Integration in Positive Cones

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Motivations

- Design a general semantics of functional programming languages with probabilistic choice.
- Based on Kozen's idea of (1st order) programs as probability distribution transformers.
- Extending probabilistic coherence spaces, which work only for "discrete" data-types.
- And featuring *continuous* data-types (like $\mathbb{R}$).
- This requires a general notion of integration of "paths" valued in any object to represent sampling.

Selinger's positive cones

Cones are objects which have at the same time

- algebraic and analytic features: sum, scalar multiplication, norm (similar to normed vector spaces)
- order-theoretic features: positivity assumptions $\rightsquigarrow$ order structure. + Scott completeness assumption.

Cones can be shown to be Cauchy-complete metric spaces, but we use Scott completeness to interpret general recursion $\rightsquigarrow$ *no need to restrict to contractive maps*.

Cones and measure theory

Cones fit very well with the basic ideas of Measure Theory:

- Basic measures are $\overline{\mathbb{R}_{\geq 0}}$-valued, and satisfy a Scott-continuity requirement wrt. inclusion of measurable sets (besides finite additivity).
- Central result of MT, the monotone convergence theorem deals with $\mathbb{R}_{\geq 0}$-valued measurable maps and is very much order-theoretic.
- Any $\mathbb{R}_{\geq 0}$-valued measurable function is the pointwise lub of an ω-indexed sequence of simple functions: this gives a very easy definition of integration (Lebesgue).

Simple function: measurable function taking only finitely many different values.

We consider $\mathbb{R}_{\geq 0}$ as a commutative monoid for 0 and $+$.

A $\mathbb{R}_{\geq 0}$-semimodule is a set P with:

- a commutative monoid structure $(0, +)$
- a bilinear scalar multiplication $\mathbb{R}_{\geq 0} \times P \to P$ mapping (λ, x) to λx with $1x = x$ and $(\lambda\mu)x = \lambda(\mu x)$.

Definition

P is

- positive if $x + y = 0 \Rightarrow x = 0$
- cancellative if $x + y = x' + y \Rightarrow x = x'$.

Algebraic order, partially defined subtraction

If P is positive and cancellative, it is (partially) ordered by: $x \leq y$ if $\exists z \in P\ x + z = y$. This z is unique: $z = y - x$.

Definition of cones

A cone is a positive and cancellative $\mathbb{R}_{\geq 0}$-semimodule P equipped with a function $\|\_\|_P : P \to \mathbb{R}_{\geq 0}$ (its norm) such that

- $\|\lambda x\| = \lambda \|x\|$ (hence $\|0\| = 0$)
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|x\| = 0 \Rightarrow x = 0$
- $\|x\| \leq \|x + y\|$, that is $x \leq y \Rightarrow \|x\| \leq \|y\|$
- if $(x_n)_{n=1}^{\infty}$ is monotone and $\forall n \, \|x_n\| \leq 1$ then $\sup_n x_n$ exists in P and $\|\sup_n x_n\| \leq 1$. NB: ω-sequences, not arbitrary directed sets, because we need the monotone conv. thm.

Fact

Addition, scalar multiplication and the norm are Scott-continuous (= monotone and commute with lubs of monotone bounded ω-sequences).

The cone of finite measures

Let $\mathcal{X}$ be a measurable space with σ-algebra $\sigma_{\mathcal{X}}$.

The set of all $\mathbb{R}_{\geq 0}$-valued measures on $\mathcal{X}$ is a cone, with

- algebraic operations defined "pointwise":
  $(\mu + \nu)(U) = \mu(U) + \nu(U)$ for all $U \in \sigma_{\mathcal{X}}$
- and $\|\mu\| = \mu(\mathcal{X})$.

Notice that $\mu \leq \nu$ means $\forall U \in \sigma_{\mathcal{X}}\ \mu(U) \leq \nu(U)$.

Notation: $\underline{\mathrm{Meas}(\mathcal{X})}$

Linear and continuous morphisms

If P, Q are cones, a function $f : P \to Q$ is linear if $f(\lambda x) = \lambda f(x)$ and $f(x_1 + x_2) = f(x_1) + f(x_2)$. Notice that f is monotone.

