|E|$.

The domain of cliques:

$$\mathrm{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in |E| \; a \frown_E a'\}$$

ordered by $\subseteq$, it is a cpo.

# Morphisms

- $|E \multimap F| = |E| \times |F|$
- $(a, b) \frown_{E \multimap F} (a', b')$ if

$$a \frown_E a' \Rightarrow (b \frown_F b' \text{ and } b = b' \Rightarrow a = a')$$

And then

$$\mathbf{Coh}(E, F) = \mathrm{Cl}(E \multimap F)$$

## Some notations for **Coh**

- Identity: $\mathrm{Id}_E = \{(a,a) \mid a \in |E|\}$
- Composition: if $s \in \mathbf{Coh}(E,F)$ and $t \in \mathbf{Coh}(F,G)$ then

$$
\begin{aligned}
t\,s = \{(a,c) \in |E| \times |G| \mid \exists b \in |F| \ (a,b) \in s \text{ and } (b,c) \in t\} \\
\in \mathbf{Coh}(E,G)
\end{aligned}
$$

- Application to a clique: if $s \in \mathbf{Coh}(E,F)$ and $x \in \mathrm{Cl}(E)$ then $s \cdot x = \{b \in |F| \mid a \in x \text{ and } (a,b) \in s\} \in \mathrm{Cl}(F)$.

# **Coh** is cartesian

- Terminal object $\top = (\emptyset, \emptyset)$.
- Cartesian product $|E_0 \,\&\, E_1| = \{0\} \times |E_0| \cup \{1\} \times |E_1|$
  $(i, a) \coh_{E_0 \& E_1} (j, b)$ if $i = j \Rightarrow a \coh_{E_i} b$.
- The projections are

  $$\mathrm{pr}_i = \{((i, a), a) \mid i \in \{0, 1\} \text{ and } a \in |E_i|\} \in \mathbf{Coh}(E_0 \,\&\, E_1, E_i)\,.$$

If $t_i \in \mathbf{Coh}(F, E_i)$ then

$$\langle t_0, t_1 \rangle = \{(b, (i, a)) \mid i \in \{0, 1\} \text{ and } (b, a) \in t_i\} \\ \in \mathbf{Coh}(F, E_0 \mathbin{\&} E_1)\,.$$

## Remark

$\mathrm{Cl}(\top) = \{\emptyset\}$ and $\mathrm{Cl}(E_0) \times \mathrm{Cl}(E_1) \simeq \mathrm{Cl}(E_0 \mathbin{\&} E_1)$ by

$$(x_0, x_1) \mapsto \{0\} \times x_0 \cup \{1\} \times x_1\,.$$

## **Coh** is monoidal closed

- Unit $1 = (\{*\}, =)$.
- Tensor product $|E_0 \otimes E_1| = |E_0| \times |E_1|$ and $(a_0, a_1) \coh_{E_0 \otimes E_1} (a'_0, a'_1)$ if $a_i \coh_{E_i} a'_i$ for $i = 0, 1$.
- If $t_i \in \mathbf{Coh}(E_i, F_i)$ for $i = 0, 1$ then

$$t_0 \otimes t_1 = \{((a_0, a_1), (b_0, b_1)) \mid (a_i, b_i) \in t_i \text{ for } i = 0, 1\} \\ \in \mathbf{Coh}(E_0 \otimes E_1, F_0 \otimes F_1)\,.$$

Monoidal closedness:

$$\mathbf{Coh}(G \otimes E, F) \simeq \mathbf{Coh}(G, E \multimap F)\,.$$

## **Coh** as a resource category

- $|!E|$ = the set of all finite multisets $m = [a_1, \ldots, a_n]$ with $a_i \in |E|$ and $\forall i, j\ a_i \coh_E a_j$. It is a uniform exponential.
- $m \coh_{!E} m'$ if $\forall a \in m, a' \in m'\ m \coh_E m'$.
- And it $t \in \mathbf{Coh}(E, F)$ then

$$
\begin{aligned}
!t = \{([a_1, \ldots, a_n], [b_1, \ldots, b_n]) \mid \\
& n \in \mathbb{N},\ [a_1, \ldots, a_n] \in |!E| \\
& \text{and } (a_i, b_i) \in t \text{ for } i = 1, \ldots, n\} \\
\in \mathbf{Coh}(!E, !F)\,.
\end{aligned}
$$

**Remark**

This is the free exponential. There is another one where $|!E|$ is made of sets instead of multisets; it is not compatible with the differential structure.