Fact

Such a function is bounded:
$\exists \lambda \in \mathbb{R}_{\geq 0} \forall x \in P \ \ \|x\| \leq 1 \Rightarrow \|f(x)\| \leq \lambda$.
We set $\|f\| = \sup_{\|x\| \leq 1} \|f(x)\|$.

We say that f is continuous if it is Scott-continuous, that is:

$((x_n)_{n=1}^{\infty}$ monotone and $\forall n \, \|x_n\| \leq 1) \Rightarrow f(\sup_n x_n) = \sup_n f(x_n)$.

Finite kernels as linear and cont. maps

$\mathcal{X}, \mathcal{Y}$ measurable spaces, $\kappa : \mathcal{X} \rightsquigarrow \mathcal{Y}$ a finite kernel, that is:

- $\kappa : \mathcal{X} \to \underline{\mathrm{Meas}(\mathcal{Y})}$
- and for all $V \in \sigma_{\mathcal{Y}}$ the function $\boldsymbol{\lambda} r \in \mathcal{X} \cdot \kappa(r)(V) : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is measurable and bounded.

We can define

$$f : \underline{\mathrm{Meas}(\mathcal{X})} \to \underline{\mathrm{Meas}(\mathcal{Y})}$$
$$\mu \mapsto \boldsymbol{\lambda} V \in \sigma_{\mathcal{Y}} \cdot \int_{\mathcal{X}} \kappa(r)(V) \mu(dr)$$

which is linear and continuous.

If $f : \underline{\mathrm{Meas}(\mathcal{X})} \to \underline{\mathrm{Meas}(\mathcal{Y})}$ is linear and continuous, we can define

$$\begin{aligned} \kappa : \mathcal{X} &\to \underline{\mathrm{Meas}(\mathcal{Y})} \\ r &\mapsto f(\underbrace{\boldsymbol{\delta}^{\mathcal{X}}(r)}_{\text{Dirac measure at } r}) \end{aligned}$$

but

- given $V \in \sigma_{\mathcal{Y}}$ the map $r \mapsto f(\boldsymbol{\delta}^{\mathcal{X}}(r))(V)$ has no reason to be measurable
- and even if it is measurable, it is not necessarily true that $f(\mu)(V) = \int f(\boldsymbol{\delta}^{\mathcal{X}}(r))(V)\mu(dr)$.

Fact

We need further measurability and integrability conditions on linear and continuous maps.

Measurability structures

A reference category of arities

We assume to be given a set $\mathbf{ar}$ of *arities* with $0 \in \mathbf{ar}$ and $+ : \mathbf{ar} \times \mathbf{ar} \to \mathbf{ar}$ and for each $a \in \mathbf{ar}$, we assume to be given a measurable space $\overline{a}$ with $\overline{0} = \{*\}$ and $\overline{a+b} = \overline{a} \times \overline{b}$.

We consider $\mathbf{ar}$ as a small cartesian category, with

$$\mathbf{ar}(a, b) = \{\varphi : \overline{a} \to \overline{b} \mid \varphi \text{ is measurable}\}\,.$$

Measurability structure on a cone

Let P be a cone. A *measurability structure* on P is a family $\mathcal{M} = (\mathcal{M}_a)_{a \in \mathrm{ar}}$ such that

- $\mathcal{M}_a \subseteq (P')^{\overline{a}}$ where P' is the set of linear and continuous maps $P \to \mathbb{R}_{\geq 0}$, in particular: $\mathcal{M}_0 \subseteq P'$, if $m \in \mathcal{M}_a$, $m : \overline{a} \times P \to \mathbb{R}_{\geq 0}$ linear and continuous in the second argument;
- if $m \in \mathcal{M}_a$ and $x \in P$ with $\|x\| \leq 1$ the function $\boldsymbol{\lambda} r \in \overline{a} \cdot m(r, x)$ is measurable $\overline{a} \to [0, 1]$;
- if $m \in \mathcal{M}_b$ and $\varphi \in \mathbf{ar}(a, b)$ then $m \circ \varphi \in \mathcal{M}_a$, in particular $\mathcal{M}_0 \subseteq \mathcal{M}_b$;