# **Coh** is canonically summable

- **Coh** has 0-morphisms: $0 = \emptyset \in \mathbf{Coh}(E, F)$.
- $\mathsf{I} = 1 \,\&\, 1$ so that $|\mathsf{I}| = \{0, 1\}$ and $0 \frown_{\mathsf{I}} 1$.
- The injections $\overline{\pi}_i = \{(*, i)\} \in \mathbf{Coh}(1, \mathsf{I})$ are jointly epic.

**Remark**

$s \in \mathbf{Coh}(\mathsf{I}, E)$ is fully determined by the pair

$$s_0 = s \cdot \{0\}, s_1 = s \cdot \{1\} \in \mathsf{Cl}(E)$$

Moreover, since $0 \frown_E 1$ (which means $0 \supset_E 1$ and $0 \neq 1$) we have

$$s_0 \cup s_1 \in \mathsf{Cl}(E) \quad \text{and} \quad s_0 \cap s_1 = \emptyset$$

Conversely if $x_0, x_1 \in \mathsf{Cl}(E)$ satisfy $x_0 \cup x_1 \in \mathsf{Cl}(E)$ and $x_0 \cap x_1 = \emptyset$ then

$$(\{0\} \times x_0) \cup (\{1\} \times x_1) \in \mathbf{Coh}(\mathsf{I}, E)$$

# Summability in **Coh**

We have seen that:

**Fact**

$x_0, x_1 \in Cl(E)$ *are summable in* $E$ *iff*

$$x_0 \cup x_1 \in Cl(E) \quad \textit{and} \quad x_0 \cap x_1 = \emptyset\,.$$

**Remark**

Each model of LL has its own notion of summability.

# The CWA holds in **Coh**

Up to iso:

$$\mathrm{Cl}(\mathbf{S}_{\mathsf{I}}E) = \{(x_0, x_1) \in \mathrm{Cl}(E)^2 \mid x_0 \cup x_1 \in \mathrm{Cl}(E) \text{ and } x_0 \cap x_1 = \emptyset\}$$

**Remark**

Up to this iso, we have

$$\emptyset = (\emptyset, \emptyset)$$
$$(x_{00}, x_{01}) \cup (x_{10}, x_{11}) = (x_{00} \cup x_{10}, x_{10} \cup x_{11})$$
$$(x_{00}, x_{01}) \cap (x_{10}, x_{11}) = (x_{00} \cap x_{10}, x_{10} \cap x_{11})\,.$$

## Summability in $\mathbf{S}_\mathsf{I} E$

So $(x_{00}, x_{01}), (x_{10}, x_{11}) \in \mathsf{Cl}(\mathbf{S}_\mathsf{I} E)$ are summable in $\mathbf{S}_\mathsf{I} E$ if

$$(x_{00} \cup x_{10}, x_{10} \cup x_{11}) \in \mathsf{Cl}(\mathbf{S}_\mathsf{I} E)$$
$$(x_{00} \cap x_{10}, x_{10} \cap x_{11}) = (\emptyset, \emptyset)$$

That is

$$x_{00} \cup x_{10} \cup x_{10} \cup x_{11} \in \mathsf{Cl}(E)$$
$$(x_{00} \cup x_{10}) \cap (x_{10} \cup x_{11}) = x_{00} \cap x_{10} = x_{10} \cap x_{11} = \emptyset$$

that is $(i, j) \neq (i', j') \Rightarrow x_{ij} \cap x_{i'j'} = \emptyset$.

Assume that

- $(x_{00}, x_{01}), (x_{10}, x_{11}) \in \mathsf{Cl}(\mathbf{S}_{\mathsf{I}} E)$ and
- $(x_{00} \cup x_{01}, x_{10} \cup x_{11}) \in \mathsf{Cl}(\mathbf{S}_{\mathsf{I}} E)$.

Then

- $x_{00} \cap x_{01} = x_{10} \cap x_{11} = \emptyset$
- $(x_{00} \cup x_{01}) \cap (x_{10} \cup x_{11}) = \emptyset$
- $x_{00} \cup x_{01} \cup x_{10} \cup x_{11} \in \mathsf{Cl}(\mathbf{S}_{\mathsf{I}} E)$

that is

- $x_{00} \cup x_{01} \cup x_{10} \cup x_{11} \in \mathsf{Cl}(\mathbf{S}_{\mathsf{I}} E)$
- $(i, j) \neq (i', j') \Rightarrow x_{ij} \cap x_{i'j'} = \emptyset$.

that is $(x_{00}, x_{01}), (x_{10}, x_{11}) \in \mathsf{Cl}(\mathbf{S}_{\mathsf{I}} E)$ are summable in $\mathbf{S}_{\mathsf{I}} E$.