- if $x, y \in P$ satisfy $\forall m \in \mathcal{M}_0 \; m(x) = m(y)$ then $x = y$ (*separation*);
- if $x \in P$ then $\|x\| = \sup\{\frac{m(x)}{\|m\|} \mid m \in \mathcal{M}_0 \setminus \{0\}\}$

$\|m\| = \sup_{\|y\|_P \leq 1} m(y)$ so $\forall y \in P \; m \in \mathcal{M}_0 \setminus \{0\} \; \frac{m(y)}{\|m\|} \leq \|y\|$.

The elements of $\mathcal{M}_a$ are the *measurability tests of arity a*.

Measurable cone

A *measurable cone* is a pair $C = (\underline{C}, \mathcal{M}^C)$ where

- $\underline{C}$ is a cone
- and $\mathcal{M}^C = (\mathcal{M}^C_a)_{a \in \mathsf{ar}}$ is a measurability structure on $\underline{C}$.

If $a \in \mathbf{ar}$, a (measurable) *a-path* of C is a bounded map $\gamma : \overline{a} \to \underline{C}$ such that, for all $b \in \mathbf{ar}$ and $m \in \mathcal{M}^C_b$, the function

$$\boldsymbol{\lambda}(s, r) \in \overline{b + a} \cdot m(s, \gamma(r)) : \overline{b + a} \to \mathbb{R}_{\geq 0}$$

is measurable.

Measurable cones as QBSs

Equipped with these paths $\underline{C}$ can be considered as a quasi Borel space, but the algebraic structure of $\underline{C}$ is also important for us.

Integrals

If $\gamma : \overline{a} \to \underline{C}$ is a measurable a-path and $\mu \in \underline{\mathrm{Meas}(\overline{a})}$ of C, an integral of γ over μ is an $x \in \underline{C}$ such that for all $m \in \mathcal{M}_0^C$, one has (NB: $m \circ \gamma : \overline{a} \to \mathbb{R}_{\geq 0}$ is measurable and bounded):

$$\overbrace{\int_{\overline{a}} m(\gamma(r))\mu(dr)}^{\text{well-defined Lebesgue integral } \in \mathbb{R}_{\geq 0}} \qquad = m(x)\,.$$

If x exists, it is unique by separation, notation

$$x = \int_{\overline{a}} \gamma(r)\mu(dr)\,.$$

Pettis integral

This is very similar to the definition of a *Pettis integral* (1938) for a function from a measurable space to a topological vector space with good separation properties, typically a locally convex tvs. Aka. Gelfand-Pettis or weak integral.

Our objects: integrable cones

A measurable cone C is integrable if for any $a \in \mathbf{ar}$, $\gamma : \overline{a} \to \underline{C}$ measurable path and $\mu \in \underline{\mathrm{Meas}(\overline{a})}$, the path γ is integrable over μ, that is

$$\int \gamma(r)\mu(dr) \in \underline{C}$$

exists.

Linear morphisms of integrable cones

Given integrable cones C, D, a linear and continuous $f : \underline{C} \to \underline{D}$ is

- *measurable* if $f \circ \gamma$ is a measurable path $\overline{a} \to \underline{D}$ for any $a \in \mathbf{ar}$ and any measurable path $\gamma : \overline{a} \to \underline{C}$
- *integrable* if, moreover, for any $\mu \in \underline{\mathrm{Meas}(\overline{a})}$, one has

$$f\left(\int \gamma(r)\mu(dr)\right) = \int f(\gamma(r))\mu(dr)\,.$$

The linear category of integrable cones

Definition

ICones is the category

- whose objects are the integrable cones
- morphisms: $f \in \mathbf{ICones}(C, D)$ if $f : \underline{C} \to \underline{D}$ is linear, continuous, integrable and $\|f\| = \sup_{\|x\|_C \leq 1} \|f(x)\|_D \leq 1$.