We already know that **Coh** has a unique differential structure wrt. !.

The commutative comonoid structure of $\mathsf{I}$ is given by

$$\mathsf{pr}_0 = \{(0,*)\} \in \mathbf{Coh}(\mathsf{I}, 1)$$
$$\widetilde{\mathsf{L}} = \{(0,(0,0)),(1,(1,0)),(1,(0,1))\} \in \mathbf{Coh}(\mathsf{I}, \mathsf{I}\otimes\mathsf{I})\,.$$

Remember it induces the monad structure $\zeta_E \in \mathbf{Coh}(E, \mathbf{S}_\mathsf{I} E)$ and $\theta_E \in \mathbf{Coh}(\mathbf{S}_\mathsf{I}^2 E, \mathbf{S}_\mathsf{I} E)$.

As expected for $((x_{00}, x_{01}),(x_{10}, x_{11})) \in \mathsf{Cl}(\mathbf{S}_\mathsf{I}^2 E)$ we have

$$\theta \cdot ((x_{00}, x_{01}),(x_{10}, x_{11})) = (x_{00}, x_{10} + x_{01}) \in \mathsf{Cl}(\mathbf{S}_\mathsf{I} E)$$

# The differential structure of **Coh**

We define $\delta \subseteq |\mathsf{I} \multimap !\mathsf{I}|$:

$$\delta = \{(0, n[0]) \mid n \in \mathbb{N}\} \cup \{(1, n[0] + [1]) \mid n \in \mathbb{N}\}$$

where $n[a] = [\overbrace{a, \ldots, a}^{n\times}]$.

$\delta \in \mathbf{Coh}(\mathsf{I}, !\mathsf{I})$ because

- $m \frown_{!\mathsf{I}} m'$ for all $m, m' \in |!\mathsf{I}|$
- and $n[0] \frown_{!E} n'[0] + [1]$ for all $n, n' \in \mathbb{N}$.

# $\delta$ is a coalgebra

The main thing to check is

$$\begin{array}{ccc} \mathsf{I} & \xrightarrow{\delta} & !\mathsf{I} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle !\delta} \\ !\mathsf{I} & \xrightarrow{\mathsf{dig}_{\mathsf{I}}} & !!\mathsf{I} \end{array}$$

that is, given $i \in \{0, 1\}$ and $M \in \mathcal{M}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{fin}}(\{0, 1\}))$,

$$(i, M) \in !\delta\, \delta \Leftrightarrow (i, M) \in \mathsf{dig}_{\mathsf{I}}\ \delta$$

where
$\mathsf{dig}_E = \{(m, [m_1, \ldots, m_k]) \in |!E| \times |!!E| \mid m = m_1 + \cdots + m_k\}$.

## main case

The main case is when $i = 1$.

$(1, M) \in {!\delta}\,\delta$ means $\exists k \in \mathbb{N}$ such that

$$(k[0] + [1], M) \in {!\delta}$$

that is:

$$M = [m_1, \ldots, m_{k+1}] \text{ with } (0, m_i) \in \delta \text{ for } i = 1, \ldots k \\ \text{and } (1, m_{k+1}) \in \delta$$

that is: $\exists k \in \mathbb{N}\, \exists n_1, \ldots, n_{k+1} \in \mathbb{N}$

$$M = [n_1[0], \ldots, n_k[0], n_{k+1}[0] + [1]]$$

And $(1, M) \in \mathsf{dig_l}\ \delta$ means $\exists k \in \mathbb{N}$ such that

$$(k[0] + [1], M) \in \mathsf{dig_l}$$

that is:

$$M = [m_1, \ldots, m_l] \text{ with } m_1 + \cdots + m_l = k[0] + [1]$$

that is: $\exists l \in \mathbb{N}^+ \, \exists n_1, \ldots, n_l \in \mathbb{N}$

$$M = [n_1[0], \ldots, n_{l-1}[0], n_l[0] + [1]]$$

The diagram commutes!

# The induced differential dist. law

Remember that $\delta$ induces a distributive law $\partial_E = \mathrm{cur}\, u \in \mathbf{Coh}(!\mathbf{S}_{\mathsf{I}} E, \mathbf{S}_{\mathsf{I}} !E)$ where

$$u : !(\mathsf{I} \multimap E) \otimes \mathsf{I} \to !E$$

is

$$!(\mathsf{I} \multimap E) \otimes \mathsf{I} \xrightarrow{\mathsf{Id} \otimes \delta} !(\mathsf{I} \multimap E) \otimes !\mathsf{I} \xrightarrow{\mu^2} !((\mathsf{I} \multimap E) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !E$$

Notice that $\mu^2_{E,F} \in \mathbf{Coh}(!E \otimes !F, !(E \otimes F))$ is

$$\mu^2_{E,F} = \{(([a_1, \ldots, a_n], [b_1, \ldots, b_n]), [(a_1, b_1), \ldots, (a_n, b_n)]) \mid n \in \mathbb{N},\ [a_1, \ldots, a_n] \in |!E| \text{ and } [b_1, \ldots, b_n] \in |!F|\}$$

$$
\begin{aligned}
\partial_E = &\{(([(0,a_1),\dots,(0,a_k)],(0,[a_1,\dots,a_k]) \mid \\
&\qquad k \in \mathbb{N} \text{ and } [a_1,\dots,a_k] \in |!E|\} \\
&\cup \{(([(0,a_1),\dots,(0,a_k),(1,a_{k+1})],(1,[a_1,\dots,a_{k+1}]) \mid \\
&\qquad k \in \mathbb{N} \text{ and } [a_1,\dots,a_{k+1}] \in |!E| \text{ and } a_{k+1} \notin \{a_1,\dots,a_k\}\} \\
&\in \mathbf{Coh}(!(\mathsf{I} \multimap E), \mathsf{I} \multimap !E)\,.
\end{aligned}
$$

# The Kleisli category $\mathbf{Coh}_!$

Object: those of $\mathbf{Coh}$ and $\mathbf{Coh}_!(E, F) = \mathbf{Coh}(!E, F)$.
$s \in \mathbf{Coh}_!(E, F)$ induces a stable function

$$\begin{aligned} \widehat{s} : \mathsf{Cl}(E) &\to \mathsf{Cl}(F) \\ x &\mapsto \{b \in |F| \mid \exists m \in \mathcal{M}_{\mathrm{fin}}(x)\ (m, b) \in s\} \end{aligned}$$

**Remark**

Different $s$'s can induce the same stable function: $\widehat{s}$ forgets about the multiplicities in multisets.

If $s_1 = \{([a], b)\}$ and $s_2 = \{([a, a], b)\}$ then $\widehat{s_1} = \widehat{s_2}$.

# Differentiation on $\mathbf{Coh}_!$

Given $t \in \mathbf{Coh}_!(E, F) = \mathbf{Coh}(!E, F)$, remember that

$$\mathbf{D}t = (\mathbf{S}t)\ \partial_E \in \mathbf{Coh}(!\mathbf{S}_\mathsf{I} E = (\mathsf{I} \multimap E), \mathbf{S}_\mathsf{I} F = (\mathsf{I} \multimap F))\,.$$

Notice that

$$\mathbf{S}t = \{((i, m), (i, b)) \mid i \in \{0, 1\} \text{ and } (m, b) \in t\}\,.$$

So for $t \in \mathcal{L}_!(E, F) = \mathbf{Coh}(!E \multimap F)$ we have

$$
\begin{aligned}
\mathbf{D}t = & \{([(0, a_1), \dots, (0, a_k)], (0, b)) \mid ([a_1, \dots, a_k], b) \in t)\} \\
& \cup \{([(0, a_1), \dots, (0, a_k), (1, a_{k+1})], (1, b)) \mid \\
& \qquad ([a_1, \dots, a_k, a_{k+1}], b) \in t) \text{ and } a_{k+1} \notin \{a_1, \dots, a_k\}\} \\
& \in \mathcal{L}_!(\mathbf{S_I}E, \mathbf{S_I}F) = \mathbf{Coh}(!(\mathbf{I} \multimap E) \multimap (\mathbf{I} \multimap F)) \,.
\end{aligned}
$$

# The stable derivative

Remember that

$$\mathsf{Cl}(\mathbf{S}_{\mathsf{I}}E) \simeq \{(x, u) \mid x \cup u \in \mathsf{Cl}(E) \text{ and } x \cap u = \emptyset\}.$$

In that way we get the stable function

$$\begin{aligned} \widehat{\mathbf{D}t} : \mathsf{Cl}(\mathbf{S}_{\mathsf{I}}E) &\to \mathsf{Cl}(\mathbf{S}_{\mathsf{I}}F) \\ (x, u) &\mapsto (\widehat{t}(x), t'(x) \cdot u) \end{aligned}$$

$$t'(x) \cdot u = \{b \in |F| \mid \exists m \in \mathcal{M}_{\mathrm{fin}}(x),\ a \in u\ (m + [a], b) \in t\}$$

## Remark

In such an $(m, a)$ we have $a \notin \mathrm{supp}(m)$ since $\mathrm{supp}(m) \subseteq x$, $a \in u$ and $x \cap u = \emptyset$.