The integrable cone of finite measures

For $a \in \mathbf{ar}$ we define the integrable cone $\mathrm{Meas}(a)$:

- the underlying cone is $\underline{\mathrm{Meas}(\overline{a})}$, the cone of finite measures on the measurable space $\overline{a}$
- for $b \in \mathbf{ar}$, $\mathcal{M}_b^{\mathrm{Meas}(a)} = \mathcal{M}_0^{\mathrm{Meas}(a)} = \{\widetilde{U} \mid U \in \sigma_{\overline{a}}\}$ where $\widetilde{U}(\mu) = \mu(U)$ for all $\mu \in \underline{\mathrm{Meas}(\overline{a})}$.

Fact

The measurable paths $\kappa : \overline{b} \to \underline{\mathrm{Meas}(a)}$ are exactly the finite kernels. All such paths are integrable, with, for all $\nu \in \underline{\mathrm{Meas}(\overline{b})}$ and $U \in \sigma_{\overline{a}}$:

$$\left(\int_{\overline{b}} \kappa(s)\nu(ds)\right)(U) = \int_{\overline{b}} \kappa(s)(U)\nu(ds)\,.$$

The integrable cone of paths

For $a \in \mathbf{ar}$ and for an integrable cone C we define $\mathrm{Path}(a, C)$ as follows:

- $\underline{\mathrm{Path}(a, C)}$ is the cone of measurable paths $\overline{a} \to \underline{C}$, operations defined pointwise and $\|\gamma\| = \sup_{r \in \overline{a}} \|\gamma(r)\| \in \mathbb{R}_{\geq 0}$ since $\gamma \in \underline{\mathrm{Path}(a, C)}$ is bounded;
- and for $b \in \mathbf{ar}$,

$$\mathcal{M}_b^{\mathrm{Path}(a,C)} = \{\varphi \triangleright m \mid \varphi \in \mathbf{ar}(b, a) \text{ and } m \in \mathcal{M}_b^C\}$$

where $\varphi \triangleright m = \boldsymbol{\lambda}(s, \gamma) \in \overline{b} \times \underline{\mathrm{Path}(a, C)} \cdot m(s, \gamma(\varphi(s)))$.

Paths of paths and their integral

So $\eta : \overline{b} \to \underline{\mathrm{Path}(a, C)}$, that is $\eta : \overline{b} \times \overline{a} \to \underline{C}$, is a measurable path if for all $c \in \mathbf{ar}$, $\varphi \in \mathbf{ar}(c, b)$ and $m \in \mathcal{M}_c^C$, the function

$$\boldsymbol{\lambda}(t, r) \in \overline{c} \times \overline{a} \cdot m(t, \eta(\varphi(t), r)) : \overline{c} \times \overline{a} \to \mathbb{R}_{\geq 0}$$

is measurable.

Given $\nu \in \underline{\mathrm{Meas}(b)}$, $\int \eta(s)\nu(ds) \in \underline{\mathrm{Path}(a, C)}$ exists and is given by

$$\left(\int \eta(s)\nu(ds)\right)(r) = \int \eta(s, r)\nu(ds) \in \underline{C}\,.$$

Fubini

So given $\mu \in \underline{\mathrm{Meas}(a)}$, the integral

$$\int_{\overline{a}} \left(\int_{\overline{b}} \eta(s)\nu(ds) \right)(r)\mu(dr) \in \underline{C}$$

is well defined.

Fubini theorem for cones

$$\begin{aligned}\int_{\overline{a}} \left(\int_{\overline{b}} \eta(s)\nu(ds) \right)(r)\mu(dr) &= \int_{\overline{b}} \left(\int_{\overline{a}} \eta'(r)\mu(dr) \right)(s)\nu(ds) \\ &= \iint_{\overline{a},\overline{b}} \eta(s,r)\mu(dr)\nu(ds)\end{aligned}$$

where $\eta' \in \underline{\mathrm{Path}(a, \mathrm{Path}(b, C))}$ given by $\eta'(r)(s) = \eta(s)(r)$.