# Local coherence space

Given $x \in \mathrm{Cl}(X)$, one defines a coherence space $E_x$ by

- $|E_x| = \{b \in |E| \mid \forall a \in x \; a \frown_E b\}$
- $a \coh_{E_x} a'$ if $a \coh_E a'$.

Then for $t \in \mathbf{Coh}(E, F)$ we have

$$t'(x) \in \mathbf{Coh}(E_x, F_{\widehat{t}(x)})$$

**Remark**

There is a dependent type intuition: the type of $t'(x)$ depends on $x$.

However this point of view hardly reflects the stability of $t'(x)$ wrt. $x$.

Whereas the compound construction $\mathbf{D}t$ does in a very simple way.

## Remark

If $t_i = \{(i[a], a)\}$ for $i = 1, 2$ we get

$$t_1'(\emptyset) \cdot \{a\} = \{a\}$$
$$t_2'(\emptyset) \cdot \{a\} = \emptyset$$

whereas $\widehat{t_1} = \widehat{t_2}$. The derivative stable function $\widehat{\mathbf{D}t}$ is associated with $t$ and not the stable function $\widehat{t}$.

In some sense this derivative "does not see mutiplicities". This is due to the uniformity of the exponential. NB: there are non-uniform coherence spaces...

# Probabilistic Coherence Spaces

# Probabilistic Coherence Spaces (PCS)

$$X = (|X|, \mathsf{P}X)$$

- $|X|$ is a set (usually at most countable)
- $\mathsf{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$
- $\forall a \in |X|\ 0 < \sup_{x \in \mathsf{P}X} x_a < \infty$
- $\mathsf{P}X$ is $\downarrow$-closed (for the pointwise order)
- $\mathsf{P}X$ contains the (pointwise) lub of any increasing $\omega$-sequence in $\mathsf{P}X$
- $x, y \in \mathsf{P}X$ and $\lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda)y \in \mathsf{P}X$

# Morphisms

$X \multimap Y$ defined by:

- $|X \multimap Y| = |X| \times |Y|$
- and $t \in (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$ is in $\mathsf{P}(X \multimap Y)$ if

$$\forall x \in \mathsf{P}X \quad t \cdot x \in \mathsf{P}Y$$

where $t \cdot x = (\sum_{a \in |X|} t_{a,b} x_a)_{b \in |Y|} \in (\mathbb{R}_{\geq 0})^{|Y|}$.

**Fact**

*$X \multimap Y$ so defined is a PCS.*

$$\mathbf{Pcoh}(X, Y) = \mathsf{P}(X \multimap Y)\,.$$

# Notations

- If $s \in \mathbf{Pcoh}(X, Y)$ and $t \in \mathbf{Pcoh}(Y, Z)$ then $t\,s \in \mathbf{Pcoh}(X, Z)$ given by

$$(t\,s)_{a,c} = \sum_{b \in |Y|} s_{a,b} t_{b,c}$$

- $\mathrm{Id}_X \in \mathbf{Pcoh}(X, X)$ is $(\delta_{a,a'})_{(a,a') \in |X \multimap X|}$.

This defines a category.

# Cartesian product

- Terminal object $\top$ such that $|\top| = \emptyset$.
- $|X_0 \mathbin{\&} X_1| = \{0\} \times |X_0| \cup \{1\} \times |X_1|$ so that $(\mathbb{R}_{\geq 0})^{|X_0 \& X_1|} \simeq (\mathbb{R}_{\geq 0})^{|X_0|} \times (\mathbb{R}_{\geq 0})^{|X_1|}$
- $\mathsf{pr}_i \in (\mathbb{R}_{\geq 0})^{|X_0 \& X_1| \times |X_i|}$ given by

$$(\mathsf{pr}_i)_{(j,a),a'} = \delta_{i,j}\delta_{a,a'}$$

- $y \in (\mathbb{R}_{\geq 0})^{|X_0 \& X_1|}$ is in $\mathsf{P}(X_0 \mathbin{\&} X_1)$ if $\mathsf{pr}_i \cdot y \in \mathsf{P}X_i$ for $i = 0, 1$.
- If $t_i \in \mathbf{Pcoh}(Y, X_i)$ for $i = 0, 1$ then $\langle t_0, t_1 \rangle \in \mathbf{Pcoh}(Y, X_0 \mathbin{\&} X_1)$ is given by $\langle t_0, t_1 \rangle_{b,(i,a)} = (t_i)_{a,b}$.

**Remark**

$\mathsf{P}(X_0 \mathbin{\&} X_1) \simeq \mathsf{P}X_0 \times \mathsf{P}X_1$

Up to this iso, the cartesian product is completely standard:

- $\mathsf{pr}_i \cdot (x_0, x_1) = x_i$
- $\langle t_0, t_1 \rangle \cdot y = (t_0 \cdot y, t_1 \cdot y)$ for $t_i \in \mathbf{Pcoh}(Y, X_i)$

# Tensor product

Given $x_i \in \mathrm{P}X_i$ for $i = 0, 1$, let $x_0 \otimes x_1 \in (\mathbb{R}_{\geq 0})^{|X_0| \times |X_1|}$ given by

$$(x_0 \otimes x_1)_{(a_0, a_1)} = {x_0}_{a_0} {x_1}_{a_1}$$

- $|X_0 \otimes X_1| = |X_0| \times |X_1|$
- $\mathrm{P}(X_0 \otimes X_1)$ minimal such that $x_0 \otimes x_1 \in \mathrm{P}(X_0 \otimes X_1)$ for all $x_i \in \mathrm{P}X_i$ for $i = 0, 1$.

**Fact**

$\mathbf{Pcoh}(Z \otimes X, Y) \simeq \mathbf{Pcoh}(Z, X \multimap Y)$.

# The object $\mathsf{I}$

$1 = (\{*\}, [0,1])$. Notice that $\mathsf{P}(1 \multimap X) \simeq \mathsf{P}X$

$\mathsf{I} = 1 \,\&\, 1$ so that $\mathsf{PI} = [0,1] \times [0,1]$

$\overline{\pi}_0, \overline{\pi}_1 \in \mathbf{Pcoh}(1, \mathsf{I}) \simeq \mathsf{PI}$, actually $\overline{\pi}_0 = (1,0)$ and $\overline{\pi}_1 = (0,1)$.

**Fact**

$\overline{\pi}_0, \overline{\pi}_1$ *are jointly monic:*

*by linearity,* $t \in \mathbf{Pcoh}(\mathsf{I}, X)$ *is fully determined by*
$t_0 = t \cdot (1,0) \in \mathsf{P}X$ *and* $t_1 = t \cdot (0,1) \in \mathsf{P}X$.

*Moreover* $t_0 + t_1 \in \mathsf{P}X$ *since* $t_0 + t_1 = t \cdot (1,1)$ *since* $(1,1) \in \mathsf{PI}$.

$\mathsf{P}(\mathsf{I} \multimap X) \simeq \{(x_0, x_1) \in \mathsf{P}X \mid x_0 + x_1 \in \mathsf{P}X\}$.

# Canonical Witness Axiom

Up to this iso we have

$$(x_{00}, x_{01}) + (x_{10}, x_{11}) = (x_{00} + x_{10}, x_{01} + x_{11})$$

and so

$$\begin{aligned}((x_{00}, x_{01}), (x_{10}, x_{11})) \in \mathsf{P}\mathbf{S}_{\mathsf{I}}^2 X &\Leftrightarrow (x_{00} + x_{10}, x_{01} + x_{11}) \in \mathsf{P}\mathbf{S}_{\mathsf{I}} X \\ &\Leftrightarrow x_{00} + x_{10} + x_{01} + x_{11} \in \mathsf{P}X \\ &\Leftrightarrow (x_{00} + x_{01}, x_{10} + x_{11}) \in \mathsf{P}\mathbf{S}_{\mathsf{I}} X\end{aligned}$$

and hence the CWA holds.