The cone of linear morphisms

Given integrable cones C, D, we define a cone $C \multimap D$ by

- $\underline{C \multimap D}$ is the cone of linear, Scott-continuous, measurable and integrable linear maps $\underline{C} \to \underline{D}$, algebraic operations defined pointwise and $\|f\|_{C \multimap D} = \sup_{\|x\|_C \leq 1} \|f(x)\|_D$;
- and

$$\mathcal{M}_a^{C \multimap D} = \{\gamma \triangleright m \mid \gamma \in \underline{\mathsf{Path}(a, C)} \text{ and } m \in \mathcal{M}_a^D\}$$

where $\gamma \triangleright m = \boldsymbol{\lambda}(r, f) \in \overline{a} \times \underline{C \multimap D} \cdot m(r, f(\gamma(r)))$.

Fact

This cone is integrable, if $\eta \in \underline{\mathsf{Path}(a, C \multimap D)}$ *then*

$$\forall \mu \in \underline{\mathsf{Meas}(a)} \ \int \eta(r)\mu(dr) = \boldsymbol{\lambda} x \in \underline{C} \cdot \int \eta(r)(x)\mu(dr) \in \underline{C \multimap D}$$

Tensor product in **ICones**

The SAFT applies!

The category **ICones** is locally small and

- is complete (it has all small products and all equalizers);
- has a cogenerator, namely the cone $1 = \mathbb{R}_{\geq 0}$, that is if $f, g \in \textbf{ICones}(C, D)$ satisfy $m\,f = m\,g$ for all $m \in \textbf{ICones}(C, 1)$, then $f = g$ by the separation condition on the measurability structure;
- and is well-powered, that is the class of subobjects of any object C is essentially small. Because a subobject of C is, up to iso, a cone structure on a subset of $\underline{C}$.

By the Special Adjoint Functor Theorem (SAFT)

Any functor **ICones** $\to \mathcal{C}$ which preserves all limits has a left adjoint.

Tensor product

Given an object C of **ICones** the functor $C \multimap \_$ preserves all limits. So it has a left adjoint $\_ \otimes C$.

Since $\_ \multimap \_ : \mathbf{ICones}^{\mathrm{op}} \times \mathbf{ICones} \to \mathbf{ICones}$, we have $\_ \otimes \_ : \mathbf{ICones} \times \mathbf{ICones} \to \mathbf{ICones}$.

Fact

The natural bijection $\mathbf{ICones}(B \otimes C, D) \to \mathbf{ICones}(B, C \multimap D)$ *induced by this adjunction is actually an iso in*

$$\mathbf{ICones}(B \otimes C \multimap D, B \multimap (C \multimap D))\,.$$

Consequence

$(\mathbf{ICones}, 1, \otimes)$ is a symmetric monoidal closed category.

Bilinear maps and the tensor product

A map $f : \underline{C} \times \underline{D} \to \underline{B}$ is bilinear if it is separately linear, Scott continuous, measurable in the sense that if $\gamma \in \underline{\mathsf{Path}(a, C)}$ and $\delta \in \underline{\mathsf{Path}(a, D)}$ then $f \circ \langle \gamma, \delta \rangle \in \underline{\mathsf{Path}(a, B)}$ and separately integrable in the sense that

$$
\begin{aligned}
f\Big(\int \gamma(r)\mu(dr), y\Big) &= \int f(\gamma(r), y)\mu(dr) \\
f\Big(x, \int \delta(r)\mu(dr)\Big) &= \int f(x, \delta(r))\mu(dr)\,.
\end{aligned}
$$

Fact

There is a bilinear $\tau : \underline{C} \times \underline{D} \to \underline{C \otimes D}$ which is universal in the sense that for any bilinear $f : \underline{C} \times \underline{D} \to \underline{B}$ there is exactly one $\widetilde{f} \in \mathbf{ICones}(C \otimes D, B)$ such that $f = \widetilde{f}\,\tau$.
Notation $x \otimes y = \tau(x, y)$. One has $\|x \otimes y\| = \|x\|\,\|y\|$.