The induced monad $\mathbf{S_I} : \mathbf{Pcoh} \to \mathbf{Pcoh}$ given by $\mathbf{S_I}X = (\mathsf{I} \multimap X)$ behaves exactly as expected:

$$\zeta_X \in \mathbf{Pcoh}(X, \mathbf{S_I}X) \qquad \zeta_X \cdot x = (x, 0)$$

$$\theta_X \in \mathbf{Pcoh}(\mathbf{S_I}^2 X, \mathbf{S_I}X) \quad \theta_X \cdot ((x_{00}, x_{01}), (x_{10}, x_{11})) = (x_{00}, x_{10} + x_{01})$$

# The differentiation coalgebra

It is defined essentially as in **Coh**, and is a coalgebra for the same reason.

$$\delta \in (\mathbb{R}_{\geq 0})^{|\mathsf{I} \multimap !\mathsf{I}|}$$

defined by

$$\delta_{i,m} = \begin{cases} 1 & \text{if } i = 0 \text{ and } \exists n \in \mathbb{N}\ m = n[0] \\ 1 & \text{if } i = 1 \text{ and } \exists n \in \mathbb{N}\ m = n[0] + [1] \\ 0 & \text{otherwise} \end{cases}$$

**Remark (Surprise)**

The case $i = 1$ implements the differential, so I expected to have stgh like $n$ as coeff. instead of 1. But it's not the case!

## The exponential functor

- $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ (no uniformity restriction).
- If $x \in \mathsf{P}X$ and $m \in |!X|$ then $x^m = \prod_{a \in |X|} x_a^{m(a)} \in \mathbb{R}_{\geq 0}$
- $x^! = (x^m)_{m \in |!X|}$
- and $\mathsf{P}(!X)$ is minimal such that $\forall x \in \mathsf{P}X \; x^! \in \mathsf{P}(!X)$.

Given $t \in \mathbf{Pcoh}(X, Y)$ we need $!t \in \mathbf{Pcoh}(!X, !Y)$ such that

$$\forall x \in \mathsf{P}X \quad !t \cdot x^! = (t \cdot x)^!$$

**Fact**

*This determines fully $!t$.*

Simple computations give, for $t \in \mathbf{Pcoh}(X, Y) \subseteq (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$ and $(m, p) \in |!X \multimap !Y| = \mathcal{M}_{\mathrm{fin}}(!X) \times \mathcal{M}_{\mathrm{fin}}(!Y)$:

$$(!t)_{m,p} = \sum_{r \in L(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} t^r$$

where

$$L(m, p) = \{r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|) \mid \sum_{b \in |Y|} r(a, b) = m(a) \text{ and } \sum_{a \in |X|} r(a, b) = p(b)\}$$

and

$$\begin{bmatrix} p \\ r \end{bmatrix} = \prod_{b \in |Y|} \frac{p(b)!}{\prod_{a \in |X|} r(a, b)!} \in \mathbb{N}$$

The evaluation morphism

$$\mathrm{ev} \in \mathbf{Pcoh}((\mathsf{I} \multimap X) \otimes \mathsf{I}, X) \quad \mathrm{ev}_{((i,a),j),b} = \delta_{a,b}\delta_{i,j}$$

Then $!\mathrm{ev}_{M,m} \neq 0$ implies

$$M = [((0,a_1),0),\ldots,((0,a_k),0),((1,b_1),1),\ldots,((1,b_n),1)] = (l,r)$$
$$m = [a_1,\ldots,a_k,b_1,\ldots,b_n] = l + r$$

Setting $l = [a_1,\ldots,a_k]$ and $r = [b_1,\ldots,b_n]$. We have

$$!\mathrm{ev}_{M,m} = \binom{l+r}{l} = \prod_{a\in|X|} \binom{l(a)+r(a)}{l(a)}$$

# The differential functor

Remember that $\partial_X = \mathrm{cur}\, f \in \mathcal{L}(!\mathbf{S}_{\mathsf{I}}X, \mathbf{S}_{\mathsf{I}}!X)$ where $f$ is

$$!(\mathsf{I} \multimap X) \otimes \mathsf{I} \xrightarrow{\mathrm{Id}\otimes\delta} !(\mathsf{I} \multimap X) \otimes !\mathsf{I} \xrightarrow{\mu^2} !((\mathsf{I} \multimap X) \otimes \mathsf{I}) \xrightarrow{!\mathrm{ev}} !X$$

Using the above computation of !ev and definition of $\delta$ we get

$$(\partial_X)_{(l,r),(i,m)} = \begin{cases} 1 & \text{if } i = 0,\ r = [\,],\ m = l \\ m(a) & \text{if } i = 1,\ r = [a],\ m = l + [a] \\ 0 & \text{otherwise.} \end{cases}$$

A $t \in \mathbf{Pcoh}_!(X, Y) = \mathrm{P}(!X \multimap Y)$ is completely characterized by the associated analytic function

$$\begin{aligned} \widehat{t} : \mathrm{P}X &\to \mathrm{P}Y \\ x &\mapsto t \cdot x^! = \left( \sum_{m \in |!X|} t_{m,b} x^m \right)_{b \in |Y|} \end{aligned}$$

Then $\mathbf{D}t \in \mathbf{Pcoh}_!(\mathbf{S}_\mathsf{I} X, \mathbf{S}_\mathsf{I} Y)$ is characterized by the analytic function (setting $f = \widehat{t} : \mathsf{P}X \to \mathsf{P}Y$)

$$\begin{aligned} \widehat{\mathbf{D}t} : \mathsf{P}(\mathbf{S}_\mathsf{I} X) &\to \mathsf{P}(\mathbf{S}_\mathsf{I} Y) \\ (x, u) &\mapsto (f(x), f'(x) \cdot u) \end{aligned}$$

where

$$f'(x) \cdot u = \Big( \sum_{a \in |X|} \Big( \sum_{l \in |!X|} (m(a) + 1) t_{m+[a],b} x^m \Big) u_a \Big)_{b \in |Y|}$$

is just the standard differential of $\widehat{t}$.

## Differential as a linear map

Given $x \in \mathsf{P}X$ we define $X_x$, the local PCS at $x$:

$$|X_x| = \{a \in |X| \mid \exists\varepsilon > 0 \; x + \varepsilon e_a \in \mathsf{P}X\}$$
$$\mathsf{P}X_x = \{u \in (\mathbb{R}_{\geq 0})^{|X_x|} \mid x + u \in \mathsf{P}X\}$$

and then $f'(x) \in \mathbf{Pcoh}(X_x, Y_{f(x)})$ satisfies (for $b \in |Y_{f(x)}|$)

$$(f'(x) \cdot u)_b = \left(\frac{d}{dt} f(x + tu)_b\right)_{t=0} \qquad \text{standard Gateaux derivative.}$$

The fact that $\mathbf{D}t \in \mathbf{Pcoh}_!(\mathbf{S_l}X, \mathbf{S_l}Y)$ also tells us that this derivative is analytic in $x$.

# Strong similarity with Tangent Categories

**Mfd**: category of smooth manifolds and smooth maps.

There is a *tangent bundle* functor

$$\begin{aligned} \mathbf{T} : \mathbf{Mfd} &\to \mathbf{Mfd} \\ X &\mapsto \{(x, u) \mid x \in X \text{ and } u \in \mathbf{T}_x X\} \end{aligned}$$

$\mathbf{T}_x X =$ tangent space at $x$ to $X$. A vector space.

And if $f \in \mathbf{Mfd}(X, Y)$,

$$\begin{aligned} \mathbf{T}f : \mathbf{T}X &\to \mathbf{T}Y \\ (x, u) &\mapsto (f(x), f'(x) \cdot u) \end{aligned}$$

Looks very much like our **D** functor!

## Discrepancies

Is $\mathbf{T}$ a special case of $\mathbf{D}$?

Of course not: given $(x, u) \in \mathbf{T}X$, it makes no sense to consider $u$ alone (no 2nd projection $\mathbf{T}X \to X$) nor to compute $x + u \in X$ in general.

Is $\mathbf{D}$ a special case of $\mathbf{T}$?

No, because our "tangent spaces" are only partial commutative monoids whereas $\mathbf{T}_x X$ is crucially a commutative monoid.

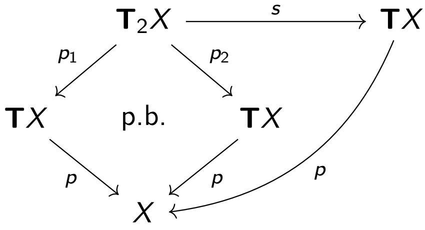
**Remark**

More philosophically, our approach is based on $\mathbf{S}$ acting on a "linear" category (a category of algebraic objects, the linear category of a model of LL).

This is typically not the case in the tangent bundle case.

More precisely, in *tangent categories* (= categorical axiomatization of the tangent bundle functor) we have a natural transformation $p_X : \mathbf{T}X \to X$, intuitively $p_X(x, u) = x$.

It is required that there is a pull-back and an *addition morphism* $s$




This $s$ is a total addition operation in the fibers of $p$.