Integration is a bilinear map

The function

$$\mathcal{I} : \underline{\mathrm{Path}(a, C)} \times \underline{\mathrm{Meas}(a)} \to \underline{C}$$
$$(\gamma, \mu) \mapsto \int \gamma(r)\mu(dr)$$

is bilinear. Let $I \in \mathbf{ICones}(\mathrm{Path}(a, C) \otimes \mathrm{Meas}(a), C)$ be such that

$$I(\gamma \otimes \mu) = \int \gamma(r)\mu(dr)\,.$$

Fact

The "Curry transpose" of I,
$\mathrm{cur}(I) \in \mathbf{ICones}(\mathrm{Path}(a, C), \mathrm{Meas}(a) \multimap C)$, *is an iso.*

If $f \in \underline{\mathrm{Meas}(a) \multimap C}$, the associated $\gamma \in \mathrm{Path}(a, C)$ is given by

$$\gamma(r) = f(\boldsymbol{\delta}^{\overline{a}}(r)).$$

Because f is integrable, we have

$$\int \gamma(r)\mu(dr) = \int f(\boldsymbol{\delta}^{\overline{a}}(r))\mu(dr) = f\left(\int \boldsymbol{\delta}^{\overline{a}}(r)\mu(dr)\right) = f(\mu)$$

for all $\mu \in \underline{\mathrm{Meas}(a)}$.

As a consequence an element of **ICones**(Meas(a), Meas(b)) is the same thing as a sub-probability kernel $\overline{a} \rightsquigarrow \overline{b}$.

Consequence

The category whose objects are the mesurable spaces $\overline{a}$ (for $a \in \mathbf{ar}$) and whose morphisms are the subprobability kernels is a full subcategory of **ICones**.

Analytic morphisms

Homogeneous polynomials

We define n-linear maps

$$g : \underline{C} \times \cdots \times \underline{C} \to \underline{D}$$

as we did for bilinear maps.

A function $h : \underline{C} \to \underline{D}$ is n-homogeneous polynomial if there is such a g, with

$$h(x) = g(\overbrace{x, \ldots, x}^{n \text{ times}}) .$$

Fact

If we assume that g is symmetric, it is possible to recover it from h by the Polarization Formula.

Analytic functions

$\mathcal{B}\underline{C} = \{x \in \mathcal{B}\underline{C} \mid \|x\|_C \leq 1\}$ the "unit ball".

A function $f : \mathcal{B}\underline{C} \to \underline{D}$ is analytic if

- f is bounded, that is $\exists \lambda \in \mathbb{R}_{\geq 0} \, \forall x \in \mathcal{B}C \;\; \|f(x)\| \leq \lambda$;
- there is a family $(f_n : \underline{C} \to \underline{D})_{n \in \mathbb{N}}$ of functions such that f_n is n-homogeneous polynomial and

$$\forall x \in \mathcal{B}\underline{C} \quad f(x) = \sum_{n=0}^{\infty} f_n(x) = \sup_{N \in \mathbb{N}} \sum_{n=0}^{N} f_n(x)\,;$$

- and for any $a \in \mathbf{ar}$ and $\gamma \in \mathcal{B}\underline{\mathrm{Path}(a, C)}$, one has $f \circ \gamma \in \underline{\mathrm{Path}(a, D)}$.

Taylor expansion of an analytic function

Fact

If f is analytic then the f_n's are uniquely determined by f, so there are uniquely determined symmetric multilinear functions $\mathrm{D}_0^{(n)} f : \underline{C}^n \to \underline{D}$ *such that*

$$\forall n \in \mathbb{N}\ f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{D}_0^{(n)} f(x, \ldots, x)\,.$$

Examples of analytic and non-analytic function

$1 = \mathrm{Meas}(0)$ is the integrable cone $\mathbb{R}_{\geq 0}$ with $\|u\| = u$. So $\mathcal{B}1 = [0,1]$.

$f : [0,1] \to \mathbb{R}_{\geq 0}$ is analytic if $f(u) = \sum_{n=0}^{\infty} a_n u^n$ with $a_n \in \mathbb{R}_{\geq 0}$ and $\sum_{n=0}^{\infty} a_n < \infty$.

e^{u-1} and $1 - \sqrt{1-u}$ are analytic functions. The second one cannot be extended to $\mathbb{R}_{\geq 0}$.

$2u - u^2$, $\sin(\frac{\pi}{2} u)$ and $k(u) = \sqrt{u}$ are not.

All these functions are smooth and monotone $[0,1] \to [0,1]$.

The cone of analytic functions

The set of all analytic functions $\mathcal{B}\underline{C} \to \underline{D}$ with

- algebraic operations defined pointwise
- $\|f\| = \sup_{x \in \mathcal{B}\underline{C}} \|f(x)\|_D$
- measurability structure defined as for $C \multimap D$

is an integrable cone $C \Rightarrow_{\mathrm{a}} D$. Integrals are defined "pointwise" as in $C \multimap D$.

Warning: stable order

If $f, g : \mathcal{B}\underline{C} \to \underline{D}$ are analytic then $f \leq g \Rightarrow \forall x \in \mathcal{B}\underline{C}\ f(x) \leq g(x)$ but the converse is not true. It is true for linear morphisms.

The CCC of analytic functions

Fact

If $f \in \underline{C \Rightarrow_{\mathrm{a}} D}$ *with* $\|f\| \leq 1$ *and* $g \in \underline{D \Rightarrow_{\mathrm{a}} E}$ *then* $g \circ f \in \underline{C \Rightarrow_{\mathrm{a}} E}$.

One defines a category **ACones**:

- objects are the integrable cones
- $\mathbf{ACones}(C, D) = \{f \in \underline{C \Rightarrow_{\mathrm{a}} D} \mid \|f\| \leq 1\}$
- identities and composition as in **Set**.

Fact

ACones *is a CCC, evaluation and curryfication defined as in* **Set**.

The analytic exponential comonad

If $f \in \mathbf{ICones}(C, D)$ then by restricting f to $\mathcal{B}\underline{C}$ one gets an analytic function $f \in \mathbf{ACones}(C, D)$.

This defines a functor $\mathrm{Der} : \mathbf{ICones} \to \mathbf{ACones}$.

Der preserves all limits and hence has a left adjoint $\mathrm{E_a} : \mathbf{ACones} \to \mathbf{ICones}$, by the SAFT.

$!^{\mathrm{a}}\_ = \mathrm{E_a} \circ \mathrm{Der} : \mathbf{ICones} \to \mathbf{ICones}$ is therefore a resource comonad.

Fact

ICones *is a model of ILL, with and* $\mathbf{ICones}_{!^{\mathrm{a}}} \simeq \mathbf{ACones}$.

Scott continuity of analytic maps

Any $f \in \mathbf{ACones}(C, D)$ is monotone and Scott-continuous:

- if $x_1, x_2 \in \mathcal{B}\underline{C}$ and $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$;
- and if $(x_n)_{n=1}^{\infty}$ monotone in $\mathcal{B}\underline{C}$ then $f(\sup_{n=1}^{\infty} x_n) = \sup_{n=1}^{\infty} f(x_n)$.

As a consequence (using cartesian closedness) for any integrable cone C there is

$$\mathcal{Y} \in \mathbf{ACones}(C \Rightarrow_{\mathrm{a}} C, C) \quad \text{with} \quad \mathcal{Y}(f) = \sup_{n=0}^{\infty} f^n(0)\,.$$

And hence we have a model of general recursion.

Another consequence of the SAFT

The category **ICones** has also all small colimits.

They seem difficult to describe explicitly (especially for coequalizers), contrariliy to limits which are easy.

Measurable spaces as $!^a$-coalgebras

The universal analytic function

By the adjunction $\mathsf{E_a} \dashv \mathsf{Der}$ we have

$$\mathsf{an}_C \in \mathbf{ACones}(C, !^{\mathsf{a}} C)$$

such that for all $f \in \mathbf{ACones}(C, D)$ there is exactly one $\widetilde{f} \in \mathbf{ICones}(!^{\mathsf{a}} C, D)$ such that

$$f = \widetilde{f} \circ \mathsf{an}_C$$

Notation: if $x \in \mathcal{B}\underline{C}$, then $x^{!_{\mathsf{a}}} = \mathsf{an}_C(x)$, so that $f(x) = \widetilde{f}(x^{!})$.

Let $a \in \mathbf{ar}$.

We have a measurable Dirac path $\boldsymbol{\delta}^a \in \mathcal{B}\underline{\mathsf{Path}(a, \mathsf{Meas}(a))}$ such that $\boldsymbol{\delta}^a(r)$ is the Dirac measure at r.

Hence $\mathsf{an}_{\mathsf{Meas}(a)} \circ \boldsymbol{\delta}^a \in \underline{\mathsf{Path}(a, !^{\mathsf{a}}\mathsf{Meas}(a))}$.

Remember that we have an iso
$\Phi \in \mathbf{ICones}(\mathsf{Path}(a, !^{\mathsf{a}}\mathsf{Meas}(a)), \mathsf{Meas}(a) \multimap !^{\mathsf{a}}\mathsf{Meas}(a))$ given by

$$\Phi(\alpha)(\mu) = \int_{\overline{a}} \alpha(r)\mu(dr)\,.$$

Let $\mathsf{h}_a = \Phi(\mathsf{an}_{\mathsf{Meas}(a)} \circ \boldsymbol{\delta}^a) \in \mathbf{ICones}(\mathsf{Meas}(a), !^{\mathsf{a}}\mathsf{Meas}(a))$

$$\mathsf{h}_a(\mu) = \int \boldsymbol{\delta}^a(r)^! \mu(dr)$$

A functor from measurable spaces to coalgebras

Fact

For any $a \in \mathbf{ar}$, $(\underline{\mathrm{Meas}(a)}, \mathrm{h}_a)$ *is a coalgebra for the* $!^{\mathrm{a}}\_$ *comonad.*

If $\varphi : \overline{a} \to \overline{b}$ is a measurable function we have the push-forward map on measures

$$\varphi_* : \underline{\mathrm{Meas}(a)} \to \underline{\mathrm{Meas}(b)}$$
$$\mu \mapsto \boldsymbol{\lambda} V \in \sigma_{\overline{V}} \cdot \mu(\varphi^{-1}(V))\,.$$

Fact

φ_* *is a coalgebra morphism* $(\underline{\mathrm{Meas}(a)}, \mathrm{h}_a) \to (\underline{\mathrm{Meas}(b)}, \mathrm{h}_b)$.

Conversely, if $f : (\mathrm{Meas}(a), \mathrm{h}_a) \to (\mathrm{Meas}(b), \mathrm{h}_b)$ is a coalgebra morphism, it is not always true that $f = \varphi_*$ for some measurable $\varphi : \overline{a} \to \overline{b}$. But

Fact

If b is a Polish space equipped with its Borelian σ-algebra then, for any coalgebra morphism $f : (\mathrm{Meas}(a), \mathrm{h}_a) \to (\mathrm{Meas}(b), \mathrm{h}_b)$ there is exacly one measurable function $\varphi : \overline{a} \to \overline{b}$ such that $f = \varphi_*$.

So, under the reasonable assumption that all the $\overline{a}$'s are Polish spaces, the Eilenberg-Moore category of $!^{\mathrm{a}}\_$ contains as a full subcategory the category **ar**

- whose objects are the $a \in$ **ar**
- and morphisms are the measurable functions.

Intuitively the Eilenberg-Moore category $\mathbf{ICones}^{!^a}$ is the category of data-types or positive formulas.

$\mathbf{ICones}^{!^a}$ is cartesian with $\otimes$ as cartesian product.

The full and faithful embedding of **ar** into $\mathbf{ICones}^{!^a}$ respects cartesian products because

$$\mathrm{Meas}(a + b) \simeq \mathrm{Meas}(a) \otimes \mathrm{Meas}(b)\,.$$

So if all the $\overline{a}$'s are Polish spaces we can consider **ar** (with measurable functions as morphisms) as a category of data-types and value preserving functions.

Concluding remarks

- **ICones** also contains the category of probabilistic coherence spaces and linear morphisms as a full subcategory.
- We conjecture that $!^{a}\_$ is the free exponential.
- Integrable cones seem to provide a very flexible setting for the semantics of probabilistic programming languages: completeness, cocompleteness, ILL, fixpoints operators at all types, Polish spaces as data-types etc.